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SOME PROPERTIES OF THE SCATTERING AMPLITUDE AND THE INVERSE SCATTERING PROBLEM

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0. Introduction

In [14] we studied some properties of the S-matrix for the Schrödinger operator

$$T = -\Delta + Q(y) \tag{0.1}$$

in R^N. The first purpose of this work is to investigate some more properties of the S-matrix for the Schrödinger operator (0.1) in R^3 with a short-range potential $Q(y)$ (§2~§4). Especially the scattering amplitude $F(k, \omega, \omega')$ will be studied under the assumption that $Q(y)=O(|y|^{-2-\varepsilon})$, $\varepsilon>0$. Next these results will be used to show the uniqueness of the inverse scattering problem for general short-range potentials $Q(y)=O(|y|^{-1-\varepsilon})$, $\varepsilon>0$.

As in [14], our methods are based on the spectral representation theory for the Schrödinger operator which has been developed by many authors (e.g., Ikebe [6], Agmon [1], Saitō [10], [11], [13]).

§1 is a preliminary section and is devoted to stating some results on the Schrödinger equation

$$(T-k^2)u = f\,. \tag{0.2}$$

One of them is the limiting absorption principle and the others are the asymptotic behavior of the solution u at infinity and the spectral representation theorem for (the self-adjoint realization of) T. The scattering operator S and the scattering matrix $\hat{S}(k)$ will be discussed in §2. In §3 and §4 we shall study the properties of the scattering amplitude $F(k, \omega, \omega')$. First we shall give some representation formulas for $F(k, \omega, \omega')$, where $F(k, \omega, \omega')$ is represented by the potential $Q(y)$ and the generalized Fourier transform associated with T. And then several properties of $F(k, \omega, \omega')$ will be derived from the representation formulas. Among others, we shall show some similar results to the ones in the recent papers of Enss and Simon [3] and Jensen [8]. The main results of §5 is

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an asymptotic formula

$$\lim_{k\to\infty} k^2(\hat{F}(k)x_{k,z}, x_{k,z})_{S^2} = -2\pi\int_{R^3} Q(y)\,|y-z|^{-2}\,dy \tag{0.3}$$

for $z\in R^3$, where $\hat{F}(k)=-2\pi ik^{-1}(\hat{S}(k)-I)$ with the S-matrix $\hat{S}$ for T (see §2), $x_{k,z}(\omega)=\exp(-ik\omega z)\in L^2(S^2)$ and $(\ ,\ )_{S^2}$ is the inner product of $L^2(S^2)$. (0.3) is valid for any short-range potential $Q(y)$ and the uniqueness of the inverse scattering problem will be shown by the use of (0,3).

(0.3) can be also used to reconstruct the potential $Q(y)$ by its S-matrix $\hat{S}(k)$. We shall discuss it elsewhere.[1] In this paper we restrict ourselves to the three dimensional case for the sake of simplicity. But all the results can be easily extended to the higher dimensional case. It will be discussed elsewhere, too.

1. Preliminaries

We consider the differential operator

$$T=-\Delta+Q(y) \tag{1.1}$$

in R^3. Here Δ is the Laplacian in R^3 and $Q(y)$ is a real-valued function which satisfies the following

(A_μ) *$Q(y)$ is a real-valued, continuous function such that*

$$|Q(y)|\leqq C(1+|y|)^{-\mu} \qquad (y\in R^3) \tag{1.2}$$

with constants $\mu>1$ and $C>0$.

For $u\in H_2(R^3)_{loc}$ Tu is well-defined as a distribution and we have $Tu\in L^2(R^3)_{loc}$, where $H_2(R^3)$ is the Sobolev space of the second order. In order to discuss the S-matrix and the scattering amplitude for T we shall recall some results about the equation

$$\begin{cases}(T-k^2)u=f,\\ u\in H_2(R^3)_{loc}\cap L^2_{-\delta}(R^3),\end{cases} \tag{1.3}$$

with the radiation condition

$$\|\mathcal{D}u\|_{\delta-1}<\infty, \tag{1.4}$$

where δ is a constant such that

$$1/2<\delta\leqq\min\{1, (\mu+1)/2\}, \tag{1.5}$$

$L^2_\beta(R^3)$ is a weighted Hilbert space defined by

$$L^2_\beta(R^3)=\{u\in L^2(R^3)_{loc}/(1+|y|)^\beta u(y)\in L^2(R^3)\} \tag{1.6}$$

1) See Saitō [15].

with its inner product

$$(u, v)_\beta = \int_{R^3} (1+|y|)^{2\beta} u(y)\cdot\overline{v(y)}dy \tag{1.7}$$

and norm

$$\|u\|_\beta = [(u, u)_\beta]^{1/2}, \tag{1.8}$$

and we set

$$\begin{cases} \mathcal{D}u = \mathcal{D}^{(k)}u = (\mathcal{D}_1 u, \mathcal{D}_2 u, \mathcal{D}_3 u), \\ \mathcal{D}_j u = \dfrac{\partial u}{\partial y_j} + \dfrac{\tilde{y}_j}{|y|} u - ik\tilde{y}_j u \qquad (j = 1, 2, 3) \end{cases} \tag{1.9}$$

with $\tilde{y}_j = y_j/|y|$. For the literature about the equations (1.3)–(1.4) see, e.g., References of Saitō [13]. Let us denote by $B(X, Y)$ all bounded linear operators from X into Y.

Theorem 1.1. *Let $Q(y)$ satisfy (A_μ) with $\mu>1$ and let δ be as in* (1.5). *Let a be a positive constant.*

(i) *Then there exists a unique solution $u=u(k, f)$ of the equations* (1.3)–(1.4) *for any real k, $k\neq 0$, and $f\in L_\delta^2(R^3)$.*

(ii) *The solution $u=u(k, f)$ satisfies the estimate*

$$\|u\|_{-\delta} \leqq \frac{C}{|k|}\|f\|_\delta \qquad (|k|\geqq a, f\in L_\delta^2(R^2)), \tag{1.10}$$

where C is a constant depending only on a, δ and $Q(y)$.

(iii) *If we define an operator $(T-k^2)^{-1}$ by*

$$(T-k^2)^{-1}f = u(k, f) \tag{1.11}$$

for real k, $k\neq 0$, then the operator $(T-k^2)^{-1}$ is a $B(L_\delta^2(R^3), L_{-\delta}^2(R^3))$-valued, continuous function on $R-\{0\}$, and we have

$$\|(T-k^2)^{-1}\| \leqq \frac{C}{|k|} \qquad (|k|\geqq a) \tag{1.12}$$

with the same C as in (1.10), *where $\|(T-k^2)^{-1}\|$ is operator norm of $(T-k^2)^{-1}$ in $B(L_\delta^2(R^3), L_{-\delta}^2(R^3))$.*

(iv) *For each real k, $k\neq 0$, $(T-k^2)^{-1}$ is a compact operator from $L_\delta^2(R^3)$ into $L_{-\delta}^2(R^3)$.*

This theorem is a special case of Theorem 1.5 of Saitō [12]. Let Σ_R be all $k\in R$ such that the equation (1.3)–(1.4) with $f=0$ has a non-trivial solution. Then it should be noted that Σ_R is at most $\{0\}$ in our case, because $Q(y)$ is real-valued.

The next theorem is concerned with the asymptotic behavior of the solution $u(k, f)$.

Theorem 1.2. *Let $Q(y)$ satisfy (A_μ) with $\mu>1$. Denote by $u=u(y, k, f)$ the solution of the equations (1.3)–(1.4) with $k\in R-\{0\}$ and $f\in L^2_\delta(R^3)$.*

(i) *Then, the strong limt,*

$$\Phi(k)f = \Phi(k, Q)f = s-\lim_{n\to\infty} e^{-ir_nk}r_nu(r_n\omega, k, f)\,, \tag{1.13}$$

exists in $L^2(S^2)$, S^2 denoting the unit sphere in R^3 and $\{r_n\}$ being a sequence such that $r_n\uparrow\infty$ and

$$\lim_{n\to\infty} r_n^2\int_{S^2}|\mathcal{D}u(r_n\omega, k, f)|^2\,d\omega = 0\,. \tag{1.14}$$

The limit $\Phi(k)f\in L^2(S^2)$ is independent of the choice of such $\{r_n\}$.

(ii) *$\Phi(k)=\Phi(k, Q)$ defines a $B(L^2_\delta(R^3), L^2(S^2))$-valued, continuous function on $R-\{0\}$ with the estimate*

$$||\Phi(k)||\leqq\frac{C}{|k|}\qquad(|k|\geqq a)\,, \tag{1.15}$$

where a is an arbitrary positive number, C depends on a and $Q(y)$ and $||\Phi(k)||$ denotes the operator norm of $\Phi(k)$ in $B(L^2_\delta(R^3), L^2(S^2))$.

(iii) *$\Phi(k)$ is a compact operator from $L^2_\delta(R^3)$ into $L^2(S^2)$ for each $k\in R-\{0\}$.*

(iv) *Let $Q_n(y)$, $n=1, 2, \cdots$, be a sequence of real-valued, continuous functions on R^3 such that* $\sup(1+|y|)^\mu|Q_n(y)|<\infty$ *with $\mu>1$ and $Q_n(y)$ converges to $Q(y)$ for each $y\in R^3$ as $n\to\infty$. Then we have the strong limit*

$$\Phi(k, Q) = \text{s}-\lim_{n\to\infty}\Phi(k, Q_n) \tag{1.16}$$

for each $k\in R-\{0\}$ and the operator norm $||\Phi(k, Q_n)||$ in $B(L^2_\delta(R^3), L^2(S^2))$ is uniformly bounded for $n=1, 2, \cdots$.

(v) *Let $Q(y)=0$. Then $\Phi(k, 0)$ takes the from*

$$(\Phi(k, 0)f)\,(\omega) = (4\pi)^{-1}s-\lim_{R\to\infty}\int_{|y|<R} e^{-ik\omega y}f(y)\,dy \tag{1.17}$$

in $L^2(S^2)$ for $k\in R-\{0\}$ and $f\in L^2_\delta(R^2)$.

Proof. (i), (iii) and (iv) of Theorem 1.2 directly follow from Proposition 12.2 of Saitō [13]. But we should note that we discuss in [13] the operator L with operator-valued coefficients which is unitarily equivalent to T by the unitary operator

$$U: L^2(R^2)\ni f(y)\mapsto rf(r\omega)\in L^2((0, \infty), L^2(S^2), dr)\qquad(r=|y|,\ \omega=y/|y|\in S^2)\,, \tag{1.18}$$

and that we should set $\mu(y,k)=|y|k$ in Proposition 12.2 of [13], since our $Q(y)$ does not contain a long-range part. Let us give a proof of (1.15). Applying Proposition 12.2, (ii) of [13] to our case, we get

$$\|\Phi(k)f\|_{S^2}^2 = k^{-1}\,\mathrm{Im}\,(u(k,f),f)_0 \tag{1.19}$$

(cf. (8.15) of Theorem 8.4 of [13], too), where $\|\ \|_{S^2}$ is the norm in $L^2(S^2)$, $(\ ,\ )_0$ is the usual $L^2(R^3)$-norm, and $\mathrm{Im}\,\lambda$ denotes the imaginary part of λ. (1.15) follows from (1.19) and the estimate (1.12). (v) can be seen from the well-known relation

$$u(y,k,f) = (4\pi)^{-1}\int_{R^3}|y-z|^{-1}e^{ik|y-z|}f(z)\,dz \tag{1.20}$$

in the case of $Q(y)=0$ and the asymptotic relation

$$\lim_{r\to\infty} r|r\omega-y|^{-1}e^{ik(|r\omega-y|-r)} = e^{-ik\omega y}\,. \tag{1.21}$$

Q.E.D.

In §4 we shall need a result from the spectral representation theory for the self-adjoint realization of $T=-\Delta+Q(y)$ ([1], [11], [13]). Let H_0 and H denote the self-adjoint realizations of $T_0=-\Delta$ and T in $L^2(R^3)$, respectively, i.e.,

$$\begin{cases} H_0u = -\Delta u\,, \\ D(H_0) = H_2(R^3)\,, \end{cases} \tag{1.22}$$

and

$$\begin{cases} Hu = Tu\,, \\ D(H) = D(H_0)\,, \end{cases} \tag{1.23}$$

where $D(A)$ is the domain of A and $H_2(R^3)$ is as above. We set for $k>0$

$$\begin{cases} \Phi_+(k) = (2/\pi)^{1/2}\Phi(k,Q)\,, \\ \Phi_0(k) = (2/\pi)^{1/2}\Phi(k,0)\,. \end{cases} \tag{1.24}$$

Then we can easily see that

$$(\Phi_0(k)f)(\omega) = (2\pi)^{-3/2}s\text{-}\lim_{R\to\infty}\int_{|y|<R} e^{-ik\omega y}f(y)\,dy \tag{1.25}$$

in $L^2(S^2)$ for $f\in L^2_\delta(R^3)$.

Theorem 1.3. *Let $Q(y)$ satisfy (A_μ) with $\mu>1$. Let $E(\cdot)$ be the spectral measure associated with H. Then we have*

$$(E(B)f,f)_0 = \int_{\sqrt{B}}\|\Phi_+(k)f\|_{S^2}^2k^2dk \qquad (f\in L^2_\delta(R^3))\,, \tag{1.26}$$

where B is a Borel set in $(0, \infty)$ *and* $\sqrt{B}=\{k>0/k^2\in B\}$.

For the proof see, e.g., §11 and §12 of [13].

2. S-matrix

In this section we shall discuss the properties of the S-matrix for the Schrödinger operator $T=-\Delta+Q(y)$. Let $Q(y)$ satisfy (A_μ) with $\mu>1$ and H and H_0 be as in §1. Then it is well-known that the wave operators $W_\pm=W_\pm(H, H_0)$ are well-defined by

$$W_\pm = s-\lim_{t\to\pm\infty} \exp(itH)\exp(-itH_0) \tag{2.1}$$

(Kuroda [9]). The scattering operator S is defined by

$$S = W_+^* W_-, \tag{2.2}$$

W_+^* denotiong the adjoint of W_+. The scattering matrix (S-matrix) $\hat{S}(k)$, $k>0$, is an operator on $L^2(S^2)$ determined by the relation

$$(\mathcal{F}S\mathcal{F}^*F)(\xi) = \{\hat{S}(|\xi|)F(|\xi|\cdot)\}(\tilde{\xi}) \qquad (\tilde{\xi}=\xi/|\xi|) \tag{2.3}$$

for $F(\xi)\in C_0^\infty(R^3)$, where $\mathcal{F}$ is the usual Fourier transform

$$(\mathcal{F}f)(\xi) = (2\pi)^{-3/2}\underset{R\to\infty}{\text{l.i.m.}} \int_{|y|<R} e^{-i\xi y}f(y)\,dy \tag{2.4}$$

in $L^2(R^3_\xi)$, and $\mathcal{F}^*$ is the adjoint operator of $\mathcal{F}$ from $L^2(R^3_\xi)$ onto $L^2(R^3_y)$.

Theorem 2.1. *Let* $Q(y)$ *satisfy* (A_μ) *with* $\mu>1$. *Let* $\Phi_+(k)$ *and* $\Phi_0(k)$ *be as in* (1,24) *and let* $\Phi_0^*(k)$ *be the adjoint operator of* $\Phi_0(k)$ *from* $L^2(S^2)$ *into* $L^2_{-\delta}(R^3)$, *i.e.*,

$$\{\Phi_0^*(k)x\}(y) = (2\pi)^{-3/2}\int_{S^2} e^{ik\omega y}x(\omega)\,d\omega \qquad (x\in L^2(S^2)). \tag{2.5}$$

Let us define a linear operator $\hat{S}(k)=\hat{S}(k, Q)$, $k>0$, *on* $L^2(S^2)$ *by*

$$\hat{S}(k) = I-i\pi k\Phi_+(k)Q\Phi_0^*(k). \tag{2.6}$$

where I denotes the identity operator on $L^2(S^2)$. *Then* $\hat{S}(k)\in B(L^2(S^2))$ *and* (2.3) *hold. Here* $B(X)$ *means* $B(X, X)$.

Proof. Since the multiplication operator $Q=Q(y)\times$ is a bounded linear operator from $L^2_{-\delta}(R^3)$ into $L^2_\delta(R^3)$, $\Phi_0^*(k)\in B(L^2(S^2), L^2_{-\delta}(R^3))$ and $\Phi_+(k)\in B(L^2_\delta(R^3), L^2(S^2))$, we can see that $\Phi_+(k)Q\Phi_0^*(k)\in B(L^2(S^2))$, and hence $\hat{S}(k)$ is well-defined as an element of $B(L^2(S^2))$. For the proof of (2.3) see Theorem 3.2 and (i) of Remark 3.3 of [14]. Q.E.D.

Let us list some properties of S-matrix $\hat{S}(k)$.

Theorem 2.2. *Let $Q(y)$ satisfy (A_μ) with $\mu>1$ and let $\hat{S}(k)=\hat{S}(k, Q)$ be the S-matrix.*

(i) *The $\hat{S}(k)$ is a unitary operator on $L^2(S^2)$.*

(ii) *Set*

$$\hat{F}(k) = -2\pi i k^{-1}(\hat{S}(k)-I) = -2\pi^2\Phi_+(k)Q\Phi_0^*(k) \qquad (k>0). \tag{2.7}$$

Then the estimate

$$||\hat{F}(k)|| = O(k^{-2}) \qquad (k>0) \tag{2.8}$$

holds. Here $||\hat{F}(k)||$ denotes the operator norm of $\hat{F}(k)$ in $B(L^2(S^2))$.

(iii) *For each $k>0$ $\hat{F}(k)$ is a compact operator on $L^2(S^2)$.*

(iv) *Let $Q_n(y)$, $n=1, 2, \cdots$, be as in* (iv) *of Theorem* 1.2. *Then we have*

$$\hat{F}(k, Q) = s-\lim_{n\to\infty} \hat{F}(k, Q_n) \tag{2.9}$$

for each $k>0$.

Proof. For the proof of (i) see §4 of [14]. Since we have $||\Phi_+(k)||$, $||\Phi_0^*(k)|| =O(k^{-1})$ from (1.15), (2.8) immediately follows. (iii) can be shown by the compactness of $\Phi_+(k)$ ((iii) of Theorem 1.2) and the boundedness of $Q\Phi_0^*(k)$. (iv) follows from (iv) of Theorem 1.2 and the fact that $Q_n\Phi_0^*(k)$ converges strongly to $Q\Phi_0^*(k)$ in $B(L^2(S^2), L_\delta^2(R^3))$ as $n\to\infty$. Q.E.D.

3. The scattering amplitude

In this and the following sections we shall assume that $Q(y)$ statisfies (A_μ) with $\mu>2$. Then it is known (Amerin et al. [2], §10.2) that $\hat{F}(k)$, defined by (2.7), is a Hilbert-Schmidt operator on $L^2(S^2)$ with its Hilbert-Schmidt kernel $F(k, \omega, \omega')$, $k>0$, $\omega, \omega'\in S^2$, i.e., we have

$$\begin{cases} (\hat{F}(k)x)(\omega) = \int_{S^2} F(k, \omega, \omega')x(\omega')\,d\omega' \qquad (x\in L^2(S^2)), \\ \int_{S^2}\int_{S^2} |F(k, \omega, \omega)|^2\,d\omega d\omega' < \infty \end{cases} \tag{3.1}$$

$F(k, \omega, \omega')$ is called the scattering amplitude. We shall give a proof of (3.1) and show a formula where the scattering amplitude $F(k, \omega, \omega')$ is represented by the potential $Q(y)$. Some properties of $F(k, \omega, \omega')$ will be obtained from the representation.

We shall start with a well-known formula for $F(k, \omega, \omega')$.

Theorem 3.1. (Ikebe [7]). *Let $Q(y)$ satisfy (A_μ) with $\mu>3$ and let $Q(y)$ be Hölder continuous. Then we have*

(3.2) $$F(k, \omega, \omega') = -(4\pi)^{-1}\int_{R^3} \varphi(y, -k\omega)Q(y)e^{ik\omega' y}\, dy \qquad (k>0,\ \omega,\ \omega' \in S^2),$$

where $\varphi(y, \xi), y, \xi \in R^3$, *is the generalized eigenfunction associated with* $H=H_0+Q$ *which is a unique solution of the Lippmann-Schwinger equation*

(3.3) $$\varphi(y, \xi) = e^{i\xi y} - (4\pi)^{-1}\int_{R^3} |y-z|^{-1}e^{i|\xi||y-z|}Q(z)\varphi(z, \xi)\, dz.$$

Further, $F(k, \omega, \omega')$ *is continuous on* $(0, \infty)\times S^2 \times S^2$ *and we have*

(3.4) $$F(k, \omega, \omega') = F(k, -\omega', -\omega) \qquad (k>0,\ \omega,\ \omega' \in S^2).$$

For the proof see Ikebe [7] (Theorem 1 and the foot-note 7)) or Amerin et al. [2], §10.2.

Now multiply both sides of (3.3) by $Q(y)$ and set $\xi=-k\omega$. Then we can rewrite (3.3) as

(3.5) $$\{I+Q(T_0-k^2)^{-1}\}(Q(\cdot)\varphi(\cdot, -k\omega)) = e^{-ik\omega y}Q(y),$$

where we should note that $(4\pi)^{-1}|y-z|^{-1}e^{ik|y-z|}$ is the resolvent kernel of $T_0=-\Delta$ and that $Q(y)\varphi(y, \xi)\in L^2_\delta(R^3)$ since $\varphi(y, \xi)$ is a bounded function on R^3 for each fixed ξ as is known in Ikebe [6]. It follows from (iv) of Theorem 1.2 that $Q(T_0-k^2)^{-1}$ is a compact operator on $L^2_\delta(R^3)$. On the other hand, the equation

(3.6) $$\{I+Q(T_0-k^2)^{-1}\}u = 0$$

has only the trivial solution $u=0$ in $L^2_\delta(R^3)$. In fact, let $u_0\in L^2_\delta(R^3)$ be a solution of the equation (3.6). Then, setting $v_0=(T_0-k^2)^{-1}u_0$, we can easily see that v_0 is a solution of the equation $(T-k^2)v_0=0$ with the radiation condition (1.4), and hence $v_0=0$ by the uniqueness of the equations (1.3)–(1.4), whence $u_0=0$ follows. These two facts are enough to show the existence of $\{I+Q(T_0-k^2)^{-1}\}^{-1}\in B(L^2_\delta(R^3))$ and we have from (3.5)

(3.7) $$Q(y)\varphi(y, -k\omega) = \{I+Q(T_0-k^2)^{-1}\}^{-1}(e^{ik\omega\cdot}Q(\cdot)).$$

Thus we arrive at

Proposition 3.2. *Let* $Q(y)$ *be as in Theorem* 3.1. *Then the scattering amplitude* $F(k, \omega, \omega')$ *for* $H=H_0+Q$ *has the expression*

(3.8) $$F(k, \omega, \omega') = -(\pi/2)^{1/2}\overline{\Phi_0(k)}\{(I+B(k))^{-1}(e^{-ik\omega\cdot}Q)\}(\omega') \qquad (k>0,\ \omega,\ \omega' \in S^2),$$

where we set

(3.9) $$B(k) = B(k, Q) = Q(T_0-k^2)^{-1}$$

and

$$(3.10)\qquad \{\overline{\Phi_0(k)}f\}(\omega) = (2\pi)^{-3/2} s\text{-}\lim_{R\to\infty} \int_{|y|<R} e^{ik\omega y} f(y)\, dy$$

for $f \in L^2_\delta(R^3)$.

Now let us assume that $Q(y)$ satisfy (A_μ) with $\mu>2$. Take a sequence $Q_n(y)$, $n=1, 2, \cdots$, such that $Q_n(y)$ converges to $Q(y)$ in the sense of (iv) of Theorem 1.2 and each $Q_n(y)$ satisfies (A_μ) with $\mu>3$ and is Hölder continuous. Let $F_n(k, \omega, \omega')$ be the scattering amplitude for $H_n=H_0+Q_n$, $n=1, 2, \cdots$. Then it follows from Proposition 3.2 that

$$(3.11)\qquad F_n(k, \omega, \omega') = -(\pi/2)^{1/2}\overline{\Phi_0(k)}\{(I+B_n(k))^{-1}(e^{-ik\omega\cdot}Q_n)\}(\omega')$$

with $B_n(k)=Q_n(T_0-k^2)^{-1}$. Since $Q_n(y)$ converges to $Q(y)$ in $L^2_\delta(R^3)$, we have for each pair $(k, \omega)\in(0, \infty)\times S^2$

$$(3.12)\qquad s\text{-}\lim_{n\to\infty} F_n(k, \omega, \cdot) = -(\pi/2)^{1/2}\overline{\Phi_0(k)}\{(I+B(k))^{-1}(e^{-ik\omega\cdot}Q)\}$$

in $L^2(S^2)$. On the other hand, it follows from (iv) of Theorem 2.2 that $\hat{F}(k, Q_n)$ converges to $\hat{F}(k, Q)$ strongly as $n\to\infty$. Therefore, denoting the right-hand side of (3.12) by $F(k, \omega, \omega')$, we get

$$(3.13)\qquad \hat{F}(k, Q)x = \int_{S^2} F(k, \omega, \omega')x(\omega')\, d\omega'$$

for each $x\in L^2(S^2)$.

Theorem 3.3. *Let $Q(y)$ satisfy (A_μ) with $\mu>2$.*

(i) *Then the operator $\hat{F}(k)$ has the kernel which has the expressin*

$$(3.14)\qquad F(k, \omega, \omega') = -(\pi/2)^{1/2}\overline{\Phi_0(k)}\{(I+B(k))^{-1}(e^{-ik\omega\cdot}Q)\}(\omega'),$$

$\overline{\Phi_0(k)}$ *and $B(k)$ being given as in* (3.10) *and* (3.9), *respectively.*

(ii) *$F(k, \omega, \cdot)$ is an $L^2(S^2)$-valued, uniformly continuous function on $[a, \infty)\times S^2$ for each $a>0$ with the estimate*

$$(3.15)\qquad \|F(k, \omega, \cdot)\|_{S^2} \leqq Ck^{-1}\|(I+B(k))^{-1}\|\,\|Q\|_\delta \qquad (k\geqq a,\ \omega\in S^2),$$

where C is a constant depending only on $a>0$ and $Q(y)$, and $\|(I+B(k))^{-1}\|$ is the operator norm of $(I+B(k))^{-1}$ in $B(L^2_\delta(R^3))$. Further, $F(k, \cdot, \cdot)$ is a Hilbert-Schmidt kernel and its Hilbert-Schmidt norm is $O(k^{-1})$ as $k\to\infty$.

(iii) *Set*

$$(3.16)\qquad F(k, \omega, \omega') = -(\pi/2)^{1/2}\overline{\Phi_0(k)}(e^{-ik\omega\cdot}Q)(\omega') + J(k, \omega, \omega').$$

Then for each $a>0$ $J(k, \omega, \omega')$ is a uniformly continuous function on $[a, \infty)\times S^2\times S^2$ with the estimate

(3.17) $$|J(k, \omega, \omega')| \leqq Ck^{-1}\|(I+B(k))^{-1}\| \, \|Q\|_\delta^2 .$$

The L_2-estimate for $J(k, \omega, \cdot)$ is given as

(3.18) $$\|J(k, \omega, \cdot)\|_{S^2} \leqq Ck^{-2}\|(I+B(k))^{-1}\| \, \|Q\|_{\delta,\infty}\|Q\|_\delta$$
$$(\|Q\|_{\delta,\infty} = \sup_y (1+|y|)^\delta |Q(y)|)$$

for $k \geqq a$, $\omega \in S^2$. Here C is a constant depending only on $a>0$ and $Q(y)$.

Proof. The existence of the integral kernel $F(k, \omega, \omega')$ of $\hat{F}(k)$ and (3.14) have been shown already in the argument before Theorem 3.3. (3.15) directly follows from the estimate (1.15) with $\Phi(k)$ replaced by $\Phi_0(-k)$, and hence $F(k, \omega, \omega')$ is a Hilbert-Schmidt kernel for each $k>0$. Since $(I+B(k))^{-1}$ is uniformly bounded on $[a, \infty)$ and $\Phi_0(k)$ and $B(k)$, $k \in [a, \infty)$, are uniformly continuous in $B(L_\delta^2(R^3), L^2(S^2))$ and $B(L_\delta^2(R^3))$, respectively, we can show the uniform continuity of $F(k, \omega, \cdot)$ on $[a, \infty) \times S^2$. Let us turn into the proof of (iii). It follows from the relation

(3.19) $$(I+B(k))^{-1} = I - B(k)(I+B(k))^{-1}$$

that

(3.20) $$J(k, \omega, \omega') = (\pi/2)^{1/2}\overline{\Phi_0(k)}\,\{B(k)(I+B(k))^{-1}(e^{-ik\omega\cdot}Q)\}(\omega') .$$

Therefore, noting the definition of $\overline{\Phi_0(k)}$ and $B(k)$ ((3.10) and (3.9), respectively) and using the Schwarz inequality, we have

(3.21) $$\begin{aligned} &|J(k, \omega, \omega')| \\ &\leqq (4\pi)^{-1}\int_{R^3} |Q(y)| \, |(T_0-k^2)^{-1}(I+B(k))^{-1}(e^{-ik\omega\cdot}Q)| \, dy \\ &\leqq (4\pi)^{-1}\|Q\|_\delta \|(T_0-k^2)^{-1}(I+B(k))^{-1}(e^{-ik\omega\cdot}Q)\|_{-\delta} . \end{aligned}$$

It is easy to see that (3.17) is obtained from (3.21). (3.18) can be shown quite in a similar way, though we have to use (1.15) (with $\Phi(k)$ replaced by $\Phi_0(-k)$) in addition. Starting with (3.20) again and proceeding as in the proof of (ii), we can easily show thw uniform continuity of $J(k, \omega, \omega')$. Q.E.D.

REMARK 3.4. Recently Jensen [8] has investigated the asymptotic behavior of the total scattering cross-section. His potential admits some local singularities. In the case that $Q(y)$ has no local singularities, His results can be derived from ours.

From (iii) of Theorem 3.3 we can see that the singularities of $F(k, \omega, \omega')$ can arise only from the term $-(\pi/2)^{1/2}\overline{\Phi_0(k)}\,(e^{-ik\omega\cdot}Q)(\omega')$. For example, let us show the following two theorems. For the first one cf. Villarroel [17].

Theorem 3.5. *Let $Q(y)$ satisfy (A_μ) with $2<\mu\leqq 3$ and let $Q(y)$ be spherically* *symmetric, i.e.,* $Q(y)=Q(|y|)$*, Then the singularities of* $F(k, \omega, \omega')$ *lie only on* $\omega=\omega'$*, i.e.,* $F(k, \omega, \omega')$ *is a continuous function for* $k>0$, $\omega\neq\omega'$*. Further, we have the estimate*

$$(3.22) \qquad F(k, \omega, \omega') = O(|\omega-\omega'|^{\tilde{\mu}-3}) \qquad (|\omega-\omega'|\to 0)\,,$$

where $\tilde{\mu}$ *is an arbitrary number such that* $2<\tilde{\mu}<\mu$.

Proof. By a simple calculation we have for $\omega\neq\omega'$

$$(3.23) \qquad -(\pi/2)^{1/2}\Phi_0(k)\,(e^{-ik\omega\cdot}Q)\,(\omega') = -\lim_{R\to\infty}(4\pi)^{-1}\int_{|y|<R} e^{ik(\omega'-\omega)y}Q(|y|)\,dy$$
$$= -(k|\omega-\omega'|)^{-1}\int_0^\infty rQ(r)\sin(kr|\omega-\omega'|)\,dr\,.$$

The right-hand side of (3.23) is continuous for $k>0$ and $\omega\neq\omega'$. By the use of the condition (A_μ) we get

$$(3.24) \qquad |(\pi/2)^{1/2}\overline{\Phi_0(k)}\,(e^{-ik\omega\cdot}Q)\,(\omega')|$$
$$\leqq C(k|\omega-\omega'|)^{\tilde{\mu}-3}\int_0^\infty r(1+r)^{-2-(\mu-\tilde{\mu})}p(kr|\omega-\omega'|)^{\tilde{\mu}-2}\,dr$$

with $p(t)=t^{-1}|\sin t|$. (3.22) directly follows from (3.24). Q.E.D.

Theorem 3.6. *Let* $Q(y)$ *satisfy* (A_μ) *with* $2<\mu\leqq 3$ *and let* $\partial Q(y)/\partial y_j$, $j=1, 2, 3$, *satisfy* $(A_{\tilde{\mu}})$ *with* $\tilde{\mu}>3$. *Then the singularities of* $F(k, \omega, \omega')$ *lie only on* $\omega=\omega'$. *We have the estimate, with any* μ' *such that* $3<\mu'<\tilde{\mu}$, $\mu'\leqq 4$,

$$(3.25) \qquad F(k, \omega, \omega') = O(|\omega-\omega'|^{\mu'-4}) \qquad (|\omega-\omega'|\to 0)\,.$$

Proof. Let $\omega\neq\omega'$. With no loss of generality we shall assume that $\omega_1\neq\omega_1'$, where $\omega=(\omega_1, \omega_2, \omega_3)$ and $\omega'=(\omega_1', \omega_2', \omega_3')$. Then we have by partial integration

$$(3.26) \qquad -(\pi/2)^{1/2}\overline{\Phi_0(k)}\,(e^{-ik\omega\cdot}Q)\,(\omega')$$
$$= (4\pi ik(\omega_1'-\omega_1))^{-1}\int_{R^3}\{e^{ik(\omega-\omega')y}-1\}\,(\partial Q/\partial y_1)\,dy\,.$$

We proceed to get (3.25) as in the proof of Theorem 3.5. Q.E.D.

Finally let us show that the scattering amplitude $F(k.\ \omega, \omega')$ can be expanded in series when k is large enough. It follows from (1.12) that for each $\rho\in(0, 1)$ there exists $k_\rho>0$ such that

$$(3.27) \qquad ||B(k)||\leqq\rho \qquad (k\geqq k_\rho)\,,$$

$||B(k)||$ being the operator norm of $B(k)=Q(T_0-k^2)^{-1}$ in $B(L^2_\delta(R^3))$. Then, making use of the relation

$$(I+B(k))^{-1}=\sum_{j=1}^{\infty}(-B(k))^j \tag{3.28}$$

and the estimate

$$|(\pi/2)^{1/2}\overline{\Phi_0(k)}\{(-B(k))^j(e^{-ik\omega\cdot}Q)\}(\omega')|\leqq Ck^{-1}\rho^{j-1}\|Q\|_\delta^2 \quad (k\geqq k_\rho,\ j=1,2,\cdots), \tag{3.29}$$

which can be obtained by proceeding as in the proof of (3.17) in Theorem 3.3, we have the series expansion

$$J(k,\omega,\omega')=-(\pi/2)^{1/2}\sum_{j=1}^{\infty}\overline{\Phi_0(k)}\{(-B(k))^j(e^{-ik\omega\cdot}Q)\}(\omega'), \tag{3.30}$$

where the right-hand side of (3.30) converges uniformly for $k\geqq k_\rho$ and $\omega, \omega'\in S^2$. Thus we arrive at the Born series (cf. Chapter 12 of Amerin et al. [2]).

Theorem 3.7. *Let $Q(y)$ satisfy (A_μ) with $\mu>2$. Then we have the series expansion*

$$\begin{aligned}F(k,\omega,\omega')&+(\pi/2)^{1/2}\overline{\Phi_0(k)}(e^{-ik\omega\cdot}Q)(\omega')\\&=-(\pi/2)^{1/2}\sum_{j=1}^{\infty}\overline{\Phi_0(k)}\{(-B(k)^j(e^{-ik\omega\cdot}Q)\}(\omega'),\end{aligned} \tag{3.31}$$

where the right-hand side of (3.31) *converges uniformly for $(k,\omega,\omega')\in[\tilde{k},\infty)\times S^2\times S^2$ with $\tilde{k}>0$ large enough.*

4. Some estimate on the total cross-section

Let us set

$$\sigma(k,\omega)=\sigma(k,\omega,Q)=\|F(k,\omega,\cdot)\|_{S^2}^2. \tag{4.1}$$

$\sigma(k,\omega)$ is called the total scattering cross-section. Recently Enss and Simon [3] got some interesting results about the estimates on the term

$$I(\alpha,\gamma)=I(\alpha,\gamma,Q)=\int_{\alpha-\gamma}^{\alpha+\gamma}\sigma(k,\omega,Q)\,dk \tag{4.2}$$

by uniting their time-dependent methods and geometrical methods. Among others, they have shown that, roughly speaking,

$$I(\alpha,\gamma,gQ)\leqq C\alpha^{-2}g^2\|Q\|_\delta^2 \quad (g>0), \tag{4.3}$$

where $\delta>\frac{1}{2}$ and C depends only on $\gamma>0$, and that

$$I(\alpha,\gamma,Q)\leqq C(R^2+R^{-1}), \tag{4.4}$$

where the support of $Q(y)$ is assumed to be compact and is contained in a ball $\{y\in R^3/|y|<R\}$, and C depends only on $\alpha,\gamma>0$.

In this section we shall show the results similar to (4.3) and (4.4) by starting with our definition of $F(k, \omega, \omega')$ and using the time-independent methods. Let us begin with an another interpretation of the classical formula (3.2) for the scattering amplitude $F(k, \omega, \omega')$. By the use of (3.4) we can rewrite (3.2) as

$$F(k, \omega, \omega') = -(4\pi)^{-1}\int_{R^3}\varphi(y, k\omega')Q(y)e^{-ik\omega y}\,dy\,. \tag{4.5}$$

Lemma 4.1. *Let* $Q(y)$ *be as in Theorem* 3.1. *Let* $\Phi_+(k)$ *be as in* (1.24). *Then we have*

$$\{J\Phi_+(k)f\}(\omega) = s-\lim_{R\to\infty}(2\pi)^{-3/2}\int_{|y|<R}\varphi(y, k\omega)f(y)\,dy \qquad (k>0,\ f\in L^2_\delta(R^3)) \tag{4.6}$$

in $L^2(S^2)$, *where* J *is a unitary operator on* $L^2(S^2)$ *defined by*

$$(Jx)(\omega) = x(-\omega) \qquad (x\in L^2(S^2))\,, \tag{4.7}$$

and $\varphi(y, \xi)$ *is the generalized eigenfunction associated with* $H=H_0+Q$ (*see Theorem* 3.1).

Proof. Set

$$u(y) = (2\pi)^{-3/2}\int_{S^2}\overline{\varphi(y, k\omega)}x(\omega)\,d\omega \tag{4.8}$$

with $x\in L^2(S^2)$. Then it follows from the Limpmann-Schwinger equation that

$$u = \overline{\Phi_0^*(k)}x-(T_0-(-k)^2)^{-1}Qu\,, \tag{4.9}$$

where

$$\overline{\Phi_0^*(k)}x = (2\pi)^{-3/2}\int_{S^2}e^{-ik\omega y}x(\omega)\,d\omega \tag{4.10}$$

and $(T_0-(-k)^2)^{-1}f$ ($f\in L^2_\delta(R^3)$) is the solution of the equations (1.3)–(1.4) with T and k replaced by T_0 and $-k$, respectively. We next obtain from Theorem 2.6 and Remark 3.3, (ii) of [14]

$$\Phi_+^*(k)x = \Phi_0^*(k)x-(T-(-k)^2)^{-1}Q\Phi_0^*(k)x\,, \tag{4.11}$$

$\Phi_+^*(k)$ being the adjoint of $\Phi_+(k)$ and $(T-(-k)^2)^{-1}$ being defined quite similarly to $(T_0-(-k)^2)^{-1}$ above. Set $v(y)=\Phi_+^*(k)x-\Phi_0^*(k)x$. Then, by (4.11), $v(y)$ satisfies the radiation condition (1.4) with k replaced by $-k$. On the other hand, by a simple calculation, we have

$$(T_0-k^2)v = -Q\Phi_+^*(k)x\,, \tag{4.12}$$

where we have used (4.11) again. Thus we get

$$\Phi_+^*(k)x = \Phi_0^*(k)x-(T_0-(-k)^2)^{-1}Q\Phi_+^*(k)x\,. \tag{4.13}$$

Replace x by Jx in (4.13) and notice that

$$\Phi_0^*(k)Jx = \overline{\Phi_0^*(k)x} \qquad (k>0,\ x\in L^2(S^2)). \tag{4.14}$$

Then we get from (4.9) and (4.13) (with x replaced by Jx) $u(y)=\Phi_+^*(k)Jx$, and hence (4.6) follows just by taking the adjoint. Q.E.D.

Theorem 4.2. *Let $Q(y)$ satisfy (A_μ) with $\mu>2$. Then we have the expression for $F(k, \omega, \omega')$*

$$F(k, \omega, \omega') = -(\pi/2)^{1/2} J\Phi_+(k)\,(e^{-ik\omega\cdot}Q)\,(\omega'). \tag{4.15}$$

Proof. We can obtain (4.15) by approximating $Q(y)$ by a sequence $Q_n(y)$, $n=1, 2, \cdots$, such that each $Q_n(y)$ satisfies (A_μ) with $\mu>3$ and is smooth enough. Then it follows from (4.5) and Lemma 4.1 that we obtain (4.15) with $F(k, \omega, \omega')$ and $Q(y)$ replaced by $F_n(k, \omega, \omega')$ and $Q_n(y)$, respectively, where $F_n(k, \omega, \omega')$ is the scattering amplitude associated with $Q_n(y)$. By letting $n\to\infty$ in the relation obtained above, we arrive at (4.15). Q.E.D.

The following theorem is a slight modification of Theorem 4.2. But it will be useful later.

Theorem 4.3. *Let $Q(y)$ satisfy (A_μ) with $\mu>2$. Let $\psi(y)$ be a C^2 function on R^3 such that $\psi(y)=1$ for $|y|>R$ with $R>0$. Then we have*

$$F(k, \omega, \omega') = -(\pi/2)^{1/2} J\Phi_+(k)\,\{(T-k^2)\,(\psi e^{-ik\omega\cdot})\}\,(\omega'). \tag{4.16}$$

Proof. It follows from Theorem 4.2 and the decomposition $e^{-ik\omega y}=\psi(y)e^{-ik\omega y}+(1-\psi(y))e^{-ik\omega y}$ that it is enough to show

$$\Phi(k)f = 0 \tag{4.17}$$

for $f=(T-k^2)u$ with $u(y)\in H_2(R^3)_{\mathrm{loc}}$ which has compact support in R^3. In fact, $u(y)$ is the solution of the equations (1.3)–(1.4), and hence, by the definition of $\Phi(k)$ ((1.13)),

$$\Phi(k)f = s-\lim_{r\to\infty} e^{-ikr} r u(r\cdot) = 0\,. \tag{4.18}$$

Q.E.D.

Now let us give an estimate on the total cross-section $\sigma(k, \omega)=\sigma(k, \omega, Q)$. Here our potential $Q(y)$ is more restricted than that in Enss and Simon [3].

Theorem 4.4. *Let $Q(y)$ satisfy (A_μ) with $\mu>5/2$. Let $\gamma>0$ and let $\alpha-2\gamma>0$. Then we have*

$$I(\alpha, \gamma, Q) = \int_{\alpha-\gamma}^{\alpha+\gamma} \sigma(k, \omega, Q)\, dk \leqq C\alpha^{-2}\|Q\|_1\|Q\|_0 \qquad (\omega\in S^2) \tag{4.19}$$

with a constant $C=C(\gamma)$ depending only on $\gamma>0$.

For the definition of the norm $\|\ \|_\beta$ see (1.6)~(1.8).

REMARK 4.5. If we replace $Q(y)$ by $gQ(y)$ with $g>0$, then we have

$$I(\alpha, \gamma, gQ) \leqq C\alpha^{-2}g^2\|Q\|_1\|Q\|_0 \qquad (\omega\in S^2)\,. \tag{4.20}$$

Proof of Theorem 4.4. The proof is divided into three steps.

(I) Let n be a positive integer and set

$$\begin{cases} m_j = \alpha-\gamma+(2\gamma/n)j & (j=0,1,\cdots,n)\,, \\ \Delta_j = [m_{j-1}, m_j] & (j=1,2,\cdots,n)\,, \\ \tilde{\Delta}_j = [m_{j-1}^2, m_j^2] & (j=1,2,\cdots,n)\,, \\ f(k) = f(k,\omega) = -(\pi/2)^{1/2}Q(y)e^{-ik\omega y}\,, \end{cases} \tag{4.21}$$

$f(k)$ being regarded as an $L^2(R^3)$-valued function on $[\alpha-\gamma\ \ \alpha+\gamma]$. Then, taking note of the continuity of $\Phi_+(k)$ in $B(L^2_\delta(R^3), L^2(S^2))$ ((ii) of Theorem 1.2) and the continuity of $f(k)$ in $L^2(R^3)$, and using Theorem 1.3, we obtain

$$\begin{aligned} \int_{\alpha-\gamma}^{\alpha+\gamma} \|F(k,\omega,\cdot)\|_{S^2}^2 k^2 dk &= \lim_{n\to\infty}\sum_{j=1}^n \int_{\Delta_j} \|\Phi_+(k)f(m_j)\|_{S^2}^2 k^2 dk \\ &= \lim_{n\to\infty}\sum_{j=1}^n \int \|E(\tilde{\Delta}_j)f(m_j)\|_0^2\,, \end{aligned} \tag{4.22}$$

where $\|\ \|_0$ is the usual $L^2(R^3)$-norm and $E(\cdot)$ is the spectral measure assciated with $H=H_0+Q$.

(II) Let $\alpha-\gamma\leqq k<m\leqq\alpha+\gamma$. Then we have

$$\|f(m)-f(k)\|_0 \leqq (\pi/2)^{1/2}\||y|Q\|_0(m-k)\,. \tag{4.23}$$

In fact, (4.23) follows from the relation

$$f(m)-f(k) = (\pi/2)^{1/2}Q(y)i\omega y\int_k^m e^{it\omega y}\,dt\,. \tag{4.24}$$

(III) Set $G(k)=E([m_0^2, k^2])$. Then the right-hand side of (4.22) can be rewritten as

$$\begin{aligned} S \equiv \sum_{j=1}^n \|E(\tilde{\Delta}_j)f(m_j)\|_0^2 &= \sum_{j=1}^n (\{G(m_j)-G(m_{j-1})\}f(m_j), f(m_j))_0 \\ &= \sum_{j=1}^{n-1} [(f(m_j)-f(m_{j+1}), G(m_j)f(m_j))_0 \\ &\qquad +(G(m_j)f(m_{j+1}), f(m_j)-f(m_{j+1}))_0] \\ &\qquad\qquad +(G(m_n)f(m_n), f(m_n))_0\,. \end{aligned} \tag{4.25}$$

Make use of the estimate (4.23), $\|G(k)\|\leqq 1$ and $\|f(k)\|_0\leqq(\pi/2)^{1/2}\|Q\|_0$. Then it follows from (4.25) that

$$S \leqq \sum_{j=1}^{n-1} [(\pi/2)^{1/2}\| |y| Q\|_0 (2\gamma/n) \times 2(\pi/2)^{1/2} \|Q\|_0] + (\pi/2) \|Q\|_0^2 \tag{4.26}$$
$$\leqq \{\pi(1+4\gamma)/2\} \|Q\|_1 \|Q\|_0 ,$$

which is combined with (4.22) to give (4.19) with $C(\gamma)=2\pi(1+4\gamma)$. Q.E.D.

The next theorem corresponds to (4.4).

Theorem 4.6. *Let $Q(y)$ be a continuous function whose support is contained in a ball $\{y \in R^3 / |y| \leqq R\}$ with $R>0$. Let $0<a<b<\infty$. Then*

$$\int_a^b \sigma(k, \omega, Q)\, dk \leqq C(R^2+R^{-1}) , \tag{4.27}$$

where C depends only on a and b.

Proof. Let us start with the expression (4.16) in Theorem 4.3. Let $\rho(t)$ be a C^2 function on $(-\infty, \infty)$ such that $\rho(t)=0$ $(t \leqq 0)$, $\rho(t)=1$ $(t \geqq 1)$. Set

$$\psi(y) = \psi_R(y) = \rho((|y|-R)/R) . \tag{4.28}$$

Then we have, noting that $Q(y)\psi(y) \equiv 0$ $(y \in R^3)$,

$$\begin{cases} F(k, \omega, \omega') = \{J\Phi_+(k) f(k)\}(\omega') , \\ f(k) = -(\pi/2)^{1/2}(T-k^2)(\psi(y)e^{-ik\omega y}) \\ \qquad = (\pi/2)^{1/2}\{\Delta\psi + 2ik(\nabla\psi)\cdot\omega\} e^{-ik\omega y} . \end{cases} \tag{4.29}$$

Since we have the estimates

$$\begin{cases} |\nabla\psi| \leqq C_1 R^{-1} & (C_1 = \max_t |\rho'(t)|) , \\ |\Delta\psi| \leqq (2C_1+C_2)R^{-2} & (C_2 = \max_t |\rho''(t)|) , \end{cases} \tag{4.30}$$

and the supports of $\nabla\psi$ and $\Delta\psi$ are contained in $\{R \leqq |y| \leqq 2R\}$, we get

$$\begin{cases} |f(k)| \leqq C_3(R^{-1}+R^{-2}) & (a \leqq k \leqq b) , \\ |f(m)-f(k)| \leqq C_4(1+R^{-1})(m-k) & (a \leqq k \leqq m \leqq b) \end{cases} \tag{4.31}$$

with $C_j = C_j(b, C_1, C_2)$, $j=3, 4$. Therefore, proceeding as in the proof of Theorem 4.4, we obtain

$$\int_a^b \sigma(k, \omega, Q) k^2 dk \leqq C_5(R^2+R^{-1}) \tag{4.32}$$

with $C_5 = C_5(b, C_1, C_2)$. (4.27) is immediate from (4.32). Q.E.D.

5. Asymptotic behavior of $\hat{F}(k)$

This section is devoted to showing two asymptotic formulas for the operator

$\hat{F}(k)=-2\pi ik^{-1}(\hat{S}(k)-I)$ as $k\to\infty$. From these formulas we shall get two theorems on the uniqueness of the inverse scattering problem.

First let us assume that the potential $Q(y)$ satisfies (A_μ) with $\mu>2$. Then the existence of the scattering amplitude $F(k, \omega, \omega')$ is guaranteed (Theorem 3.3). Let $\xi\in R^3$, $\xi\neq 0$, and set $m=|\xi|>0$, $\omega_0=\xi/|\xi|\in S^2$, i.e., $\xi=m\omega_0$. For each positive integer n let $k_n(m)$ be a function on $(0, \infty)$ such that $k_n(m)\geqq m$ and $k_n(m)\uparrow\infty$ as $n\to\infty$ for each $m>0$. We shall adopt polar coordinates (φ, θ), $0\leqq\varphi<2\pi$, $0\leqq\theta\leqq\pi$, and let ω_0 be represented as (φ_0, θ_0). Set

$$(5.1)\qquad \begin{cases}\omega_n(m, \omega_0)=(\varphi_0, \theta_0+\theta_1)\in S^2\,,\\ \omega_n'(m, \omega_0)=\omega_n(m, \omega_0)-(m/k_n(m))\omega_0\in S^2\,,\end{cases}$$

where θ_1 is determined by $\cos\theta_1=m/(2k_n(m))$. Then we have $k_n(\omega_n'-\omega_n)=-\xi$. Further we set

$$(5.2)\qquad F_n(\xi)=F(k_n(m), \omega_n(m, \omega_0), \omega_n'(m, \omega_0))\quad (\xi=m\omega_0\in R^3, \xi\neq 0, n=1, 2, \cdots)\,.$$

The following theorem is an extesion of Faddeev [4].

Theorem 5.1. *Let $Q(y)$ satisfy (A_μ) with $\mu>2$. Let $F(k, \omega, \omega')$ be the scattering amplitude for $H=H_0+Q$. Then for each positive integer n $F_n(\xi)$, defined by* (5.2), *is well-defined for almost all $\xi\in R^3$ and we have*

$$(5.3)\qquad \lim_{n\to\infty} F_n(\xi)=-(\pi/2)^{1/2}(\mathcal{F}Q)(\xi)\,,$$

where $\mathcal{F}$ is the usual Fourier transform and the lef-hand side exists and is equal to the right-hand side for almost all $\xi\in R^3$.

The relation (5.3) can be written, symbolically, as

$$(5.4)\qquad \lim_{\substack{\xi=k(\omega-\omega')\\ k\to\infty}} F(k, \omega, \omega')=-(\pi/2)^{1/2}(\mathcal{F}Q)(\xi)\,.$$

Corollary 5.2. *Let $Q_1(y)$ and $Q_2(y)$ satisfy (A_μ) with $\mu>2$. Let $F_j(k, \omega, \omega')$, $j=1, 2$, be the scattering amplitude for $H_j=H_0+Q_j$. If $F_1(k, \omega, \cdot)=F_2(k, \omega, \cdot)$ in $L^2(S^2)$ for each pair $(k, \omega)\in(0, \infty)\times S^2$, then we have $Q_1(y)\equiv Q_2(y)$ for all $y\in R^3$.*

Proof of Theorem 5.1. It follows from (iii) of Theorem 3.3 that

$$(5.5)\qquad F(k, \omega, \omega')=-(4\pi)^{-1}s\text{-}\lim_{R\to\infty}\int_{|y|<R} e^{ik(\omega'-\omega)y}Q(y)\,dy + J(k, \omega, \omega')$$

in $L^2(S^2_{\omega'})$. Take a sequence R_p, $p=1,2,\cdots$, such that $R_p\uparrow\infty$ as $p\to\infty$ and there exists the limit

$$(5.6) \qquad (2\pi)^{-3/2}\lim_{p\to\infty}\int_{|y|<R_p} e^{-i\xi y}Q(y)\,dy = (\mathcal{F}Q)(\xi)$$

for almost all $\xi\in R^3$. Set $\xi=m\omega_0$, $m>0$ and $\omega_0\in S^2$, and set $k=k_n(m)$, $\omega=\omega_n(m, \omega_0)$, $\omega'=\omega_n'(m, \omega_0)$ in (5.5), where k_n, ω_n and ω_n' are as above. Then we have

$$(5.7) \qquad F_n(\xi) = -(\pi/2)(\mathcal{F}Q)(\xi)+J(k_n, \omega_n, \omega_n'),$$

and, since the right-hand side is well-defined for almost all $\xi\in R^3$, $F_n(\xi)$ is also well-defined for almost all $\xi\in R^3$. The estimate (3.17) can be applied to show that

$$(5.8) \qquad F_n(\xi)+(\pi/2)(\mathcal{F}Q)(\xi) = o(1)$$

as $n\to\infty$, which implies that $\lim_{n\to\infty} F_n(\xi)$ exists for almost all $\xi\in R^3$ and we get (5.3). Q.E.D.

Next let us assume that $Q(y)$ is a general short-range potential. In this case the Hilbert-Schmidt kernel $F(k, \omega, \omega')$ does not exist in general. But the operator $\hat{F}(k)$ is well-defined as was shown in §2. For $z\in R^3$ and $k>0$ we set

$$(5.9) \qquad x_{k,z}(\omega) = e^{-ik\omega z}\in L^2(S^2) \qquad (\omega\in S^2).$$

Theorem 5.3. *Let $Q(y)$ satisfy (A_μ) with $\mu>1$. Let $\hat{F}(k)$ and $x_{k,z}(\omega)$ be as above. Then we have for any $z\in R^3$*

$$(5.10) \qquad \lim_{k\to\infty} k^2(\hat{F}(k)x_{k,z}, x_{k,z})_{S^2} = -2\pi\int_{R^3} Q(y)|y-z|^{-2}\,dy.$$

Here $(\ ,\ )_{S^2}$ denotes the inner product of $L^2(S^2)$.

Proof. From the definition of $\hat{F}(k)$ ((2.7)) and (4.11) it follows that

$$(5.11) \qquad \begin{aligned}&(\hat{F}(k)x_{k,z}, x_{k,z})_{S^2}\\ &= -2\pi^2(Q\Phi_0^*(k)x_{k,z}\ \Phi_+^*(k)x_{k,z})_0\\ &= -2\pi^2(Q\Phi_0^*(k)x_{k,z}\ \Phi_0^*(k)x_{k,z})_0\\ &\quad +2\pi^2(Q\Phi_0^*(k)x_{k,z}, (T-(-k)^2)^{-1}Q\Phi_0^*(k)x_{k,z})_0\\ &\equiv f_1(k, z)+f_2(k, z).\end{aligned}$$

The definition of $\Phi_+^*(k)$ and $(T-(-k)^2)^{-1}$ are given in the proof of Lemma 4.1. Since

$$(5.12) \qquad \begin{aligned}(\Phi_0^*(k)x_{k,z})(y) &= (2\pi)^{-3/2}\int_{S^2} e^{ik\omega(y-z)}\,d\omega\\ &= (2/\pi)^{1/2}(k|y-z|)^{-1}\sin(k|y-z|),\end{aligned}$$

we have

$$k^2 f_1(k, z) = -4\pi \int_{R^3} Q(y) |y-z|^{-2} \{\sin(k|y-z|)\}^2 \, dy \tag{5.13}$$
$$\to -2\pi \int_{R^3} Q(y) |y-z|^{-2} \, dy \qquad (k \to \infty),$$

where we have used the Riemann-Lebesgue theorem. On the other hand, by Theorem 1.1, we get

$$\begin{aligned} &|k^2 f_2(k, z)| \\ &\leqq 4\pi |(Q|y-z|^{-1} q_{k,z}(y), (T-(-k)^2)^{-1} Q|y-z|^{-1} q_{k,z}(y))_0| \\ &\qquad\qquad\qquad (q_{k,z}(y) = \sin(k|y-z|)) \\ &\leqq 4\pi \| |y-z|^{-1} Q \|_\delta^2 \| (T-(-k)^2)^{-1} \| . \end{aligned} \tag{5.14}$$

(5.14) is combined with the estimate (1.12) (with k replaced by $-k$) to give

$$k^2 f_2(k, z) = O(k^{-1}) \qquad (k \to \infty) \tag{5.15}$$

for each $z \in R^3$. (5.10) follows from (5.13) and (5.15). Q.E.D.

By the use of the formula (5.10) we can show the following

Theorem 5.4 (the uniqueness of the inverse scattering problem for the short-range potentials). *Let $Q_1(y)$ and $Q_2(y)$ satisfy (A_μ) with $\mu > 1$ and let $\hat{S}_1(k)$ and $\hat{S}_2(k)$ be the S-matrices for $H_1 = H_0 + Q_1$ and $H_2 = H_0 + Q_2$, respectively. If $\hat{S}_1(k) = \hat{S}_2(k)$ for $k > 0$ (or more exactly, $\hat{S}_1(k_n) = \hat{S}_2(k_n)$ for a sequence $\{k_n\}$ such that $k_n \uparrow \infty$ as $n \to \infty$), then $Q_1(y) \equiv Q_2(y)$ for all $y \in R^3$.*

Proof. It follows from Theorem 5.3 that we have only to show the following: Assume that $Q(y)$ satisfies (A_μ) with $\mu > 1$ and

$$g(z) = \int_{R^3} Q(y) |y-z|^{-2} \, dy = 0 \tag{5.16}$$

holds for all $z \in R^3$. Then $Q(y) \equiv 0$.

Denote the dual form between the spase $S(R^3)$ of rapidly decreasing functions and the dual space $S'(R^3)$ by $\langle \ , \ \rangle$ and let $S_0(R^3_\xi)$ be all $G(\xi) \in S(R^3_\xi)$ such that $G(\xi) = 0$ in a neighborhood of the origin $\xi = 0$. Then we obtain from the definition of the Fourier transform in $S'(R^3)$

$$\begin{aligned} 0 &= \langle \mathcal{F} g, G \rangle = \langle g, \overline{\mathcal{F}}^{-1} G \rangle \\ &= \int_{R^3} dz \int_{R^3} dy \, Q(y) |y-z|^{-2} (\overline{\mathcal{F}}^{-1} G)(z) \\ &= \langle Q, |z|^{-2} * (\overline{\mathcal{F}}^{-1} G) \rangle \end{aligned} \tag{5.17}$$

for $G \in S_0(R^3_\xi)$. Here $\overline{\mathcal{F}}^{-1} G$ is defined by

$$(\mathcal{F}^{-1}G)(z) = (2\pi)^{-3/2}\int_{R^3} e^{-i\xi z}G(\xi)\, d\xi \tag{5.18}$$

and we should note that $|z|^{-2}*(\mathcal{F}^{-1}G)\in S(R^3)$ for $G\in S_0(R^3_\xi)$ ($*$ denotes the convolution), because

$$\mathcal{F}(|z|^{-2}*(\mathcal{F}^{-1}G))(\xi) = 2\pi^2|\xi|^{-1}G(\xi)\in S(R^3)\,, \tag{5.19}$$

where we used the formula

$$\mathcal{F}(|z|^{-2})(\xi) = 2\pi^2|\xi|^{-1} \tag{5.20}$$

(Gel' fand and Shilov [5], p. 194). From (5.17) and (5.19) it can be seen that

$$\langle \mathcal{F}Q,\ |\xi|^{-1}G\rangle = 0 \qquad (G\in S_0(R^3_\xi))\,, \tag{5.21}$$

which implies that

$$\langle \mathcal{F}Q, H\rangle = 0 \qquad (H\in S_0(R^3_\xi))\,, \tag{5.22}$$

because $|\xi|H(\xi)\in S_0(R^3_\xi)$ for any $H(\xi)\in S_0(R^3_\xi)$. Therefore the support of $\mathcal{F}Q$ is at most the origin $\xi=0$. Thus, applying a theorem in the theory of distributions (see, e.g., Schwartz [16], p. 100), $\mathcal{F}Q$ is represented as

$$\mathcal{F}Q = P(D)\delta\,, \tag{5.23}$$

where $P(y)=P(y_1, y_2, y_3)$ is a polynomial, δ is Dirac's δ-function and $D=(-i\partial/\partial\xi_1, -i\partial/\partial\xi_2, -i\partial/\partial\xi_3)$. Since $\mathcal{F}^{-1}(P(D)\delta)=P(y)$, finally we get

$$Q(y) = P(y)\,. \tag{5.24}$$

But any polynomial other than $P=0$ can not be short-range, and hence $Q(y)\equiv 0$. Q.E.D.

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