Carroll, R.
Osaka J. Math.
19 (1982), 815-831

TRANSMUTATION, GENERALIZED TRANSLATION, AND TRANSFORM THEORY. PART I

ROBERT CARROLL

(Received July 22, 1980)

1. Introduction. The development of transmutation theory initiated in Carroll [8; 9] has proved to be very productive and we continue this line of investigation here (see [8] for extensive references). Let us think of $P(D)$ and $Q(D)$ as (usually second order) linear differential operators acting in spaces E and F. An operator B: $E \rightarrow F$, usually some kind of integral operator, transmutes P into Q if $QB=BP$ acting on suitable objects. In [8; 10] we displayed a framework of spaces and maps linking the various transforms arising from P and Q together with B, B^{-1}, B^*, etc. In this series of papers (I and II) we will deploy this framework in developing further the ideas of [8; 9], and proving some results announced in [10], as well as generating, in a sort of canonical way, some general Parseval formulas of Gasymov-Marčenko type (cf. [26; 27; 37]). Our framework also guided us to formulate the link between some of Fadeev's scattering technique in [24] and various connecting formulas for special functions involving Riemann-Liouville and Weyl type integrals (cf. [31]) and we have written this up as joint work in [12; 13]. Further extension of this scattering technique allows us also to give versions of the Gelfand-Levitan and Marčenko equations in a context based on harmonic analysis in symmetric spaces (see [11] and cf. also [29]): such Gelfand-Levitan equations are also derivable via a technique indicated here in Part II in our general transmutation framework.

2. General formulas. One way of finding a transmutation operator B: $P \rightarrow Q$ is to consider a Cauchy problem (*) $P(D_x)\varphi(x,y)=Q(D_y)\varphi(x,y)$ with $\varphi(x,0)=f(x)$ and $\varphi_y(x,0)=0$, where f is extended to $(-\infty,0)$ as an even function for example; then, given a unique solution to such problems, $Bf(y)$ can be defined as $\varphi(0,y)$ (cf. Lions [35]). A Cauchy problem is not always suitable here and unique solutions can often be assured by an appropriate combination $\mathcal{R}$ of data at $y=0$ plus growth conditions on φ. When $Q=P$ we speak of a generalized translation associated with P and write $\varphi(x,y)=T_x^y f(x)$; similarly for $P=Q$ we write $\varphi(x,y)=S_x^y f(x)$ so that $P \sim T$ and $Q \sim S$. It is important to note that not all transmutations need arise in the above manner (cf. [11; 13; 24]). As in Carroll [8; 9] let us consider generalized eigenfunctions of the form

$$(2.1)\qquad P(D_x)H(x,\mu)=\mu H(x,\mu);\ H(0,\mu)=1;$$
$$Q(D_y)\Theta(y,\nu)=\nu\Theta(y,\nu);\ \Theta(0,\nu)=1;$$
$$(2.2)\qquad P^*(D_x)\Omega(x,\mu)=\mu\Omega(x,\mu);$$
$$Q^*(D_y)W(y,\nu)=\nu W(y,\nu)$$

where P^* and Q^* denote formal adjoints. Here we are using $x\in[0,\infty)$ and $y\in[0,\infty)$ as a model situation but other intervals (e.g. $(-\infty,\infty)$) can also be envisioned. We will proceed somewhat formally at first. Thus assume whatever requirements $\mathcal{R}_H$ or $\mathcal{R}_\Theta$ of the form $D_x^kH(0,\mu)=0$ or $D_y^k\Theta(0,\mu)=0$ plus possible growth conditions are necessary in order to make H and Θ unique in suitable spaces $\boldsymbol{E}$ and $\boldsymbol{F}$. This is discussed in [8: 9] and examples are given; in the present situation simply take $H'(0,\mu)=0$ and $\Theta'(0,\nu)=0$ as a model (for higher order operators see e.g. [4; 5; 6; 7; 22; 33; 38; 41; 42; 46; 47; 48]). Similarly we will define bracket operations $\langle H(x,\mu),f(x)\rangle$ and $\langle H(x,\mu),F(\mu)\rangle$ for example without spelling out details. The first bracket is usually a (weighted) distribution pairing and the second a spectral type integral involving a suitable measure (see Section 3 for examples). Then define

$$(2.3)\qquad \beta(y,x)=\langle\Omega(x,\mu),\Theta(y,\mu)\rangle_\sigma.$$

Note here that μ and ν may be different in (2.2) and the σ pairing refers basically to a measure or distribution over the spectrum $\sigma(P)$ of P. In order for expressions such as (2.3) to make sense and be useful we must have some coincidence of $\sigma(P)$ and $\sigma(Q)$. Our examples in Section 3 below are illustrative here and provide situations where $\mu=-\lambda^2-\rho_P^2$ and $\nu=-\lambda^2-\rho_Q^2$; the pairing involves a measure $d\sigma$ over a λ range $(0,\infty)$. Also in such a case we shift notation and speak of transmuting $\hat{P}(D)=P(D)+\rho_P^2$ into $\hat{Q}(D)=Q(D)+\rho_Q^2$ and then the two spectra $\sigma(\hat{P})$ and $\sigma(\hat{Q})$ are identical; we basically assume this kind of situation holds here in general. It could be interesting to study cases where the two spectra are substantially different but we have not done this. Now let $U(x,\xi)$ be the solution of

$$(2.4)\qquad P(D_x)U=P(D_\xi)U;\ U(x,0)=f(x)$$

so that $U(x,\xi)=T_x^\xi f(x)$ (with $U(0,\xi)=f(\xi)$). Then formally

$$(2.5)\qquad \phi(x,y)=\langle\beta(y,\xi),U(x,\xi)\rangle$$

satisfies $(*)$ with $\phi(x,0)=f(x)$ and from $\phi(0,y)=(Bf)(y)=\langle\beta(y,\xi),f(\xi)\rangle$ we have $\beta(y,\xi)$ desplayed as the kernel of B. This can be made rigorous by assuming

$$(2.6)\qquad \langle\Omega(x,\mu),1\rangle_\sigma=\delta(x)$$

which is known to be true in many abstract and concrete situations (cf. Section

3). If one writes further

$$(2.7)\qquad \hat{f}(\mu) = \langle \Omega(x, \mu), f(x)\rangle = \mathbf{P}f(\mu)$$

then rearranging (2.5) it can be shown that

$$(2.8)\qquad \phi(x, y) = \langle \Theta(y, \mu), \hat{f}(\mu)H(x, \mu)\rangle_\sigma$$

which is in a form generally associated with separation of variables techniques for $(*)$ and this implies, setting $y=0$,

$$(2.9)\qquad f(x) = \langle \hat{f}(\mu), H(x, \mu)\rangle_\sigma = \mathbf{P}\hat{f}(x)$$

while the transmutation operator B is displayed as (setting $x=0$)

$$(2.10)\qquad Bf(y) = \langle \Theta(y, \mu), \hat{f}(\mu)\rangle_\sigma = \mathcal{Q}\hat{f}(y)\,.$$

Thus assuming $\mathcal{R}$ and $\mathcal{R}_H$ we can state (cf. Carroll [8; 9])

Theorem 2.1. *Given* (2.6) *the kernel of B is given by $\beta(y, x)$ of* (2.3) *and ϕ is given by* (2.5) *or* (2.8). *Further $\mathbf{P}=\mathbf{P}^{-1}$ and $B=\mathcal{Q}\mathbf{P}$.*

Proof. That ϕ satisfies $(*)$ follows from (2.6) and the uniqueness insured by $\mathcal{R}$ gives a well defined B with kernel β (provided the ϕ of (2.5) satisfies $\mathcal{R}$). The rearrangement of (2.5) involves looking at $\langle \Omega(\xi, \mu), U(x, \xi)\rangle = \psi(x, \mu)$ and $\mathcal{R}_H$ will give $\psi(x, \mu)=\hat{f}(\mu)H(x, \mu)$ (provided $\psi(\cdot, \mu)\in \mathbf{E}$). The inversion formula (2.9) is then immediate and (2.10) defines $\mathcal{Q}$. Q.E.D.

REMARK 2.2. This presentation is somewhat "neater" than that of Carroll [8; 9] and shows that the inversion (2.9) can be based on the uniqueness insured by $\mathcal{R}$ and $\mathcal{R}_H$.

Let us assume also that, working in suitable spaces, B is 1–1 with inverse $\mathcal{B}$ which can be then characterized as follows. Let $V(y, \eta)=S^y_\eta Bf(\eta)$ $(=S^\eta_y Bf(y))$ satisfy

$$(2.11)\qquad Q(D_y)V = Q(D_\eta)V;\; V(y, 0) = Bf(y)\,.$$

(so that $V(0, \eta)=Bf(\eta)$). Consider, in analogy with (2.3), but with a generally different bracket operation (cf. Section 3)

$$(2.12)\qquad \gamma(x, y) = \langle H(x, \nu), W(y, \nu)\rangle_\omega\,.$$

Then we can show that

$$(2.13)\qquad \tilde{\phi}(x, y) = \langle \gamma(x, \eta), V(y, \eta)\rangle = \phi(x, y)\,.$$

The brackets here involve generally different pairings than (2.3) and (2.5) (e.g. a different spectral measure arises in (2.12)). Indeed $P(D_x)\tilde{\phi}=\langle P(D_x)\gamma, V\rangle$

while $Q(D_y)\tilde{\phi}=\langle\gamma, Q(D_y)V\rangle=\langle\gamma, Q(D_\eta)V\rangle=\langle Q^*(D_\eta)\gamma, V\rangle$ and $P(D_x)\gamma=Q^*(D_\eta)\gamma$ so that $\tilde{\phi}$ satisfies $(*)$ with

$$\tilde{\phi}(0, y)=\langle\gamma(0, \eta), V(y, \eta)\rangle=\langle\langle 1, W(\eta, \nu)\rangle_\omega, V(y, \eta)\rangle= \langle\delta(\eta), V(y, \eta)\rangle=V(y, 0)=Bf(y) \tag{2.14}$$

where it is required that (cf. (2.6))

$$\langle W(y, \nu), 1\rangle_\omega=\delta(y). \tag{2.15}$$

Now if Bf is extended suitably for $y<0$ and natural conditions $\hat{\mathcal{R}}$, compatible with $\mathcal{R}$, can be imposed on $D_x^p\phi(0,y)$ which together with possible growth conditions guarantee a unique solution of (1.1) for $x\geq 0$ with data $Bf(y)$ then $\tilde{\phi}=\phi$ as asserted (provided $\tilde{\phi}$ as constructed, and ϕ, satisfy $\hat{\mathcal{R}}$). Since $V(0, \eta)=Bf(\eta)$ we have then

$$\tilde{\phi}(x, 0)=\langle\gamma(x, \eta)V(0, \eta)\rangle=f(x)=\langle\gamma(x, \eta), Bf(\eta)\rangle= \langle\langle H(x, \nu), W(\eta, \nu)\rangle_\omega, Bf(\eta)\rangle=\langle H(x, \nu), \langle W(\eta, \nu), Bf(\eta)\rangle\rangle_\omega \tag{2.16}$$

which we display as

$$B\tilde{f}(\nu)=\langle W(\eta, \nu), Bf(\eta)\rangle=\mathbf{Q}Bf(\nu) \tag{2.17}$$

$$f(x)=\mathcal{B}Bf(x)=\langle H(x, \nu), \widetilde{Bf}(\nu)\rangle_\omega=\boldsymbol{\rho}\mathbf{Q}Bf(x) \tag{2.18}$$

so that $\mathcal{B}=\boldsymbol{\rho}\mathbf{Q}$ ($\boldsymbol{\rho}$ involves a different pairing than $\mathbf{P}$ in general). Further from $Q(D_y)\langle W(\eta, \nu), V(y, \eta)\rangle=\nu\langle W(\eta, \nu), V(y, \eta)\rangle$ with $\langle W(\eta, \nu), V(0, \eta)\rangle=\widetilde{Bf}(\nu)$ and the condition $\mathcal{R}_\Theta$ one has by uniqueness $\langle W(\eta, \nu), V(y, \eta)\rangle=\widetilde{Bf}(\nu)\Theta(y, \nu)$ provided $\langle W(\eta, \nu), V(y, \eta)\rangle\in\mathbf{F}$. Hence (cf. (2.8)–(2.9)) from (2.13) we obtain

$$\phi(x, y)=\langle H(x, \nu), \widetilde{Bf}(\nu)\Theta(y, \nu)\rangle_\omega \tag{2.19}$$

and as inversion for $\mathbf{Q}$ set $x=0$ to get (with a different bracket than for $\mathcal{Q}$)

$$Bf(y)=\langle\widetilde{Bf}(\nu), \Theta(y, \nu)\rangle_\omega=\mathbf{Q}\widetilde{Bf}(y). \tag{2.20}$$

Theorem 2.3. *Given* $\hat{\mathcal{R}}$ *and* $\mathcal{R}_\Theta$ *as indicated plus* (2.15) *it follows that* $\mathcal{B}=\boldsymbol{\rho}\mathbf{Q}$ *has the kernel* $\gamma(x,y)$ *of* (2.12) *with* $\mathcal{B}=B^{-1}$ *and* ϕ *is given by* (2.13) *or* (2.19). *Further* $\mathbf{Q}=\mathbf{Q}^{-1}$.

Let us define next the transforms

$$\mathcal{P}f(\mu)=\langle H(x, \mu), f(x)\rangle;\ \boldsymbol{P}F(x)=\langle\Omega(x, \mu), F(\mu)\rangle_\sigma \tag{2.21}$$

$$\mathcal{Q}f(\nu)=\langle\Theta(y, \nu), f(y)\rangle;\ \boldsymbol{Q}F(y)=\langle W(y, \nu), F(\nu)\rangle_\omega. \tag{2.22}$$

Now (cf. Carroll [8; 9]) the eigenfunctions Θ of $Q(D)$ can be characterized as $\Theta(y, \nu)=B[H(x, \nu)](y)$ since if one looks at $w(x,y,\nu)=H(x, \nu)\Theta(y, \nu)$ then

$P(D_x)w=Q(D_y)w$ with $w(x,0,\nu)=H(x,\nu)$. Given that w satisfies $\mathcal{R}$, which involves some compatibility of $\mathcal{R}_H$ and $\mathcal{R}_\Theta$ with $\mathcal{R}$, it follows that $w(0,y,\nu)=\Theta(y,\nu)=B[w(x,0,\nu)](y)$. Given that B has the kernel $\beta(y,x)$ of (2.3) we have then

$$(2.23)\qquad \Theta(y,\nu)=\langle\beta(y,\xi),H(\xi,\nu)\rangle=\mathcal{P}\beta(y,\cdot)(\nu)$$

whereas the form of β says that

$$(2.24)\qquad \beta(y,x)=\boldsymbol{P}\Theta(y,\cdot)(x)$$

so that $\mathcal{P}\boldsymbol{P}\Theta=\Theta$. Suppose $\mathcal{Q}\colon\mathcal{H}\to\check{\mathcal{H}}$ is an isomorphism for suitably large spaces $\mathcal{H}$ and $\check{\mathcal{H}}$. Then for $f\in\mathcal{H}$ one has

$$(2.25)\qquad \mathcal{P}\boldsymbol{P}\check{f}=\langle\mathcal{P}\boldsymbol{P}\Theta(y,\nu),f(y)\rangle=\check{f}\,.$$

On the other hand $\beta=\boldsymbol{P}\mathcal{P}\beta$ so that for $g=B^*f=\langle\beta(y,x),f(y)\rangle$ we have

$$(2.26)\qquad \boldsymbol{P}\mathcal{P}g=\langle\boldsymbol{P}\mathcal{P}\beta(y,x),f(y)\rangle=g\,.$$

But we observe that

$$(2.27)\qquad B^*f(x)=\langle\beta(y,x),f(y)\rangle=\langle\Omega(x,\mu),\langle\Theta(y,\mu),f(y)\rangle\rangle_\sigma=\boldsymbol{P}\mathcal{Q}f(x)$$

so $g\in\boldsymbol{P}\check{\mathcal{H}}$ and $\mathcal{P}\boldsymbol{P}\check{\mathcal{H}}=\check{\mathcal{H}}$ by (2.25). Thus $\mathcal{P}\colon\boldsymbol{P}\check{\mathcal{H}}\to\check{\mathcal{H}}$ and $\boldsymbol{P}\colon\check{\mathcal{H}}\to\boldsymbol{P}\check{\mathcal{H}}$ are inverse.

Theorem 2.4. Let $\mathcal{Q}\colon\mathcal{H}\to\check{\mathcal{H}}$ be an isomorphism. Then $\mathcal{P}\colon\boldsymbol{P}\check{\mathcal{H}}\to\check{\mathcal{H}}$ and $\boldsymbol{P}\colon\check{\mathcal{H}}\to\boldsymbol{P}\check{\mathcal{H}}$ are inverses. Similarly let $\mathcal{P}\colon\mathcal{K}\to\check{\mathcal{K}}$ be an isomorphism. Then $\mathcal{Q}\colon\boldsymbol{Q}\check{\mathcal{K}}\to\check{\mathcal{K}}$ and $\boldsymbol{Q}\colon\check{\mathcal{K}}\to\boldsymbol{Q}\check{\mathcal{K}}$ are inverses.

Proof. Only the last part remains and we observe first that from

$$(2.16)\qquad \mathcal{B}h(x)=\langle\gamma(x,\eta),h(\eta)\rangle\quad\text{with}$$
$$(2.28)\qquad \mathcal{B}^*k(y)=\langle W(y,\nu),\langle H(x,\nu),k(x)\rangle\rangle_\omega=\boldsymbol{Q}\mathcal{P}k(y)\,.$$

Now by (2.12) $\gamma=\boldsymbol{Q}H$ while upon using the function $w(x,y,\nu)=H(x,\nu)\Theta(y,\nu)$ again and assuming it satisfies $\check{\mathcal{R}}$ we obtain $H(x,\nu)=\mathcal{B}\Theta(\cdot,\nu)(x)$ so that $H=\mathcal{Q}\gamma$. Hence $H=\mathcal{Q}\boldsymbol{Q}H$ and $\gamma=\boldsymbol{Q}\mathcal{Q}\gamma$. Consequently for $\check{h}\in\check{\mathcal{K}}$ $\mathcal{Q}\boldsymbol{Q}\check{h}=\langle\mathcal{Q}\boldsymbol{Q}H,\check{h}\rangle=\check{h}$ while for $g=\mathcal{B}^*f$, $\boldsymbol{Q}\mathcal{Q}g=\langle\boldsymbol{Q}\mathcal{Q}\gamma,f\rangle=g$ with $g\in\boldsymbol{Q}\check{\mathcal{K}}$. Thus as before $\mathcal{Q}\colon\boldsymbol{Q}\check{\mathcal{K}}\to\check{\mathcal{K}}$ and $\boldsymbol{Q}\colon\check{\mathcal{K}}\to\boldsymbol{Q}\check{\mathcal{K}}$ are inverse. Q.E.D.

3. Models based on selfadjointness. There are numerous examples where all of this fits together. Classical cases are $P(D)=D^2-q(x)$ and $Q(D)=D^2$, or vice versa, which arise in quantum mechanics (cf. [15; 16; 17; 24; 33; 37; 39]). More generally one deals in quantum mechanics with $D^2-(m^2-\frac{1}{4})/x^2$. with or without a potential $q(x)$, and we prefer to treat this in the form D^2+

$[(2m+1)/x]D$ (cf. [1; 2; 8; 9; 10; 17; 24; 26; 27; 34; 35; 39; 44; 45; 49]). Indeed, treated in this manner it provides a natural link to many other intersting operators which arise in the context of harmonic analysis on symmetric spaces (cf. [8; 9; 18; 19; 25; 31]); we have studied this latter situation in more detail in [11; 12; 13] but will give some information here to give body to the general theory.

EXAMPLE 3.1. Consider $P(D)u=(Au')'/A$ where A has properties, spelled out in [13], modeled on the radial part of the Laplace-Beltrami operator on a noncompact Riemannian symmetric space of rank one. In particular one can take $A(x)=\Delta_P(x)=\Delta_{\alpha\beta}(x)=(e^x-e^{-x})^{2\alpha+1}(e^x+e^{-x})^{2\beta+1}$ with $\rho_A=\alpha+\beta+1$ or $A(x)=\Delta^{pq}(x)=(e^x-e^{-x})^p(e^{2x}-e^{-2x})^q$ with $\rho_A=\frac{1}{2}(p+2q)$. The relevant eigenfunction equation is $P(D)u=(-\lambda^2-\rho_A^2)u$ where $\rho_A=\frac{1}{2}\lim A'/A$ as $x\to\infty$. At this point let us shift to $\hat{P}=P+\rho_A^2$ and speak of transmutations $\hat{P}\to\hat{Q}=Q+\rho_B^2$ etc. Thus $\hat{P}(D)u=-\lambda^2 u$ is the eigenfunction equation *for* $\hat{P}$. Note for $A(x)=x^{2m+1}$ we get $D^2+((2m+1)/x)D$ with $\rho_A=0$. Let $\varphi_\lambda^P(x)$ be a "spherical function", $\varphi_\lambda^P(0)=1$, and $D_x\varphi_\lambda^P(0)=0$, satisfying the eigenfunction equation for $\hat{P}$ so that $\varphi_\lambda^P(x)=H(x,\mu)$ for $\mu=-\lambda^2$. Set $\Omega_\lambda^P(x)=\Delta_P(x)\varphi_\lambda^P(x)$; $P^*(D)\Omega_\lambda^P=\mu\Omega_\lambda^P$ where $P^*(D)\psi=(A(\psi/A)')'$ for $A=\Delta_p$. Thus $\Omega_\lambda^P(x)=\Omega(x,\mu)$ in the notation of Section 2. For example when $\Delta_p=\Delta_{\alpha\beta}$ then $\varphi_\lambda^P(x)=\varphi_\lambda^{\alpha\beta}(x)=F(\frac{1}{2}(\rho_A+i\lambda),\frac{1}{2}(\rho_A-i\lambda),\alpha+1,-sh^2x)$ is a Jacobi function of the first kind (cf. [23]). The case $A(x)=sh^{2m+1}x$ with $A'/A=(2m+1)\coth x$ arises in working with $SL(2,\boldsymbol{R})/SO(2)$ and is particularly useful for illustrative purposes. Operators $P(D)$ as indicated are selfadjoint in $L^2(Adx)$ but we prefer to work in $E=E_A=\{f;A^{1/2}f\in L^2\}$ with $P^*(D)$ acting in $E'_A=\{f;A^{-1/2}f\in L^2\}$. In fact our general framework of spaces and maps in [3; 5] was based upon "spreading out" a selfadjoint situation in this manner (cf. Section 4 for some further remarks on these spaces). For P of this type we write $d\nu_P$ for the associated spectral measure ($\nu\sim\sigma$ in Section 2). Let us remark also that explicit formulas for $d\nu_P$ exist in terms of $|c(\lambda)|^{-2}$ where $c(\lambda)$ is the Harish-Chandra (or Jost) function (see [7; 8; 9] for details). Then write with a slight change of notation ($\hat{f}(\lambda)$ in place of $\hat{f}(\mu)$, etc.)

$$(3.1)\qquad \hat{f}(\lambda)=\mathbf{P}f(\lambda)=\int_0^\infty f(x)\varphi_\lambda^P(x)\Delta_P(x)dx;$$

$$f(x)=\mathbf{P}\hat{f}(x)=\int_0^\infty \hat{f}(\lambda)\varphi_\lambda^P(x)d\nu_P(\lambda).$$

Let $Q(D)$ arise from $\Delta_Q(x)=B(x)$ as above so we have $\varphi_\lambda^Q(y)\sim\Theta(y,\mu)$ (i.e. $Q(D)\varphi_\lambda^Q=(-\lambda^2-\rho_B^2)\varphi_\lambda^Q$ and $\hat{Q}(D)\varphi_\lambda^Q=-\lambda^2\varphi_\lambda^Q$ so $\nu\sim-\lambda^2$) and $W(y,\nu)\sim\Omega_\lambda^Q(y)=\Delta_Q(y)\varphi_\lambda^Q(y)$. $\mathbf{Q}$ and $\mathbf{Q}$ will have a form similar to (3.1), which we can write also as $\tilde{g}(\lambda)=\mathbf{Q}g(\lambda)=\langle g(y),\Omega_\lambda^Q(y)\rangle$ and $\mathbf{Q}G(y)=\langle G(\lambda),\varphi_\lambda^Q(y)\rangle_\omega$; ω of Section 2 corresponds here to $d\omega_Q(\lambda)$. The $\mathcal{P}$ and $\mathcal{Q}$ transforms are written then as $\mathcal{P}F(x)=\langle F(\lambda),\varphi_\lambda^P(x)\rangle_\omega$ and $\mathcal{Q}G(y)=\langle G(\lambda),\varphi_\lambda^Q(y)\rangle_\nu$ while $\mathcal{P}f(\lambda)=\langle f(x),$

$\varphi_\lambda^P(x)\rangle$, $\boldsymbol{P}F(x)=\langle F(\lambda), \Omega_\lambda^P(x)\rangle_\nu$, etc. The various inversions $\mathbf{P}=\boldsymbol{P}^{-1}$ etc. from Section 2 are then also known from results in [13; 18; 23] for example. We note that such inversions based on uniqueness requirements $(\mathcal{R}, \mathcal{R}_H)$ and $(\hat{\mathcal{R}}, \mathcal{R}_\Theta)$ provide new derivations of integral transforms such as Hankel, generalized Mehler, etc. modulo the spectral pairings.

Now using this background we can generalize some results of [8; 10] as follows. First let us recall from [8; 18; 19; 21; 30; 31; 32; 40] that if T_x^y is the generalized translation associated with $P(D)$, or equivalently with $\hat{P}(D)$ (!) then formally $T^y=H(y,P)=\int_\sigma H(y,\mu)dE_\mu$, where dE_μ is a spectral resolution of I over $\sigma(\hat{P})$ $(\lambda\in(0,\infty)$, $\mu=-\lambda^2)$, and $T_x^yH(x,\mu)=H(y,\mu)H(x,\mu)$. Further $[T_x^yf(x)]^\wedge(\mu)=H(y,\mu)\hat{f}(\mu)$ which we also write in our altered notation as $[T_x^yf(x)]^\wedge(\lambda)=\varphi_\lambda^P(y)\hat{f}(\lambda)$. A generalized convolution is defined by $(f*g)(x)=\int_0^\infty T_x^yf(x)g(y)A(y)dy$ and we note that φ in (2.5) can be written as $(T_x^\xi f(x)=U(x,\xi)=T_\xi^x f(\xi)$; $\Omega=AH)$

$$\begin{aligned}(3.2)\qquad \varphi(x,y)&=\langle\beta(y,\xi),\,U(x,\xi)\rangle=\\ &\int_0^\infty A(\xi)\langle H(\xi,\mu),\,\Theta(y,\mu)\rangle_\nu T_x^\xi f(x)d\xi=[f(\cdot)*\tilde{\beta}(y,\cdot)](x)\end{aligned}$$

where $\tilde{\beta}(y,\xi)=\langle H(\xi,\mu),\Theta(y,\mu)\rangle_\nu$. Now define $\delta_A(x)=\delta(x)/A(x)$ (working on suitable functions). Then $\hat{\delta}_A(\lambda)=\langle\delta_A(x),\Omega_\lambda^P(x)\rangle=\langle\delta(x),\varphi_\lambda^P(x)\rangle=1$ and $\delta_A(x)=\langle\varphi_\lambda^P(x),1\rangle$ (i.e. $\langle\Omega_\lambda^P(x),1\rangle=\delta(x)$ as desired in (2.6) for example). Further $[T_x^y\delta_A(x)]^\wedge=\varphi_\lambda^P(y)$ and similarly for Q arising from $B(x)$, $\tilde{\delta}_B(\lambda)=\langle\delta_B(y),\Omega_\lambda^Q(y)\rangle=\langle\delta(y),\varphi_\lambda^Q(y)\rangle=1$ with $\delta_B(y)=\langle\varphi_\lambda^Q(y),1\rangle$ and $[S_x^y\delta_B(x)]^\sim=\varphi_\lambda^Q(y)$. Then from $Bf(y)=\langle\beta(y,x),f(x)\rangle$ with $\beta(y,x)=\langle\Omega_\lambda^P(x),\varphi_\lambda^Q(y)\rangle_\nu$ (cf. (2.3)) we have (recall $B\colon\hat{P}\to\hat{Q}$)

$$\begin{aligned}(3.3)\qquad \beta(y,x)&=A(x)\int_0^\infty\varphi_\lambda^P(x)\varphi_\lambda^Q(y)d\nu_p(\lambda)=\\ &A(x)\int_0^\infty(T_\xi^x\delta_A(\xi))^\wedge(S_\xi^y\delta_B(\xi))^\sim d\nu_P(\lambda)\,.\end{aligned}$$

Similarly $\mathcal{B}g(x)=\langle\gamma(x,y),g(y)\rangle$ where $\gamma(x,y)$ is given by (2.12) as

$$\begin{aligned}(3.4)\qquad \gamma(x,y)&=B(y)\int_0^\infty\varphi_\lambda^P(x)\varphi_\lambda^Q(y)d\omega_Q(\lambda)=\\ &B(y)\int_0^\infty(T_\xi^x\delta_A(\xi))^\wedge(S_\xi^y\delta_B(\xi))^\sim d\omega_Q(\lambda)\,.\end{aligned}$$

Note here in (3.3) that formally $[\Theta(y,P_x)\delta_A(x)]^\wedge=\Theta(y,\mu)\hat{\delta}_A=\Theta(y,\mu)=\varphi_\lambda^Q(y)$ while in (3.4) $[H(x,Q_y)\delta_B(y)]^\sim=\varphi_\lambda^P(x)$. Hence formally

$$\begin{aligned}(3.5)\qquad \beta(y,x)&=A(x)\Theta(y,P_x)\delta_A(x);\\ \gamma(x,y)&=B(y)H(x,Q_y)\delta_B(y)\,.\end{aligned}$$

Theorem 3.2. *Given $P\sim A=\Delta_P$ and $Q\sim B=\Delta_Q$ as above the kernels β and γ are given by* (3.3)–(3.4) *and* (3.5) *holds formally* ($B\colon \hat{P}\to\hat{Q}$, $\mathcal{B}\colon \hat{Q}\to\hat{P}$).

REMARK 3.3. Note that if brackets $Bf(y)=\langle\beta_A(y,x),f(x)\rangle=\int_0^\infty \beta_A(y,x)f(x)\times A(x)dx$ and $\mathcal{B}g(x)=\langle\gamma_B(x,y),g(y)\rangle=\int_0^\infty \gamma_B(x,y)g(y)B(y)dy$ are used then $\beta_A(y,x)=\beta(y,x)/A(x)$ and $\gamma_B(x,y)=\gamma(x,y)/B(y)$ are respectively given by the formulas $\langle\varphi_\lambda^P(x),\varphi_\lambda^Q(y)\rangle_\nu$ and $\langle\varphi_\lambda^P(x),\varphi_\lambda^Q(y)\rangle_\omega$.

Let us observe next that (cf. (2.8)), in the present context.

$$(3.6)\qquad T_x^y f(x)=\langle H(y,\mu),\hat{f}(\mu)H(x,\mu)\rangle_\nu=\phi(x,y)$$

since this ϕ satisfies $\hat{P}(D_x)\phi=\hat{P}(D_y)\phi$ with $\phi(x,0)=\langle 1,f(\mu)H(x,\mu)\rangle_\nu=\langle\hat{f}(\mu),H(x,\mu)\rangle_\nu=\mathbf{PP}f(x)=f(x)$ while $\phi(0,y)=\langle H(y,\mu),f(\mu)\rangle_\nu=f(y)$ so that $U(x,y)=\phi(x,y)$. Similarly (cf. (2.19); $\mu=\nu=-\lambda^2$)

$$(3.7)\qquad S_y^x g(y)=V(x,y)=\langle\Theta(x,\mu),\tilde{g}(\mu)\Theta(y,\mu)\rangle_\omega\,.$$

Writing $U(x,y)=T_x^y f(x)=\langle\beta(x,y,\xi),f(\xi)\rangle$ and $V(x,y)=S_y^x g(y)=\langle\gamma(x,y,\eta),g(\eta)\rangle$ we obtain formally

$$(3.8)\qquad \beta(x,y,\xi)=\int H(x,\mu)H(y,\mu)\Omega(\xi,\mu)d\nu$$

$$(3.9)\qquad \gamma(x,y,\eta)=\int \Theta(x,\mu)\Theta(y,\mu)W(\eta,\mu)d\omega\,.$$

We record this in

Theorem 3.4. *The generalized translation operators T_x^y and S_y^x have kernels $\beta(x,y,\xi)$ and $\gamma(x,y,\xi)$ given by* (3.8) *and* (3.9).

EXAMPLE 3.5. Consider (3.8) with H and Ω given by

$$(3.10)\qquad \begin{aligned} H(x,\mu)&=2^m\Gamma(m+1)(\lambda x)^{-m}J_m(\lambda x)=\hat{R}^m(x,\lambda)\\ \Omega(x,\mu)&=2^{-2m}\Gamma(m+1)^{-2}(\lambda x)^{2m+1}H(x,\mu)\,. \end{aligned}$$

This is the standard example from [8;9] and we note that Ω is chosen differently than before; this was done for symmetry and is explained in Section 4. Then we have

$$(3.11)\qquad \begin{aligned} \beta(x,y,\xi)=2^m\Gamma(m+1)(xy)^{-m}\xi^{m+1/2}\,.\\ \cdot\int_0^\infty \lambda^{-m+1/2}(\xi\lambda)^{1/2}J_m(\lambda x)J_m(\lambda y)J_m(\lambda\xi)d\lambda\,. \end{aligned}$$

By known formulas (cf. [3; 36]) one has then $\beta(x,y,\xi)=0$ for $0<\xi<|x-y|$ and $\xi>x+y$ while for $|x-y|<\xi<x+y$

$$(3.12)\qquad \beta(x,y,\xi)=\frac{\Gamma(m+1)}{\sqrt{\pi}\,\Gamma(m+1/2)}\left(\frac{\xi}{xy}\right)(1-z^2)^{m-1/2}$$

where $z=(x^2+y^2-\xi^2)/2xy$. Thus

$$T_x^y f(x) = \int_{|x-y|}^{x+y} f(\xi)\beta(x, y, \xi)d\xi \tag{3.13}$$

and hence we can write

$$T_x^y f(x) = \frac{\Gamma(m+1)}{\sqrt{\pi}\,\Gamma(m+1/2)} \int_{-1}^{1} (1-z^2)^{m-1/2} f(\xi)dz \tag{3.14}$$

with $\xi=\sqrt{x^2+y^2-2xyz}$. This is a new derivation of a formula of Copson-Erdelyi [20] and Levitan [34].

4. Explicit formulas for a model case. Let us record some formulas here for a model problem based on $P_m(D)=D^2+((2m+1)/x)D$. This operator will be used in developing the framework for general Parseval formulas in Part II based on which analogous results for other operators $P(D)u=(Au')'/A$ will follow (here $A=x^{2m+1}$). Thus recall (3.10) for H and Ω (satisfying $P_m(D)u=-\lambda^2 u$). One can write $\mathsf{P}f(\lambda)=(\lambda^{m+1/2}/2^m\Gamma(m+1))\boldsymbol{H}_m[x^{m+1/2}f(x)]$ and $\mathbf{P}F(x)=(2^m\Gamma(m+1)/x^{m+1/2})\boldsymbol{H}_m[\lambda^{-m-1/2}F(\lambda)]$ where $\boldsymbol{H}_m$ denotes a Hankel transform. We emphasize that Ω is not AH here so P is not the same as in Section 3. Standard theorems on Hankel transforms say that $\boldsymbol{H}_m\colon L^2\to L^2$ is a selfinverse isometric isomorphism for suitable m and here, modulo constants, $x^{m+1/2}f(x)\leftrightarrow\lambda^{-m-1/2}\hat{f}(\lambda)$ under $\boldsymbol{H}_m$. Thus the choice of $E=E_A=\{f;\, x^{m+1/2}f(x)\in L^2\}$ is indicated (cf. Example 3.1) and we take $\hat{E}=\mathsf{P}E=\{\hat{f};\, \lambda^{-m-1/2}\hat{f}(\lambda)\in L^2\}$ (recall also $E'=\boldsymbol{E}$. The basic duality between $\hat{E}$ and $\hat{E}'=\hat{\hat{E}}$ is expressed as $\langle\hat{f},\hat{\hat{f}}\rangle=\int_0^\infty \hat{f}(\lambda)\hat{\hat{f}}(\lambda)d\lambda$ ($\hat{\hat{E}}=\{\hat{\hat{f}};\, \lambda^{m+1/2}\hat{\hat{f}}(\lambda)\in L^2\}$). Note here that for this model $1/2\pi|c(\lambda)|^2=c_m^2\lambda^{2m+1}=R_0$ where $c_m=1/2^m\Gamma(m+1)$ (cf. [11; 13]); this can be written $d\nu_P=R_0 d\lambda$ where $R_0^{1/2}=c_m\lambda^{m+1/2}$. Now in (3.10) a normalizing factor involving λ has been inserted into Ω (i.e. $\Omega\neq\Delta H$). In fact what we have is exactly the relation $\Omega(x,\mu)=R_0(\lambda)\Delta(x)H(x,\mu)$! Hence it was possible in [8; 9; 10] to omit the measures $d\nu$ and set for example $\mathbf{P}_0\hat{f}(x)=\int_0^\infty \hat{f}(\lambda)H(x,\mu)d\lambda$ with $\mathsf{P}_0 f(\lambda)=\hat{f}(\lambda)=\int_0^\infty f(x)\Omega(x,\mu)dx$ since $\mathsf{P}_0 f=R_0(\lambda)\mathsf{P}f$ and $\mathbf{P}_0\hat{f}=\int R_0(\lambda)\mathsf{P}fH d\lambda=\int \mathsf{P}fH d\nu=\mathbf{P}\mathsf{P}f=f$.

REMARK 4.1. The procedure just indicated could be followed more generally when $d\nu=\hat{\nu}^2(\lambda)d\lambda$ (which will *not* usually be the case when a potential is present) by choosing $\Omega(x,\mu)=\Omega_\lambda^P(x)=\hat{\nu}^2(\lambda)\Delta_P(x)\varphi_\lambda^P(x)$. Then a certain symmetry could be introduced into the spaces E, $\hat{E}$, etc. in which P, $\mathbf{P}$, etc. are taken to have their "basic" action. The symmetry in $E=E_A$ and E_A' was indicated in Example 3.1. For $\hat{E}_A$ we could take now $\hat{E}_A=\{\hat{f};\, \hat{\nu}^{-1}(\lambda)\hat{f}\in L^2\}$ with $\hat{E}_A'=\{\hat{\hat{f}};\, \hat{\nu}(\lambda)\hat{\hat{f}}\in L^2\}$. Then $\hat{\nu}^{-1}\hat{f}=\hat{\nu}(\lambda)\int_0^\infty \Delta_P\varphi_\lambda^P f dx=\hat{\nu}(\lambda)\int_0^\infty \Delta_P^{1/2}\varphi_\lambda^P(\Delta_P^{1/2}f)dx$ and one is in the position of asking that the kernel $\hat{\nu}(\lambda)\Delta_P^{1/2}(x)\varphi_\lambda^P(x)$ map $L^2 \to L^2$. In our model this is $c_m\lambda^{m+1/2}x^{m+1/2}c_m^{-1}(\lambda x)^{-m}J_m(\lambda x)=(\lambda x)^{1/2}J_m(\lambda x)$ which is the standard Hankel form. One expects this situation to prevail more generally but we leave this for now.

REMARK 4.2. In general an emphasis on symmetry of the form discussed in Remark 4.1 for E_A, $\hat{E}_A$, etc. is probably misdirected effort. The operators $\mathbf{P}$, $\mathcal{P}$, etc. with $\Omega=AH$ will have realizations in various spaces (e.g. one will want to talk about $\mathcal{P}T^x$, $\mathcal{P}\beta(y, \cdot)$, etc. and deal with various distribution spaces). Also in general the spectral pairings will be effected by means of a generalized spectral function (cf. Part II and [5; 26; 27; 37]) so a weight function $\hat{\nu}^2(\lambda)$ does not exist. Thus at this stage we will concentrate more on the form of our operators and not on their domains.

Now retaining the formula (3.10) for Ω and the spaces E, $\hat{E}$, etc. take $Q(D)=D^2$ in $L^2(0, \infty)=F$ with $\Theta(y, \mu)=\text{Cos}\,\lambda y$ (satisfying $D^2\Theta=\mu\Theta$ for $\mu=-\lambda^2$). Then $W(y, \mu)=\dfrac{2}{\pi}\text{Cos}\,\lambda y$ so that $\mathcal{Q}=\mathcal{F}_c$ and $\boldsymbol{Q}=\mathcal{F}_c^{-1}$ where $\mathcal{F}_c$ denotes the Fourier cosine transform. It is natural to take $\boldsymbol{F}=L^2(0, \infty)$, $\bar{\boldsymbol{F}}=L^2(0, \infty; d\lambda)$, and $\tilde{F}=\bar{\boldsymbol{F}}$; here we will identify F and F', etc. Let us write out $\mathcal{Q}\mathbf{P}=B$ as

$$(4.1)\qquad \mathcal{Q}\mathbf{P}f(y)=\int_0^\infty \hat{f}(\lambda)\,\text{Cos}\,\lambda y d\mu=\int_0^\infty \lambda^{m+1/2}F(\lambda)\,\text{Cos}\,\lambda y d\lambda$$

where $F(\lambda)\in L^2$. Then it is natural to take $D(\mathcal{Q})=\{\hat{f}\in\hat{E}; \hat{f}\in L_\lambda^2\}=\hat{E}\cap\tilde{F}$ so that map $\mathcal{Q}$ will into F. Similarly consider $\mathcal{B}=\boldsymbol{\rho}\mathbf{Q}$ written as

$$(4.2)\qquad \mathcal{B}f(x)=\int_0^\infty F(\lambda)\hat{R}^m(x, \lambda)d\lambda=\boldsymbol{\rho}F(x)=\frac{2^m\Gamma(m+1)}{x^{m+1/2}}\boldsymbol{H}_m[\lambda^{-m-1/2}F(\lambda)]$$

where $F(\lambda)=\mathbf{Q}f(\lambda)\in L^2=\tilde{F}$. In order to insure that $R(\boldsymbol{\rho})\subset E$ we take $D(\boldsymbol{\rho})=\{F\in\tilde{F}; \lambda^{-m-1/2}F(\lambda)\in L_\lambda^2\}=\tilde{F}\cap\hat{E}$. Observe also

$$(4.3)\qquad \langle\mathbf{P}f, \hat{\hat{g}}\rangle=\int\langle\Omega(x, \mu), f(x)\rangle\hat{\hat{g}}(\lambda)d\lambda=$$
$$\langle f(x), \int\Omega(x, \mu)\hat{\hat{g}}(\lambda)d\lambda\rangle=\langle f(x), \boldsymbol{P}\hat{\hat{g}}(x)\rangle$$

for example which displays $\boldsymbol{P}$ as $\mathbf{P}^*: \hat{E}'\to E'$ ($\hat{\hat{\boldsymbol{E}}}\to\boldsymbol{E}$). Similarly for $\check{e}\in D(\mathcal{Q})$ and suitable $f'\in F'$

$$(4.4)\qquad \langle\mathcal{Q}\hat{e}, f'\rangle=\langle\int\hat{e}(\lambda)\Theta(y, \mu)d\lambda, f'\rangle$$
$$=\int\hat{e}(\lambda)\langle\Theta(y, \mu), f'(y)\rangle d\lambda=\langle\hat{e}, \mathcal{Q}f'\rangle$$

provided we have $F'=\boldsymbol{F}$ so that $f'\sim\boldsymbol{f}\in\boldsymbol{F}$ and $\mathcal{Q}f'\sim\mathcal{Q}\boldsymbol{f}\in\bar{\boldsymbol{F}}$ which we want to intersect $\hat{\hat{\boldsymbol{E}}}=\hat{E}'$ so that $\mathcal{Q}\boldsymbol{f}\in\hat{\hat{\boldsymbol{E}}}\cap\bar{\boldsymbol{F}}$. Then (4.4) says $\boldsymbol{\mathcal{Q}}^*=\mathcal{Q}$. Let us observe also in equation (4.4) that $\hat{e}\in\hat{E}\cap\tilde{F}$ with $\mathcal{Q}\boldsymbol{f}\in\hat{\hat{\boldsymbol{E}}}\cap\bar{\boldsymbol{F}}=\hat{E}'\cap\tilde{F}'$ and one has $\langle\hat{e},\mathcal{Q}\boldsymbol{f}\rangle=\int\lambda^{-m-1/2}\hat{e}(\lambda)\lambda^{m+1/2}\mathcal{Q}\boldsymbol{f}(\lambda)d\lambda=\int\hat{e}(\lambda)\mathcal{Q}\boldsymbol{f}(\lambda)d\lambda=\langle\hat{e},\mathcal{Q}\boldsymbol{f}\rangle$. For $\boldsymbol{\mathscr{P}}$ we have a formula analogous to (4.4) for $\tilde{f}\in D(\boldsymbol{\mathscr{P}})\subset\tilde{F}$ and $e'\sim\boldsymbol{e}\in\boldsymbol{E}$ with $\mathscr{P}\boldsymbol{e}\in\bar{\boldsymbol{F}}\cap\hat{\hat{\boldsymbol{E}}}-\tilde{F}'\cap\hat{E}'$, namely

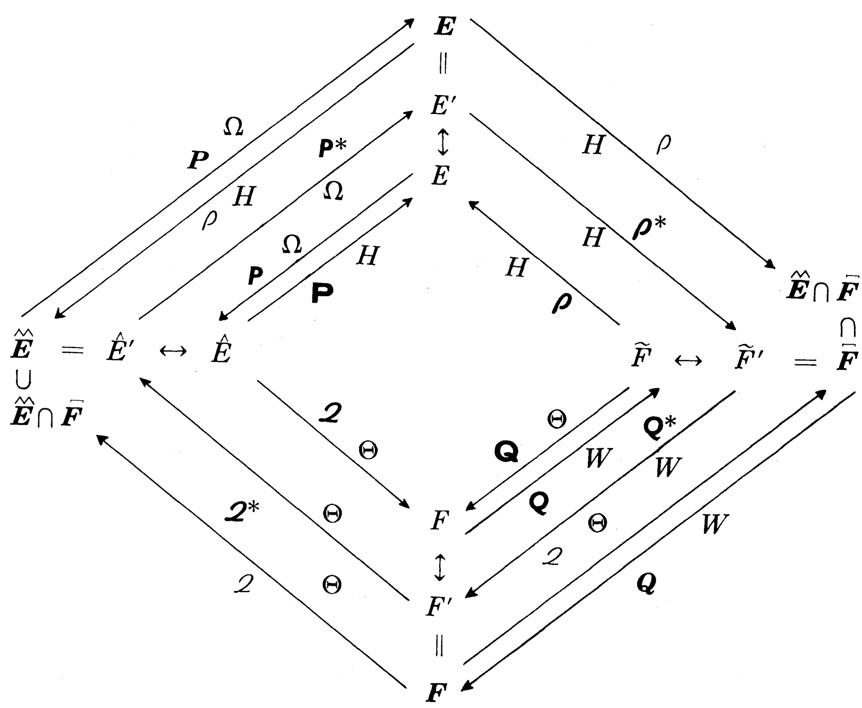
$$\text{(4.5)}\qquad \langle\boldsymbol{\mathscr{P}}\tilde{f},e'\rangle=\left\langle\int\tilde{f}(\lambda)H(x,\mu)d\lambda,\boldsymbol{e}\right\rangle=$$

$$\int\tilde{f}(\lambda)\langle H(x,\mu),\boldsymbol{e}(x)\rangle d\lambda=\langle\tilde{f},\mathscr{P}\boldsymbol{e}\rangle$$

This displays $\mathscr{P}$ as $\boldsymbol{\mathscr{P}}^*$ and as above for $\mathcal{Q}$ one has $\langle\tilde{f},\mathscr{P}\boldsymbol{e}\rangle=\langle\boldsymbol{\mathscr{P}}\tilde{f},\boldsymbol{e}\rangle$.

The above example furnishes a typical model of a transform theory linked to transmutation which we display in the following diagram.

(4.6)



Theorem 4.3. *The diagram* (4.6) *indicates the relations* $\mathbf{P}=\mathbf{P}^{-1}$, $\mathbf{Q}=\mathbf{Q}^{-1}$, $\boldsymbol{P}=\mathscr{P}^{-1}$, $\boldsymbol{Q}=\mathcal{Q}^{-1}$, $\boldsymbol{P}^*=\mathbf{P}$, $\mathbf{Q}^*=\boldsymbol{Q}$, $\boldsymbol{\mathscr{P}}^*=\mathscr{P}$, $\boldsymbol{\mathcal{Q}}^*=\mathcal{Q}$, $B^*=(\boldsymbol{\mathcal{Q}}\mathbf{P})^*=\boldsymbol{P}\mathcal{Q}$, *and* $\mathcal{B}^*=(\boldsymbol{\mathscr{P}}\mathbf{Q})^*=\boldsymbol{Q}\mathscr{P}$. *Here* $D(\boldsymbol{\mathcal{Q}})=D(\boldsymbol{\mathscr{P}})=\hat{E}\cap\tilde{F}$ *and* $R(\boldsymbol{\mathcal{Q}}^*)$, $R(\boldsymbol{\mathscr{P}}^*)\subset\hat{\hat{\boldsymbol{E}}}\subset\bar{\boldsymbol{F}}=\hat{E}'\cap\tilde{F}'$. *From* $\mathcal{B}=\boldsymbol{\mathscr{P}}\mathbf{Q}=B^{-1}=(\boldsymbol{\mathcal{Q}}\mathbf{P})^{-1}=\mathbf{P}^{-1}\boldsymbol{\mathcal{Q}}^{-1}$ *we obtain* $\boldsymbol{\mathcal{Q}}^{-1}=\mathbf{P}\boldsymbol{\mathscr{P}}\mathbf{Q}$ *and* $\boldsymbol{\mathscr{P}}^{-1}=\mathbf{Q}\boldsymbol{\mathcal{Q}}\mathbf{P}$.

REMARK 4.4. As a further guide to understanding $D(\mathcal{P})$ and $D(\mathcal{Q})$, or equivalently $D(\mathcal{B})$ and $D(B)$, recall that from $Bf(y)=\langle\beta(y, x), f(x)\rangle$ with $\beta(y, x)=\langle\Omega(x, \mu), \theta(y, \mu)\rangle$ one obtains

$$(4.7)\qquad \beta(y, x)=\frac{x^{m+1/2}}{2^m\Gamma(m+1)}\boldsymbol{H}_m[\lambda^{m+1/2}\operatorname{Cos}\lambda y]$$

and this was examined in Carroll [8; 9]. Using a technique of Lions [35] an explicit formula for $\beta(y, x)$ was obtained as a distribution having a determination for $-1/2<m<n-1/2$ of the form $\beta(y, x)=\beta_m^n(y, x)$ with

$$(4.8)\qquad \langle\beta_m^n(y, x), f(x)\rangle=y^{-2n+1}\sum_{k=1}^{n+1}c_{nk}\langle x^{2m+k}(y^2-x^2)_+^{-m+n-3/2}, D^{k-1}f(x)\rangle_+ .$$

Thus in particular one needs n derivatives of f in order to define Bf. Similarly for $\mathcal{B}g(x)=\langle\gamma(x, y), g(y)\rangle$ with $\gamma(x, y)=\langle H(x, \mu), W(y, \mu)\rangle$ one can write $\gamma(x, y)=R_+^m(x, y)=2R^m(x, y)\in\mathcal{E}'_y$ where R^m is the resolvant distribution of *EPD* theory (cf. Carroll-Showalter [14]) which can be displayed in the form

$$(4.9)\qquad R^m(x, y)=\frac{\Gamma(m+1)x^{-2m}}{\Gamma(1/2)\Gamma(m+1/2)}(x^2-y^2)_+^{m-1/2} .$$

(cf. Carroll [8]). We can actually obtain however a somewhat nicer expression for β by recalling a formula used in the solution of Euler-Poisson-Darboux equations (cf. Carroll-Showalter [14]). Thus for $-1/2<m<n-1/2$

$$(4.10)\qquad \operatorname{Cos}\lambda y=\gamma_m^n y\left(\frac{1}{y}D_y\right)^n\left[y^{2n-1}\int_0^1\hat{R}^m(\xi y, \lambda)\xi^{2m+1}(1-\xi^2)^{n-m-3/2}d\xi\right]$$

where $\gamma_m^n=\Gamma(1/2)/2^{n-1}\Gamma(m+1)\Gamma(n-m-1/2)$ and $\hat{R}^m(\xi y, \lambda)=H(\xi y, \mu)$ for $\mu=-\lambda^2$. This can be rewritten as

$$(4.11)\qquad \Theta(y, \mu)=\gamma_m^n y\left(\frac{1}{y}D_y\right)^n\int_0^y H(x, \mu)x^{2m+1}(y^2-x^2)^{n-m-3/2}dx$$

Then since $\Theta=BH$ we have

$$(4.12)\qquad B\varphi(y)=\gamma_m^n y\left(\frac{1}{y}D_y\right)^n\int_0^y\varphi(x)x^{2m+1}(y^2-x^2)^{n-m-3/2}dx .$$

REMARK 4.5. In Part II it will be necessary to deal with $B\varphi$ for $\varphi=\delta(x)/x^{2m+1}$ and it seems appropriate to make a few preliminary observations here. From (4.12) we have formally

$$(4.13)\qquad \begin{aligned} B\varphi(y) &= \gamma_m^n y\left(\frac{1}{y}D_y\right)^n y^{2n-2m-3} \\ &= [2\Gamma(1/2)/\Gamma(m+1)\Gamma(-1/2-m)]y^{-2m-2} . \end{aligned}$$

Now $\beta(y, x) = c_m^2 \int (\lambda x)^{2m+1} H(x, \mu) \operatorname{Cos} \lambda y d\lambda$ with $Bf(y) = \langle \beta(y, x), f(x) \rangle$ and thus

$$(4.14) \qquad x^{-2m-1}\beta(y, x) \to c_m^2 \int_0^\infty \lambda^{2m+1} \operatorname{Cos} \lambda y d\lambda$$

as $x \to 0$. Hence $B\varphi$ should equal $c_m^2 \int \lambda^{2m+1} \operatorname{Cos} \lambda y d\lambda$. To see that this makes sense we can use the appropriate pseudofunctions following Gelfand-Šilov [28] and Schwartz [43]. Thus recall

$$(4.15) \qquad \langle x_+^\alpha, \varphi \rangle = \int_0^\infty x^\alpha \left[\varphi(x) - \sum_{l=0}^{n-1} \varphi^{(l)}(0) \frac{x^l}{l!} \right] dx$$

for $-n-1 < \operatorname{Re} \alpha < -n$ (so $\operatorname{Re}(n+\alpha) > -1$). Since

$$(4.16) \qquad \int_\varepsilon^\infty x^\alpha \varphi(x) dx = \int_\varepsilon^\infty x^\alpha \left[\varphi(x) - \sum_0^{n-1} \varphi^{(l)}(0) \frac{x^l}{l!} \right] dx - \sum_{l=0}^{n-1} \frac{\varepsilon^{\alpha+l+1} \varphi^{(l)}(0)}{l!(\alpha+l+1)}$$

one can write $\langle x_+^\alpha, \varphi \rangle$ as

$$(4.17) \qquad \langle Pf(x^\alpha), \varphi \rangle = \lim_{\varepsilon \to 0} \left[\int_\varepsilon^\infty x^\alpha \varphi(x) dx + \sum_{l=0}^{n-1} \varphi^{(l)}(0) \varepsilon^{\alpha+l+1} / l!(\alpha+l+1) \right].$$

This distributions $Y_\beta \in \mathcal{D}'_+$ are defined by $Y_\beta = (1/\Gamma(\beta)) Pf(x^{\beta-1})$ for $\beta \neq a$ negative integer or zero with $Y_{-n} = \delta^{(n)}$ for $n \geq 0$ an integer. One has $Y_p * Y_q = Y_{p+q}$ and $D^m T = Y_{-m} * T$ with $I^m T = Y_m * T$. The Fourier transforms of these pseudofunctions are given in Gelfand-Šilov [28]. First recall that $x_-^\alpha = |x|^\alpha$ for $x < 0$ and is 0 for $x \geq 0$; then

$$(4.18) \qquad \langle x_-^\alpha, \varphi(x) \rangle = \langle x_+^\alpha, \varphi(-x) \rangle = \int_{-\infty}^0 |x|^\alpha \varphi(x) dx .$$

The distributions $(x \pm i0)^\alpha$ are defined by

$$(4.19) \qquad (x+i0)^\alpha = x_+^\alpha + e^{i\alpha\pi} x_-^\alpha ; \quad (x-i0)^\alpha = x_+^\alpha + e^{-i\alpha\pi} x_-^\alpha .$$

The following formulas then apply

$$(4.20) \qquad \mathcal{F}(x_+^\alpha) = ie^{i\alpha\pi/2} \Gamma(\alpha+1)(\sigma+i0)^{-\alpha-1} ; \quad \mathcal{F}(x_-^\alpha) = -ie^{-i\alpha\pi/2} \Gamma(\alpha+1)(\sigma-i0)^{-\alpha-1} .$$

In particular since $Y_\beta = (1/\Gamma(\beta)) x_+^{\beta-1}$

$$(4.21) \qquad \mathcal{F} Y_{\alpha+1} = ie^{i\alpha\pi/2}(\sigma+i0)^{-\alpha-1} .$$

Now we can write

$$(4.22)\qquad \int_0^\infty \lambda^\alpha \operatorname{Cos} \lambda y d\lambda = \frac{1}{2}\left[\int_0^\infty \lambda^\alpha e^{i\lambda y} d\lambda + \int_{-\infty}^0 (-\mu)^\alpha e^{i\mu y} d\mu\right]$$
$$= \frac{1}{2}[\mathcal{F}\lambda_+^\alpha + \mathcal{F}\lambda_-^\alpha]$$
$$= \frac{1}{2}\Gamma(\alpha+1)[ie^{i\alpha\pi/2}(y+i0)^{-\alpha-1} - ie^{-i\alpha\pi/2}(y-i0)^{-\alpha-1}]$$
$$= -\Gamma(\alpha+1) \operatorname{Sin} \frac{\pi\alpha}{2}[y_+^{-\alpha-1} + y_-^{-\alpha-1}]\,.$$

Consider the identification $y^{-\alpha-1} = [y_+^{-\alpha-1} + y_-^{-\alpha-1}]$ and set $\beta_m = 2\Gamma(1/2)/\Gamma(m+1) \times\Gamma(-1/2-m)$. Then for $\alpha = 2m+1$ we compare (from (4.22)) $-c_m^2\Gamma(2m+2)$ $\operatorname{Sin}(m+1/2)\pi$ with β_m. Now recall that (cf. [36]) $\Gamma(2m+2) = 2^{2m+1}\Gamma(m+1) \times\Gamma(m+3/2)/\sqrt{\pi}$ while $1/\Gamma(1/2-m) = \Gamma(m+3/2)\operatorname{Sin}(m+1/2)\pi/\pi(m+1/2)$ so we have $-c_m^2\Gamma(2m+2)\operatorname{Sin}(m+1/2)\pi = -2\Gamma(m+3/2)\operatorname{Sin}(m+1/2)\pi/\Gamma(m+1)\sqrt{\pi}$ while $\beta_m = -(2m+1)\Gamma(1/2)/\Gamma(m+1)\Gamma(1/2-m) = -2(m+1/2)\Gamma(1/2)\Gamma(m+3/2)$ $\operatorname{Sin}(m+1/2)\pi/\Gamma(m+1)\pi(m+1/2) = -2\Gamma(m+3/2)\operatorname{Sin}(m+1/2)\pi/\Gamma(m+1)/\sqrt{\pi}$. Thus we have proved

Theorem 4.6. *The formula* (4.13) *for* $\varphi(x) = \delta(x)/x^{2m+1}$ *is valid in the form below and* $B\varphi$ *is representable as indicated* ($2m \neq$ *integer*)

$$(4.23)\qquad B\varphi(y) = \beta_m y^{-2m-2} = \beta_m[y_+^{-2m-2} + y_-^{-2m-2}]$$
$$= c_m^2 \int_0^\infty \lambda^{2m+1} \operatorname{Cos} \lambda y d\lambda = \langle R_0, \Theta(y, \mu)\rangle_\lambda$$

REMARK 4.7. The model diagram (4.6) was constructed via the model operator $P_m(D)$ and the choice (3.10) for Ω. Let us point out that the same kind of diagram holds more generally if we take $\Omega = AH$ as in Section 3. Thus consider an equation such as (4.3) and recall (3.1) etc. for the operators $\boldsymbol{P}$, $\mathbf{P}$, etc. Let us write

$$(4.24)\qquad \langle \boldsymbol{P}f, \hat{\hat{g}}\rangle = \int \hat{f}(\lambda)\hat{\hat{g}}(\lambda)d\nu_p(\lambda) =$$
$$\langle\langle\Omega_\lambda^P(x), f(x)\rangle, \hat{\hat{g}}(\lambda)\rangle_\nu = \langle f(x), \langle\Omega_\lambda^P(x), \hat{\hat{g}}(\lambda)\rangle_\nu\rangle = \langle f(x), \boldsymbol{P}\hat{\hat{g}}(x)\rangle\,.$$

Thus take $\hat{E} = \boldsymbol{P}E \sim L^2(\lambda; d\nu)$ and $\hat{E}' = \hat{\hat{E}} \sim \hat{E}$. It would probably be most natural to couple this with $E = L^2(Adx)$ but this would be contrary to our desire to "spread out" a selfadjoint situation using $P^*(D)$ in $L^2(dx)$ etc. This directive is of course founded in the fact that when complex potentials $q(x)$ are present for example we do not have a selfadjoint situation and in general $d\nu$ will be replaced by a generalized spectral function R acting on suitable elements. This is examined in Part II. Note also here that an equation like $B^* = (\mathcal{Q}\boldsymbol{P})^* = \boldsymbol{P}\mathcal{Q}$ involves the kernel $\beta^*(y, x)$ where $\beta(y, x) = \langle\Omega_\lambda^P(x), \varphi_\lambda^Q(y)\rangle_\nu =$

$\int\Omega_\lambda^P(x)\varphi_\lambda^Q(y)d\nu_P(\lambda)$ arises from $\mathcal{Q}\boldsymbol{P}$ $(Bf(y)=\langle\beta(y,x),f(x)\rangle)$. But $\boldsymbol{P}\mathcal{Q}f(x)=\boldsymbol{P}\langle f(y),\varphi_\lambda^Q(y)\rangle=\int\Omega_\lambda^P(x)\langle f(y),\varphi_\lambda^Q(y)\rangle d\nu=\langle\beta(y,x),f(y)\rangle$ so everything fits together.

References

[1] B. Braaksma: *A singular Cauchy problem and generalized translations,* Int. Conf. Diff. Eqs., Academic Press, N.Y., 1975, 40–52.

[2] B. Braaksma and H. deSnoo: *Generalized translation operators associated with a singular differential operator,* Springer Lect. Notes 415, 1974, 62–77.

[3] Yu. Bryčkov and A. Prudnikov: Integral transformations of generalized functions. Moscow, 1977.

[4] J. Butler: *On the inverse problem for ordinary differential operators of even order,* J. Math. Anal. Appl. **26** (1969), 142–158.

[5] J. Butler: *On the inverse problem for selfadjoint operators defined on certain rigged Hilbert spaces,* SIAM J. Math. Anal. **10** (1979), 281–291.

[6] J. Butler: *On the inverse problem for differential equations of fourth order with rational reflection coefficients.* J. Differential Equations **4** (1968), 573–589.

[7] J. Butler: *On smooth perturbations of selfadjoint operators defined on rigged Hilbert spaces,* Math. Nachr. **53** (1972), 1–12.

[8] R. Carroll: Transmutation and operator differential equations. Notes de Matematica 67, North-Holland, Amsterdam, 1979.

[9] R. Carroll: *Transmutation and separation of variables,* Applicable Anal. **8** (1979), 253–263.

[10] R. Carroll: *Some remarks on transmutation,* Applicable Anal. **9** (1979), 291–294.

[11] R. Carroll: *Remarks on the Gelfand-Levitan and Marčenko equations,* Rocky Mountain J. Math., to appear.

[12] R. Carroll and J. Gilbert: *Scattering techniques in transmutation and some connection formulas for special functions,* Proc. Japan Acad. **57** (1981), 34–37.

[13] R. Carroll and J. Gilbert: *Transmutation, scattering theory, and special functions,* Math. Ann. **258** (1981), 39–54.

[14] R. Carroll and R. Showalter: Singular and degenerate Cauchy problems, Academic Press, N.Y., 1976.

[15] R. Carroll and F. Santosa: *Inverse scattering techniques in geophysics,* Applicable Anal. **11** (1980), 79–81.

[16] R. Carroll and F. Santosa: *Scattering techniques for a one dimensional inverse problem in geophysics,* Math. Methods Appl. Sci. **3** (1981), 145–171.

[17] K. Chadan and P. Sabatier: Inverse problems in quantum scattering theory, Springer. N.Y., 1977.

[18] H. Chebli: *Opérateurs de translation généralisés et semigroupes de convolution.* Springer Lect. Notes 404, 1974, 35–59.

[19] H. Chebli: *Théorème de Paley-Wiener associé à un opérateur différentiel singulier sur* (0, ∞), J. Math. Pures Appl. **58** (1979). 1–19.

[20] E. Copson and A. Erdelyi: *On a partial differential equation with two singular*

lines. Arch. Rational Mech. Anal. **2** (1958), 76–86.

[21] C. Coudray and M. Coz: *Generalized translation operators and the construction of potentials at fixed energy,* Ann. Physics **61** (1970), 488–529.

[22] J. Delsarte and J. Lions: *Transmutation d'opérateurs différentiels dans le domaine complexe,* Comment. Math. Helv. **32** (1957), 113–128.

[23] H. deSnoo: *Inversion theorems for some generalized Laplace transforms,* I and II, Proc. Kon. Ned. Ak. Wet. **73** (1970), 222–244.

[24] L. Fadeev: *The inverse problem of quantum scattering theory,* Uspekhi Mat. Nauk **14** (1959), 57–119.

[25] M. Flensted-Jensen: *Paley-Wiener type theorems for a differential operator connected with symmetric spaces,* Ark. Mat. **10** (1972), 143–162.

[26] M. Gasymov: *The expansion in eigenfunctions of a nonselfadjoint second order differential operator with a singularity at zero,* Trudy Letnej. Šk. Spektral. Teorii Operator. ..., Izd. Elm, Baku, 1975, 20–45.

[27] M. Gasymov: *On the eigenfunction expansion of a nonselfadjoint boundary problem for a differential equation with a singularity at zero,* Dokl. Akad. Nauk SSSR **165** (1965), 261–264.

[28] I. Gelfand and G. Šilov: Generalized functions, Vols. 1–3, Moscow, 1958.

[29] J. Geronimo and K. Case: *Scattering theory and polynomials orthogonal on the real line,* Trans. Amer. Math. Soc. **258** (1980), 467–494.

[30] V. Hutson and J. Pym: *Generalized translations associated with a differential operator,* Proc. London Math. Soc. **24** (1972), 548–576.

[31] T. Koornwinder: *A new proof of a Paley-Wiener type theorem for the Jacobi transform,* Ark Mat. **31** (1975)., 145–159.

[32] N. Leblanc: *Algèbres de Banach associées à un opérateru différentiel de Sturm-Liouville,* Springer Lect. Notes 336, 1973, 40–50.

[33] B. Levitan: The theory of generalized translation operators, Izd. Nauka, Moscow, 1973.

[34] B. Levitan: *The expansion in Bessel functions for Fourier series and integrals,* Uspekhi Mat. Nauk **6** (1951), 102–143.

[35] J. Lions: *Opérateurs de Delsarte et problèmes mixtes,* Bull. Soc. Math. France **84** (1956), 9–95.

[36] W. Magnus, F. Oberhettinger, and R. Soni: Formulas and theorems for the special functions of mathematical physics, Springer, N.Y., 1966.

[37] V. Marčenko: Sturm-Liouville operators and their applications, Izd. Nauk. Dumka, Kiev, 1977.

[38] J. McLaughlin: *An inverse eigenvalue problem of fourth order,* SIAM J. Math. Anal. **7** (1976), 646–661.

[39] R. Newton: Scattering theory of waves and particles, McGraw-Hill, N.Y., 1966.

[40] A. Povzner: *Differential equations of Sturm-Liouville type on a half axis,* Mat. Sb. **23** (1948), 3–52.

[41] L. Sakhnovič: *The inverse problem for differential operators of order $n>2$ with analytic coefficients,* Mat. Sb. **46** (1958), 61–76.

[42] L. Sakhnovič: *The method of the transformation operator for equations of higher order*, Mat. Sb. **55** (1961), 347–360.

[43] L. Schwartz: Théorie des distributions, Edition "Papillon", Hermann, Paris, 1966.
[44] J. Siersma: *On a class of singular Cauchy problems,* Thesis, Groningen, 1979.
[45] V. Staševskaya: *The inverse problem of spectral analysis for a certain class of differential equations,* Dokl. Akad. Nauk SSSR **93** (1953), 409–412.
[46] W. Symes: *An integral representation for eigenfunctions of linear ordinary differential equations,* J. Math. Anal. Appl. **70** (1979), 101–110.
[47] V. Tkačenko: *On a transformation operator in spaces of entire functions of finite order,* Theory of functions, functional anal. and appl. **22** (1975), 154–159.
[48] V. Tkačenko: *Transformation operators in spaces of entire functions,* Sibirsk. Mat. Ž. **20** (1979), 152–163.
[49] V. Volk: *On inversion formulas for differential operators with a singularity at* $x=0$, Uspekhi Mat. Nauk **8** (1953), 141–151.

Department of Mathematics
University of Illinois
at Urbana-Champaign
Urbana, Illinois 61801
U.S.A.