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VANISHING THEORMS FOR TYPE $(0, q)$ COHOMOLOGY OF LOCALLY SYMMETRIC SPACES II

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1. Introduction

Let G/K be a Hermitian symmetric space where G is a connected non-compact semisimple Lie group and $K \subset G$ is a maximal compact subgroup. We fix a discrete subgroup Γ of G which acts freely on G/K and for which the quotient $X=\Gamma\backslash G/K$ is compact. Let $E_\tau \to G/K$ be a homogeneous C^∞ vector bundle over G/K induced by a finite-dimensional irreducible representation τ of K. Then E_τ has a holomorphic structure and one can define a presheaf by assigning to an open set U in X the abelian group of Γ-invariant holomorphic sections of E_τ on the inverse image (under the map $G/K \to X$) of U in G/K. Let $\theta_\tau \to X$ be the sheaf generated by this presheaf and let $H^q(X, \theta_\tau)$ denote the qth cohomology space of X with coefficients in θ_τ. In this paper we continue the program initiated in [23] of obtaining some general vanishing theorems for the spaces $H^q(X, \theta_\tau)$ by the application of recent representation-theoretic results. This allows for a unified view-point and one by which, in particular, the classical vanishing theorems of [3], [4], [5], [6], [7], [12], and [13] may be deduced.

Following Hotta and Murakami [4] we represent $H^q(X, \theta_\tau)$ as a space of automorphic forms. Then its dimension can be expressed by a formula of Matsushima and Murakami [14] in terms of certain irreducible unitary representations π of G, the multiplicity of π in $L^2(\Gamma\backslash G)$, and the K intertwining number of π with $\mathrm{Ad}_+^q \otimes \tau$ where Ad_+^q is the qth exterior power of the adjoint representation of K on the space of holomorphic tangent vectors at the origin of G/K. Based on results of Kumaresan [9], Parthasarathy [17], and Vogan [21], we have been able to obtain in [23] and [24] a clearer understanding of the structure of the unitary representations π of G in the Matsushima-Murakami formula; also see Theorem 3.3 of the present paper. We apply this new knowledge in conjunction with the Matsushima-Murakami formula to deduce the main result of this paper, which is Theorem 4.3. We can deduce, in particular, results of [23] from Theorem 4.3 *without assuming the linearity of G.* Thus we drop the linearity assumption in the present paper, which was enforced in [23].

2. Unitary representations intertwining $\chi^{\pm}\otimes\tau_{\Lambda+\delta_n}$

In this section G will denote a non-compact connected semisimple Lie group with finite center and $K\subset G$ will denote a maximal compact subgroup of G. However, proceeding more generally, we shall *not* assume that G/K is Hermitian symmetric (until later). Let $\mathfrak{g}_0=\mathfrak{k}_0+\mathfrak{p}_0$ be a Cartan decomposition of the Lie algebra $\mathfrak{g}_0$ of G, where $\mathfrak{k}_0$ is the Lie algebra of K and $\mathfrak{p}_0$ is the orthogonal complement of $\mathfrak{k}_0$ relative to the Killing form $(\ ,\ )$ of $\mathfrak{g}_0$. Let $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{p}$ denote, respectively, the complexifications of $\mathfrak{g}_0$, $\mathfrak{k}_0$, $\mathfrak{p}_0$. We shall assume throughout that $\mathfrak{k}$ contains a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$; i.e. we assume G and K have the same rank. This will be the case in particular when G/K is Hermitian. Let Δ be the set of non-zero roots of $(\mathfrak{g}, \mathfrak{h})$, let Δ_k, Δ_n denote the compact, non-compact roots respectively in Δ, let $\Delta^+\subset\Delta$ be an arbitrary choice of a system of positive roots, let $\Delta_k^+=\Delta^+\cap\Delta_k$, $\Delta_n^+=\Delta^+\cap\Delta_n$, and let $2\delta=\langle\Delta^+\rangle$, $2\delta_k=\langle\Delta_k^+\rangle$, $2\delta_n=\langle\Delta_n^+\rangle$, where we write $\langle\Phi\rangle=\sum_{\alpha\in\Phi}\alpha$ for $\Phi\subset\Delta$. Let $\mathscr{F}$ denote the integral linear forms Λ on $\mathfrak{h}$; i.e. $\Lambda\in\mathfrak{h}^*$ (the dual space of $\mathfrak{h}$) satisfies: $\frac{2(\Lambda, \alpha)}{(\alpha, \alpha)}$ is an integer for each α in Δ. We define

$$(2.1)\qquad \mathscr{F}_0' = \{\Lambda\in\mathscr{F}\,|\,(\Lambda+\delta, \alpha)\neq 0 \text{ for } \alpha \text{ in } \Delta \text{ and } (\Lambda+\delta, \alpha)>0 \text{ for } \alpha \text{ in } \Delta_k^+\}\ .$$

Let $\mathfrak{g}_\alpha$ be the (one dimensional) root space of $\alpha\in\Delta$. Given $\Lambda\in\mathscr{F}_0'$ $\Lambda+\delta_n$ is the highest weight with respect to Δ_k^+ of an irreducible representation $\tau_{\Lambda+\delta_n}$ of $\mathfrak{k}$. The Killing form of $\mathfrak{g}_0$ induces a real inner product on $\mathfrak{p}_0$ and since $\mathfrak{p}_0$ is even-dimensional (because G and K are of equal rank) the spin representation σ of $\mathfrak{so}(\mathfrak{p}_0)$ has a decomposition $\sigma=\sigma^+\oplus\sigma^-$ into two irreducible representations $\sigma^{\pm}$. Let

$$(2.2)\qquad \chi^{\pm} = \sigma^{\pm}\circ(\mathrm{ad}\,\mathfrak{k}_0)\,|\,\mathfrak{p}_0$$

where $(\mathrm{ad}\,\mathfrak{k}_0)\,|\,\mathfrak{p}_0$ is the adjoint representation of $\mathfrak{k}_0$ on $\mathfrak{p}_0$. Then $\chi^{\pm}\otimes\tau_{\Lambda+\delta_n}$ always integrates to a representation of K (which we shall denote by the same symbol) for $\Lambda\in\mathscr{F}_0'$ even though $\tau_{\Lambda+\delta_n}$ may not. Let Ω denote the Casimir operator of G and let $\hat{G}$ denote the equivalence classes of irreducible unitary representations (π, H_π) of G on a Hilbert space H_π. Given $\Lambda\in\mathscr{F}_0'$ we shall want to pin down the structure of a $(\pi, H_\pi)\in\hat{G}$ such that $\pi(\Omega)=(\Lambda, \Lambda+2\delta)1$ and such that $\mathrm{Hom}_K(\pi, \chi^{\pm}\otimes\tau_{\Lambda+\delta_n})\neq 0$. Here H_π also denotes the space of K finite vectors in H_π which is regarded as a $U\mathfrak{g}$ module where $U\mathfrak{g}$ is the universal enveloping algebra of $\mathfrak{g}$; thus $\pi(\Omega)$ is well-defined. We shall need the following additional notation. If $\theta\subset\mathfrak{g}$ is a parabolic subalgebra we shall write $\theta=\mathfrak{m}+\mathfrak{u}$ for its Levi decomposition where $\mathfrak{m}$ and $\mathfrak{u}$ denote the reductive and nilpotent parts respectively of θ, $\Delta(\mathfrak{m})$ for the roots of $\mathfrak{m}$, $\theta_{u,n}$ for the set of non-compact roots in the nilpotent radical $\mathfrak{u}$, M for the closed Lie subgroup of G whose complexified Lie algebra is $\mathfrak{m}$, and we shall write $2\delta_{u,n}=\langle\theta_{u,n}\rangle$. Let c: $\mathfrak{g}_0\to\mathfrak{g}_0$ denote the Cartan involution for the Cartan decomposition $\mathfrak{g}_0=\mathfrak{k}_0+\mathfrak{p}_0$ above. Let F be a finite-dimensional irreducible $\mathfrak{g}$ module and let $\theta=\mathfrak{m}+\mathfrak{u}\supset\mathfrak{h}$ be a c stable parabolic subalgebra of $\mathfrak{g}$ such that the space F^u of $\mathfrak{u}$ invariants is a one dimensional unitary M module. If $\lambda\in\mathfrak{m}^*$ is the differential of F^u then $\lambda(\Delta(\mathfrak{m}))=0$ and we shall write $A_\theta(\lambda)$ for the unique (up to equivalence) irreducible $\mathfrak{g}$ module with minimal $\mathfrak{k}$ type $\lambda|_{\mathfrak{h}}+2\delta_{u,n}$. This means that $A_\theta(\lambda)$ is the only irreducible $\mathfrak{g}$ module such that (i) $A_\theta(\lambda)|_{\mathfrak{k}}$ contains the irreducible $\mathfrak{k}$ module with Δ_k^+-highest weight $\lambda|_{\mathfrak{h}}+2\delta_{u,n}$ and (ii) the Δ_k^+-highest weight of any irreducible $\mathfrak{k}$ submodule of $A_\theta(\lambda)|_{\mathfrak{k}}$ is of the form $\lambda|_{\mathfrak{h}}+2\delta_{u,n}+\sum_{\beta\in\theta_{u,n}} n_\beta\beta$ where $n_\beta\geq 0$. For the existence and construction of the $\mathfrak{g}$ modules $A_\theta(\lambda)$ the reader may consult [16], [25]. One knows that the special $\mathfrak{k}$ type $\lambda|_{\mathfrak{h}}+2\delta_{u,n}$ occurs exactly once in $A_\theta(\lambda)|_{\mathfrak{k}}$. Now let W be the Weyl group of $(\mathfrak{g}, \mathfrak{h})$ and let W_K be the subgroup of W generated by reflections corresponding to compact roots. For $\Lambda\in\mathcal{F}_0'$ let

$$P^{(\Lambda)}=\{\alpha\in\Delta\,|\,(\Lambda+\delta, \alpha)>0\} \tag{2.3}$$

be the system of positive roots corresponding to the regular element $\Lambda+\delta$, let

$$\begin{aligned} &Q_\Lambda=\{\alpha\in\Delta_n^+\,|\,(\Lambda+\delta, \alpha)>0\} \\ &P_n^{(\Lambda)}=P^{(\Lambda)}\cap\Delta_n\,,\quad 2\delta^{(\Lambda)}=\langle P^{(\Lambda)}\rangle\,,\quad 2\delta_n^{(\Lambda)}=\langle P_n^{(\Lambda)}\rangle \end{aligned} \tag{2.4}$$

and for $w_1\in W$, $\tau_1\in W_K$ let

$$\begin{aligned} &\Phi_{w_1}^{(\Lambda)}=w_1(-P^{(\Lambda)})\cap P^{(\Lambda)}\,,\quad \Phi_{w_1}=w_1(-\Delta^+)\cap\Delta^+ \\ &\Phi_{\tau_1}^k=\tau_1(-\Delta_k^+)\cap\Delta_k^+. \end{aligned} \tag{2.5}$$

Proposition 2.6. *Let $\tau\in W_K$ and let $w\in W$ be such that $\Delta_k^+\subset wP^{(\Lambda)}$. Then $\Phi_{\tau^{-1}w}^{(\Lambda)}=\Phi_{\tau^{-1}}^k\cup(\Phi_{\tau^{-1}w}^{(\Lambda)}-\Phi_{\tau^{-1}}^k)$, $\Phi_{\tau^{-1}w}^{(\Lambda)}-\Phi_{\tau^{-1}}^k=\{\alpha\in P_n^{(\Lambda)}\,|\,w^{-1}\tau\alpha\in -P^{(\Lambda)}\}$. Also $\Phi_w^{(\Lambda)}\subset P_n^{(\Lambda)}$.*

Proof. If $\alpha\in\Phi_{\tau^{-1}}^k$ then $\alpha\in\Delta_k^+\subset P^{(\Lambda)}$ and $\tau\alpha\in -\Delta_k^+\subset w(-P^{(\Lambda)})\Rightarrow w^{-1}\tau\alpha\in -P^{(\Lambda)}\Rightarrow\Phi_{\tau^{-1}}^k\subset\Phi_{\tau^{-1}w}^{(\Lambda)}$ and hence $\Phi_{\tau^{-1}w}^{(\Lambda)}=\Phi_{\tau^{-1}}^k\cup(\Phi_{\tau^{-1}w}^{(\Lambda)}-\Phi_{\tau^{-1}}^k)$. If $\alpha\in\Phi_{\tau^{-1}w}^{(\Lambda)}-\Phi_{\tau^{-1}}^k$ then $\alpha\in P^{(\Lambda)}$, $w^{-1}\tau\alpha\in -P^{(\Lambda)}$ and we claim $\alpha\notin\Delta_k^+$. For otherwise $\tau\alpha\in\Delta_k^+$ since $\alpha\notin\Phi_{\tau^{-1}}^k$. Then $\tau\alpha\in wP^{(\Lambda)}\Rightarrow w^{-1}\tau\alpha\in P^{(\Lambda)}$ is a contradiction. Thus we must have $\alpha\in P^{(\Lambda)}-\Delta_k^+=P_n^{(\Lambda)}$; i.e. $\Phi_{\tau^{-1}w}^{(\Lambda)}-\Phi_{\tau^{-1}}^k\subset\{\alpha\in P_n^{(\Lambda)}\,|\,w^{-1}\tau\alpha\in -P^{(\Lambda)}\}$. Conversely $\{\alpha\in P_n^{(\Lambda)}\,|\,w^{-1}\tau\alpha\in -P^{(\Lambda)}\}\subset\Phi_{\tau^{-1}w}^{(\Lambda)}-\Phi_{\tau^{-1}}^k$ since $\Phi_{\tau^{-1}}^k\subset\Delta_k^+$ and since $\Delta_k\cap\Delta_n=\phi$. Clearly $\Phi_w^{(\Lambda)}\subset P_n^{(\Lambda)}$ since $\Delta_k^+\subset wP^{(\Lambda)}\cap P^{(\Lambda)}$. Q.E.D.

Using Proposition 2.6 we can now state the following theorem whose proof is given in [24] (see Theorem 2.15 there).

Theorem 2.7. *Let $\Lambda\in\mathcal{F}_0'$ in (2.1), let $P^{(\Lambda)}$ be the corresponding positive system in (2.3), and let $\sigma\in W$ be the unique Weyl group element such that $\sigma\Delta^+=P^{(\Lambda)}$.*

Let $(\pi, H_\pi)\in\hat{G}$ *be such that* $\pi(\Omega)=(\Lambda, \Lambda+2\delta)1$ *and such that* $\mathrm{Hom}_K(\pi, \chi^\pm\otimes\tau_{\Lambda+\delta_n})\neq 0$. *Then there is a pair* $(\tau, w)\in W_K\times W$ *and a c stable parabolic subalgebra* $\theta=\mathfrak{m}+\mathfrak{u}$ *of* $\mathfrak{g}$ *containing a Borel subalgebra* $\mathfrak{h}+\sum_{\alpha\in\Delta_1^+}\mathfrak{g}_\alpha$ *where* $\Delta_1^+\supset\Delta_k^+$ *such that*

(i) $H_\pi=A_\theta(\lambda)$ *and the minimal* $\mathfrak{k}$ *type* $\lambda|_\mathfrak{h}+2\delta_{u,n}$ (*which characterizes* H_π) *has the form* $\lambda|_\mathfrak{h}+2\delta_{u,n}=\Lambda+\delta_n+\tau^{-1}(w\delta^{(\Lambda)}-\delta_k)$

(ii) (τ, w) *satisfy* $\Delta_k^+\subset wP^{(\Lambda)}$, $\tau(\Lambda+\delta-\delta^{(\Lambda)})=w(\Lambda+\delta-\delta^{(\Lambda)})=\Lambda+\delta-\delta^{(\Lambda)}$, $\Phi_w^{(\Lambda)}$, $\Phi_{\tau^{-1}w}^{(\Lambda)}-\Phi_{\tau^{-1}}^k$, and $\{\alpha\in P_n^{(\Lambda)}|\tau\alpha\in -P_n^{(\Lambda)}\}$ are contained in $\{\alpha\in P_n^{(\Lambda)}|(\Lambda+\delta-\delta^{(\Lambda)}, \alpha)=0\}$, *and* $(-1)^{|\Phi_\sigma|}=\pm(-1)^{|\Phi_w^{(\Lambda)}|}=\pm(-1)^{n+|\theta_{u,n}|}$ *where* $|S|$ *denotes the cardinality of a set* S *and* $n=\frac{1}{2}\dim_R G/K$[1] (see (2.5)); *also* $\Phi_{\tau^{-1}}^k\subset\{\alpha\in\Delta_k^+|(\Lambda+\delta-\delta^{(\Lambda)}, \alpha)=0\}$

(iii) *the relative Lie algebra cohomology* $H^j(\mathfrak{m}, \mathfrak{m}\cap\mathfrak{k}, \boldsymbol{C})$ (*for the trivial module* $\boldsymbol{C}$=*the complex numbers*) *is non-zero for* $j=n-|\theta_{u,n}|-|\{\alpha\in P_n^{(\Lambda)}|w^{-1}\tau\alpha\in -P^{(\Lambda)}\}|$. *Hence the latter number is even.*

REMARKS. (i) If F is the finite-dimensional irreducible $\mathfrak{g}$ module with $P^{(\Lambda)}$-highest weight $\Lambda+\delta-\delta^{(\Lambda)}$ then H_π in Theorem 2.7 satisfies

$$\mathrm{Hom}_K(H_\pi, \wedge^i\mathfrak{p}\otimes F)=H^i(\mathfrak{g}, \mathfrak{k}, H_\pi\otimes F^*)=H^{i-|\theta_{u,n}|}(\mathfrak{m}, \mathfrak{m}\cap\mathfrak{k}, \boldsymbol{C})\qquad \text{for } i\geqslant 0$$

(ii) $\Lambda+\delta_n+\tau^{-1}(w\delta^{(\Lambda)}-\delta_k)$ is the only $\mathfrak{k}$ type which occurs both in $\pi|K$ and in $\chi^\pm\otimes\tau_{\Lambda+\delta_n}$

(iii) If $\sigma_1\in W$ is the unique Weyl group element such that $\sigma_1\Delta_1^+=P^{(\Lambda)}$ then $\sigma_1\lambda|\mathfrak{h}=\Lambda+\delta-\delta^{(\Lambda)}$ (see [24]).

(iv) The proof of Theorem 2.7 leans heavily on the recent unpublished results of D. Vogan [21]. Vogan's results depend in part on the important theorem of S. Kumaresan [9] which specifies the structure of an irreducible $\mathfrak{k}$ component of $\wedge\mathfrak{p}$ that can occur in an irreducible unitary $\mathfrak{g}$ module H_π when $\pi(\Omega)=0$.

(v) $\Phi_\sigma=\Delta_n^+-Q_\Lambda$.

3. Unitary representations intertwining $\mathrm{Ad}_+^q\otimes\tau_\Lambda$

We now assume that for G, K in section 2, the quotient G/K admits a G invariant complex structure; i.e. G/K is a Hermitian symmetric domain. We choose the positive system Δ^+ above to be compatible with the complex structures on G/K. This means that

$$\mathfrak{p}^\pm=\sum_{\pm\alpha\in\Delta_n^+}\mathfrak{g}_\alpha \tag{3.1}$$

where $\mathfrak{p}=\mathfrak{p}^+\oplus\mathfrak{p}^-$ is the splitting of $\mathfrak{p}$ into the spaces of holomorphic and anti-

1) Hence $|\theta_{u,n}|=\dim\mathfrak{u}\cap\mathfrak{p}$.

holomorphic tangent vectors $\mathfrak{p}^+$, $\mathfrak{p}^-$ respectively at the origin in G/K. The spaces $\mathfrak{p}^{\pm}$ are K and $\mathfrak{k}$ stable abelian subalgebras of $\mathfrak{g}$. The condition of the compatibility of Δ^+ with a G invariant complex structure is equivalent to the following: every $\alpha\in\Delta_n^+$ is *totally positive*; i.e. for each α in Δ_n^+ we have $\alpha+\beta\in\Delta_n^+$ for any $\beta\in\Delta_k$ such that $\alpha+\beta\in\Delta$. If $\mu\in\mathfrak{h}^*$ is integral and Δ_k^+ dominant we write (τ_μ, V_μ) for the corresponding irreducible of representation of $\mathfrak{k}$ (or of K if $(\tau_\mu, V_\mu)\in\hat{K}$). Let $L^{\pm}$ denote the representation space of $\chi^{\pm}$. Then we have

$$(3.2)\qquad \sum_{(-1)^j=\pm 1}\oplus\Lambda^{n-j}\mathfrak{p}^+ = L^{\pm}\otimes V_{\delta_n}$$

as K modules. Here note that $\dim V_{\delta_n}=1$ by Weyl's formula since $(\delta_n, \alpha)=0$ for $\alpha\in\Delta_k^+$ in the Hermitian symmetric case. Again $n=\frac{1}{2}\dim_R G/K=\dim_C G/K=|\Delta_n^+|$. We now prove the following Hermitian analogue of Theorem 2.7.

Theorem 3.3. *Let Λ, $P^{(\Lambda)}$ σ be as in Theorem 2.7 where Λ is the Δ_k^+-highest weight of $(\tau_\Lambda, V_\Lambda)\in\hat{K}$. Let $(\pi, H_\pi)\in\hat{G}$ be such that $\pi(\Omega)=(\Lambda, \Lambda+2\delta)1$ and such that $\mathrm{Hom}_K(H_\pi, \wedge^q\mathfrak{p}^+\otimes V_\Lambda)\neq 0$ where $q\geqslant 0$ is fixed. Then there is a pair $(\tau, w)\in W_K\times W$ and a c stable parabolic subalgebra $\theta=\mathfrak{m}+\mathfrak{u}$ of $\mathfrak{g}$ containing a Borel subalgebra $\mathfrak{h}+\sum_{\alpha\in\Delta_1^+}\mathfrak{g}_\alpha$ where $\Delta_1^+\supset\Delta_k^+$ such that H_π, (τ, w), θ satisfy conditions* (i), (ii), (iii) *of Theorem* 2.7 *where in* (ii) $\pm$ *is chosen according as* $(-1)^{n-q}=\pm 1$. *If $A_{\Lambda,\tau,w}=\{\alpha\in P_n^{(\Lambda)}\,|\,w^{-1}\tau\alpha\in -P^{(\Lambda)}\}$ (see Proposition 2.6), then q satisfies $q=|A_{\Lambda,\tau,w}|-2|Q_\Lambda\cap A_{\Lambda,\tau,w}|+|Q_\Lambda|$ where Q_Λ is given by* (2.4).

Proof. Suppose that $\mathrm{Hom}_K(H_\pi, \Lambda^q\mathfrak{p}^+\otimes V_\Lambda)\neq 0$. Writing $q=n-(n-q)$ and using (3.2) we have for $(-1)^{n-q}=\pm 1$ the K module inclusion $\Lambda^q\mathfrak{p}^+\otimes V_\Lambda\subset L^{\pm}\otimes V_{\delta_n}\otimes V_\Lambda=L^{\pm}\otimes V_{\Lambda+\delta_n}$ so that $\mathrm{Hom}_K(H_\pi, L^{\pm}\otimes V_{\Lambda+\delta_n})\neq 0$ since $H_\pi|_K$ and $\Lambda^q\mathfrak{p}^+\otimes V_\Lambda$ contain a common K type V_μ. Thus Theorem 2.2 applies. Tbe Δ_k^+-highest weight μ satisfies $\mu=\Lambda+\langle Q_1\rangle$ where $Q_1\subset\Delta_n^+$ such that $|Q_1|=q$. Let $Q_2=\Delta_n^+-Q_1$ so that $\mu=\Lambda+2\delta_n-\langle Q_2\rangle$. Define $Q_3=(Q_\Lambda-Q_2)\cup-(Q_2\cap Q'_\Lambda)\subset P_n^{(\Lambda)}=Q_\Lambda\cup -Q'_\Lambda$ where $Q'_\Lambda=\Delta_n^+-Q_\Lambda$. Then one easily checks that

$$(3.4)\qquad \begin{aligned}&|Q_3|=|Q_2|-2|Q_2\cap Q_\Lambda|+|Q_\Lambda| \quad\text{and}\\ &\langle Q_3\rangle=\langle Q_\Lambda\rangle-\langle Q_2\rangle.\end{aligned}$$

Let $Q_4=P_n^{(\Lambda)}-Q_3$. One has $\delta_n+\delta_n^{(\Lambda)}=\langle Q_\Lambda\rangle$ so that using (3.4) $\mu=\Lambda+\delta_n+\delta_n-\langle Q_2\rangle=\Lambda+\delta_n+\delta_n-\langle Q_\Lambda\rangle+\langle Q_3\rangle=\Lambda+\delta_n+\delta_n^{(\Lambda)}-\langle Q_4\rangle$. On the other hand by remark (ii) above $\Lambda+\delta_n+\tau^{-1}(w\delta^{(\Lambda)}-\delta_k)$ is the only $\mathfrak{k}$ type occurring both in $\pi|_K$ and $\chi^{\pm}\otimes\tau_{\Lambda+\delta_n}$ which means that $\mu=\Lambda+\delta_n+\tau^{-1}(w\delta^{(\Lambda)}-\delta_k)=\Lambda+\delta_n+\delta_n^{(\Lambda)}-\langle Q_4\rangle$ and hence $\tau^{-1}(w\delta^{(\Lambda)}-\delta_k)=\delta_n^{(\Lambda)}-\langle Q_4\rangle$. Therefore $\langle Q_4\cup\Phi_{\tau^{-1}}^k\rangle$ (see (2.5)) $=\langle Q_4\rangle+\langle\Phi_{\tau^{-1}}^k\rangle=\langle Q_4\rangle+\delta_k-\tau^{-1}\delta_k=\delta_k+\delta_n^{(\Lambda)}-\tau^{-1}w\delta^{(\Lambda)}=\delta^{(\Lambda)}-\tau^{-1}\delta w^{(\Lambda)}=\langle\Phi_{\tau^{-1}w}^{(\Lambda)}\rangle$. Thus by (5.10.2) of Kostant [8] $Q_4\cup\Phi_{\tau^{-1}}^k=\Phi_{\tau^{-1}w}^{(\Lambda)}$. Then $Q_4=\Phi_{\tau^{-1}w}^{(\Lambda)}-\Phi_{\tau^{-1}}^k=$

$A_{\Lambda,\tau,w}$ (by Proposition 2.6) and since $Q_4=P_n^{(\Lambda)}-Q_3$, $Q_2=\Delta_n^+-Q_1$ we get $|A_{\Lambda,\tau,w}|=n-|Q_3|=n-|Q_2|+2|Q_2\cap Q_\Lambda|-|Q_\Lambda|$ (by (3.4)) $=|Q_1|+2|Q_2\cap Q_\Lambda|-|Q_\Lambda|$ $=q+2|Q_2\cap Q_\Lambda|-|Q_\Lambda|$. But by definition of Q_3 we have $Q_2\cap Q_\Lambda=Q_\Lambda-Q_3=Q_\Lambda\cap Q_4=Q_\Lambda\cap A_{\Lambda,\tau,w}$ and hence $|A_{\Lambda,\tau,w}|=q+2|Q_\Lambda\cap A_{\Lambda,\tau,w}|-|Q_\Lambda|$. This proves Theorem 3.3.

In the statement of Tbeorem 3.3 no conditions are imposed on $\Lambda\in\mathcal{F}_0'$. However suppose for example that we impose the following condition: we assume every $\alpha\in P_n^{(\Lambda)}$ is totally positive. Then we have the following refinement of Theorem 3.3.

Corollary 3.5. *Let $(\tau_\Lambda, V_\Lambda)$, $P^{(\Lambda)}$, σ, (π, H_π) be as in Theorem 3.3 with q fixed. Suppose in addition that $P^{(\Lambda)}$ is compatible with a G invariant complex structure on G/K; i.e. assume every non-compact root in $P^{(\Lambda)}$ is totally positive. Then there is a Weyl group element w and a c stable parabolic subalgebra $\theta=\mathfrak{m}+\mathfrak{u}$ satisfying the conditions of Theorem 2.7 where in* (i), (ii), (iii) *$\tau\in W_K$ may be assumed equal to the identity element (thus for example H_π is characterized by the minimal $\mathfrak{k}$ type $\Lambda+\delta_n+w\delta^{(\Lambda)}-\delta_k$ and $j=n-|\theta_{u,n}|-|\Phi_w^{(\Lambda)}|$) and in* (ii) *$\pm$ is chosen according as $(-1)^{n-q}=\pm 1$. q satisfies $q=|\Phi_w^{(\Lambda)}|-2|Q_\Lambda\cap\Phi_w^{(\Lambda)}|+|Q_\Lambda|$.*

Proof. Choose (τ, w), $\theta=\mathfrak{m}+\mathfrak{u}$ as in Theorem 2.7 or Theorem 3.3. Since every non-compact root in $P^{(\Lambda)}$ is totally positive and since $\tau\in W_K$ we have $\tau P_n^{(\Lambda)}=P_n^{(\Lambda)}$. This implies that

$$A_{\Lambda,\tau,w}=\tau^{-1}\Phi_w^{(\Lambda)} \tag{3.6}$$

Also one has $\tau Q_\Lambda=Q_\Lambda$ and hence by (3.6)

$$\tau(Q_\Lambda\cap A_{\Lambda,\tau,w})=Q_\Lambda\cap\Phi_w^{(\Lambda)}. \tag{3.7}$$

Thus in Theorem 3.3 we have $q=|A_{\Lambda,\tau,w}|-2|Q_\Lambda\cap A_{\Lambda,\tau,w}|+|Q_\Lambda|=|\Phi_w^{(\Lambda)}|-2|Q_\Lambda\cap\Phi_w^{(\Lambda)}|+|Q_\Lambda|$. Also by (3.6) we see that in statement (iii) of Theorem 2.7 we have $j=n-|\theta_{u,n}|-|A_{\Lambda,\tau,w}|=n-|\theta_{u,n}|-|\Phi_w^{(\Lambda)}|$. To complete the proof of Corollary 3.4 we must show that in statement (i) of Theorem 2.7 $\tau^{-1}(w\delta^{(\Lambda)}-\delta_k)=w\delta^{(\Lambda)}-\delta_k$. Now since the positive system $P^{(\Lambda)}$ is compatible with a G invariant complex structure on G/K we have $(\delta_n^{(\Lambda)}, \alpha)=0$ for α in Δ_k^+ so that $\pm\delta_n^{(\Lambda)}$ is Δ_k^+-dominant. Also since $\Delta_k^+\subset wP^{(\Lambda)}$ we have $(w\delta^{(\Lambda)}, \alpha)=(\delta^{(\Lambda)}, w^{-1}\alpha)>0$ for α in Δ_k^+ so that $w\delta^{(\Lambda)}-\delta_k$ is Δ_k^+-dominant. Similarly $\Lambda+\delta-\delta^{(\Lambda)}$ is $P^{(\Lambda)}$ dominant (since $(\Lambda+\delta, \alpha)>0$ for α in $P^{(\Lambda)}$) and in particular $\Lambda+\delta-\delta^{(\Lambda)}+w\delta^{(\Lambda)}-\delta_k=\Lambda+\delta_n-\delta_n^{(\Lambda)}+w\delta^{(\Lambda)}-\delta_k$ is Δ_k^+-dominant. Moreover $\tau^{-1}(\Lambda+\delta-\delta^{(\Lambda)}+w\delta^{(\Lambda)}-\delta_k)=\Lambda+\delta-\delta^{(\Lambda)}+\tau^{-1}(w\delta^{(\Lambda)}-\delta_k)$ (since $\tau^{-1}(\Lambda+\delta-\delta^{(\Lambda)})=\Lambda+\delta-\delta^{(\Lambda)}$ by statement (ii) of Theorem 2.7) $=\Lambda+\delta_n-\delta_n^{(\Lambda)}+\tau^{-1}(w\delta^{(\Lambda)}-\delta_k)=\lambda|\mathfrak{h}+2\delta_{u,n}-\delta_n^{(\Lambda)}$ which is also Δ_k^+-dominant since $-\delta_n^{(\Lambda)}$ is Δ_k^+-dominant. But since only one transform of $\Lambda+\delta-\delta^{(\Lambda)}+w\delta^{(\Lambda)}-\delta_k$ under the Weyl group W_K can be Δ_k^+-dominant we conclude that $\Lambda+\delta-\delta^{(\Lambda)}+w\delta^{(\Lambda)}-\delta_k=\tau^{-1}(\Lambda+\delta-\delta^{(\Lambda)}+w\delta^{(\Lambda)}-\delta_k)=\Lambda+\delta-\delta^{(\Lambda)}$

$+\tau^{-1}(w\delta^{(\Lambda)}-\delta_k)$ and hence $w\delta^{(\Lambda)}-\delta_k=\tau^{-1}(w\delta^{(\Lambda)}-\delta_k)$ as desired.

Proposition 3.8. *Suppose in Theorem* 3.3 *the parabolic subalgebra* $\theta=\mathfrak{m}+\mathfrak{u}$ *is* $\mathfrak{g}$ *itself. Then* $\Lambda=\delta^{(\Lambda)}-\delta$ *and* $q=n-|Q_\Lambda|$.

Proof. $\theta=\mathfrak{g}$ means that $\mathfrak{u}=0$, $\mathfrak{m}=\mathfrak{g}$. Then $\theta_{u,n}=\phi$ and $\Delta(\mathfrak{m})=\Delta$. Recalling that $\lambda(\Delta(\mathfrak{m}))=0$ (see section 2) we have $\lambda(\Delta)=0$ and hence $\lambda|\mathfrak{h}=0$. By remark (iii) following Theorem 2.7 $\sigma_1\lambda|\mathfrak{h}=\Lambda+\delta-\delta^{(\Lambda)}$; hence $\Lambda+\delta-\delta^{(\Lambda)}=0$ $\Rightarrow\Lambda=\delta^{(\Lambda)}-\delta$. Also since $\theta_{u,n}=\phi$ the equality of $\mathfrak{k}$ types $\lambda|\mathfrak{h}+2\delta_{u,n}=\Lambda+\delta_n+\tau^{-1}(w\delta^{(\Lambda)}-\delta_k)$ in (i) of Theorem 2.7 reduces to $0=\delta_n^{(\Lambda)}+\tau^{-1}(w\delta^{(\Lambda)}-\delta_k)$, since $\Lambda=\delta^{(\Lambda)}-\delta=\delta_n^{(\Lambda)}-\delta_n$ and so $\Lambda+\delta_n=\delta_n^{(\Lambda)}$. But this says that $\langle\Phi_{\tau^{-1}w}^{(\Lambda)}\rangle=\delta^{(\Lambda)}-\tau^{-1}w\delta^{(\Lambda)}=\delta^{(\Lambda)}+\delta_n^{(\Lambda)}-\tau^{-1}\delta_k=2\delta_n^{(\Lambda)}+\delta_k-\tau^{-1}\delta_k=\langle P_n^{(\Lambda)}\rangle+\langle\Phi_{\tau^{-1}}^k\rangle=\langle P_n^{(\Lambda)}\cup\Phi_{\tau^{-1}}^k\rangle$ (see (2.5)) and hence $\Phi_{\tau^{-1}w}^{(\Lambda)}=P_n^{(\Lambda)}\cup\Phi_{\tau^{-1}}^k$ by (5.10.2) of [8]; i.e. $\Phi_{\tau^{-1}w}^{(\Lambda)}-\Phi_{\tau^{-1}}^k=P_n^{(\Lambda)}$ or $A_{\Lambda,\tau,w}=P_n^{(\Lambda)}$ by Proposition 2.6. Then by Theorem 3.3 $q=|A_{\Lambda,\tau,w}|-2|Q_\Lambda\cap A_{\Lambda,\tau,w}|+|Q_\Lambda|=n-2|Q_\Lambda|+|Q_\Lambda|=n-|Q_\Lambda|$.

Proposition 3.9. *Let* $\Lambda\in\mathcal{F}_0'$ *be such that every non-compact root in* $P^{(\Lambda)}$ *is totally positive. Let*

$$\mathfrak{p}^{(\Lambda)+}=\sum_{\alpha\in P_n^{(\Lambda)}}\mathfrak{g}_\alpha \tag{3.10}$$

be the $\mathfrak{k}$ *module of holomorphic tangent vectors for the corresponding G invariant complex structure on G/K compatible with* $P^{(\Lambda)}$; *cf.* (3.1). *Suppose* $w\in W$ *is a Weyl group element such that* $\Delta_k^+\subset wP^{(\Lambda)}$. *Then we have a* $\mathfrak{k}$ *module inclusion* $V_{\delta_n^{(\Lambda)}+w\delta^{(\Lambda)}-\delta_k}\subset\wedge^{n-|\Phi_w^{(\Lambda)}|}\mathfrak{p}^{(\Lambda)+}$.

Proof. In the proof of Corollary 3.5 we observed that indeed $\delta_n^{(\Lambda)}+w\delta^{(\Lambda)}-\delta_k$ is Δ_k^+-dominant. Of course

$$\delta_n^{(\Lambda)}+w\delta^{(\Lambda)}-\delta_k=2\delta_n^{(\Lambda)}-(\delta^{(\Lambda)}-w\delta^{(\Lambda)})=\langle P_n^{(\Lambda)}-\Phi_w^{(\Lambda)}\rangle. \tag{3.11}$$

Write $P_n^{(\Lambda)}-\Phi_w^{(\Lambda)}=\{\alpha_1,\cdots,\alpha_t\}$, $t=n-|\Phi_w^{(\Lambda)}|$, and let

$$X=X_{\alpha_1}\wedge\cdots\wedge X_{\alpha_t}\quad\text{where } X_{\alpha_j}\in\mathfrak{g}_{\alpha_j}-\{0\}. \tag{3.12}$$

We claim that $X\in\wedge^t\mathfrak{p}^{(\Lambda)+}$ is a Δ_k^+-highest weight vector. By (3.11) X is clearly a weight vector of the weight $\delta_n^{(\Lambda)}+w\delta^{(\Lambda)}-\delta_k$. Let $\beta\in\Delta_k^+$ be arbitrary and choose $X_\beta\in\mathfrak{g}_\beta-\{0\}$. We must show that

$$\mathrm{ad}_{X_\beta}X=\sum_{j=1}^t X_{\alpha_1}\wedge\cdots\wedge[X_\beta,X_{\alpha_j}]\wedge\cdots\wedge X_{\alpha_t}=0. \tag{3.13}$$

If $\beta+\alpha_j$ is not a root $[X_\beta,X_{\alpha_j}]=0$. Assume $\beta+\alpha_j$ is a root. Then $\beta+\alpha_j\in P_n^{(\Lambda)}$ since $\alpha_j\in P_n^{(\Lambda)}$ is totally positive. On the other hand $\alpha_j\notin\Phi_w^{(\Lambda)}$ implies $w^{-1}\alpha_j\in P^{(\Lambda)}$. Also by hypothesis $\Delta_k^+\subset wP^{(\Lambda)}$ so $w^{-1}\beta\in P^{(\Lambda)}$. Hence $w^{-1}(\beta+\alpha_j)=$

$w^{-1}\beta+w^{-1}\alpha_j\in P^{(\Lambda)}$; i.e. $\beta+\alpha_j\in P_n^{(\Lambda)}-\Phi_w^{(\Lambda)}$ which implies that $\beta+\beta_j=$some α_i, $i\neq j$. Then $[X_\beta, X_{\alpha_j}]=$a multiple of X_{α_i}. We conclude that (3.13) is valid and $U(\mathfrak{k})X$ is a $\mathfrak{k}$ submodule of $\wedge^t\mathfrak{p}^{(\Lambda)+}$ $\mathfrak{k}$-equivalent to $V_{\delta_n^{(\Lambda)}+w\delta^{(\Lambda)}-\delta_k}$.

Corollary 3.14. *Let Λ, $P^{(\Lambda)}$, and w be as in Proposition 3.9. Then we have the k module inclusion $V_{\Lambda+\delta_n+w\delta^{(\Lambda)}-\delta_k}\subset V_{\Lambda+\delta-\delta^{(\Lambda)}}\otimes V_{\delta_n^{(\Lambda)}+w\delta^{(\Lambda)}-\delta_k}\subset V_{\Lambda+\delta-\delta^{(\Lambda)}}\otimes \wedge^t\mathfrak{p}^{(\Lambda)+}$ where $t=n-|\Phi_w^{(\Lambda)}|$.*

Proof.
$$\begin{aligned}\Lambda+\delta_n+w\delta^{(\Lambda)}-\delta_k &= \Lambda+\delta_n-\delta_n^{(\Lambda)}+\delta_n^{(\Lambda)}+w\delta^{(\Lambda)}-\delta_k\\ &= \Lambda+\delta-\delta^{(\Lambda)}+\delta_n^{(\Lambda)}+w\delta^{(\Lambda)}-\delta_k\,.\end{aligned}$$

Corollary 3.15. *Let $(\tau_\Lambda, V_\Lambda)\in\hat{K}$ where $\Lambda\in\mathcal{F}_0'$ and every non-compact root in $P^{(\Lambda)}$ is totally positive. Let $(\pi, H_\pi)\in\hat{G}$ be such that $\pi(\Omega)=(\Lambda, \Lambda+2\delta)1$ and $\mathrm{Hom}_K(H_\pi, \wedge^q\mathfrak{p}^+\otimes V_\Lambda)\neq 0$. Let $\mu=\Lambda+\delta_n+w\delta^{(\Lambda)}-\delta_k$ be the minimal $\mathfrak{k}$ type of H_π given by Corollary 3.5. Then relative to the positive system $\bar{P}^{(\Lambda)}=P_k^{(\Lambda)}\cup -P_n^{(\Lambda)}=\Delta_k^+\cup -P_n^{(\Lambda)}$, H_π is a highest weight $\mathfrak{g}$ module with highest weight μ.*

Proof. We have $\mathfrak{k}$ module inclusions $V_\mu\subset H_\pi$ and (by Corollary 3.14) $V_\mu\subset V_{\Lambda+\delta-\delta^{(\Lambda)}}\otimes\wedge^t\mathfrak{p}^{(\Lambda)+}$ where $t=n-|\Phi_w^{(\Lambda)}|$ and where $\Lambda+\delta-\delta^{(\Lambda)}$ is $P^{(\Lambda)}$-dominant. Since $|(\Lambda+\delta-\delta^{(\Lambda)})+\delta^{(\Lambda)}|^2-|(\delta^{(\Lambda)}, \delta^{(\Lambda)})|^2=|\Lambda+\delta|^2-|(\delta, \delta)|^2=\pi(\Omega)$ Corollary 3.15 follows from Lemma 3.7 of [6] or from the proof of Lemma 2 of [4].

The fact that any $(\pi, H_\pi)\in\hat{G}$ as in Corollary 3.15 has to be a $\bar{P}^{(\Lambda)}$-highest weight $\mathfrak{g}$ module is also proved in [23] (see the proof of Lemma 2.4 there) by different means.

4. Vanishing theorems

In this section we again assume, as in section 3, that G/K is a Hermitian symmetric domain and that the positive system Δ^+ is compatible with the G invariant complex structure on G/K. We fix a discrete subgroup Γ of G which acts freely on G/K and for which the quotient $X=\Gamma\backslash G/K$ is compact. Let $\tau=\tau_\Lambda\in\hat{K}$ be a fixed finite-dimensional irreducible representation of K acting on a complex vector space V_Λ where $\Lambda\in\mathcal{F}_0'$ is the Δ_k^+-highest weight of τ. The induced C^∞ vector bundle $E_\tau\to G/K$ has a holomorphic structure. To prove this one usually assumes that G is a real form of a complex Lie group G^C (i.e. G is linear). Since we are not imposing the latter assumption on G we appeal to the more general criteria of [19], [20] for the existence of holomorphic structures on homogeneous bundles. The induced sheaf $\theta_\tau\to X$ of abelian groups over X given in the introduction will also be denoted by θ_Λ. Let Ad_+^q denote the adjoint representation of K on $\wedge^q\mathfrak{p}^+$. Then as in [4] the sheaf cohomology $H^q(X, \theta_\Lambda)$ can be identified with the space $A(\mathrm{Ad}_+^q\otimes\tau_\Lambda, (\Lambda, \Lambda+2\delta), \Gamma)$ of automorphic forms of type $(\mathrm{Ad}_+^q\otimes\tau_\Lambda, (\Lambda,\Lambda+2\delta), \Gamma)$; i.e.

$$H^q(X,\theta_\Lambda)=\{f\colon G\to\wedge^q\mathfrak{p}^+\otimes V_\Lambda\mid f \text{ is } C^\infty,\ f(\gamma a)=f(a),\ f(ak^{-1})=(\mathrm{Ad}_q^+\otimes\tau_\Lambda)(k)f(a) \text{ for } (\gamma,a,k) \text{ in } \Gamma\times G\times K \text{ and } \Omega f=(\Lambda,\Lambda+2\delta)f\}\ . \tag{4.1}$$

By the formula of Matsushima-Murakami [14] we therefore have

$$\dim H^q(X,\theta_\Lambda)=\sum_{\substack{(\pi,H_\pi)\in\hat{G}\\ \pi(\Omega)=(\Lambda,\Lambda+2\delta)1}} m_\pi(\Gamma)\dim\mathrm{Hom}_K(H_\pi,\wedge^q\mathfrak{p}^+\otimes V_\Lambda) \tag{4.2}$$

where $m_\pi(\Gamma)$ is the multiplicity of π in the right regular representation of G on $L^2(\Gamma\backslash G)$. Using (4.2) we immediately deduce from Theorem 3.3 the following main theorem.

Theorem 4.3. *Let $\Lambda\in\mathscr{F}_0'$ in* (2.1) *be the Δ_k^+-highest weight of $(\tau_\Lambda, V_\Lambda)\in\hat{K}$. Let $\sigma\in W$ be the unique Weyl group element such that $\sigma\Delta^+=P^{(\Lambda)}$ where $P^{(\Lambda)}$ is the system of positive roots in* (2.3). *Suppose that $H^q(\Gamma\backslash G/K,\theta_\Lambda)\neq 0$. Then there is a pair (τ, w) in $W_K\times W$ and a c stable parabolic subalgebra $\theta=\mathfrak{m}+\mathfrak{u}$ of $\mathfrak{g}$ containing the Borel subalgebra $\mathfrak{h}+\sum_{\alpha\in\Delta_+^1}\mathfrak{g}_\alpha$ for some positive system $\Delta_1^+\supset\Delta_k^+$ (cf. earlier notation) such that*

(i) $q=|A_{\Lambda,\tau,w}|-2|Q_\Lambda\cap A_{\Lambda,\tau,w}|+|Q_\Lambda|$ *where* $A_{\Lambda,\tau,w}=\{\alpha\in P_n^{(\Lambda)}\mid w^{-1}\tau\alpha\in -P^{(\Lambda)}\}$ *and where Q_Λ is given by* (2.4)

(ii) $\Delta_k^+\subset wP^{(\Lambda)}$ *(so that by Proposition* 2.6 $A_{\Lambda,\tau,w}=\Phi_{\tau^{-1}w}^{(\Lambda)}-\Phi_{\tau^{-1}}^k$), $\tau(\Lambda+\delta-\delta^{(\Lambda)})=w(\Lambda+\delta-\delta^{(\Lambda)})=\Lambda+\delta-\delta^{(\Lambda)}$, *and* $A_{\Lambda,\tau,w}$, $\Phi_w^{(\Lambda)}$, *and* $\{\alpha\in P_n^{(\Lambda)}\mid\tau\alpha\in -P_n^{(\Lambda)}\}$ *are all contained in* $\{\alpha\in P_n^{(\Lambda)}\mid(\Lambda+\delta-\delta^{(\Lambda)},\alpha)=0\}$; $\Phi_{\tau^{-1}}^k\subset\{\alpha\in\Delta_k^+\mid(\Lambda+\delta-\delta^{(\Lambda)},\alpha)=0\}$; *see notation of* (2.5)

(iii) *the relative Lie algebra cohomology* $H^j(\mathfrak{m},\mathfrak{m}\cap\mathfrak{k},C)\neq 0$ *for* $j=n-|\theta_{u,n}|-|A_{\Lambda,\tau,w}|$ *(hence the latter is an even number) where, as above, $\theta_{u,n}$ is the set of non-compact roots in the nilradical $\mathfrak{u}$ of θ and* $n=\frac{1}{2}\dim_R G/K$

(iv) *For* $(-1)^{n-q}=\pm 1$ *we have* $(-1)^{|\Phi_\sigma|}=\pm(-1)^{|\Phi_w^{(\Lambda)}|}=\pm(-1)^{n+|\theta_{u,n}|}$.

As has been noted $\Phi_\sigma=\Delta_n^+-Q_\Lambda$, and if $\sigma_1\in W$ is the unique Weyl group element such that $\sigma_1\Delta_1^+=P^{(\Lambda)}$ then $(\Lambda+\delta-\delta^{(\Lambda)},\sigma_1(\Delta(\mathfrak{m}))=0$ where $\Delta(\mathfrak{m})$ is the set of roots for the reductive part $\mathfrak{m}$ of θ. From Corollary 3.4 we obtain

Corollary 4.4. *Let $\Lambda\in\mathscr{F}_0'$ in Theorem* 4.3 *satisfy the condition that every non-compact root in $P^{(\Lambda)}$ is totally positive. Then if $H^q(\Gamma\backslash G/K,\theta_\Lambda)\neq 0$ we can choose $w\in W$ satisfying $\Delta_k^+\subset wP^{(\Lambda)}$ and a c stable parabolic subalgebra $\theta=\mathfrak{m}+\mathfrak{u}\supset\mathfrak{h}+\sum_{\alpha\in\Delta_1^+\supset\Delta_k^+}\mathfrak{g}_\alpha$ such that*

(i) $q=|\Phi_w^{(\Lambda)}|-2|Q_\Lambda\cap\Phi_w^{(\Lambda)}|+|Q_\Lambda|$

(ii) $H^{n-|\theta_{u,n}|-|\Phi_w^{(\Lambda)}|}(\mathfrak{m},\mathfrak{m}\cap\mathfrak{k},C)\neq 0$

(iii) $\Phi_w^{(\Lambda)}\subset\{\alpha\subset P_n^{(\Lambda)}\mid(\Lambda+\delta-\delta^{(\Lambda)},\alpha)=0\}$.

Statement (iv) *of Theorem* 4.3 *holds.*

Consider for example the special case when Λ is actually Δ^+-dominant. Then $P^{(\Lambda)}=\Delta^+$ so that Λ indeed satisfies Corollary 4.4. Also in this case $Q_\Lambda=\Delta_n^+$ so that $Q_\Lambda\cap\Phi_w^{(\Lambda)}=\Phi_w^{(\Lambda)}$. Thus by (i) of Corollary 4.4 $H^q\neq 0\Rightarrow q=|\Phi_w^{(\Lambda)}|-2|\Phi_w^{(\Lambda)}|+n=n-|\Phi_w^{(\Lambda)}|$ and hence by (ii) $H^{q-|\theta_{u,n}|}(\mathfrak{m}, \mathfrak{m}\cap\mathfrak{k}, C)\neq 0$. Thus we have proved the following conjecture of R. Parthasarathy.

Corollary 4.5. *Suppose the Δ_k^+-highest weight Λ of τ is actually Δ^+-dominant. Then if $H^q(\Gamma\backslash G/K, \theta_\Lambda)\neq 0$ so is $H^{q-|\theta_{u,n}|}(\mathfrak{m}, \mathfrak{m}\cap\mathfrak{k}, C)$ for some c stable parabolic subalgebra $\theta=\mathfrak{m}+\mathfrak{u}$ of $\mathfrak{g}$.*

Our argument shows moreover that in Corollary 4.5 $q\overset{(a.)}{=}n-|w(-\Delta^+)\cap\Delta^+|$ for some $w\in W$ with $\Delta_k^+\subset w\Delta^+$, $w(-\Delta^+)\cap\Delta^+\overset{(b.)}{\subset}\{\alpha\in\Delta_n^+\,|\,(\Lambda, \alpha)=0\}$; $w\Lambda=\Lambda$. Let $l(w)=|w(-\Delta^+)\cap\Delta^+|$ (=length of w) and let

$$n_\Lambda = |\{\alpha\in\Delta_n^+\,|\,(\Lambda, \alpha)>0\}|. \tag{4.6}$$

Then $|\{\alpha\in\Delta_n^+\,|\,(\Lambda, \alpha)=0\}|=n-n_\Lambda$ so that by (b.) $l(w)\leqslant n-n_\Lambda$ and by (a.) $q=n-l(w)\geqslant n_\Lambda$. That is

Corollary 4.7 (Hotta-Murakami [4]). *Suppose Λ is Δ^+-dominant. Then $H^q(\Gamma\backslash G/K, \theta_\Lambda)=0$ for $q<n_\Lambda$ in* (4.6). *More generally for $H^q(\Gamma\backslash G/K, \theta_\Lambda)\neq 0$ $q=n-l(w)$ for some $w\in W$ satisfying $w(-\Delta^+)\cap\Delta^+\subset\{\alpha\in\Delta_n^+\,|\,(\Lambda, \alpha)=0\}$, $w\Lambda=\Lambda$.*

We define

$$R = R(Q) = \min\{|\theta_{u,n}|\,|\,\theta=c \text{ stable parabolic subalgebra of } \mathfrak{g},\ \theta\neq\mathfrak{g}\}\ . \tag{4.8}$$

Again note that for $\theta=\mathfrak{g}$ $\mathfrak{u}=0$ and hence $|\theta_{u,n}|=\dim\mathfrak{u}\cap\mathfrak{p}=0$. The values $R(G)$ have been computed by Vogan for general symmetric spaces. Specializing his results to the Hermitian case we have the following table for the irreducible Hermitian symmetric spaces.

TABLE 4.9

G	$R(G)$	real rank of G/K	$\frac{1}{2}\dim_R G/K$
$Su(n,m)$, $n\geqslant m$	m	m	nm
$Sp(n,R)$	n	n	$\frac{n(n+1)}{2}$
$SO_0(n,2)$, $n>2$	2	2	n
$SO^*(2n)$, $n>3$	$n-1$	$[\frac{n}{2}]$	$\frac{n(n-1)}{2}$
real form of E_6	8	2	16
real form of E_7	11	3	17

In Theorem 4.3 $H^j(\mathfrak{m}, \mathfrak{m}\cap\mathfrak{k}, \boldsymbol{C})\neq 0$ for $j=n-|\theta_{u,n}|-|A_{\Lambda,\tau,w}|$ by (iii); hence $j\geqslant 0$. That is $|A_{\Lambda,\tau,w}|\leqslant n-|\theta_{u,n}|$ and if $\theta\neq\mathfrak{g}$ $|A_{\Lambda,\tau,w}|\leqslant n-R(G)$. Thus applying Proposition 3.8 we get

Proposition 4.10. *Suppose in Theorem* 4.3 *that either* $\Lambda\neq\delta^{(\Lambda)}-\delta$ *or* $q\neq n-|Q_\Lambda|$. *Then* $A_{\Lambda,\tau,w}$ *there satisfies* $|A_{\Lambda,\tau,w}|\leqslant n-R(G)$. *Similarly* w *in Corollary* 4.4 *satisfies* $|\Phi_w^{(\Lambda)}|\leqslant n-R(G)$.

Note that, in general, by Theorem 4.3 we always have $|A_{\Lambda,\tau,w}|$, $|\Phi_w^{(\Lambda)}|\leqslant |\{\alpha\in P_n^{(\Lambda)}\,|\,(\Delta+\delta-\delta^{(\Lambda)}, \alpha)=0\}|$. In Corollary 4.7 $q=n-l(w)$ for $H^q\neq 0$. By Proposition 4.10. $l(w)\leqslant n-R(G)$ if either $\Lambda\neq 0$ or $q\neq 0$; i.e. $q=n-l(w)\geqslant R(G)$ which establishes

Corollary 4.11. *Suppose* Λ *is* Δ^+*-dominant. If* $\Lambda\neq 0$ *then* $H^q(\Gamma\backslash G/K, \theta_\Lambda)=0$ *for* $0\leqslant q<R(G)$. *If* $\Lambda=0$ *then* $H^q(\Gamma\backslash G/K, \theta_\Lambda)=0$ *for* $1\leqslant q<R(G)$.

In particular we see that since for G in Table 4.9 rank of $G/K\leqslant R(G)$ the following weaker version of Corollary 4.11 holds.

Corollary 4.12. *If* G/K *is irreducible then* $H^q(\Gamma\backslash G/K, \theta_\Lambda)=0$ *for* $0\leqslant q<$ *rank fo* G/K, Λ Δ^+*-dominant,* $\Lambda\neq 0$. *The* $(0, q)$ *Betti number of* $\Gamma\backslash G/K$ *vanishes for* $1\leqslant q<$ *rank of* G/K.

Corollary 4.12 is of course well-known; see Theorem 4.2 of [6] and Theorem 4 of [4]. In the case where G/K is irreducible a slight improvement of Corollary 4.11 is given by Theorem 3.5 of [23]. Another extreme case is the case $Q_\Lambda=\phi$; i.e. $(\Lambda+\delta, \alpha)<0$ for $\alpha\in\Delta_n^+$, $P^{(\Lambda)}=\Delta'_+\overset{\text{def.}}{=}\Delta_k^+\cup-\Delta_n^+$. If $H^q\neq 0$ then from Corollary 4.4 $q=|\Phi_w^{(\Lambda)}|$ for some $w\in W$ such that $\Delta_k^+\subset w\Delta'_+$, $\Phi_w^{(\Lambda)}\subset\{\alpha\in-\Delta_n^+\,|\,(\Lambda+2\delta_n, \alpha)=0\}$ and (by (ii) of Corollary 4.4) $H^{n-q-|\theta_{u,n}|}(\mathfrak{m},\mathfrak{m}\cap\mathfrak{k}, \boldsymbol{C})\neq 0$ for some c stable parabolic $\theta=m+u$. By Proposition 3.8 $\theta\neq\mathfrak{g}$ unless $\Lambda=-2\delta_n$ or $q=n$. Barring the latter two cases we have $|\Phi_w^{(\Lambda)}|\leqslant n-R(G)$ by Proposition 4.10 so that $q\leqslant n-R(G)$. This gives

Corollary 4.13. *Suppose* $(\Lambda+\delta, \alpha)<0$ *for* α *in* Δ_n^+. *If* $\Lambda\neq-2\delta_n$ *then* $H^q(\Gamma\backslash G/K, \theta_\Lambda)=0$ *for* $q>n-R/(G)$. *If* $\Lambda=-2\delta_n$ *then* $H^q(\Gamma\backslash G/K, \theta_\Lambda)=0$ *for* $n-R(G)<q<n$. *In any case we always have* $H^q(\Gamma\backslash G/K, \theta_\Lambda)=0$ *for* $q>|\{\alpha\in-\Delta_+^n\,|\,(\Lambda+2\delta_n, \alpha)=0\}|$.

The last statement of Corollary 4.13 is statement (i) of Theorem 3.12 of [23]. However in [23] G is assumed to be linear. We now indicate how the main result of [23] (Theorem 2.3) can be deduced with the aid of Corollary 3.5; see Theorem 4.16.

Proposition 4.14 *Let* $\Lambda\in\mathcal{F}'_0$ *and let* $w\in W$ *be a Weyl group element which* *satisfies* $\Delta_k^+ \subset wP^{(\Lambda)}$, $w(\Lambda+\delta-\delta^{(\Lambda)})=\Lambda+\delta-\delta^{(\Lambda)}$, *and* $\Phi_w^{(\Lambda)} \subset \{\alpha\in P_n^{(\Lambda)} \mid (\Lambda+\delta-\delta^{(\Lambda)}, \alpha)=0\}$ (*cf.* (ii) *of Theorem* 4.3) *Then* $\Lambda+\delta-\delta^{(\Lambda)}+w\delta^{(\Lambda)}$ *is a regular element* (*i.e.* $(\Lambda+\delta-\delta^{(\Lambda)}+w\delta^{(\Lambda)}, \alpha)\neq 0$ *for every* α *in* Δ) *and the corresponding positive system*

(4.15) $P' = \{\alpha\in\Delta \mid (\Lambda+\delta-\delta^{(\Lambda)}+w\delta^{(\Lambda)}, \alpha)>0\}$ *coincides with* $wP^{(\Lambda)}$.
Also $P_n^{(\Lambda)}-\Phi_w^{(\Lambda)} = P'\cap P_n^{(\Lambda)}$.

Proof. For $\alpha\in\Delta_k^+$ $(\Lambda+\delta-\delta^{(\Lambda)}+w\delta^{(\Lambda)}, \alpha)=(\Lambda+\delta-\delta^{(\Lambda)}, \alpha)+(\delta^{(\Lambda)}, w^{-1}\alpha)>0$ since $w^{-1}\Delta_k^+\subset P^{(\Lambda)}$. Suppose $\alpha\in P_n^{(\Lambda)}$. If $(\Lambda+\delta-\delta^{(\Lambda)}, \alpha)=0$ then $(\Lambda+\delta-\delta^{(\Lambda)}+w\delta^{(\Lambda)}, \alpha)=(\delta^{(\Lambda)}, w^{-1}\alpha)\neq 0$. Assume $(\Lambda+\delta-\delta^{(\Lambda)}, \alpha)>0$. Then $\alpha\notin\Phi_w^{(\Lambda)}$ since by hypothesis $\Phi_w^{(\Lambda)}\subset\{\alpha\in P_n^{(\Lambda)} \mid (\Lambda+\delta-\delta^{(\Lambda)}, \alpha)=0\}$. Thus we must have $w^{-1}\alpha\in P^{(\Lambda)}$. Since $\Lambda+\delta-\delta^{(\Lambda)}$ is $P^{(\Lambda)}$-dominant $(\Lambda+\delta-\delta^{(\Lambda)}, \alpha)+(\delta^{(\Lambda)}, w^{-1}\alpha)>0$. Thus we have shown $(\Lambda+\delta-\delta^{(\Lambda)}+w\delta^{(\Lambda)}, \alpha)\neq 0$ for $\alpha\in P^{(\Lambda)}$ which proves $\Lambda+\delta-\delta^{(\Lambda)}+w\delta^{(\Lambda)}$ is regular. Let $\alpha\in P^{(\Lambda)}$ be arbitrary. Then $(\Lambda+\delta-\delta^{(\Lambda)}+w\delta^{(\Lambda)}, w\alpha)=(w^{-1}(\Lambda+\delta-\delta^{(\Lambda)}+w\delta^{(\Lambda)}), \alpha)=(\Lambda+\delta, \alpha)$ (since $w^{-1}(\Lambda+\delta-\delta^{(\Lambda)})=\Lambda+\delta-\delta^{(\Lambda)}$) which is positive. That is $w\alpha\in P'\Rightarrow wP^{(\Lambda)}\subset P'\Rightarrow wP^{(\Lambda)}=P'$. Now $\Phi_w^{(\Lambda)} \overset{\text{def.}}{=} w(-P^{(\Lambda)})\cap P^{(\Lambda)}=-P'\cap P^{(\Lambda)}$ and since $\Phi_w^{(\Lambda)}\subset P_n^{(\Lambda)}$ the last equation implies that $P_n^{(\Lambda)}-\Phi_w^{(\Lambda)}=P'\cap P_n^{(\Lambda)}$ since $\Delta=P'\cup -P'$.

REMARK. In Proposition 4.14 (and hence in Theorem 4.3) the condition $\Phi_w^{(\Lambda)}\subset\{\alpha\subset P_n^{(\Lambda)} \mid (\Lambda+\delta-\delta^{(\Lambda)}, \alpha)=0\}$ is automatically satisfied. Indeed for $\alpha\in\Phi_w^{(\Lambda)}\subset P_n^{(\Lambda)}$ $0\leqslant(\Lambda+\delta-\delta^{(\Lambda)}, \alpha)=(w^{-1}(\Lambda+\delta-\delta^{(\Lambda)}), w^{-1}\alpha)=(\Lambda+\delta-\delta^{(\Lambda)}, w^{-1}\alpha)\leqslant 0$ (since $w^{-1}\alpha\in P^{(\Lambda)}$) and so $(\Lambda+\delta-\delta^{(\Lambda)}, \alpha)=0$.

Theorem 4.16. *Assume that G is linear and its complexification G^C is simply connected.* (*In particular if $\Lambda\in\mathfrak{h}^*$ is Δ_k^+-dominant integral the irreducible finite-dimensional representation of $\mathfrak{k}$ defined by Λ integrates to a representation of K.*) *Let $\Lambda\in\mathcal{F}_0'$ be such that every non-compact root in $P^{(\Lambda)}$ is totally positive. If $H^q(\Gamma\backslash G/K, \theta_\Lambda)\neq 0$ then there is a parabolic subalgebra $\theta_1=\mathfrak{m}_1+\mathfrak{u}_1$ of $\mathfrak{g}$ which contains the specific Borel subalgebra $\mathfrak{h}+\sum_{\alpha\in P^{(\Lambda)}}\mathfrak{g}_\alpha$ such that $q=2|\theta_{1,n}\cap Q_\Lambda|+|\Delta_n^+-Q_\Lambda|-|\theta_{u_1,n}|$. Also $(\Lambda+\delta-\delta^{(\Lambda)}, \Delta(\mathfrak{m}_1))=0$.*

Proof. If $H^q(\Gamma\backslash G/K, \theta_\Lambda)\neq 0$ then by (4.2) $\mathrm{Hom}_K(H_\pi, \wedge^q\mathfrak{p}^+\otimes V_\Lambda)\neq 0$ for some $(\pi, H_\pi)\in\hat{G}$ such that $\pi(\Omega)=(\Lambda, \Lambda+2\delta)1$. By Corollary 3.5 H_π has minimal $\mathfrak{k}$ type $\mu=\Lambda+\delta_n+w\delta^{(\Lambda)}-\delta_k$ for some Weyl group element w such that $\Delta_k^+\subset wP^{(\Lambda)}$ and $q=|\Phi_w^{(\Lambda)}|-2|Q_\Lambda\cap\Phi_w^{(\Lambda)}|+|Q_\Lambda|$; $w(\Lambda+\delta-\delta^{(\Lambda)})=\Lambda+\delta-\delta^{(\Lambda)}$. By Corollary 3.15 H_π is a highest weight $\mathfrak{g}$ module with highest weight μ relative to the positive system $\bar{P}^{(\Lambda)}=P_k^{(\Lambda)}\cup -P_n^{(\Lambda)}=\Delta_k^+\cup -P_n^{(\Lambda)}$. Also $\mu+\delta_k-\delta_n^{(\Lambda)}=\Lambda+\delta_n-\delta_n^{(\Lambda)}+w\delta^{(\Lambda)}=\Lambda+\delta-\delta^{(\Lambda)}+w\delta^{(\Lambda)}$ is regular by Proposition 4.14 (see remark following Proposition 4.14). Thus since G is assumed to be linear we can apply Parthasarathy's Theorem A of [17] to conclude the following:

$\mu=\Lambda_0+\langle\theta_{u_1,n}\rangle$ for some parabolic subalgebra $\theta_1=\mathfrak{m}_1+\mathfrak{u}_1$ of $\mathfrak{g}$ where $\theta_1\supset\mathfrak{h}+\sum_{\alpha\in P^{(\Lambda)}}\mathfrak{g}_\alpha$ and where $\Lambda_0\in\mathfrak{h}^*$ is $P^{(\Lambda)}$-dominant integral, and $(\Lambda_0, \Delta(\mathfrak{m}_1))=0$. Moreover by (3.49) of [17] $\theta_{u_1,n}=P'\cap P_n^{(\Lambda)}$ where P' is the positive system defined by the regular element $\mu+\delta_k-\delta_n^{(\Lambda)}$. Hence by Proposition 4.14 $\theta_{u_1,n}=P_n^{(\Lambda)}-\Phi_w^{(\Lambda)}$. Then $\Lambda+\delta_n+w\delta^{(\Lambda)}-\delta_k=\mu=\Lambda_0+\langle\theta_{u_1,n}\rangle=\Lambda_0+\langle P_n^{(\Lambda)}-\Phi_w^{(\Lambda)}\rangle=\Lambda_0+\delta_n^{(\Lambda)}+w\delta^{(\Lambda)}-\delta_k$ (by (3.11))$\Rightarrow\Lambda_0=\Lambda+\delta_n-\delta_n^{(\Lambda)}=\Lambda+\delta-\delta^{(\Lambda)}\Rightarrow(\Lambda+\delta-\delta^{(\Lambda)}, \Delta(\mathfrak{m}_1))=0$. We also have $|\theta_{u_1,n}|=n-|\Phi_w^{(\Lambda)}|$ so that $q=|\Phi_w^{(\Lambda)}|-2|Q_\Lambda\cap\Phi_w^{(\Lambda)}|+|Q_\Lambda|=n-|\theta_{u_1,n}|-2|Q_\Lambda-\theta_{u_1,n}|+|Q_\Lambda|=n-|\theta_{u_1,n}|-2(|Q_\Lambda|-|Q_\Lambda\cap\theta_{u_1,n}|)+|Q_\Lambda|=2|Q_\Lambda\cap\theta_{u_1,n}|-|\theta_{u_1,n}|+|\Delta_n^+-Q_\Lambda|$.

REMARK. If additional information on the Weyl group element σ_1 above (where $\sigma_1\Delta_1^+=P^{(\Lambda)}$) were available the preceding proof might not require the appeal to Theorem A of [17]. For example if it were known that $\langle P_n^{(\Lambda)}-\sigma_1\Delta(\mathfrak{m})\rangle$ ⓐ $=\delta_n^{(\Lambda)}+w\delta^{(\Lambda)}-\delta_k$ for $\theta=\mathfrak{m}+\mathfrak{u}$ in Theorem 4.3 then Theorem 4.16 would follow (even for G non-linear) by taking $\theta_1=\sigma_1\theta$. However ⓐ is true only when certain additional restrictions on Λ are imposed.

Another classical vanishing theorem for the spaces $H^q(\Gamma\backslash G/K, \theta_\Lambda)$ is the following one of Hotta and Parthasarathy; see Proposition 1 of [5].

Theorem 4.17. *Let $\Lambda\in\mathcal{F}_0'$ be the Δ_k^+-highest weight of $(\tau_\Lambda, V_\Lambda)\in\hat{K}$. Suppose that $(\Lambda+\delta-\delta^{(\Lambda)}, \alpha)>0$ for every α in $P_n^{(\Lambda)}$. Then $H^q(\Gamma\backslash G/K, \theta_\Lambda)=0$ for $q\neq|Q_\Lambda|$.*

Here G is not assumed to be linear. Theorem 4.17 follows from a trivial application of Theorem 4.3. Namely if $H^q(\Gamma\backslash G/K, \theta_\Lambda)\neq 0$ then $q=|A_{\Lambda,\tau,w}|-2|Q_\Lambda\cap A_{\Lambda,\tau,w}|+|Q_\Lambda|$ where $A_{\Lambda,\tau,w}\subset\{\alpha\subset P_n^{(\Lambda)}|(\Lambda+\delta-\delta^{(\Lambda)}, \alpha)=0\}$. But $(\Lambda+\delta-\delta^{(\Lambda)}, \alpha)>0$ for $\alpha\in P_n^{(\Lambda)}$ by hypothesis so $A_{\Lambda,\tau,w}=\phi$. Thus $q=|Q_\Lambda|$.

References

[1] A. Borel: *On the curvature tensor of the Hermitian symmetric manifold,* Ann. of Math. **71** (1960), 508–521.

[2] A. Borel and N. Wallach: Continuous cohomology, discrete subgroups, and representations of reductive groups, Ann. of Math. Studies, 94, Princeton Univ. Press and Univ. Tokyo Press, 1980.

[3] E. Calabi and E. Vesentini: *On compact locally symmetric Kähler manifolds,* Ann. of Math. **71** (1960), 472–507.

[4] R. Hotta and S. Murakami: *On a vanishing theorem for certain cohomology groups,* Osaka J. Math. **12** (1975), 555–564.

[5] R. Hotta and R. Parthasarathy: *A geometric meaning of the multiplicities of integrable discrete classes in $L^2(\Gamma\backslash G)$,* Osaka J. Math. **10** (1973), 211–234.

[6] R. Hotta and N. Wallach: *On Matsushima's formula for the Betti numbers of a locally symmetric space,* Osaka J. Math. **12** (1975), 419–431.

[7] M. Ise: *Generalized automorphic forms and certain homomorphic vector bundles,* Amer. J. Math. **86** (1964), 70–108.

[8] B. Kostant: *Lie algebra cohomology and the generalized Borel-Weil Theorem,* Ann. of Math. **74** (1961), 329–387.

[9] S. Kumaresan: *On the canonical k types in the irreducible unitary g modules with non-zero relative cohomology,* Invent. Math. **59** (1980), 1–11.

[10] Y. Matsushima: *On the first Betti number of compact quotient spaces of higher dimentional symmetric spaces,* Ann. of Math. **75** (1962), 312–330.

[11] ————: *A formula for the Betti numbers of compact locally symmetric Riemannian manifolds,* J. Differential Geom. **1** (1967), 99–109.

[12] Y. Matsushima and S. Murakami: *On vector bundle valued harmonic forms and automorphic forms on symmetric Riemannian manifolds,* Ann. of Math. **78** (1963), 365–416.

[13] ————: *On certain cohomology groups attached to Hermitian symmetric spaces,* Osaka J. Math. **2** (1965), 1–35.

[14] ————: *On certain cohomology groups attached to Hermitian symmetric spaces* (II), Osaka J. Math. **5** (1968), 223–241.

[15] S. Murakami: Cohomology groups of vector-valued forms on symmetric spaces, Lecture notes, Univ. Chicago, 1966.

[16] R. Partharathy: *A generalization of the Enright-Varadarajan modules,* Compositio Math. **36** (1978), 53–73.

[17] ————: *Criteria for the unitarizability of some highest weight modules,* Proc. Indian Acad. Sci. **89** (1980), 1–24.

[18] ————: *Holomorphic forms on* $\Gamma\backslash G/K$ *and Chern classes,* to appear.

[19] J. Tirao and J. Wolf: *Homogeneous holomorphic vector bundles,* Indiana Univ. Math. J. **20** (1970), 15–31.

[20] J. Tirao: *Square integrable representations of semisimple Lie groups,* Trans. Amer. Math. Soc. **190** (1974), 57–75.

[21] D. Vogan: *Manuscript on the classification of unitary representations with relative Lie algebra cohomology,* Dept. Math., M.I.T.

[22] ————: Cohomology of Riemannian locally symmetric spaces, a lecture given at Brown Univ. and the Univ. of Utah.

[23] F. Williams: *Vanishing theorems for type* (0,*q*) *cohomology of locally symmetric spaces,* Osaka J. Math. **18** (1981), 147–160.

[24] ————: *Remark on the unitary representations appearing in the Matsushima-Murakami formula,* Lecture Notes in Math. 880: Non-Commutative Harmonic Analysis, from the Marselle Conference, June 1980, Springer-Verlag.

[25] G. Zuckerman: *Unitary representations in complex homogeneous spaces,* Manuscript, Dept. Math. Yale Univ.

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