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ON THE ZETA FUNCTIONS OF THE VARIETIES $X(w)$ OF THE SPLIT CLASSICAL GROUPS AND THE UNITARY GROUPS

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0. Introduction

Let G be one of the split classical groups SO_{2n}^{+}, SO_{2n+1}, Sp_{2n} or a unitary group defined over the finite field $\boldsymbol{F}_q$ of q elements. Let F be the Frobenius mapping, G^F the subgroup of F-stable elements, W the Weyl group of G and let δ be the smallest positive integer such that F^δ acts trivially on W. For $w\in W$, Deligne-Lusztig [3] has defined the F^δ-stable variety $X(w)$ for any connected reductive group. If w is a Coxeter element of W, the zeta function of $X(w)$ was obtained by Lusztig [9] as a by-product when he determined the Green polynomial associated with w. In this paper we shall determine the zeta function of $X(w)$ for any $w\in W$.

To state our result more explicitly, let B be a fixed F-stable Borel subgroup of G, $\mathfrak{A}^K(W)$ the Hecke algebra of the representation of G^{F^m} induced from the trivial representation of B^{F^m} and let $\{a_w^K;\ w\in W\}$ be the natural basis of $\mathfrak{A}^K(W)$. When δ divides m the number of F^m-stable points of $X(w)$ is expressed in terms of the dimensions of the unipotent representations of G^F and the trace of a_w^K on each irreducible representation of $\mathfrak{A}^K(W)$.

The crucial point of our arguments depends on the lifting theory due to Shintani-Kawanaka ([15], [7], [8]) and a result of Lusztig ([12], Corollary 3.9), which says that for any unipotent representation ρ of G^F, the eigenvalues of F^δ on the ρ-isotypic component of $H_c^i(X(w))$ are independent of i and w up to a multiple factor of the form $q^{i\delta}$, $i\in\boldsymbol{Z}$.

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1. General results

1.1. First we summarize the known results (Shintani [14], Kawanaka [7], [8]) to apply for our use.
Let m be a positive integer (maybe 1), $k=\boldsymbol{F}_q$, $K=\boldsymbol{F}_{q^m}$, G a connected algebraic

group defined over k, F the Frobenius over k, $\sigma=F|_{G^{F^m}}$ and A the cyclic group (of order m) generated by σ. Let $x_1, x_2\in G^{F^m}$. $x_1\sigma$ and $x_2\sigma$ are conjugate in $G^{F^m}A$ (semi-direct) if and only if there exists $h\in G^{F^m}$ such that $x_1=h^{-1}x_2^{\sigma}h$. If this is the case, we say x_1 and x_2 are σ-conjugate and we write $x_1\underset{\sigma}{\sim}x_2$. If $m=1$, we simply write $x_1\sim x_2$ instead of $x_1\underset{\sigma}{\sim}x_2$. The following lemma is proved in [7].

Lemma 1.1.1. *For $x\in G^{F^m}$, take $a\in G$ such that $x=a^{-1F}a$. Let $y={}^{F^m}a\,a^{-1}$. Then $y\in G^F$, and the conjugacy class of y in G^F is uniquely determined by the σ-conjugacy class of x in G^{F^m}. And the mapping $x\mapsto y$ defines a bijection: $G^{F^m}/\underset{\sigma}{\sim}\to G^F/\sim$.*

Definition 1.1.2. We denote the bijection $G^{F^m}/\underset{\sigma}{\sim}\to G^F/\sim$ in the above lemma by $n_{K/k}$. (Notice $n_{K/k}$ is defined even if $m=1$.) Define $\mathfrak{N}_{K/k}=n_{k/k}^{-1}\,n_{K/k}$. This also is a bijection from $G^{F^m}/\underset{\sigma}{\sim}$ onto $G^F/\sim$.

Remark 1.1.3. The reader should refer Kawanaka [8] for the relation between the norm mapping in [loc. cit.] and our norm mapping $\mathfrak{N}_{K/k}$.

The following lemma features some property of the mapping $\mathfrak{N}_{K/k}$, which is not used in this paper. The proof is omitted.

Lemma 1.1.4. *Let G be a connected reductive group and $Z(G)$ the center of G. Let $s\in Z(G)^F$ and $u\in G^F$. Let r be the order of s. Assume $m\equiv 1 \bmod r$. Then $\mathfrak{N}_{K/k}^{-1}(su)=s\mathfrak{N}_{K/k}^{-1}(u)$.*

For $\chi_K\in \widehat{G^{F^m}}^{\sigma}$ (=the set of σ-invariant irreducible characters of G^{F^m}), there exists $\tilde{\chi}_K\in \widehat{G^{F^m}A}$ such that $\tilde{\chi}_K|_{G^{F^m}}=\chi_K$. Let $\chi_k\in\widehat{G^F}$.

Definition 1.1.5. Let $m>1$. We say χ_K is the lifting of χ_k in $\widehat{G^{F^m}}$ if there exists a constant c such that $\tilde{\chi}_K(y\sigma)=c\chi_k(\mathfrak{N}_{K/k}\,y)$ for any $y\in G^{F^m}$. (The lifting of χ_k is uniquely determined by χ_k if it exists. See [7].)

Theorem 1.1.6 ([7], [8], [15]).
Let $m>1$. Assume one of the following.

(1) $G=GL_n$.

(2) $G=U_n$, $(m, p)=1$.

(3) $G=SO_{2n+1}$, Sp_{2n} *or* $SO_{2n}^{\pm}$, $(m, 2p)=1$.

Then any $\chi_k\in\widehat{G^F}$ has the lifting $\chi_K\in\widehat{G^{F^m}}$. And the mapping $\chi_k\mapsto\chi_K$ defines a bijection between $\widehat{G^F}$ and $\widehat{G^{F^m}}^{\sigma}$.

Remark 1.1.7. The theorem is proved by Shintani [15] in case (1), by Kawanaka [7] in case (2) and by Kawanaka [8] in case (3).

The following lemmas can be extracted from [7].

Lemma 1.1.8. *Let f_1 and f_2 be class functions on $G^{F^m}A$. Define class functions g_1 and g_2 on G^F by: $g_i(\mathfrak{N}_{K/k}y)=f_i(y\sigma)$ for any $y\in G^{F^m}$. Then*

$$|G^{F^m}|^{-1}\sum_{y\in G^{F^m}} f_1(y\sigma)\overline{f_2(y\sigma)} = |G^F|^{-1}\sum_{x\in G^F} g_1(x)\overline{g_2(x)}$$

Lemma 1.1.9. *Let H be an F-stable closed subgroup. Let f and g be class functions on $H^{F^m}A$ and H^F respectively. If $g(\mathfrak{N}_{K/k}y)=f(y\sigma)$ for any $y\in H^{F^m}$, then* $(\mathrm{Ind}\,_{H^F}^{G^F} g)(\mathfrak{N}_{K/k}y)=(\mathrm{Ind}\,_{H^{F^m}A}^{G^{F^m}A} f)(y\sigma)$ *for any $y\in G^{F^m}$.*

1.2. Henceforth G is a connected reductive group defined over $k=\boldsymbol{F}_q$, B is an F-stable Borel subgroup, U is the unipotent radical of B, T is an F-stable maximal torus of B and $W=N_G(T)/T$.
Let $w\in W^{F^m}$ and $\dot{w}$ its representative in $N_G(T)^{F^m}$.
Let $X(w)$, $S_{\dot{w}}$, $T(w)^F$ and $R^1_{T_w}$ be as in [3]. They are as follows.

$$S_{\dot{w}} = \{g\in G; g^{-1F}g\in \dot{w}U\},\ T(w)^F = \{t\in T;\ \dot{w}^F t\dot{w}^{-1}=t\}\ ,$$

$X(w)=S_{\dot{w}}/T(w)^F U\cap \dot{w}U\dot{w}^{-1}$ and $R^1_{T_w}$ is the virtual character of G^F such that $\mathrm{Tr}(x, R^1_{T_w})=\mathrm{Tr}(x^{*-1}, \sum_{i\geqslant 0}(-1)^i H^i_c(X(w))$.
Then we have

Lemma 1.2.1 (cf. Remark 1.4.2). *Let $x\in G^F$. Take $a\in G$ such that $x={}^{F^m}a^{-1}a$. Let $y=a^Fa^{-1}\in G^F$* (cf. Lemma 1.1.1). *Then*

$$\mathrm{Tr}((x^{-1}F^m)^*, \sum_{i\geqslant 0}(-1)^i H^i_c(X(w)) = (|T^{F^m}|q^{md})^{-1}\#\{h\in G^{F^m};\ h^{-1}y^\sigma h\in \dot{w}B\}\ ,$$

where $d=\dim U\cap \dot{w}U\dot{w}^{-1}$.

1.3. Let $Z^K=\mathrm{Ind}_{B^{F^m}}^{G^{F^m}}1$(=the representation of G^{F^m} induced from the trivial representation of B^{F^m}). Then $Z^K=\sum_{g\in G^{F^m}/B^{F^m}}\overline{\boldsymbol{Q}}_l gv$ as vector spaces with B^{F^m} acting trivially on $\overline{\boldsymbol{Q}}_l v$. As is known, $\mathrm{End}\,_{G^{F^m}}Z^K=\sum_{w\in W^{F^m}}\overline{\boldsymbol{Q}}_l a^K_w$, where a^K_w is defined by: $a^K_w v=\sum_{u\in U_w^{-,F^m}} u\dot{w}^{-1}v$ with $U^-_w=U\cap \dot{w}U^-\dot{w}^{-1}$ (U^- is the maximal unipotent subgroup opposite to U). Define the linear mapping I_σ on Z^K by:
I_σ: $\sum_{g\in G^{F^m}/B^{F^m}} c_g gv\mapsto \sum_{g\in B^{F^m}/B^{F^m}} c^\sigma_g gv$ $(c_g\in\overline{\boldsymbol{Q}}_l)$. Then for any $g\in G^{F^m}$ and $z\in Z$, $I_\sigma(gz)={}^\sigma g I_\sigma z$.
Then we have

Lemma 1.3.1 (cf. Remark 1.4.2). *For $g\in G^{F^m}$ and $w\in W^{F^m}$,* $\mathrm{Tr}(ya^K_w I_\sigma, Z^K)$ $=(q^{md}|T^{F^m}|)^{-1}\#\{g\in G^{F^m};\ g^{-1}y^\sigma g\in \dot{w}B\}$, *where $d=\dim U\cap \dot{w}U\dot{w}^{-1}$.*

1.4. For any $x\in G^F$, write $x={}^{F^m}a^{-1}a$ with $a\in G$ and let $y=a^Fa^{-1}\in G^{F^m}$. By Lemma 1.2.1 and 1.3.1, $\mathrm{Tr}((x^{-1}F^m)^*, \sum_i(-1)^i H^i_c(X(w)))=\mathrm{Tr}(ya^K_w I_\sigma, Z^K)$.

Since $\mathrm{Tr}(ya_w^K I_\sigma, Z^K)$ does not depend on the σ-conjugacy class of y, we have

Theorem 1.4.1. *For any* $y \in G^{F^m}$, $\mathrm{Tr}((n_{K/k}(y)^{-1}F^m)^*, \sum_i (-1)^i H_c^i(X(w)))$ $= \mathrm{Tr}(ya_w^K I_\sigma, Z^K)$.

Remark 1.4.2. (i) The above formula (and also Lemma 1.2.1, 1.3.1) were first appeared in [2]. This was informed to the author by Kawanaka.

(ii) It should be noted here that there are similar formulae to that of the theorem. If F^m acts canonically on R_T^θ or $R_{L\subset P}(\pi)$, the analogy of the theorem is also true as is easily checked.

1.5. Let δ be the smallest integer $\geqslant 1$ such that F^δ acts trivially on W. Let $\rho \in \mathcal{E}(G^F, \{1\})$ (=the set of all (equivalence classes of) unipotent representations of G^F). By Lusztig [12], Coro. 3.9, if $\rho \in H_c^i(X(w))_\mu$ (=the generalized μ-eigenspace of $F^{\delta *}$ on $H_c^i(X(w))$), then μ is uniquely determined (up to an integral power of q^δ) by ρ (not depending on i or w).

Definition 1.5.1. For $\rho \in \mathcal{E}(G^F, \{1\})$, let μ be as above. Define λ_ρ to be the constant such that $\lambda_\rho = \mu q^{\delta r}$ for some $r \in \boldsymbol{Z}$ and $1 \leqslant |\lambda_\rho| < q^\delta$.

For $\rho \in \mathcal{E}(G^F, \{1\})$, let $H_c^i(X(w))_\rho$ be the largest subspace of $H_c^i(X(w))$ on which G^F acts by a multiple of ρ. Then

Lemma 1.5.2. *For any* $\rho \in \mathcal{E}(G^F, \{1\})$ *and* $w \in W$, *there exists* $f_{\rho,w}(X) \in \boldsymbol{Z}[X, X^{-1}]$ *such that if* δ *divides* m,
$\mathrm{Tr}((x^{-1}F^m)^*, \sum_i (-1)^i H_c^i(X(w))_\rho) = f_{\rho,w}(q^m)\lambda_\rho^{m/\delta}\rho(x)$ *for any* $x \in G^F$ *and* $f_{\rho,w}(1) = \langle \rho, R_{T_w}^1 \rangle$.

2. Split case

2.1. In introducing the notation we only assume that G splits over K. Let $\mathfrak{A}^K(W) = \mathrm{End}_{G^{F^m}} Z^K$ and S the set of simple reflections of W (corresponding to B). Let $\mathfrak{A}(W)$ be the generic algebra of $\mathfrak{A}^K(W)$ over the extension field of $\boldsymbol{Q}(X)$ (X: indeterminate) and $\{a_w; w \in W\}$ be its basis. ($\mathfrak{A}^K(W)$ is obtained from $\mathfrak{A}(W)$ by the specialization $X \mapsto q^m$ or more precisely by the homomorphism from the integral closure of $\boldsymbol{Q}[X]$ to $\boldsymbol{Q}$ which maps X to q^m.) Let $\hat{W}$ be the set of equivalence classes of the irreducible representation of W. For any $\chi \in \hat{W}$, let ν_χ, ν_χ^K, ρ_χ^K be the corresponding irreducible representation (or its character) of $\mathfrak{A}(W)$, $\mathfrak{A}^K(W)$, G^{F^m} respectively. Then Z^K can be written in the form: $Z^K = \bigoplus_{\chi \in \hat{W}} \nu_\chi^K \otimes \rho_\chi^K$. For an F-stable subset $J \subseteq S$, let W_J be the subgroup of W generated by J, P_J the corresponding standard parabolic subgroup of G, L_J its standard Levi subgroup and $Z_J^K = \mathrm{Ind}_{B^{F^m}}^{P_J^{F^m}} 1 (= \mathrm{Ind}_{(B\cap L_J)}^{L_J^{F^m}} 1$ as $L_J^{F^m}$-modules). Z_J^K is canonically regarded as a subspace of Z^K and $\mathrm{End}_{P_J^{F^m}} Z_J^K = \sum_{w \in W_J} \overline{\boldsymbol{Q}}_l\, a_w |_{Z_J^K}$. The following are also defined: $\mathfrak{A}^K(W_J)$, $\mathfrak{A}(W_J)$, $\{\nu_\chi, \nu_\chi^K, \rho_\chi^K;\ \chi \in \hat{W}_J\}$. Since W_J is a parabolic subgroup of W, $\mathfrak{A}(W_J)$ (resp. $\mathfrak{A}^K(W_J)$) is regarded as a subalgebra of $\mathfrak{A}(W)$ (resp. $\mathfrak{A}^K(W)$). For any $\chi' \in \hat{W}_J$ and $\chi \in \hat{W}$, define the non-negative integer $n_{\chi,\chi'}$ by: $\mathrm{Ind}_{W_J}^W \chi' = \sum_{\chi \in \hat{W}} n_{\chi,\chi'} \chi$. For $\chi' \in \hat{W}_J$, let $Z_{\chi'}^K$ (resp. $Z_{J,\chi'}^K$) be the largest subspace of Z^K (resp. Z_J^K) on which $\mathfrak{A}^K(W_J)$ acts by a multiple of $\nu_{\chi'}^K$. For $\chi \in \hat{W}$, Z_χ^K is defined similarly. The following are checked easily: for $\chi' \in \hat{W}_J$, $\mathrm{Ind}_{P_J^{F^m}}^{G^{F^m}} Z_{J,\chi'}^K = Z_{\chi'}^K$, $Z_{J,\chi'}^K = \nu_{\chi'} \otimes \rho_{\chi'}^K$, $Z_{\chi'}^K = \sum_{\chi \in \hat{W}} n_{\chi,\chi'} \nu_{\chi'}^K \otimes \rho_\chi^K$, and for $\chi' \in \hat{W}_J$ and $\chi \in \hat{W}$, $Z_\chi^K \cap Z_{\chi'}^K = n_{\chi,\chi'} \nu_{\chi'}^K \otimes \rho_\chi^K$.

2.2. Henceforth in this section we assume G to be split over k. Then the mapping I_σ commutes with any $a_w^K (w \in W)$, thus with $\mathfrak{A}^K(W)$. Therefore each ρ_χ^K is regarded as an irreducible G^{F^m} A-modules which is denoted by $\tilde{\rho}_\chi^K$. By Theorem 1.4.1, we have

Lemma 2.2.1. *For any* $y \in G^{F^m}$,
$\mathrm{Tr}((n_{K k/}(y)^{-1} F^m)^*, \sum_i (-1)^i H_c^i(X(w))) = \sum_{\chi \in \hat{W}} \nu_\chi^K(a_w^K) \tilde{\rho}_\chi^K(y\sigma)$.

Let $J \subset S$ be F-stable. $\tilde{\rho}_{\chi'}^k (\chi' \in \hat{W}_J)$ are similiarly defined as $\tilde{\rho}_\chi^K (\chi \in \hat{W})$. Now, for any $z \in Z_J^K$ and $g \in G^{F^m}$, $I_\sigma(gz) = {}^\sigma g I_\sigma(z)$. Thus for $\chi' \in \hat{W}_J$, $\mathrm{Ind}_{P_J^{F^m} A}^{G^{F^m} A} Z_{J,\chi'}^K = Z_{\chi'}^K$ as G^{F^m} A-modules. Hence

Lemma 2.2.2. *Assume* $\mathrm{Ind}_{W_J}^W \chi' = \sum_{\chi \in \hat{W}} n_{\chi,\chi'} \chi\, (\chi' \in \hat{W}_J,\ n_{\chi,\chi'} \geqslant 0)$. *Then* $\mathrm{Ind}_{P_J^{F^m}}^{G^{F^m}} \rho_{\chi'}^K = \sum_{\chi \in \hat{W}} n_{\chi,\chi'} \rho_\chi^K$ and $\mathrm{Ind}_{P_J^{F^m} A}^{G^{F^m} A} \tilde{\rho}_{\chi'}^K = \sum_{\chi \in \hat{W}} n_{\chi,\chi'} \tilde{\rho}_\chi^K$.

Lemma 2.2.3. *Assume the Dynkin graph of G does not have irreducible components of type E_7 or E_8. Assume that for any $J \subseteqq S$ and $\chi' \in \hat{W}_J$, there exists the lifting of $\rho_{\chi'}^k$ in $\widehat{L_J^{F^m}}$. Then for any $\chi \in \hat{W}$ and $y \in G^{F^m}$, $\rho_\chi^k(\mathfrak{N}_{K/k}(y)) = \tilde{\rho}_\chi^K(y\sigma)$.*

Proof. By Lemma 1.1.9, $(\mathrm{Ind}_{B^F}^{G^F} 1)\, (\mathfrak{N}_{K/k} y) = (\mathrm{Ind}_{B^{F^m} A}^{G^{F^m} A} 1)\, (y\sigma)$ for any $y \in G^{F^m}$. Thus

$$\text{(a)} \qquad \sum_{\chi \in \hat{W}} \dim \chi\, \chi_{\rho_\chi^k}(\mathfrak{N}_{K/k} y) = \sum_{\chi \in \hat{W}} \dim \chi\, \chi_{\tilde{\rho}_\chi^K}(y\sigma) \quad \text{for any } y \in G^{F^m}.$$

The existence of the lifting of each ρ_χ^k shows for each $\chi \in \hat{W}$ there exists $\chi' \in \hat{W}$ such that $\rho_\chi^k(\mathfrak{N}_{K/k} y) = c\, \tilde{\rho}_{\chi'}^K(y\sigma)$ for any $y \in G^{F^m}$ and $c = 1$. (This is checked by taking the inner product with the relation (a). See Lemma 1.1.8.) If $\chi = 1$, the statement of the lemma is obvious. If $\chi = St_W$ (=the sign character of W), it is also obvious. This proves the case when the semisimple rank of G is 1. Assume the semisimple rank of $G \geqslant 2$ and the statement holds for any L_J with $J \subsetneqq S$.

Let $J \subsetneqq S$. Then for any $\chi' \in \hat{W}_J$ and $y \in G^{F^m}$, $\rho^k_{\chi'}(\mathfrak{N}_{K/k}y) = \tilde{\rho}^K_{\chi'}(y\sigma)$. Write $\mathrm{Ind}^W_{W_J}\chi' = \sum_{\chi \in \hat{W}} n_{\chi,\chi'}\chi$. Then by Lemma 2.2.2, $\sum_{\chi \in \hat{W}} n_{\chi,\chi'}\tilde{\rho}^K_\chi(\mathfrak{N}_{K/k}y) = \sum_{\chi \in \hat{W}} n_{\chi,\chi'}\tilde{\rho}^K_\chi(y\sigma)$ for any $y \in G^{F^m}$. Thus the lemma is an easy consequence of the following well known result (cf. Benson-Curtis [1]):

Let (W, S) be the Weyl group which does not have the irreducible factors of type G_2, E_7 or E_8 and assume rank $(W, S) \geqslant 2$. For $\chi_1, \chi_2 \in \hat{W}$, if $\chi_1|_{W_J} = \chi_2|_{W_J}$ for any $J \subsetneqq S$, then $\chi_1 = \chi_2$.

By Lemma 2.2.1 and 2.2.3 we have

Lemma 2.2.4. *Assume the assumption of* Lemma 2.2.3. *Then*

$$\mathrm{Tr}((x^{-1}F^m)^*, \sum_i (-1)^i H^i_c(X(w)) = \sum_{\chi \in \hat{W}} \nu^K_\chi(a^K_w)\rho^k_\chi(n^{-1}_{k/k}x) \quad \textit{for any } x \in G^F .$$

2.3. If $G = GL_n$, we can easily check the following theorem, which is proved in [2] and also by Lusztig independently.

Theorem 2.3.1. *Assume* $G = GL_n$. *Then*

(i) $\rho^k_\chi(n_{K/k}y) = \tilde{\rho}^K_\chi(y\sigma)$ *for any* $\chi \in \hat{W}$ *and* $y \in G^{F^m}$,

(ii) $f_{\rho_\chi, w}(X) = \nu_\chi(a_w)$ *for any* $\chi \in \hat{W}$ *and* $w \in W$,

(iii) $|X^{F^m}_w| = \sum_{\chi \in \hat{W}} \nu^K_\chi(a^K_w) \dim \rho^k_\chi$.

2.4. In 2.4 we assume $G = Sp_{2n}$, SO_{2n+1} or SO^+_{2n}.

Lemma 2.4.1. *If* $(m, 2p) = 1$, *then*

(i) $\sum_\rho f_{\rho,w}(q^m)\lambda^m_\rho \rho = \sum_{\chi \in \hat{W}} \nu^K_\chi(a^K_w)\rho^k_\chi \cdot n^{-1}_{k/k}$,

(ii) $\sum_\rho f_{\rho,w}(q^m)\lambda^m_\rho \dim \rho = \sum_{\chi \in \hat{W}} \nu^K_\chi(a^K_w) \dim \rho^k_\chi$,

where ρ *ranges over* $\mathcal{E}(G^F, \{1\})$.

Proof. By Lemma 1.5.2 and 2.2.1, $\sum_\rho f_{\rho,w}(q^m)\lambda^m_\rho \rho(n_{K/k}y) = \sum_{\chi \in \hat{W}} \nu^K_\chi(a^K_w)\tilde{\rho}^K_\chi(y\sigma)$ for any $y \in G^{F^m}$. By Theorem 1.1.6 and Lemma 2.2.3, $\tilde{\rho}^K_\chi(y\sigma) = \rho^k_\chi(\mathfrak{N}_{K/k}y) = \rho^k_\chi(n^{-1}_{k/k}n_{K/k}y)$ for any $y \in G^{F^m}$. Thus we have (i). Since $n_{k/k}(\{1\}) = \{1\}$, we have (ii).

To proceed further we need some lemmas. The following one is obvious.

Lemma 2.4.2. *Let* $c_1, \cdots, c_r, x_1, \cdots, x_r \in \bar{\boldsymbol{Q}}^\times_l$. *Assume* $\sum_{1 \leqslant i \leqslant r} c_i x^t_i = 0$ *for* $t = 1, \cdots, r$. *Then there exist* $1 \leqslant i \neq j \leqslant r$ *such that* $x_i = x_j$.

Lemma 2.4.3. *Let* $f(X), g(X) \neq 0 \in \bar{\boldsymbol{Q}}_l[X]$, t *a positive integer* (*maybe* 1) *and* $\lambda \in \boldsymbol{Q}^\times_l$. *Assume* $f(q^m)\lambda^m = g(q^m)$ *for any positive integer* m *such that* $(m, t) = 1$. *Then* $\lambda = \zeta q^\alpha$ *with* ζ *a t-th root of unity and* α *an integer*.

Proof. Write $f(X)=\sum_{0\leqslant i\leqslant r} a_i X^i$, $g(X)=\sum_{0\leqslant i\leqslant s} b_i X^i (a_i,\ b_i\in \bar{\boldsymbol{Q}}_l)$. By the assumption, $f(q^{mt+1})\lambda^{mt+1}=g(q^{mt+1})$ for any $m\in \boldsymbol{N}$. Thus $\sum_{0\leqslant i\leqslant r} a_i q^i \lambda(q^{ti}\lambda^t)^m=\sum_{0\leqslant i\leqslant s} b_i q^i (q^{ti})^m$ for any $m\in \boldsymbol{N}$. If $i\neq j$, $q^{ti}\neq q^{tj}$ and $q^{ti}\lambda^t\neq q^{tj}\lambda^t$. Thus, by Lemma 2.4.2, $q^{ti}\lambda^t=q^{tj}$ for some $0\leqslant i\leqslant r$, $0\leqslant j\leqslant s$. Therefore $\lambda=\zeta q^{\alpha}$ with ζ a t-th root of unity and α a positive integer.

The following proposition is known when q is larger than the Coxeter number of G (cf. Lusztig [12], p. 25, (d)).

Proposition 2.4.4. For any $\rho\in\mathcal{E}(G^F,\ \{1\})$, $\lambda_\rho=1$ or -1.

Proof. If ρ is not cuspidal, the computation of λ_ρ is reduced to the groups of smaller ranks. Thus it remains to check for the cuspidal $\rho_0\in\mathcal{E}(G^F,\ \{1\})$. Take $w\in W$ such that $\langle\rho_0,\ R^1_{T_w}\rangle\neq 0$. Then $f_{\rho_0,w}(X)\neq 0$ (cf. 1.5). If $(m,\ 2p)=1$, $\sum_\rho f_{\rho,w}(q^m)\lambda_\rho^m \dim\rho=\sum_{\chi\in\hat{W}} \nu_\chi^K(a_w^K)\dim\rho_\chi^k$ by Lemma 2.4.1, (ii). We may assume if $\rho\neq\rho_0$, $\lambda_\rho=1$ or -1. Thus, for any positive integer m such that $(m,\ 2p)=1$, we have $f_{\rho_0,w}(q^m)\lambda^m_{\rho_0}\dim\rho_0+\sum_{\rho\neq\rho_0} f_{\rho,w}(q^m)\lambda_\rho \dim\rho=\sum_{\chi\in\hat{W}}\nu_\chi^K(a_w^K)\dim\rho_\chi^k$. Applying Lemma 2.4.3 we have $\lambda_{\rho_0}^{2p}=1$ (since $0\leqslant|\lambda_{\rho_0}|<q$). Thus it suffices to prove $\lambda_{\rho_0}\in\boldsymbol{Q}$. But for any positive integer m, $f_{\rho_0,w}(q^m)\lambda^m_{\rho_0}\dim\rho_0+\sum_{\rho\neq\rho_0} f_{\rho,w}(q^m)\lambda_\rho\dim\rho=\mathrm{Tr}(F^{m*},\ \sum_i(-1)^i H^i_c(X(w)))=|X(w)^{F^m}|$. Thus $f_{\rho_0,w}(q^m)\lambda^m_{\rho_0}\in\boldsymbol{Q}$ for any positive integer m. Since $f_{\rho_0,w}(X)\neq 0$, there exists an integer m_0 such that if $m\geqslant m_0$, $f_{\rho_0,w}(q^m)\neq 0$. Thus if $m\geqslant m_0$, $\lambda^m_{\sigma_0}\in\boldsymbol{Q}$. Therefore $\lambda_{\rho_0}=(\lambda_{\rho_0})^{m_0+1}\lambda_{\rho_0}^{-m_0}\in\boldsymbol{Q}$.

Lemma 2.4.5. $\sum_\rho f_{\rho,w}(X)\lambda_\rho\rho=\sum_{\chi\in\hat{W}}\nu_\chi(a_w)\rho_\chi^k\cdot n_{k/k}^{-1}$ *as* $\boldsymbol{Q}[X]$*-linear combinations of class functions of* G^F.

Proof. Fix $y\in G^F$. By Lemma 2.4.1 and Proposition 2.4.4, if $(m,\ 2p)=1$, then $\sum_\rho f_{\rho,w}(q^m)\lambda_\rho\rho(y)=\sum_{\chi\in\hat{W}}\nu_\chi^K(a_w^K)\rho_\chi^k(n_{k/k}^{-1}y)$. Since there exist infinitely many positive integers m such that $(m,\ 2p)=1$, $\sum_\rho f_{\rho,w}(X)\lambda_\rho\rho(y)=\sum_{\chi\in\hat{W}}\nu_\chi(a_w)\rho_\chi^k(n_{k/k}^{-1}y)$ as polynomials in X (with $y\in G^F$ being fixed). This proves the lemma.

For $\chi\in\hat{W}$, let $R_\chi=|W|^{-1}\sum_{w\in W}\chi(w)\,R^1_{T_w}$. Then

Lemma 2.4.6. $\rho_\chi^k\cdot n_{k/k}^{-1}=\sum_\rho\langle R_\chi,\ \rho\rangle\lambda_\rho\rho$.

Proof. By the specialization $X\mapsto 1$, the relation in Lemma 2.4.5 is specialized to: $\sum_\rho\langle R^1_{T_w},\ \rho\rangle\lambda_\rho\rho=\sum_{\chi\in\hat{W}}\chi(w)\rho_\chi^k\cdot n_{k/k}^{-1}$. Hence

$$\rho_\chi^k\cdot n_{k/k}^{-1}\ (=|W|^{-1}\sum_{w\in W}\chi(w)\sum_{\chi_1\in\hat{W}}\chi_1(w)\rho_{\chi_1}^k\cdot n_{k/k}^{-1})$$
$$=|W|^{-1}\sum_{w\in W}\chi(w)\sum_\rho\langle R^1_{T_w},\ \rho\rangle\lambda_\rho\rho=\sum_\rho\langle R_\chi,\ \rho\rangle\lambda_\rho\rho\ .$$

Lemma 2.4.7. (i) *For any* $w\in W$ *and* $\rho\in\mathcal{E}(G^F,\{1\})$, $f_{\rho,w}(X)=\sum_{\chi\in\hat{W}}\nu_\chi(a_w)\langle R_\chi,\rho\rangle$.

(ii) $\sum_\rho f_{\rho,w}(X)\rho=\sum_{\chi\in\hat{W}}\nu_\chi(a_w)R_\chi$.

Proof. (i) $\lambda_\rho f_{\rho,W}(X)=\langle\sum_{\rho_1}f_{\rho_1,w}(X)\lambda_{\rho_1}\rho_1,\rho\rangle=\sum_{\chi\in\hat{W}}\nu_\chi(a_w)\langle\rho_\chi^k\cdot n_{k/k}^{-1},\rho\rangle$ (by Lemma 2.4.5)$=\sum_{\chi\in\hat{W}}\nu_\chi(a_w)\langle R_\chi,\rho\rangle\lambda_\rho$ (by Lemma 2.4.6). This proves (i). (ii) is an easy consequence of (i).

Theorem 2.4.8. *Let* $w\in W$.

(i) *If m is odd,* $|X(w)^{F^m}|=\sum_{\chi\in\hat{W}}\nu_\chi^K(a_w^K)\dim\rho_\chi^k$.

(ii) *If m is even,* $|X(w)^{F^m}|=\sum_{\chi\in\hat{W}}\nu_\chi^K(a_w^K)\dim R_\chi$.

Proof. $|X(w)^{F^m}|=\sum_\rho f_{\rho,w}(q^m)\lambda_\rho^m\dim\rho$. Assume m is odd. Then $|X(w)^{F^m}|=\sum_\rho f_{\rho,w}(q^m)\lambda_\rho\dim\rho$ (since $\lambda_\rho=1$ or -1)$=\sum_{\chi\in\hat{W}}\nu_\chi^K(a_w^K)\,\rho_k^\chi(n_{k/k}^{-1}\{1\})$ (by Lemma 2.4.5)$=\sum_{\chi\in\hat{W}}\nu_\chi^K(a_w^K)\dim\rho_\chi^k$. Assume m is even. Then $|X(w)^{F^m}|=\sum_\rho f_{\rho,w}(q^m)\dim\rho$ $=\sum_{\chi\in\hat{W}}\nu_\chi^K(a_w^K)\dim R_\chi$ (by Lemma 2.4.7, (ii)).

The following lemma is well known (cf. [4]).

Lemma 2.4.9. *Let* $\mathfrak{A}$ *be a semisimple and symmetric algebra over the algebraic closed field of characteristic* 0. *Let* $\{e_1,\cdots,e_r\}$ *be a basis of* $\mathfrak{A}$ *and* $\{e_1^*,\cdots,e_r^*\}$ *be its dual basis. Let* χ_1,χ_2 *be the irreducible characters of* $\mathfrak{A}$. *Then* $\sum_i\chi_1(e_i)\chi_2(e_i^*)=0$ *if and only if* $\chi_1\neq\chi_2$.

Theorem 2.4.10. (i) *If m is odd,* $\tilde{\rho}_\chi^K(y\sigma)=\rho_\chi^k(\mathfrak{A}_{K/k}y)$ *for any* $\chi\in\hat{W}$ *and* $y\in G^{F^m}$.

(ii) *If m is even,* $\tilde{\rho}_\chi^K(y\sigma)=R_\chi(n_{K/k}y)$ *for any* $\chi\in\hat{W}$ *and* $y\in G^{F^m}$.

Proof. For any $y\in G^{F^m}$ and $w\in W$, $\sum_{\chi\in\hat{W}}\nu_\chi^K(a_w^K)\tilde{\rho}_\chi^K(y\sigma)=\mathrm{Tr}(((n_{K/k}y)^{-1}F^m)^*,\sum_i(-1)^iH_c^i(X(w))$ (by Lemma 2.2.1)$=\sum_\rho f_{\rho,w}(q^m)\lambda_\rho^m\rho(n_{K/k}y)$. Assume m is odd. Then $\sum_{\chi\in\hat{W}}\nu_\chi^K(a_w^K)\tilde{\rho}_\chi^K(y\sigma)=\sum_\rho f_{\rho,w}(q^m)\lambda_\rho\rho(n_{k/k}\mathfrak{N}_{K/k}y)=\sum_{\chi\in\hat{W}}\nu_\chi^K(a_w^K)\rho_\chi^k(\mathfrak{N}_{K/k}y)$ (by Lemma 2.4.5). Thus $\sum_{\chi\in\hat{W}}\nu_\chi^K(a_w^K)\tilde{\rho}_\chi^K(y\sigma)=\sum_{\chi\in\hat{W}}\nu_\chi^K(a_w^K)\rho_\chi^k(\mathfrak{N}_{K/k}y)$. Hence we have (i) by the orthogonality relations in Lemma 2.4.9. (ii) is proved similarly.

REMARK 2.4.11. If char $\boldsymbol{F}_q\neq2$, then $R_\chi=R_\chi\cdot n_{k/k}^{-1}$ by the following lemma. Therefore "$n_{k/k}$" in (ii) of Theorem 2.4.10 can be replaced by "$\mathfrak{N}_{K/k}$" if char $\boldsymbol{F}_q\neq2$. This seems to be true even if char $\boldsymbol{F}_q=2$.

Lemma 2.4.12. *Let G be a connected reductive group over $\boldsymbol{F}_q$. (We do not assume the assumption imposed on G in 2.4.) Let $x \in G^F$ and $x = su$ be the Jordan decomposition (s: a semisimple element, u: a unipotent element). Assume u is contained in the identity component of the centralizer of u in $Z_G(s)^0$. (Notice $u \in Z_G(s)^0$ by* [16], Corollary 4.4.) *Then for any F-stable maximal torus T of G and linear character θ of T^F, $R_T^\theta(n_{k/k}(x)) = R_T^\theta(x)$.*

Proof. Let $H = Z_G(s)^0$. Let T' be an F-stable maximal torus of H. Take $a \in T'$ such that $s = a^{-1F}a$. Take $b \in Z_H(u)^0$ such that $u = b^{-1F}b$. Then $x = su = sb^{-1F}b = b^{-1}s^Fb = b^{-1}a^{-1F}a^Fb = (ab)^{-1F}(ab)$. Thus $n_{k/k}(x) = {}^F(ab)\,(ab)^{-1} = {}^Fa^Fbb^{-1}a^{-1}$ ${}^Fabub^{-1}a^{-1} = {}^Faua^{-1}$ (b commutes with u) $= {}^Faa^{-1}aua^{-1} = saua^{-1}$. Therefore $n_{k/k}(x) = saua^{-1}$. Since s commutes with aua^{-1} and aua^{-1} is a unipotent element, $n_{k/k}(x) = s(aua^{-1})$ is the Jordan decomposition of $n_{k/k}(x)$. Let $\{g_1, \cdots, g_r\}$ be the representatives of $H^F \backslash \{g \in G^F; g^{-1}sg \in T\}$. Then by [2], Theorem 4.2, we have $R_T^\theta(x) = \sum_{1 \leqslant i \leqslant r} Q_{g_iTg_i^{-1},H}(u)\theta(g_i^{-1}sg_i)$. Similarly, $R_T^\theta(n_{k/k}(x)) = \sum_{1 \leqslant i \leqslant r} Q_{g_iTg_i^{-1},H}(aua^{-1})$ $\theta(g_i^{-1}sg_i)$. Let H_{ad} be the adjoint group of H and $\pi: H \to H_{ad}$ be the canonical mapping. Since $a^{-1F}a = s \in Z(H)$, $\pi(a) \in H_{ad}^F$. Thus $\pi(u)$ and $\pi(aua^{-1})$ are conjugate in H_{ad}^F. Therefore $Q_{g_iTg_i^{-1},H}(u) = Q_{\pi(g_iTg_i^{-1}),H_{ad}}(\pi(u)) = Q_{g_iTg_i^{-1},H}(aua^{-1})$. Hence $R_T^\theta(x) = R_T^\theta(n_{k/k}(x))$.

2.5. In 2.5, we wish to describe some conjectural statements flourishing from Lemma 2.4.6, if we assume Conjecture 4.3 of Lusztig [12]. To do this we need to recall some results of [11], [12]. For $\Lambda \in \Phi_n$ (resp. Φ_n^+), let ρ_Λ be the corresponding unipotent representations of Sp_{2n}^F or SO_{2n+1}^F (resp. $SO_{2n}^{+,F}$). For $\chi \in \hat{W}_n$ (resp. $\hat{W}_n$), let Λ be the corresponding symbol class in Φ_n (resp. Φ_n^+) and we put $R_\Lambda = R_\chi$. For $\Lambda \in \Phi_n$ (resp. Φ_n^+), write $\Lambda = (X \cup (Y - I), X \cup I)$, where X, Y are finite subsets of $\{0, 1, 2, \cdots\}$, $X \cap Y = \phi$, I is a subset of Y such that $2\,|I| + 1 \equiv |Y| \bmod 4$ (resp. $2\,|I| \equiv |Y| \bmod 4$). Now, fix X and Y. We put $|Y| = 2s$ or $2s+1$, and assume $s > 0$ if $|Y| = 2s$. Let $Y = \{\lambda_0 < \lambda_1 < \lambda_2 \cdots\}$, $Y^0 = \{\lambda_0, \lambda_2, \lambda_4, \cdots\}$ and $Y^1 = \{\lambda_1, \lambda_3, \lambda_5, \cdots\}$. Let $\mathcal{P}$ be the set of all subsets of Y and $\mathcal{P}_s = \{I \in \mathcal{P}: |I| \equiv s \bmod 2\}$. Then $\mathcal{P}$ is regarded as a vector space over $\boldsymbol{F}_2$ by the addition: $I, J \in \mathcal{P} \mapsto IJ = I \cup J - I \cap J$ and $\mathcal{P}_0$ is regarded as a subspace. By the bijection $\mathcal{P}_s \to \mathcal{P}_0$ ($I \mapsto IY^1$), we can regard $\mathcal{P}_s$ as a vector space over $\boldsymbol{F}_2$. Define $Q: \mathcal{P}_s \to \{\pm 1\}$ ($I \mapsto (-1)^{(|I|-s)/2}$). If we identify $\boldsymbol{F}_2$ canonically with $\{\pm 1\}$, the mapping Q is regarded as a quadratic form on $\mathcal{P}_s$ whose associated bilinear form B is: $I, J \in \mathcal{P}_s \mapsto B(I, J) = (-1)^{|I \cap X^0| + |J \cap Y^1| + |I \cup J|}$. Thus the Fourier transform of Lusztig [11], [12] takes the form:

$$\hat{\rho}_{(X \cup I', X \cup I)} = 2^{-s} \sum_{J \in \mathcal{P}_s} B(I, J)\rho_{(X \cup J', X \cup J)} \quad \text{for } I \in \mathcal{P}_s\,.$$

DEFINITION 2.5.1. (i) For a class function f on G^F, let $f^\Delta = f \cdot n_{k/k}$. $\Delta^2 = 1$ and $\dim f^\Delta = \dim f$. (Notice that for any connected algebraic group G over $\boldsymbol{F}_q$, if $x^r \in Z_G(x)^0$, then $n^r_{k/k}(\{x\})=\{x\}$.)

(ii) Let $\mathfrak{R}_{X,Y}=\sum_{J\in\mathcal{P}_s}\bar{Q}_I\rho_{(X\cup J',X\cup J)}$. Define the linear automorphims $\tilde{\Delta}$ of $\mathfrak{R}_{X,Y}$ by the condition $\hat{\rho}^{\tilde{\Delta}}_{(X\cup I',X\cup I)}=Q(I)\hat{\rho}_{(X\cap I',X\cup I)}$ for $I\in\mathcal{P}_s$. Since $\dim\hat{\rho}_{(X\cup I',X\cup I)}=0$ if $|I|\neq s$, we have $\dim f^{\tilde{\Delta}}=\dim f$ for any $f\in\mathfrak{R}_{X,Y}$.

It can easily be checked the following

Lemma 2.5.2. *For any* $J\in\mathcal{P}_s$,

$$Q(I)\rho^{\tilde{\Delta}}_{(X\cup I',X\cup I)}=2^{-s}\sum_{J\in\mathcal{P}_s}B(I,J)Q(J)\rho_{(X\cup J',X\cup J)}\,.$$

Theorem 2.5.3. *Assume Conjecture* 4.3 *in* [12] *is true. Then for* $I\in\mathcal{P}_s$, $\lambda_{\rho(X\cup I',X\cup I)}=Q(I)\ (=(-1)^{(|I|-s)/2})$.

Proof. By the induction of the semisimple rank of G, it is needed to check only for the cuspidal $\rho_{(X\cup I'_0,X\cup I_0)}$ (I_0 or $I'_0=Y$). Thus we may assume the statement is true if $I\neq I_0, I'_0$. Since we have assumed Conjecture 4.3 in [12], $R_{(X\cup I',X\cup I)}=\hat{\rho}_{(X\cup I',X\cup I)}$ if $|I|=s$. Thus by Lemma 2.4.6, $R_{(X\cup I',X\cup I)}=2^{-r}\sum_{J\in\mathcal{P}_s}B(I,J)Q(J)\,\rho_{(X\cup J',X\cup J)}$. But $\dim R^{\Delta}_{(X\cup I',X\cup I)}=\dim R_{(X\cup I',X\cup I)}=\dim R^{\tilde{\Delta}}_{(X\cup I',X\cup I)}$. Thus $2^{-s}\sum_{J\in\mathcal{P}_s}B(I,J)Q(J)\dim\rho_{(X\cup J',X\cup J)}=2^{-s}\sum_{J\in\mathcal{P}_s}B(I,J)\lambda_{\rho(X\cup J',X\cup J)}\dim\rho_{(X\cup J',X\cup J)}$ by Lemma 2.5.2. This relation shows our statement.

REMARK 2.5.4. (i) The statement of Theorem 2.5.3 is a counterpart of the statements for some families of the unipotent representations of the exceptional groups given in Lusztig [12], p. 45 and [13], p. 335.

(ii) Assume char $\boldsymbol{F}_q\neq 2$. Lemma 2.4.12 and the proof of Theorem 2.5.3 show that Δ and $\tilde{\Delta}$ coincide on the subspace $\mathfrak{S}$ of $\mathfrak{R}_{X,Y}$ which is spanned by $\{\rho_{(X\cup I',X\cup I)},\ \rho^{\Delta}_{(X\cup I',X\cup I)};I\in\mathcal{P}_s,|I|=s\}$ and $\{R_{(X\cup I',X\cup I)};I\in\mathcal{P}_s,|I|=s\}$ under the assumption of Theorem 2.5.3. If $|Y|=1$, 3 or 4, $\mathfrak{S}=\mathfrak{R}_{X,Y}$. If $|Y|=5$ or 6, $\dim\mathfrak{S}=\dim\mathfrak{R}_{X,Y}-1$. We may ask if the following is true (cf. Remark 2.4.11).

CONJECTURE 2.5.5. $\tilde{\Delta}=\Delta$.

3. Unitary case

The method which we applied in the case of split classical groups is also effective for the unitary groups. Let G be the unitary group U_n over $\boldsymbol{F}_q$ and we assume m is an even integer. The Weyl group W is canonically identified with the symmetric group S_n and we assume the generic algebra $\mathfrak{A}(W)$ (cf. 2.1) is over the extension field of $\mathfrak{A}(X)$ which contains $X^{1/2}$ ($X^{1/2}$ being fixed). Let $\mathcal{P}(n)$ be the set of all partitions of n. For $\alpha\in\mathcal{P}(n)$, let χ_α (resp. ν_{χ_α}) be the corresponding irreducible representation (or its character) of W (resp. $\mathfrak{A}(W)$). The following lemma is easily checked by the induction on n.

Lemma 3.1. *Let w_0 be the longest element of W and $\alpha=(\alpha_1\geqslant\cdots\geqslant\alpha_s>0)$ $\in\mathcal{P}(n)$. Define $C_\alpha=\binom{n}{2}+\sum_{\substack{1\leqslant i\leqslant s\\ 1\leqslant j\leqslant\alpha_i}}(j-i)$. Then $a_{w_0}^2$ acts as a scalar X^{C_α} on the representation v_{χ_α} of $\mathfrak{A}(W)$.*

Let the notation be as in 2.1. We write ρ_α^K instead $\rho_{\chi_\alpha}^K$ of for $\alpha\in\mathcal{P}(n)$ to simplify the notation. Since $a_{w_0}^K I_\sigma$ commutes with $\mathfrak{A}^K(W)$, each irreducible component ρ_α^K of Z^K is regarded as a G^{F^m} A-module ρ_α^K by the mapping $\sigma\mapsto(q^m)^{-C_\alpha/m}$ $a_{w_0}^K I_\rho$.

For $\alpha\in\mathcal{P}(n)$, let $\rho_\alpha^k=|W|^{-1}\sum_{w\in W}\chi_\alpha(ww_0)R_{T_w}^1$. If we put η_α=the signature of dim ρ_α^k, then by [14], $\eta_\alpha\rho_\alpha^k$ is the irreducible representation of G^F and all the unipotent representations of G^F are of this form. For the simplification of the notation we let $f_{\alpha,w}(X)=\eta_\alpha f_{\eta_\alpha\rho_\alpha^k,w}(X)$ $(\alpha\in\mathcal{P}(n),\ w\in W)$, $\lambda_\alpha=\lambda_{\eta_\alpha\rho_\alpha^k}$. Then

Theorem 3.2. *Assume* char $\boldsymbol{F}_q\neq 2$. *Let $\alpha\in\mathcal{P}(n)$ and $w\in W$.* Then

(i) $\rho_\alpha^k(n_{K/k}y)=(-1)^{mC_\alpha/2}\tilde{\rho}_\alpha^K(y\sigma)$ *for any even integer m and $y\in G^{F^m}$,*

(ii) $f_{\alpha,w}(q^m)\lambda_\alpha^{m/2}=v_{\chi_\alpha}^K(a_w^K a_{w_0}^K)\,(-q)$ *for any even integer m,*

(iii) $|X(w)^{F^m}|=\sum_{\alpha\in\mathcal{P}(n)}v_{\chi_\alpha}^K(a_w^K a_{w_0}^K)\,(-q)^{-mC_\alpha/2}$ dim ρ_α^k *for any even integer m.*

Our proof is based on Kawanaka [7] as is stated in the introduction. In this respect, (i) of the theorem for cuspidal or subcuspidal ρ_α^k's is essential. The detailed arguments, which is slightly tedious, are omitted.

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