Naitoh, H. and Takeuchi, M.
Osaka J. Math.
19 (1982), 717–731

TOTALLY REAL SUBMANIFOLDS AND SYMMETRIC BOUNDED DOMAINS

HIROO NAITOH*) AND MASARU TAKEUCHI

(Received October 25, 1980)

Introduction. Let $P_n(c)$ denote the complex projective n-space endowed with the Kählerian metric of constant holomorphic sectional curvature $c>0$. We consider an n-dimensional complete totally real submanifold M of $P_n(c)$ with parallel second fundamental form σ. The first named author [6] reduced the classification of such submanifolds to that of certain cubic forms of n-variables, and he classified completely those without Euclidean factor among such submanifolds. (Note that such a submanifold is always locally symmetric.)

In this note we shall give another way of the classification of these submanifolds. Let $D\subset \boldsymbol{C}^{n+1}$ be a symmetric bounded domain of tube type realized by the Harish-Chandra imbedding. We imbed the Shilov boundary $\hat{M}$ of D into the hypersphere $S^{2n+1}(c/4)$ of the radius $2/\sqrt{c}$ with respect to a suitable hermitian inner product of $\boldsymbol{C}^{n+1}$. Let $M\subset P_n(c)$ be the image of $\hat{M}$ under the Hopf fibering $\pi\colon S^{2n+1}(c/4)\to P_n(c)$. Then M is an n-dimensional complete totally real submanifold with parallel second fundamental form (Theorem 2.1), and conversely such a submanifold is obtained in this way (Theorem 3.1). The crucial point in the argument is that $M\subset P_n(c)$ has the parallel second fundamental form if and only if $\hat{M}=\pi^{-1}(M)\subset S^{2n+1}(c/4)$ has the parallel second fundamental form (Lemma 1.1). Thus we may use the classification (Ferus [3], Takeuchi [10]) of submanifolds in spheres with parallel second fundamental form.

As an application, we give a characterization of an n-dimensional compact totally real minimal submanifold M of $P_n(c)$ with $\|\sigma\|^2=n(n+1)c/4(2n-1)$. (Recall that $\|\sigma\|^2<n(n+1)c/4(2n-1)$ implies $\sigma=0$. cf. Chen-Ogiue [1].) Such a submanifold M is unique and nothing but the flat isotropic surface $M_0^2\subset P_2(c)$ with parallel second fundamental form constructed in Naitoh [5] (Theorem 4.5).

1. Hopf fiberings

Let $\boldsymbol{R}^{n+1}$ be the real Cartesian $(n+1)$-space with the standard inner pro-

*) Partially supported by the Yukawa Foundation.

duct $\langle\ ,\ \rangle$. For a constant $k>0$, we denote by $S^n(k)$ the hypersphere of $\boldsymbol{R}^{n+1}$ with the radius $1/\sqrt{k}$ endowed with the Riemannian metric $\hat{g}$ induced from $\langle\ ,\ \rangle$.

Now we fix a positive integer m and a constant $c>0$, and denote by $P_m(c)$ the complex projective m-space $P_m(\boldsymbol{C})$ endowed with the Kählerian metric g of constant holomorphic sectional curvature c. We regard the complex Cartesian $(m+1)$-space $\boldsymbol{C}^{m+1}$ as a Euclidean $(2m+2)$-space by the inner product: $\langle z,w\rangle = \mathcal{R}e\,{}^t z\bar{w}$ for $z,w\in\boldsymbol{C}^{m+1}$. Then the Hopf fibering $\pi: S^{2m+1}(c/4)\to P_m(c)$ defined by $\pi(z)=[z]$, $[z]$ being the point of $P_m(\boldsymbol{C})$ with the homogeneous coordinate z, is a Riemannian submersion in the sense of O'Neill [7]. The complex structure tensors on $\boldsymbol{C}^{m+1}$ and $P_m(\boldsymbol{C})$ are denoted by J. We write $S=S^{2m+1}(c/4)$. Define a unit normal vector field ν for the imbedding $S\hookrightarrow\boldsymbol{C}^{m+1}$ by $\nu_q=(\sqrt{c}/2)q$ for $q\in S$, and put $V_q=\boldsymbol{R}(J\nu_q)$ and

$$H_q = \{z\in\boldsymbol{C}^{m+1}; \langle z,q\rangle = \langle z,J\nu_q\rangle = 0\}$$

for $q\in S$. Then the subbundles $V(S)=\bigcup_{q\in S}V_q$ and $H(S)=\bigcup_{q\in S}H_q$ of the tangent bundle TS of S are the vertical and the horizontal distributions for the Riemannian submersion π, respectively, and thus we have an orthogonal Whitney sum: $TS=V(S)\oplus H(S)$. The complex structure J on $\boldsymbol{C}^{m+1}$ leaves each H_q invariant and $J_q|H_q$ corresponds to $J_{\pi(q)}$ on $P_m(\boldsymbol{C})$ under the linear isometry $\pi_*: H_q\to T_{\pi(q)}P_m(c)$. For a vector field X on S, its $V(S)$-component and $H(S)$-component will be denoted by $\mathcal{V}X$ and $\mathcal{H}X$, respectively. If $\mathcal{V}X=X$ (resp. $\mathcal{H}X=X$), X is said to be *vertical* (resp. *horizontal*). If X is horizontal and projectable to a vector field X_* on $P_m(\boldsymbol{C})$, it is called the *horizontal lift* of X_* and denoted by $X=h.l.\,X_*$. The Riemannian connections of S and $P_m(c)$ are denoted by ∇^S and $\overline{\nabla}$, respectively. Let A and T be the fundamental tensors for the Riemannian submersion π defined in O'Neill [7]. Then we have $T=0$, since each fibre of the Hopf fibering π is totally geodesic in S. For such a Riemannian submersion we have the following identities:

(1.1) $\nabla^S_V X = \mathcal{H}\nabla^S_V X\,,$

(1.2) $\nabla^S_V V = A_X V + \mathcal{V}\nabla^S_X V\,,$

(1.3) $\nabla^S_X Y = \mathcal{H}\nabla^S_X Y + A_X Y$

for horizontal vector fields X, Y and a vertical vector field V on S. If further $X=h.l.\,X_*$ and $Y=h.l.\,Y_*$, then we have

(1.4) $\mathcal{H}\nabla^S_V X = A_X V\,,$

(1.5) $\mathcal{H}\nabla^S_X Y = h.l.\,\overline{\nabla}_{X_*}Y_*\,.$

The fundamental tensor A for our Hopf fibering π is given by

(1.6) $A_X(J\nu)=(\sqrt{c}/2)JX$,

(1.7) $A_XY=(\sqrt{c}/2)\langle X,JY\rangle J\nu$

for horizontal vector fields X,Y on S. For these identities (1.1)~(1.7), we refer the reader to O'Neill [7].

Now let $f:(M,g)\to P_m(c)$ be an isometric immersion of a Riemannian manifold (M,g) into $P_m(c)$. The complex structure and the connection on the pull back $f^{-1}T(P_m(\boldsymbol{C}))$ induced from J and $\bar{\nabla}$ are also denoted by J and $\bar{\nabla}$. Let $\hat{M}$ be the total space of the pull back $f^{-1}S$ of the principal $U(1)$-bundle $\pi: S\to P_m(\boldsymbol{C})$. The $U(1)$-bundle map $\hat{f}:\hat{M}\to S$ which covers f is also an immersion, and so we may define a Riemannian metric $\hat{g}$ on $\hat{M}$ in such a way that $\hat{f}:(\hat{M},\hat{g})\to S$ is an isometric immersion. Then the projection $\pi:(\hat{M},\hat{g})\to(M,g)$ is also a Riemannian submersion with $T=0$. Note that we have an orthogonal Whitney sum: $\hat{f}^{-1}(TS)=\hat{f}^{-1}V(S)\oplus\hat{f}^{-1}H(S)$. The connection on $\hat{f}^{-1}(TS)$ induced from ∇^S on TS and the complex structure on $\hat{f}^{-1}H(S)$ induced from J on $H(S)$ are also denoted by ∇^S and J, respectively. We define $V(\hat{M})=\hat{f}^{-1}V(S)$, which is the vertical distribution for the Riemannian submersion $\pi:(\hat{M},\hat{g})\to(M,g)$. Then the section $J\nu$ of $V(S)$ induces a section of $V(\hat{M})$, which will be also denoted by $J\nu$. Furthermore, regarding $T\hat{M}$ as a subbundle of $\hat{f}^{-1}(TS)$, we define $H(\hat{M})=T\hat{M}\cap\hat{f}^{-1}H(S)$, which is the horizontal distribution for π. Thus we have an orthogonal Whitney sum: $T\hat{M}=V(\hat{M})\oplus H(\hat{M})$. The second fundamental forms of $f:(M,g)\to P_m(c)$ and $\hat{f}:(\hat{M},\hat{g})\to S$ will be denoted by σ and $\hat{\sigma}$, respectively.

The isometric immersion $f:(M,g)\to P_m(c)$ is said to be *totally real* if $\langle J(T_pM), T_pM\rangle=\{0\}$ for each $p\in M$. This is the case if and only if

(1.8) $\langle JH_q(\hat{M}), H_q(\hat{M})\rangle=\{0\}$

for each $q\in\hat{M}$, where $H_q(\hat{M})$ denotes the fibre of $H(\hat{M})$ over q.

Lemma 1.1. *Let $f:(M,g)\to P_m(c)$ be a totally real isometric immersion and $\hat{f}:(\hat{M},\hat{g})\to S$ the isometric immersion induced from f in the above way. Then*

1) *f is minimal if and only if $\hat{f}$ is minimal;*

2) *(M,g) is complete if and only if $(\hat{M},\hat{g})$ is complete;*

3) *$f(M)$ is not contained in any complex hyperplane of $P_m(\boldsymbol{C})$ if and only if $\hat{f}(\hat{M})$ is not contained in any real hyperplane of $\boldsymbol{C}^{m+1}$;*

4) *Both $V(\hat{M})$ and $H(\hat{M})$ are parallel subbundles of $T\hat{M}$, i.e., they are invariant under the parallel translation of $(\hat{M},\hat{g})$ along any curve of $\hat{M}$;*

5) *Assume that the linear span $N_p^1(M)$ of $\sigma(T_pM,T_pM)$ is contained in $J(T_pM)$ for each $p\in M$. Then, σ is parallel if and only if $\hat{\sigma}$ is parallel.*

Proof. We shall prove first the following: Let ∇ and $\hat{\nabla}$ denote the Riemannian connections of (M,g) and $(\hat{M},\hat{g})$, respectively. Let X,Y be vector fields on

$\hat{M}$ which are horizontal lifts of vector fields X_*, Y_* on M, respectively. Then

(1.9) $\hat{\nabla}_X Y = h.l.\, \nabla_{X_*} Y_*$;

(1.10) $\hat{\sigma}(X, Y) = h.l.\sigma(X_*, Y_*)$;

(1.11) $\hat{\nabla}_X(J\nu) = \mathcal{V}\nabla^S_X(J\nu)$;

(1.12) $\hat{\sigma}(X, J\nu) = (\sqrt{c}/2)JX$;

(1.13) $\hat{\nabla}_{J\nu} X = 0$;

(1.14) $\hat{\nabla}_{J\nu}(J\nu) = 0$;

(1.15) $\hat{\sigma}(J\nu, J\nu) = 0$.

We have

$$\begin{aligned} \nabla^S_X Y &= \mathcal{H}\nabla^S_X Y + A_X Y && \text{by (1.3)} \\ &= \mathcal{H}\nabla^S_X Y + (\sqrt{c}/2)\langle X, JY\rangle J\nu && \text{by (1.7)} \\ &= \mathcal{H}\nabla^S_X Y = h.l.\bar{\nabla}_{X_*} Y_* && \text{by (1.8), (1.5)}. \end{aligned}$$

This implies (1.9), (1.10). We have

$$\begin{aligned} \nabla^S_X(J\nu) &= A_X(J\nu) + \mathcal{V}\nabla^S_X(J\nu) && \text{by (1.2)} \\ &= (\sqrt{c}/2)JX + \mathcal{V}\nabla^S_X(J\nu) && \text{by (1.6)}. \end{aligned}$$

This together with (1.8) implies (1.11), (1.12). We have

$$\nabla^S_{J\nu} X = \mathcal{H}\nabla^S_{J\nu} X = A_X(J\nu) \qquad \text{by (1.1), (1.4)}.$$

Thus, by (1.6) we obtain

(1.16) $\nabla^S_{J\nu} X = (\sqrt{c}/2)JX$.

This together with (1.8) implies (1.13). The equalities (1.14), (1.15) follow from $\nabla^S_{J\nu}(J\nu) = 0$.

1) Let η and $\hat{\eta}$ denote the mean curvature vectors of f and $\hat{f}$, respectively. Let $\dim M = n$ and so $\dim \hat{M} = n+1$. For an arbitrary $q \in \hat{M}$, choose an orthonormal basis $\{x_1, \cdots, x_n\}$ of $H_q(\hat{M})$ and put $x_{i*} = \pi_* x_i$, $1 \leqq i \leqq n$. Extend each x_{i*} to a vector field X_{i*} on M and let $X_i = h.l.\, X_{i*}$. Then, by (1.10), (1.15) we have

$$\begin{aligned} (n+1)\hat{\eta}_q &= \textstyle\sum_{i=1}^n \hat{\sigma}(X_i, X_i)_q + \hat{\sigma}(J\nu, J\nu)_q \\ &= \textstyle\sum_{i=1}^n (h.l.\sigma(X_{i*}, X_{i*}))_q = n(h.l.\,\eta)_q. \end{aligned}$$

This implies the assertion 1).

2) This follows from the compactness of the fibre $U(1)$ of π.

3) Assume that $\hat{f}(\hat{M})$ is contained in a real hyperplane of $\boldsymbol{C}^{m+1}$. Then

there exist $a\in \boldsymbol{C}^{m+1}-\{0\}$ and $k\in \boldsymbol{R}$ and that $\langle \hat{f}(\hat{M}),a\rangle=\{k\}$. Take a point $q\in\hat{M}$ and let ${}^t(\hat{f}(q))\bar{a}=re^{\sqrt{-1}\phi}$ so that $k=r\cos\phi$. For each $\varepsilon=e^{\sqrt{-1}\theta}\in U(1)$, $\theta\in\boldsymbol{R}$, we have

$$k=\langle \hat{f}(q\varepsilon),a\rangle=\langle \hat{f}(q)\varepsilon,a\rangle=\mathcal{R}e\{{}^t(\hat{f}(q))\bar{a}\varepsilon\}$$
$$=\mathcal{R}e(re^{\sqrt{-1}(\phi+\theta)})=r\cos(\phi+\theta)\,.$$

We have therefore $r=0$, and hence $\langle \hat{f}(\hat{M}),a\rangle=\{0\}$. Now for each $\varepsilon\in U(1)$ we have $\langle \hat{f}(\hat{M}),a\varepsilon\rangle=\langle \hat{f}(\hat{M})\bar{\varepsilon},a\rangle=\langle \hat{f}(\hat{M}\bar{\varepsilon}),a\rangle=\{0\}$. Thus $f(M)$ is contained in the complex hyperplane $\{[z]\in P_m(\boldsymbol{C});\,{}^tza=0\}$ of $P_m(\boldsymbol{C})$. If conversely $f(M)$ is contained in a complex hyperplane $\{[z]\in P_m(\boldsymbol{C});\,{}^tza=0\}$, $a\in\boldsymbol{C}^{m+1}-\{0\}$, then $\hat{f}(\hat{M})$ is contained in the real hyperplane $\{z\in\boldsymbol{C}^{m+1};\,\langle z,a\rangle=0\}$ of $\boldsymbol{C}^{m+1}$.

4) Equalities (1.11), (1.14) and (1.9), (1.13) imply that $V(\hat{M})$ and $H(\hat{M})$ are invariant, respectively, under the covariant differentiation by any vector field on M. Thus the assertion 4) follows.

5) Let $\nabla^{\perp}$ and $\hat{\nabla}^{\perp}$ be the normal connections on the normal bundles NM and $N\hat{M}$, respectively, and let ∇^* and $\hat{\nabla}^*$ be the coveriant derivations on $T^*M\otimes T^*M\otimes NM$ and $T^*\hat{M}\otimes T^*\hat{M}\otimes N\hat{M}$, respectively, where T^*M and $T^*\hat{M}$ denote the cotangent bundles. Let X,Y,Z be the horizontal lifts of vector fields X_*, Y_*,Z_* on M, respectively. Then

$$\text{(a)}\quad (\hat{\nabla}^*\hat{\sigma})(J\nu,J\nu,J\nu)=\hat{\nabla}^{\perp}_{J\nu}\hat{\sigma}(J\nu,J\nu)-2\hat{\sigma}(\hat{\nabla}_{J\nu}(J\nu),J\nu)$$
$$=0\qquad\text{by (1.15), (1.14)}\,.$$

$$\text{(b)}\quad (\hat{\nabla}^*\hat{\sigma})(X,J\nu,J\nu)=\hat{\nabla}^{\perp}_{X}\hat{\sigma}(J\nu,J\nu)-2\hat{\sigma}(\hat{\nabla}_{X}(J\nu),J\nu)$$
$$=-2\hat{\sigma}(\mathcal{V}\nabla^{S}_{X}(J\nu),J\nu)\qquad\text{by (1.15), (1.11)}$$
$$=0\qquad\text{by (1.15)}\,.$$

$$(\hat{\nabla}^*\hat{\sigma})(J\nu,X,Y)=\hat{\nabla}^{\perp}_{J\nu}\hat{\sigma}(X,Y)-\hat{\sigma}(\hat{\nabla}_{J\nu}X,Y)-\hat{\sigma}(X,\hat{\nabla}_{J\nu}Y)$$
$$=\hat{\nabla}^{\perp}_{J\nu}\hat{\sigma}(X,Y)\qquad\text{by (1.13)}\,.$$

Here $\hat{\sigma}(X,Y)=h.l.\,\sigma(X_*,Y_*)$ by (1.10). Therefore we have

$$\nabla^{S}_{J\nu}\hat{\sigma}(X,Y)=(\sqrt{c}/2)J\hat{\sigma}(X,Y)=(\sqrt{c}/2)h.l.\,J\sigma(X_*,Y_*)\,,$$

since (1.16) holds also for the horizontal lift X of a vector field X_* on $P_m(\boldsymbol{C})$. Now the assumption $N^1_p(M)\subset J(T_p(M))$ implies that $J\sigma(X_*,Y_*)$ is tangent to M, and hence $\nabla^{S}_{J\nu}\hat{\sigma}(X,Y)$ is tangent to $H(M)$. Thus $\hat{\nabla}^{\perp}_{J\nu}\hat{\sigma}(X,Y)=0$, and hence

$$\text{(c)}\quad (\hat{\nabla}^*\hat{\sigma})(J\nu,X,Y)=0\,.$$

Moreover, by (1.9) ,(1.10) we have

$$(\hat{\nabla}^*\hat{\sigma})(X,Y,Z)=\hat{\nabla}^{\perp}_{X}\hat{\sigma}(Y,Z)-\hat{\sigma}(\hat{\nabla}_XY,Z)-\hat{\sigma}(Y,\hat{\nabla}_XZ)$$
$$=\hat{\nabla}^{\perp}_{X}\hat{\sigma}(Y,Z)-h.l.\,\sigma(\nabla_{X_*}Y_*,Z_*)-h.l.\,\sigma(Y_*,\nabla_{X_*}Z_*)\,.$$

Here $\hat{\sigma}(Y,Z)=h.l.\sigma(Y_*,Z_*)$ by (1.10), and thus $\nabla^S_X\hat{\sigma}(Y,Z)=h.l.\bar{\nabla}_{X_*}\sigma(Y_*,Z_*)$ by (1.5). Therefore $\hat{\nabla}^\perp_X\hat{\sigma}(Y,Z)=h.l.\nabla^\perp_{X_*}\sigma(Y_*,Z_*)$. Thus we obtain

(d) $(\hat{\nabla}^*\hat{\sigma})(X,Y,Z)=h.l.(\nabla^*\sigma)(X_*,Y_*,Z_*)$.

Now (a), (b), (c), (d) imply the assertion 5), since $\hat{\nabla}^*\hat{\sigma}$ is symmetric trilinear in virtue of the Codazzi equation. q.e.d.

2. Shilov boundaries of symmetric bounded domains of tube type

We fix a positive integer n and a constant $c>0$. Let us consider an object $\mathfrak{d}=(D_1,\cdots,D_s;c_1,\cdots,c_s)$, $s\geqq 1$, where

(i) D_i, $1\leqq i\leqq s$, is an irreducible symmetric bounded domain of tube type, and $\Sigma_i \dim_C D_i=n+1$;

(ii) c_i, $1\leqq i\leqq s$, is a positive constant, and $\Sigma_i 1/c_i=1/c$.

We shall associate to such an object $\mathfrak{d}$ a totally real isometric imbedding f: $(M,g)\to P_m(c)$ of an n-dimensional complete connected Riemannian manifold (M,g) with parallel second fundamental form.

Let $D=D_1\times\cdots\times D_s$ be the direct product of the D_i's, $1\leqq i\leqq s$. It is also a symmetric bounded domain of tube type. Note that $\dim_C D=n+1$ in virtue of (i). The identity component G of the group of holomorphisms of D is semisimple and with the trivial center. Therefore it is identified with the group of inner automorphisms of $\mathfrak{g}=\mathrm{Lie}\, G$, the Lie algebra of G, and hence it is also identified with a closed subgroup of the group G_C of inner automorphisms of the complexification $\mathfrak{g}_C$ of $\mathfrak{g}$. Fix a point $o\in D$ and put

$$K=\{\phi\in G;\ \phi o=o\}\,,\qquad \mathfrak{k}=\mathrm{Lie}\, K\,.$$

Then the subspace

$$\mathfrak{p}=\{X\in\mathfrak{g};\ B(X,\mathfrak{k})=\{0\}\}$$

of $\mathfrak{g}$, where B denotes the Killing form of $\mathfrak{g}_C$, is invariant under the adjoint action of K, and it is identified with the tangent space T_oD of D at o. Let H be the unique element of the center of $\mathfrak{k}$ such that $ad\,H|\mathfrak{p}$ coincides with the complex structure J of D on $\mathfrak{p}=T_oD$. Then the complexification $\mathfrak{p}_C$ of $\mathfrak{p}$ is decomposed to the direct sum: $\mathfrak{p}_C=\mathfrak{p}_C^+ +\mathfrak{p}_C^-$ of K-invariant subspaces $\mathfrak{p}_C^\pm$ defined by

$$\mathfrak{p}_C^\pm=\{X\in\mathfrak{p}_C;\ [H,X]=\pm\sqrt{-1}\,X\}\,.$$

Note that the linear map $\iota\colon\mathfrak{p}\to\mathfrak{p}_C^+$ defined by $\iota(X)=(1/2)(X-\sqrt{-1}\,[H,X])$ is a K-equivariant C-linear isomorphism of $(\mathfrak{p},J)$ onto $\mathfrak{p}_C^+$. Denoting by τ the complex conjugation of $\mathfrak{g}_C$ with respect to the compact real form $\mathfrak{g}_u=\mathfrak{k}+\sqrt{-1}\,\mathfrak{p}$, we define a K-invariant hermitian inner product $(\ ,\ )_\tau$ on $\mathfrak{p}_C^+$ by

$(X,Y)_\tau=-B(X,\tau Y)$ for $X,Y\in\mathfrak{p}_C^+$. We define then a K-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{p}_C^+$, regarded as a real vector space, by $\langle X,Y\rangle=2\mathcal{R}e(X,Y)_\tau$ for $X,Y\in\mathfrak{p}_C^+$. Then we have

$$\text{(2.1)}\quad \langle\iota X,\iota Y\rangle=B(X,Y) \qquad \text{for}\quad X,Y\in\mathfrak{p}\,.$$

Let $c\in G_u$, G_u being the connected subgroup of G_C generated by $\mathfrak{g}_u$, denote the standard Cayley transform for D(cf. Takeuchi [9]), and define an involutive automorphism θ of G_C by $\theta(x)=c^2xc^{-2}$ for $x\in G_C$. The differential $Ad\,c^2$ of θ will be also denoted by θ. Then we have $\theta\tau=\tau\theta$, $\theta\mathfrak{k}=\mathfrak{k}$ and $\theta H=-H$. We may define an anti-linear endomorphism $X\to\bar{X}$ of $\mathfrak{p}_C^+$ by $\bar{X}=\tau\theta X$, so that

$$\mathfrak{p}^+=\{X\in\mathfrak{p}_C^+;\bar{X}=X\}$$

is a real form of $\mathfrak{p}_C^+$. Let now $F\colon D\hookrightarrow\mathfrak{p}_C^+$ be the Harish-Chandra imbedding for D, and $S\subset\partial D\subset\mathfrak{p}_C^+$ the Shilov boundary of D. The groups G,K or $\mathfrak{g},\mathfrak{k}$, $\mathfrak{p}$, $\mathfrak{p}_C^+$ etc. are the direct products or the direct sums of respective objects for D_i, $1\leqq i\leqq s$, which will be denoted by the same notation but with the suffix i. Then F is the product imbedding $F_1\times\cdots\times F_s$ of Harish-Chandra imbeddings $F_i\colon D_i\hookrightarrow\mathfrak{p}_{iC}^+$ for D_i, and S is the direct product $S_1\times\cdots\times S_s$ of Shilov boundaries $S_i\subset\partial D_i\subset\mathfrak{p}_{iC}^+$ of D_i. The group K acts transitively on S and S is a compact connected manifold with dim $S=\dim_C D=n+1$. Let $X_i^0\in S_i$ be the standard base point of S_i (cf. Takeuchi [9]). Then

(2.2) eigenvalues of $ad(\iota^{-1}(\sqrt{-1}\,X_i^0))$ on $\mathfrak{g}_i$ are $0,2,-2$.

Put $X^0=X_1^0+\cdots+X_s^0\in S$ and

$$K_0=\{k\in K;\,kX^0=X^0\}\,.$$

Then (K,K_0) is a symmetric pair with respect to θ and S is identified with the quotient manifold K/K_0. If we set

$$\mathfrak{s}=\{X\in\mathfrak{k};\,\theta X=-X\}\,,$$

and $\psi(X)=[X,\sqrt{-1}\,X^0]$ for $X\in\mathfrak{s}$, then ψ defines a linear isomorphism of $\mathfrak{s}$ onto $\mathfrak{p}^+$. In particular, we have

$$\text{(2.3)}\quad [\mathfrak{s},\sqrt{-1}\,X^0]=\mathfrak{p}^+\,.$$

For these properties of symmetric bounded domains of tube type, we refer the reader to Korányi-Wolf [4], Takeuchi [9].

Now let $\dim_C D_i=n_i+1$ and put $a_i=1/\sqrt{2c_i(n_i+1)}$, $1\leqq i\leqq s$. We define an $(n+1)$-dimensional compact connected submanifold $\hat{M}$ of $\mathfrak{p}_C^+$ by

$$\hat{M}=a_1S_1\times\cdots\times a_sS_s\,,$$

and endow it with the Riemannian metric $\hat{g}$ induced from $\langle\ ,\ \rangle$. We write

$\hat{M}_i = a_i S_i \subset \mathfrak{p}_{iC}^+$, $1 \leqq i \leqq s$. If we put $E_i = \sqrt{-1} a_i X_i^0 \in \mathfrak{p}_{iC}^+$ and $E = E_1 + \cdots + E_s \in \mathfrak{p}_C^+$, then E_i belongs to $\hat{M}_i$, since each D_i is a circular domain in $\mathfrak{p}_{iC}^+$, and hence E belongs to $\hat{M}$. Thus we have $\hat{M}_i = K_i E_i$ and $\hat{M} = KE$. Note that we have also

$$(2.4) \quad K_0 = \{k \in K; kE = E\},$$

and hence $\hat{M}$ is identified with K/K_0. Moreover, (2.1), (2.2) imply

$$\langle \sqrt{-1} X_i^0, \sqrt{-1} X_i^0 \rangle = 4 \dim \mathfrak{p}_i = 8 \dim_C D_i = 8(n_i + 1)$$

and hence $\langle E_i, E_i \rangle = 4/c_i$, thus $\langle E, E \rangle = \Sigma_i \langle E_i, E_i \rangle = \Sigma_i 4/c_i = 4/c$ in virtue of (ii). Therefore, identifying $\mathfrak{p}_{iC}^+$ with C^{n_i+1} by an orthonormal basis of $\mathfrak{p}_{iC}^+$ with respect to $2(\ ,\ )_\tau$, and thus identifying $\mathfrak{p}_C^+$ with C^{n+1}, we have

$$\hat{M}_i \subset S^{2n_i+1}(c_i/4), \quad 1 \leqq i \leqq s,$$

and

$$\hat{M} = \hat{M}_1 \times \cdots \times \hat{M}_s \subset S^{2n+1}(c/4).$$

Furthermore, the property (2.2) implies that each inclusion $\hat{M}_i \hookrightarrow S^{2n_i+1}(c_i/4)$ is a standard minimal isometric imbedding of an irreducible symmetric R-space $\hat{M}_i$ in the sense of Takeuchi [10]. Thus, by Takeuchi [10] the inclusion $\hat{f}$: $(\hat{M}, \hat{g}) \to S^{2n+1}(c/4)$ is an isometric imbedding with parallel second fundamental form such that $\hat{f}(\hat{M})$ is not contained in any real hyperplane of C^{n+1}. Here the identity component $I^0(\hat{M})$ of the group of isometries of $(\hat{M}, \hat{g})$ may be identified with K. Moreover, $\hat{f}$ is minimal if and only if

$$(2.5) \quad c_i(n_i + 1) = c(n+1) \qquad \text{for each } i,\ 1 \leqq i \leqq s.$$

Now let $\pi: S^{2n+1}(c/4) \to P_n(c)$ be the Hopf fibering and put $M = \pi(\hat{M})$. It is a compact connected submanifold of $P_n(C)$ since it is a K-orbit in $P_n(C)$. We endow M with the Riemannian metric g induced from that of $P_n(c)$, and denote by $f: (M,g) \to P_n(c)$ the inclusion. Since the connected subgroup Z of K generated by RH acts on $\mathfrak{p}_C^+$ by $U(1) = \{\varepsilon I;\ \varepsilon \in C,\ |\varepsilon| = 1\}$, we have $\pi^{-1}(M) = \hat{M}$. Therefore we have dim $M = n$. Thus we are in the position of **1** with $m = n$.

Theorem 2.1. *Let $f: (M,g) \to P_n(c)$ be the isometric imbedding associated to $\mathfrak{d} = (D_1, \cdots, D_s; c_1, \cdots, c_s)$ in the above way. Then*

1) *f is totally real and has the parallel second fundamental form. In particular, (M,g) is locally symmetric;*

2) *f is minimal if and only if $c_i \dim_C D_i = c(n+1)$ for each i, $1 \leqq i \leqq s$;*

3) *The dimension of the Euclidean factor of the locally symmetric space (M,g) is equal to $s-1$;*

4) *(M,g) has no Euclidean factor if and only if $s=1$ and $\dim_C D_1 \geqq 2$. In*

this case, (M,g) *is irreducible and* f *is minimal;*

5) (M,g) *is flat if and only if* $s=n+1$ *and* $\dim_C D_i=1$*, i.e.,* D_i *is the unit disk, for each* i*,* $1\leqq i\leqq n+1$.

Proof. We prove first that f is totally real. Since K acts on $P_n(c)$ as isometric holomorphisms of $P_n(c)$, f is K-equivariant and M is a K-orbit, we need only to prove the property (1.8) for $q=E$. By (2.4) the tangent space $T_E\hat{M}$ is identified with $\mathfrak{s}$. Moreover, by (2.3) we have $[\mathfrak{s},E]=\mathfrak{p}^+$, and hence $T_E\hat{M}$ is identified with $\mathfrak{p}^+$. In particular we have $\sqrt{-1}\,E=[H,E]\in\mathfrak{p}^+$, since $H\in\mathfrak{s}$. Thus, if we put

$$\mathfrak{h}=\{X\in\mathfrak{p}^+;\langle X,\sqrt{-1}\,E\rangle=\{0\}\}\,,$$

it is identified with $H_E(\hat{M})$. Now $\langle\mathfrak{p}^+,\sqrt{-1}\,\mathfrak{p}^+\rangle=\{0\}$ implies $\langle\mathfrak{h},\sqrt{-1}\,\mathfrak{h}\rangle=\{0\}$. We have therefore the required property: $\langle H_E(\hat{M}),JH_E(\hat{M})\rangle=\{0\}$.

The assertion that σ is parallel is an immediate consequence of Lemma 1.1,5), since $NM=J(TM)$ in our case. The assertion 2) follows from Lemma 1.1,1) and (2.5). The assertions 3),4),5), except for the minimality for f in 4), follow from the following observations:

(a) the dimension of Euclidean factor of M=the one of $\hat{M}-1$;

(b) the dimension of Euclidean factor of $\hat{M}_i=1$;

(c) the number of irreducible factors of $\hat{M}_i=\begin{cases}1 & \text{if } \dim_C D_i\geqq 2\,,\\ 0 & \text{if } \dim_C D_i=1\,.\end{cases}$

The minimality of f in 4) follows from 2). q.e.d.

3. Classification of totally real submanifolds with parallel second fundamental form

Let $\mathfrak{d}=(D_1,\cdots,D_s;c_1,\cdots,c_s)$ and $\mathfrak{d}'=(D_1',\cdots,D_t';c_1',\cdots,c_t')$ satisfy conditions (i), (ii) in **2**. They are said to be *equivalent,* denoted by $\mathfrak{d}\sim\mathfrak{d}'$, if $s=t$ and there exists a permutation p of s-letters $\{1,2,\cdots,s\}$ such that $D_{p(i)}'$ is isomorphic to D_i and $c_{p(i)}'=c_i$ for each i, $1\leqq i\leqq s$. The set of all equivalence classes of $\mathfrak{d}=(D_1,\cdots,D_s;c_1,\cdots,c_s)$ with (i), (ii) will be denoted by $\mathcal{D}_{n,c}$. Let $\mathrm{Aut}(P_n(c))$ denote the group of isometric holomorphisms of $P_n(c)$. It is isomorphic to the projective unitary group $PU(n+1)$ of degree $n+1$ in the natural way. We denote by $\mathcal{S}_{n,c}$ the set of all $\mathrm{Aut}(P_n(c))$-congruence classes of n-dimensional complete connected totally real submanifolds M of $P_n(c)$ with parallel second fundamental form. Then from the naturality of Harish-Chandra imbedding our correspondence $\mathfrak{d}\to M$ in **2** induces a map $\mathcal{D}_{n,c}\to\mathcal{S}_{n,c}$.

Theorem 3.1. 1) *The map* $\mathcal{D}_{n,c}\to\mathcal{S}_{n,c}$ *is a bijection.*

2) *Let* $f:(M,g)\to P_n(c)$ *be a totally real isometric immersion of an n-dimensional complete connected Riemannian manifold* (M,g) *with parallel second fundamental*

form. Then there exist an n-dimensional complete connected totally real submanifold $\iota: M' \hookrightarrow P_n(c)$ *with parallel second fundamental form and an isometric covering* $f': M \to M'$ *such that* $f=\iota\circ f'$.

Proof. 1) *Surjectivity*: Let $M\subset P_n(c)$ be an n-dimensional complete connected totally real submanifold with parallel second fundamental form. We use the notation in **1** with $m=n$. Then, by Lemma 1.1 $\hat{M}=\pi^{-1}(M)\subset S^{2n+1}(c/4)$ is complete, connected, with parallel second fundamental form and not contained in any real hyperplane of $\boldsymbol{C}^{n+1}$. Moreover we have $\pi(\hat{M})=M$. Thus, by Theorem 4.1 of Takeuchi [10]

$$\hat{M}=\hat{M}_1\times\cdots\times\hat{M}_s\subset S^{m_1}(c_1/4)\times\cdots\times S^{m_s}(c_s/4)\subset S^{2n+1}(c/4)\,,$$

where each $\hat{M}_i\subset S^{m_i}(c_i/4)$, $c_i>0$, is an irreducible symmetric R-space, $\Sigma_i\, m_i+s=2n+2$ and $\Sigma_i\, 1/c_i=1/c$. Here the group $K=I^0(\hat{M})$ is identified with the identity component of the group $\{\phi\in O(\boldsymbol{C}^{n+1});\ \phi\hat{M}=\hat{M}\}$. Since $\hat{M}$ is invariant under the subgroup $Z=\{\varepsilon I;\ \varepsilon\in\boldsymbol{C},\ |\varepsilon|=1\}$ of $O(\boldsymbol{C}^{n+1})$, Z is a closed subgroup of K. Let $p:\tilde{M}\to\hat{M}$ be the universal Riemannian covering of $\hat{M}$. Then, by Lemma 1.1,4) $\tilde{M}$ is the Riemannian product $\tilde{V}\times\tilde{H}$ of maximal integral submanifolds $\tilde{V}$ and $\tilde{H}$ in $\tilde{M}$ of distributions $p^{-1}V(\hat{M})$ and $p^{-1}H(\hat{M})$, respectively. Since $\tilde{V}$ is a flat line, it is contained in the Euclidean part of $\tilde{M}$. Thus, if we identify Lie $I^0(\hat{M})$ with a Lie subalgebra of Lie $I^0(\tilde{M})$, Lie $Z=$ Lie $I^0(\tilde{V})$ is contained in the center of Lie $I^0(\hat{M})=$ Lie K. Therefore Z is contained in the center of K, which implies that K is a subgroup of the unitary group $U(n+1)$. It follows that each irreducible symmetric pair $(\mathfrak{g}_i,\mathfrak{k}_i)$ associated to $\hat{M}_i$ is of hermitian type. Moreover, each $\mathfrak{g}_i$ has a semi-simple element E_i such that $ad\,E_i$ has just three distinct real eigenvalues. This is the case if and only if each irreducible symmetric bounded domain D_i associated to $(\mathfrak{g}_i,\mathfrak{k}_i)$ is of tube type. Here we have $2\dim_{\boldsymbol{C}} D_i=m_i+1$, and hence $\Sigma_i \dim_{\boldsymbol{C}} D_i=n+1$. Therefor, $M\subset P_n(c)$ is obtained from $\mathfrak{d}=(D_1,\cdots,D_s;c_1,\cdots,c_s)$ by the construction in **2**. This proves the surjectivity of our map.

Injectivity: Let $M\subset P_n(c)$ and $M'\subset P_n(c)$ be associated to $\mathfrak{d}=(D_1,\cdots,D_s;c_1,\cdots,c_s)$ and $\mathfrak{d}'=(D_1',\cdots,D_t';c_1',\cdots,c_t')$, respectively. Various objects in the construction of M' will be denoted by the same notation as for M but with primes. Suppose that there exists $\phi\in\mathrm{Aut}(P_n(c))=PU(n+1)$ with $\phi M=M'$. Then we have a $\boldsymbol{C}$-linear isometry $\hat{\phi}:\mathfrak{p}_{\boldsymbol{C}}^+\to\mathfrak{p}'^+_{\boldsymbol{C}}$ with respect to $\langle\ ,\ \rangle$ and $\langle\ ,\ \rangle'$ such that $\hat{\phi}\hat{M}=\hat{M}'$ and $\hat{\phi}$ induces ϕ. Then the homomorphism $\hat{\phi}_K: K=I^0(\hat{M})\to K'=I^0(\hat{M}')$ defined by $\hat{\phi}_K(k)=\hat{\phi}\circ k\circ\hat{\phi}^{-1}$ is an isomorphism. The differential $(\hat{\phi}_K)_*:\mathfrak{k}\to\mathfrak{k}'$ of $\hat{\phi}_K$ will be denoted by $\hat{\phi}_{\mathfrak{k}}$. Moreover, the $\boldsymbol{C}$-linear isomorphism $\hat{\phi}_{\mathfrak{p}}:(\mathfrak{p},J)\to(\mathfrak{p}',J')$ with $\hat{\phi}\circ\iota=\iota'\circ\hat{\phi}_{\mathfrak{p}}$ is a linear isometry with respect to B and B', and it satisfies

$$(3.1)\qquad \hat{\phi}_{\mathfrak{p}}(kX)=\hat{\phi}_K(k)\,(\hat{\phi}_{\mathfrak{p}}X)\qquad\text{for}\quad k\in K,\ X\in\mathfrak{p}\,.$$

We define an $\boldsymbol{R}$-linear isomorphism $\Phi\colon \mathfrak{g}=\mathfrak{k}+\mathfrak{p}\to\mathfrak{g}'=\mathfrak{k}'+\mathfrak{p}'$ by $\Phi=\hat{\phi}_{\mathfrak{k}}+\hat{\phi}_{\mathfrak{p}}$. Then (3.1) implies

$$(3.2)\quad \Phi\circ ad\,X=(ad\,\Phi X)\circ\Phi \qquad \text{for}\quad X\in\mathfrak{k}\,.$$

We shall show that Φ is actually a Lie isomorphism. Since (3.2) holds, we need only to show

$$(3.3)\quad \Phi[X,Y]=[\Phi X,\Phi Y] \qquad \text{for}\quad X,Y\in\mathfrak{p}\,.$$

For each $Z\in\mathfrak{k}$ we have

$$\begin{aligned} B'([\Phi X,\Phi Y],\Phi Z) &= B'(\Phi X,[\Phi Y,\Phi Z])\\ &= B'(\Phi X,\Phi[Y,Z]) && \text{by (3.2)}\\ &= B'(\hat{\phi}_{\mathfrak{p}}X,\hat{\phi}_{\mathfrak{p}}[Y,Z])=B(X,[Y,Z])\\ &= B([X,Y],Z)=B'(\Phi[X,Y],\Phi Z) && \text{by (3.2)}\,. \end{aligned}$$

This implies (3.3). Now the naturality of Harish-Chandra imbedding implies $\mathfrak{d}\sim\mathfrak{d}'$. This proves the injectivity of our map.

2) Construct an isometric immersion $\hat{f}\colon(\hat{M},\hat{g})\to S^{2n+1}(c/4)$ from f in the same way as in **1**. Then, by Lemma 1.1 $(\hat{M},\hat{g})$ is complete and $\hat{f}$ has the parallel second fundamental form. Thus, by Theorem 4.1 of Takeuchi [10] the image $\hat{M}'=\hat{f}(\hat{M})$ is a complete submanifold of $S^{2n+1}(c/4)$ and the map $\hat{f}'\colon\hat{M}\to\hat{M}'$ induced by $\hat{f}$ is an isometric covering. Therefore $M'=\pi(\hat{M}')$ is an n-dimensional complete connected submanifold of $P_n(c)$ and the induced map $f'\colon M\to M'$ is an isometric covering. It is clear that M' is a totally real submanifold of $P_n(c)$ with parallel second fundamental form. This completes the proof. q.e.d.

EXAMPLE. Let D be the irreducible symmetric bounded domain of type (IV) with $\dim_C D=n+1$, $n\geqq 2$. Then the submanifold $M\subset P_n(c)$ corresponding to $\mathfrak{d}=(D;c)$ is the naturally imbedded real projective n-space $P^n(c/4)$ with constant sectional curvature $c/4$, which is totally geodesic in $P_n(c)$.

We define a convex subset $\boldsymbol{F}_{n,c}$ of $\boldsymbol{R}^n$ by

$$\boldsymbol{F}_{n,c}=\{\alpha=(\alpha_i)\in\boldsymbol{R}^n;\ \alpha_i\geqq 0\,(1\leqq i\leqq n),\ \alpha_1+2\alpha_2+\cdots+n\alpha_n<1/c\}\,.$$

For each $\alpha\in\boldsymbol{F}_{n,c}$ we define constants $c_1,\cdots,c_{n+1}$ with $0<c_1\leqq c_2\leqq\cdots\leqq c_{n+1}$ by the relations

$$(3.4)\quad \alpha_i=1/c_i-1/c_{i+1}\ (1\leqq i\leqq n) \ \text{ and } \ \Sigma_i\,1/c_i=1/c\,,$$

and put

$$\hat{M}_\alpha^{n+1}=S^1(c_1/4)\times\cdots\times S^1(c_{n+1}/4)\subset S^{2n+1}(c/4)\,.$$

Then, by Theorem 2.1,5) $M_\alpha^n=\pi(\hat{M}_\alpha^{n+1})\subset P_n(c)$ is an n-dimensional complete connected flat totally real submanifold with parallel second fundamental form. Let $\mathcal{F}_{n,c}$ denote the set of all $\mathrm{Aut}(P_n(c))$-congruence classes of such submanifolds. Then the correspondence $\alpha \rightarrow M^n_\alpha$ induces a map $\boldsymbol{F}_{n,c} \rightarrow \mathcal{F}_{n,c}$.

Theorem 3.2. 1) *The map $\boldsymbol{F}_{n,c} \rightarrow \mathcal{F}_{n,c}$ is a bijection.*

2) *An n-dimensional complete connected flat totally real minimal submanifold of $P_n(c)$ with parallel second fundamental form is unique up to the congruence relative to the group $Aut(P_n(c))$, and it is given by $M^n_0 \subset P_n(c)$.*

Proof. 1) By Theorem 2.1,5) and Theorem 3.1, $\mathcal{F}_{n,c}$ corresponds one to one to the set of all $(n+1)$-tuples $(c_1, \cdots, c_{n+1})$ with $0 < c_1 \leqq c_2 \leqq \cdots \leqq c_{n+1}$ and $\Sigma_i 1/c_i = 1/c$. But the latter set corresponds one to one to the set $\boldsymbol{F}_{n,c}$ by the relations (3.4).

2) By Theorem 2.1,2), $M^n_\alpha \subset P_n(c)$ is minimal if and only if $c_i = c(n+1)$ for each i, $1 \leqq i \leqq n$. This is the case if and only if $\alpha = 0$. q.e.d.

REMARK. The norm $||\sigma_\alpha||$ of the second fundamental form σ_α of $M^n_\alpha \subset P_n(c)$ is given by

$$||\sigma_\alpha||^2 = \{\Sigma_i c_i - (3n+1)c\}/4 .$$

In particular, we have $||\sigma_0||^2 = n(n-1)c/4$.

4. Characterization of a flat totally real surface in $P_2(c)$

Let $f: (M,g) \rightarrow (\bar{M},\bar{g})$ be an isometric immersion of an n-dimensional Riemannian manifold (M,g) into an $(n+q)$-dimensional Riemannian manifold $(\bar{M},\bar{g})$ with $q \geqq 1$. The inner product and the norm of tensors defined by Riemannian metrics are denoted by $\langle\ ,\ \rangle$ and $||\ \ ||$, respectively. We denote by σ the second fundamental form of f, and by S_ξ the shape operator of f. They are related by $\langle S_\xi X, Y\rangle = \langle \sigma(X,Y), \xi\rangle$ for vector fields X, Y on M and a normal vector field ξ. We define a section $\tilde{\sigma}$ of the bundle $\mathrm{End}(NM)$ of endomorphisms of the normal bundle NM by $\tilde{\sigma} = \sigma \circ {}^t\sigma$, regarding σ as a homomorphism from $TM \otimes TM$ to NM. Moreover, we define a homomorphism $S^\perp$ from $TM \otimes TM$ to $\mathrm{End}(NM)$ by $S^\perp(X,Y)\xi = \sigma(X, S_\xi Y) - \sigma(Y, S_\xi X)$ for vector fields X, Y on M and a normal vector field ξ. Then we have the following

Lemma 4.1 (Simons [8], Chern-do Carmo-Kobayashi [2]). *Let p be an arbitrary point of M. Then we have an inequality*

$$||\tilde{\sigma}_p||^2 + ||S^\perp_p||^2 \leqq (2-1/q)||\sigma_p||^4 .$$

If the equality holds, then either $\sigma_p = 0$ or $\sigma_p \neq 0$, $N^1_p(M) = N_p M$ and $q \leqq 2$.

Now assume that $(\bar{M},\bar{g})$ is a Kählerian manifold $\boldsymbol{M}_m(c)$ of constant holomorphic sectional curvature c with $\dim_C \boldsymbol{M}_m(c) = m$, and that f is totally real in the sense that $\langle J(T_pM), T_pM\rangle=\{0\}$ for each $p\in M$, where J denotes the complex structure tensor of $\boldsymbol{M}_m(c)$. Then we have an orthogonal Whitney sum: $NM=J(TM)\oplus J(TM)^{\perp}$, where $J(TM)^{\perp}$ denotes the orthogonal complement of $J(TM)$ in NM. We define a homomorphism σ_J from $TM\otimes TM$ to NM by $\sigma_J(X,Y)=J(TM)$-component of $\sigma(X,Y)$ with respect to the above decomposition, for vector fields X,Y on M. Let $\Delta=Tr_g\nabla^{*2}$ denote the Laplacian on NM. Then, from Simons' formula (Simons [8]) which describes $\Delta\sigma$ for a general minimal isometric immersion, we have the following lemma.

Lemma 4.2. *Let $f:(M,g)\rightarrow \boldsymbol{M}_m(c)$ be a totally real minimal isometric immersion. Then*

$$(4.1)\quad \langle\Delta\sigma,\sigma\rangle=(n\|\sigma\|^2+\|\sigma_J\|^2)c/4-\|\tilde{\sigma}\|^2-\|S^{\perp}\|^2 .$$

Proposition 4.3. *Let $f: (M,g)\rightarrow \boldsymbol{M}_m(c)$, $c\leqq 0$, be a totally real minimal isometric immersion with parallel second fundamental form. Then f is totally geodesic.*

Proof. Since $\nabla^*\sigma=0$, we have by Lemma 4.2

$$(n\|\sigma\|^2+\|\sigma_J\|^2)c/4=\|\tilde{\sigma}\|^2+\|S^{\perp}\|^2 \quad \text{with} \quad c\leqq 0 .$$

This implies $\tilde{\sigma}=0$, and hence $\sigma=0$. q.e.d.

Lemma 4.4. *Let $f: (M,g)\rightarrow \boldsymbol{M}_n(c)$ be a totally real minimal isometric immersion of an n-dimensional Riemannian manifold (M,g). Then*

1) *We have an inequality*

$$(4.2)\quad -\langle\Delta\sigma,\sigma\rangle\leqq\{(2-1/n)\|\sigma\|^2-(n+1)c/4\}\|\sigma\|^2;$$

2) *If furthermore M is compact, then we have*

$$(4.3)\quad \int_M\|\nabla^*\sigma\|^2v_g\leqq\int_M\{(2-1/n)\|\sigma\|^2-(n+1)c/4\}\|\sigma\|^2v_g ,$$

where v_g denotes the Riemannian measure of (M,g).

Proof. 1) Since $J(TM)=NM$ in our case, we have $\sigma_J=\sigma$. Thus the equality (4.1) reduces to $\langle\Delta\sigma,\sigma\rangle=(n+1)c\|\sigma\|^2/4-\|\sigma\|^2-\|S^{\perp}\|^2$. Now (4.2) follows from Lemma 4.1.

2) Integrating the equality: $(1/2)\Delta(\|\sigma\|^2)=\langle\Delta\sigma,\sigma\rangle+\|\nabla^*\sigma\|^2$, we obtain

$$\int_M\|\nabla^*\sigma\|^2v_g=-\int_M\langle\Delta\sigma,\sigma\rangle v_g .$$

Thus (4.2) implies (4.3). q.e.d.

Theorem 4.5. *Let $f: (M,g)\rightarrow P_n(c)$, $c>0$, be a totally real minimal isometric immersion of a compact connected Riemannian manifold (M,g) with dim $M=$*

$n \geqq 2$. *Suppose that the second fundamental form* σ *of* f *satisfies an inequality*

$$||\sigma||^2 \leqq n(n+1)c/4(2n-1)$$

everywhere on M. *Then either* f *is totally geodesic and it is an isometric covering to the naturally imbedded real projective n-space in* $P_n(c)$, *or* $n=2$, $||\sigma||^2=c/2$ $(=n(n+1)c/4(2n-1))$ *everywhere on* M *and* f *is an isometric covering to the flat surface* $M_0^2 \subset P_2(c)$ *defined in* **3** (*up to the congruence relative to* $Aut(P_n(c))$).

Proof. We have

$$(2-1/n)||\sigma||^2-(n+1)c/4 = (2-1/n)\{||\sigma||^2-n(n+1)c/4(2n-1)\} \leqq 0$$

from the assumption. It follows from (4.3) that

$$\{||\sigma||^2-n(n+1)c/4(2n-1)\}||\sigma||^2 = 0$$

everywhere and that σ is parallel. Assume that f is not totally geodesic. Then $||\sigma||^2=n(n+1)c/4(2n-1)$ everywhere, and hence $n=2$ by Lemma 4.1. Now we see from Theorem 3.1 that a 2-dimensional complete connected totally real minimal submanifold M' of $P_2(c)$ with parallel second fundamental form is congruent to M_0^2 unless it is totally geodesic. On the other hand, the second fundamental form σ_0 of $M_0^2 \subset P_2(c)$ satisfies $||\sigma_0||^2=c/2$ (cf. Remark in **3**). Thus we get the theorem. q.e.d.

REMARK. Our $M_0^2 \subset P_2(c)$ is nothing but the flat isotropic surface in $P_2(c)$ with parallel second fundamental form constructed in Naitoh [5].

References

[1] B.Y. Chen-K. Ogiue: *On totally real submanifolds,* Trans. Amer. Math. Soc. **193** (1974), 257–266.

[2] S.S. Chern-M. do Carmo-S. Kobayashi: *Minimal submanifolds of a sphere with second fundamental form of constant length,* Functional Analysis and Related Fields, ed. by F.E. Browder, Springer, 1970, 59–75.

[3] D. Ferus: *Immersions with parallel second fundamental form,* Math. Z. **140** (1974), 87–93.

[4] A. Korányi-J.A. Wolf: *Realization of hermitian symmetric spaces as generalized half-planes,* Ann. of Math. **81** (1965), 265–288.

[5] H. Naitoh: *Isotropic submanifolds with parallel second fundamental forms in* $P^m(c)$, Osaka J. Math. **18** (1981), 427–464.

[6] ————: *Totally real parallel submanifolds in* $P^n(C)$, Tokyo J. Math. **4** (1981), 279–306.

[7] B.O'Neill: *The fundamental equations of submersion,* Michigan Math. J. **13** (1966), 459–469.

[8] J. Simons: *Minimal varieties in riemannian manifolds,* Ann. of Math. **88** (1968),

62–105.
[9] M. Takeuchi: *Polynomial representations associated with symmetric bounded domains,* Osaka J. Math. **10** (1973), 441–475.
[10] ————: *Parallel submanifolds of space forms,* Manifolds and Lie Groups, in honor of Yozô Matsushima, ed. by J. Hano et al., Birkhäuser, Boston, 1981, 429–447.

Hiroo Naitoh
Department of Mathematics
Yamaguchi University
Yamaguchi 753
Japan

Masaru Takeuchi
Department of Mathematics
College of General Education
Toyonaka, Osaka 560
Japan