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UNIFORM APPROXIMATION BY ENTIRE FUNCTIONS OF SEVERAL COMPLEX VARIABLES

Dedicated to Professor Yukinari Tôki on his 70th birthday

AKIRA SAKAI

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Introduction. Let G be a holomorphically convex open subset of $\boldsymbol{C}^n$ and T a closed subset of G. We say that T is *totally real,* if it is the zero set of a non-negative C^2 function ρ which is strictly plurisubharmonic on T. It is known that a real C^1 submanifold M is totally real if and only if it has no complex tangents (cf. [3]). The problem of uniform approximation on totally real submanifols was studied to a great extent by many authors (cf. Wells [9], Hörmander and Wermer [4], Nirenberg and Wells [5], Harvey and Wells [2], [3] and Nunemacher [6]). The result of [6] states that if M is a totally real submanifold then there exists a holomorphically convex open neighborhood B such that every continuous function on M is uniformly approximated on M by functions holomorphic in B. In [8], the author extended this result to the case of totally real sets with C^∞ defining functions. (A totally real set is not necessarily a submanifold. The approximation theorem for totally real sets contains one for totally real analytic subvarieties which was conjectured by Wells [9].)

In this paper, we give a sufficient condition for T and G under which every continuous function on T is uniformly approximated on T by functions holomorphic in G. The theorem we prove contains the following result which is a straight generalization to higher dimensions of Carleman's theorem [1].

Every continuous function on $\boldsymbol{R}^n$, canonically imbedded in $\boldsymbol{C}^n$, is uniformly approximated on $\boldsymbol{R}^n$ by entire functions on n complex variables.

We shall make use of the L^2-method due to Hörmander and Wermer [4] and the swelling method similar to one used in [8].

1. Statements.

Let S be a closed subset of an open set U of $\boldsymbol{C}^n$. We denote by $H(S)$ (or $H(S, U)$) the algebra of uniform limits of restrictions of functions holomorphic in a neighborhood of S (or in U, resp.).

We use an abbreviation $L[u; \xi]$ for the Levi form of a C^∞ function u:

$$L[u; \xi] = \sum_{j,k} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \xi_j \bar{\xi}_k , \qquad \xi \in \boldsymbol{C}^n .$$

By an exhaustion function σ of G we mean a positive C^∞ strictly plurisubharmonic function which maps properly G into $\boldsymbol{R}$. We define a form

$$A[\sigma;\xi]=\frac{1}{2\sigma}L[\sigma^2;\xi]$$
$$=L[\sigma;\xi]+\frac{1}{\sigma}\left|\sum_j\frac{\partial\sigma}{\partial z_j}\xi_j\right|^2,\quad \xi\in\boldsymbol{C}^n.$$

Theorem. *Let G be a holomorphically convex open subset of $\boldsymbol{C}^n$ and σ be an exhaustion function of G. If T is the zero set of a nonnegative C^∞ function ρ on G satisfying*

(1) $$L[\rho;\xi]\geq cA[\sigma;\xi],\quad \xi\in\boldsymbol{C}^n,$$

for some constant $c>0$, then $H(T,G)=C(T)$.

When G is $\boldsymbol{C}^n$, this is a uniform approximation theorem by entire functions. In this case, we can choose $\sigma(z)=|z|^2+1$ as an exhaustion function of $\boldsymbol{C}^n$ and we have $|\xi|^2\leq A[\sigma;\xi]\leq 2|\xi|^2$, $\xi\in\boldsymbol{C}^n$. Therefore, we obtain

Corollary 1. *If T is the zero set of a nonnegative C^∞ function ρ on $\boldsymbol{C}^n$ satisfying*

(2) $$L[\rho;\xi]\geq c|\xi|^2,\quad \xi\in\boldsymbol{C}^n,$$

with some constant $c>0$, then $H(T,\boldsymbol{C}^n)=C(T)$.

If we write $\boldsymbol{R}^n=\{z;y_j=0,\ j=1,\cdots,n\}$, then $\rho(z)=\sum_j|y_j|^2$ is a defining function of $\boldsymbol{R}^n$ satisfying (2). Thus we obtain the following corollary.

Corollary 2. $H(\boldsymbol{R}^n,\boldsymbol{C}^n)=C(\boldsymbol{R}^n)$.

The proof of Theorem is based on the following lemma essentially due to [4]. (For the proof, see Proposition 1 of [7].)

Lemma 1. *Let δ be a nonnegative function defined in an open set V in $\boldsymbol{C}^n$. Suppose K is a compact subset of V satisfying the following condition: There exists a constant $\eta>0$ such that for every sufficiently small $\varepsilon>0$, we can find a holomorphically convex open set V_ε satisfying*

$$\{z:\ dist\,(z,K)<\varepsilon\}\subset V_\varepsilon\subset\{z:\ \delta(z)<\varepsilon\eta\}.$$

If F is a a C^∞ function on V satisfying

$$|\bar\partial F(z)|\leq c\delta(z)^{n+1},\quad z\in V,$$

then $F|_K$ belongs to $H(K)$.

2. Construction of an exhaustion $\{K_m\}$ of G. Let σ and ρ be functions satsifying the assumption of the theorem. For every positive number r, the open set $G_r=\{z\in G_r: \sigma(z)<r\}$ is relatively compact in G.

Let λ be a C^∞ function: $\boldsymbol{R}\to[0, 1]$ such that $\lambda(t)=1\,(t<0)$ and $\lambda(t)=0$ $(t>2)$. For every positive number m, we set

$$\lambda_m(z) = \lambda(\sigma(z)/m)\,.$$

Then we have

$$\begin{aligned} L[\lambda_m: \xi] &= \frac{1}{m}\left\{\lambda' L[\sigma;\xi]+\frac{\lambda''}{m}\left|\sum_j \frac{\partial\sigma}{\partial z_j}\xi_j\right|^2\right\} \\ &\leq \frac{a}{m}A[\sigma;\xi]\,, \qquad \xi\in\boldsymbol{C}^n\,, \end{aligned}$$

with $a=\sup\{|\lambda'|+2|\lambda''|+1\}$, since $\lambda_m(z)=0$ for $z\in G\backslash G_{2m}$.

We set $\rho_0=\rho$ and $\rho_m=\rho-m\lambda_m$ for $m>1$. Since we may assume that $L[\rho;\xi]\geq 2aA[\sigma;\xi]$, $\xi\in\boldsymbol{C}^n$, multiplying ρ by a constant if necessary, we have

$$L[\rho_m;\xi]\geq aA[\sigma;\xi]\,,\quad \xi\in\boldsymbol{C}^n\,.$$

For each nonnegative integer m, we define the compact set $K_m=\{z\in\bar{G}_{2m+3}: \rho_m(z)\leq 0\}$. It is easy to show that $K_m\subset K_{m+1}$ and $\bigcup_m K_m=G$.

3. Approximation on K_m. In this section, we fix a nonnegative integer m. We shall prove the following lemma.

Lemma 2. *If f is a C^∞ function, then $f|_{K_0}\in H(K_0, G)$. If f is a C^∞ function which is holomorphic in an open neighborhood of $\bar{G}_{2m}$, $m>0$, then $f|_{K_m}\in H(K_m, G)$.*

Proof. Since ρ_m is strictly plurisubharmonic in G and since $K_m=\{\rho_m\leq 0\}$ $\cap\{\sigma\leq 2m+3\}$, K_m is $\mathcal{O}_G$-convex and therefore we have $H(K_m)=H(K_m, G)$. It suffices to prove that $f|_{K_m}\in H(K_m)$.

Let ψ be a C^∞ function satsfying $\psi=1$ in an open neighborhood of $\bar{G}_{2m}$ and $\psi=0$ in $G\backslash\bar{G}_{2m+1}$. We consider the function

$$\delta_m = \psi\rho_m+(1-\psi)\sum_\nu\left|\frac{\partial\rho}{\partial z_\nu}\right|^2.$$

If $z\in G_{2m}$, we have $L[\delta_m;\xi]=L[\rho_m:\xi]$, $\xi\neq 0$. If $z\in T\backslash G_{2m}$ then $\rho_m=\rho=0$ and $d\rho=0$. Hence we have

$$L[\delta_m;\xi]\geq\psi L[\rho;\xi]+(1-\psi)L[\rho;\xi]|\xi|^{-2}>0\,,\quad \xi\neq 0\,.$$

Therefore we can find an open neighborhood Ω_m of K_m so that δ_m is strictly plurisubharmonic in Ω_m. There exists a constant $\eta>0$ such that $\delta_m(z)\leq$ $\eta\,\mathrm{dist}(z, K_m)$ and $\sigma(z)\leq 2m+3+\eta\,\mathrm{dist}(z, G_{2m+3})$. If we set $\delta(z)=\max\{0, \delta_m(z)\}$

and $V_\varepsilon=\{z\in\Omega_m: \delta_m(z)<\varepsilon\eta\}\cap G_{2m+3+\varepsilon}$, then, for sufficiently small $\varepsilon>0$, V_ε is holomorphically convex and satisfies

$$\{z: \operatorname{dist}(z, K_m)<\varepsilon\}\subset V_\varepsilon\subset\{z: \delta(z)<\varepsilon\eta\}\ .$$

We can now find a C^∞ extension F of $f|_T$ on G which satisfies

$$|\bar\partial F(z)|\le c\delta(z)^{n+1}, \quad z\in V\backslash K_m$$

for an open neighborhood V of K_m and for some positive constant c. The way of construction of F is the same as one in Lemma 6 of [7]. We note that, if f is holomorphic in an open neighborhood U of $\bar G_{2m}$, then F is holomorphic in U. By Lemma 1, we have $f|_{K_m}=F|_{K_m}\in H(K_m)$, which proves the lemma.

4. Global approximation. Let f be an arbitrary function in $C^\infty(G)$ and let ε be any positive number. We shall construct a sequence $\{f_m\}$ of functions holomorphic in G and satisfying

$$|f_m-f_{m-1}|<2^{-m-1}\varepsilon \qquad \text{on} \quad K_m$$

and

$$|f_m-f|<\sum_{\nu=1}^{m+1}2^{-\nu}\varepsilon \qquad \text{on} \quad T\cap\bar G_{2m+3}\,.$$

We define the function $f_\varepsilon=\lim f_m$. A standard argument shows that f_ε is holomorphic in G and that $|f_\varepsilon-f|<\varepsilon$ on T.

The construction of $\{f_m\}$ is as follows. By Lemma 2, we can find a function f_0 holomorphic in G such that

$$|f_0-f|<2^{-1}\varepsilon \qquad \text{on} \quad K_0=T\cap\bar G_3$$

Suppose f_j, $j=1,\cdots,m-1$ are already defined. Let ψ be a C^∞ function: $G\to[0, 1]$ satisfying $\psi=1$ in an open neighborhood U of $\bar G_{2m}$ and $\psi=0$ in $G\backslash G_{2m+1}$. Set $g=\psi f_{m-1}+(1-\psi)f$. Then g is holomorphic in U. By Lemma 2, we can find a function f_m holomorphic in G so that

$$|f_m-g|<2^{-m-1}\varepsilon \qquad \text{on} \quad K_m\,.$$

Since $g=f_{m-1}$ in U and $K_m\subset U$, we have

$$|f_m-f_{m-1}|<2^{-m-1}\varepsilon \qquad \text{on} \quad K_m\,.$$

Since $|g-f|=\psi|f_{m-1}-f|<\sum_{\nu=1}^{m}2^{-\nu}\varepsilon$ on $T\cap\bar G_{2m+1}$ and since $g=f$ on $T\backslash G_{2m+1}$, we have

$$|f_m-f|<(2^{-m-1}+\sum_{\nu=1}^{m}2^{-\nu})\varepsilon \qquad \text{on} \quad T\cap\bar G_{2m+3}\,.$$

This completes the proof of the theorem.

REMARK 1. The question arises whether the same conclusion as Theorem can be obtained under the condition that ρ is C^∞ strictly plurisubharmonic in G. (There is a simple example of T such that every defining function of T is not strictly plurisubharmonic in G and such that $H(T, G) \neq C(T)$.) When T is compact this condition is sufficient. This follows at once from Theorem 2 of [7] and the fact that T is then $\mathcal{O}_G$-convex. We do not know whether it is true even when T is not assumed to be compact.

REMARK 2. It is reasonable to conjecture that the theorem will be valid even when a defining function ρ of T is of class C^2. In fact, when T is a submanifold, C^2-diffrentiability of ρ is sufficient to derive the approximation by functions holomorphic in a neighborhood of T (c.f. Harvey-Wells [2] and Nunemacher [6]). The C^∞ differentiability assumption in this paper was necessary because of the L^2-method we employed.

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Department of Mathematics
Himeji Institute of Technology
Shosha 2167, Himeji,
671–22, Japan