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SCATTERING THEORY FOR PSEUDO-DIFFERENTIAL OPERATORS
II. THE COMPLETENESS OF WAVE OPERATORS

TOMIO UMEDA

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1. Introduction

The present paper is a continuation of the previous work [12] and deals with a scattering theory for pseudo-differential operators.

The operators we consider are of the form

$$P(D)+A(X, D)$$

in $\boldsymbol{R}^n$, where the unperturbed operator $P(D)$ and the perturbation $A(X, D)$ are pseudo-differential operators. In the previous paper [12] we showed the existence of wave operators. In the present paper we prove the completeness of wave operators.

Our proof is based on the Enss decomposition principle. More precisely, we follow the ideas of Simon [11], which are extensions of a work of Enss [5], and construct the "decomposition operators" $D_j^\pm$ and D_j^0 for $j=1, 2, \cdots$. We give a proof of the Enss decomposition principle for these operators, while Enss [5] and Simon [11] proved this principle for vectors.

We make assumptions on the symbols of $P(D)$ and $A(X, D)$, not on the operators $P(D)$ and $A(X, D)$. In treating pseudo-differential operators, it is important to make assumptions on the symbols of pseudo-differential operators.

Throughout this paper, the same notation as in [12] will be used (the list of the notation is given in Section 2 of [12]), and formulas, lemmas, etc. given in [12] will be quoted as (I.2.3), Lemma I.2.3, etc..

Finally, we sketch the contents of the paper. Section 2 contains the main theorem. Using Cook's method, we show the existence of wave operators in Section 3. In Section 4, we prove some technical lemmas which will be used in the proofs of the Enss decomposition principle and the main theorem. In Section 5, we construct the decomposition operators $D_j^\pm$ and D_j^0 for $j=1, 2, \cdots$, and establish the Enss decomposition principle. With the aid of this principle we prove the main theorem in Section 6.

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2. The main theorem

In this section, we state the main theorem. We first introduce some notation which supplement Section 2 of [12].

$\mathcal{F}u=\hat{u}$: Fourier transform of u.
$\bar{\mathcal{F}}v$: inverse Fourier transform of v.
$C_\infty(\boldsymbol{R}^n)$: the space of complex-valued continuous functions u vanishing at infinity, equipped with the norm

$$||u||_\infty = \sup_{x\in\boldsymbol{R}^n} |u(x)|.$$

$H_\nu(\boldsymbol{R}^n)$: the Sobolev space of order ν, equipped with the norm

$$||u||_\nu=\left(\int (1+|\xi|^2)^\nu |\hat{u}(\xi)|^2 d\xi\right)^{1/2}.$$

$C^k(\boldsymbol{R}^n)$, $C^\infty(\boldsymbol{R}^n)$, $C_0^\infty(\boldsymbol{R}^n)$, $\mathcal{S}(R^n)$ etc. will be employed as usual.
$||T||$: the operator norm of a bounded operator T on $L^2(\boldsymbol{R}^n)$.
$\sigma_{\text{ess}}(H)$: the essential spectrum of an operator H.
Λ^ν $(\nu>0)$: the operator in $L^2(\boldsymbol{R}^n)$ with domain $H_\nu(\boldsymbol{R}^n)$ defined by

$$\Lambda^\nu u = \bar{\mathcal{F}}(1+|\xi|^2)^{\nu/2}\mathcal{F}u.$$

$\Lambda^{-\nu}$: the inverse of Λ^ν.
F_r $(r>0)$: the operator of multiplication by $\zeta_r(x)=\zeta(x/r)$ with $\zeta\in C_0^\infty(\boldsymbol{R}^n)$ and $\zeta(y)=1$ (resp. 0) for $|y|\leqslant 1$, (resp. $\geqslant 2$).
$\mathcal{H}_{\text{ac}}(H)$: the subspace of absolute continuity with respect to a self-adjoint operator H.
$P_{\text{ac}}(H)$: the orthogonal projection onto $\mathcal{H}_{\text{ac}}(H)$.
meas: the Lebesgus measure.

We now state the assumptions made on the symbol $p(\xi)$ of $P(D)$. We denote by C_v the set of critical values of $p(\xi)$.

(P_0) $p(\xi)$ is a real-valued C^∞ function on $\boldsymbol{R}^n$ such that the estimate

$$C_0(1+|\xi|)^k\leqslant p(\xi)\leqslant \tilde{C}_0(1+|\xi|)^{\tilde{k}}, \qquad |\xi|\geqslant R_0 \tag{2.1}$$

is valid for some positive constants k, $\tilde{k}$, C_0, $\tilde{C}_0$ and R_0.
(P_1) The closure of C_v is at most countable.

We shall denote by H_0 the closure in $L^2(\boldsymbol{R}^n)$ of $P(D)$ with domain $\mathcal{S}(\boldsymbol{R}^n)$.

To state the assumption on the symbol $a(x, \xi)$ of $A(X, D)$, we introduce the notation. Let $V(x)$ be a measurable function on $\boldsymbol{R}^n$. If there exist constants $\varepsilon>0$, $\gamma\in\boldsymbol{R}$ and $\theta\in\boldsymbol{R}$ such that for some constant C and some measurable set $\Gamma\subset\boldsymbol{R}^n$ with

(2.2) $$\operatorname{meas}(\Gamma\cap\{x\in\boldsymbol{R}^n \mid |x|\geqslant R\})=O(R^{-2\theta}) \qquad \text{as } R\to\infty,$$

the estimate

(2.3) $$|V(x)|\leqslant C(1+|x|)^{-1-\varepsilon}, \qquad x\in\boldsymbol{R}^n\backslash\Gamma,$$

(2.4) $$|V(x)|\leqslant C(1+|x|)^{\gamma}, \qquad x\in\Gamma,$$

are valid, then we write $V\in\mathcal{V}(\varepsilon, \gamma, \theta)$. Our assumption on $A(X, D)$ is

(A_0) $a(x, \xi)$ is a complex-valued measurable function on $\boldsymbol{R}^n_x\times\boldsymbol{R}^n_\xi$ which verifies the following conditions:

(i) For any fixed x, $a(x, \xi)$ is a smooth function of ξ.

(ii) There are constants $\varepsilon>0$, $\gamma\geqslant -1$, $\theta>\gamma+1$ and $m\in\boldsymbol{R}$ such that for some $V\in\mathcal{V}(\varepsilon, \gamma, \theta)$ the estimate

(2.5) $$|(\partial/\partial\xi)^{\mu}a(x, \xi)|\leqslant V(x)(1+|\xi|)^m$$

is valid for every multi-index μ with $|\mu|\leqslant 2([(2|\gamma|+n)/2]+2)$. (Here $[a]$ denotes the integral part of a real number a.)

We shall prove the following lemma in Section 3.

Lemma 2.1. *Let (A_0) be fulfilled and let $M>m+n$. There exists a constant C depending only on M and γ, such that*

(2.6) $$\|A(X, D)u\|_{L^2}\leqslant C\|u\|_M, \qquad u\in H_M(\boldsymbol{R}^n).$$

In view of this lemma, the perturbation A of H_0 is defined as follows: Let N be a positive number such that

(2.7) $$N\geqslant\tilde{k},\ N>m+n.$$

(Throughout this paper, we fix the number N.)
Define

(2.8) $$\begin{cases}\mathcal{D}(A)=H_N(\boldsymbol{R}^n)\\ Au=A(X, D)u.\end{cases}$$

Moreover, we make the following assumptions:

(A_1) A is symmetric in $L^2(\boldsymbol{R}^n)$.

(H) $(H_0+A)\restriction H_N(\boldsymbol{R}^n)$ has a self-adjoint extension H in $L^2(\boldsymbol{R}^n)$ such that $\mathcal{D}(H)\subset H_s(\boldsymbol{R}^n)$ for some $s>0$.

We shall denote by $E(B)$ the spectral measure associated with H, where B varies over all Borel sets of the real line.

Our main theorem is

Theorem 1. *Let assumptions (P_0), (P_1), (A_0), (A_1) and (H) be fulifilled. Then,*

(i) *The wave operators* $W_{\pm}=s-\lim_{t\to\pm\infty} e^{itH}e^{-itH_0}P_{\mathrm{ac}}(H_0)$ *exist.*

(ii) *The range of* $W_{\pm}$ *is the subspace* $\mathcal{H}_c(H)$ *of continuity with respect to* H.

(iii) *The only possible limit points for the point spectrum of* H *are in* $\bar{C}_v$. *Any eigenvalue not in* $\bar{C}_v$ *has finite multiplicity.*

(iv) $\sigma_{\mathrm{ess}}(H)=\sigma_{\mathrm{ess}}(H_0)=\{p(\xi)\,|\,\xi\in\boldsymbol{R}^n\}$.

REMARK. Since the range of $W_{\pm}$ must be contained in the subspace of absolute continuity with respect to H, conclusion (ii) of Theorem 1 implies that $W_{\pm}$ are complete and that H has no singular continuous spectrum.

Before proving Theorem 1, we give an example.

EXAMPLE. Let $n=3$ and let $H_0=(1-\Delta)^{1/2}$. For the perturbation we take $A=(1+|x|^2)^{-\delta/4}(1-\Delta)^{3/2}(1+|x|^2)^{-\delta/4}$ with $\delta>1$. In view of Example I.2.4 it is easily checked that assumptions (P_0), (P_1), (A_0), (A_1) are satisfied. Since H_0+A is a semibounded symmetric operator, its Friedrichs extension H can be defined. From the definition of the Friedrichs extension it follows immediately that $\mathcal{D}(H)\subset H_{1/2}(\boldsymbol{R}^3)$. Thus assumption (H) is satisfied as well. Hence all the conclusions of Theorem 1 hold.

3. Existence of wave operators

In this section we prove the existence of wave operators. We begin with the following lemma which is basic in our analysis.

Lemma 3.1. *Let* (A_0) *be fulfilled and let* $M>m+n$. *Then the following estimate holds*

$$\|A(X, D)\Lambda^{-M}u\|_{L^2}^2\leqslant C\int(1+|x|)^{-2(1+\tilde{\varepsilon})}|u(x)|^2dx \tag{3.1}$$

for any $u\in\mathcal{S}(\boldsymbol{R}^n)$, *where* $\tilde{\varepsilon}$ *is a positive constant depending only on* ε, γ *and* θ; C *is a constant depending only on* M *and* γ.

Proof. Let $u\in\mathcal{S}(\boldsymbol{R}^n)$ and let

$$K(x, w)=\int e^{i\langle w,\xi\rangle}a(x, \xi)(1+|\xi|^2)^{-M/2}d\xi\,. \tag{3.2}$$

Then, by Fubini's theorem, it follows that

$$\int e^{i\langle x,\xi\rangle}a(x, \xi)(1+|\xi|^2)^{-M/2}\hat{u}(\xi)d\xi=\int K(x, x-x')u(x')dx'\,. \tag{3.3}$$

By repeated integration by parts in (3.2), we have for $j=[(2|\gamma|+n)/2]+2$

$$(1+|w^2|)^j|K(x, w)|\leqslant CV(x)\,, \tag{3.4}$$

where C is a constant depending only on M and γ. Combining (3.3) and (3.4), we see that

$$(3.5) \qquad \|A(X, D)\Lambda^{-M}u\|^2 \leqslant C\int |V(x)|^2 \{\int (1+|x-x'|^2)^{-j}|u(x')|dx'\}^2 dx\,.$$

Since $2j \geqslant n+1$, it follows (by Schwarz' inequality and Fubini's theorem) that

$$(3.6) \qquad \begin{aligned}&\|A(X, D)\Lambda^{-M}u\|^2\\ &\leqslant C\int |u(x')|^2\{\int |V(x)|^2(1+|x-x'|^2)^{-j}dx\}dx'\\ &\equiv C\int |u(x')|^2 I(x')dx'\,.\end{aligned}$$

To prove (3.1), we write

$$\begin{aligned}I(x') &= \Big(\int_{B(|x'|/2)} + \int_{B^c(|x'|/2)\cap\Gamma^c} + \int_{B^c(|x'|/2)\cap\Gamma}\Big)|V(x)|^2(1+|x-x'|^2)^{-j}dx\\ &\equiv I_1(x')+I_2(x')+I_3(x')\,.\end{aligned}$$

Here $B(|x'|/2)$ denotes the ball of radius $|x'|/2$ with its center at $0\in \boldsymbol{R}^n$ and $B^c(|x'|/2)$ denotes the complement of $B(|x'|/2)$ in $\boldsymbol{R}^n$,

$$B^c(|x'|/2) = \boldsymbol{R}^n \backslash B(|x'|/2)\,.$$

Γ^c also denotes the complement of Γ in $\boldsymbol{R}^n$. Since $|x-x'| \geqslant |x'|/2$ for any $x\in B(|x'|/2)$, we get

$$I_1(x') \leqslant C(1+|x'|)^{2|\gamma|-2j+n}\,.$$

Since $|V(x)| \leqslant C(1+|x|)^{-1-\varepsilon}$ for any $x\in\Gamma^c$, we have

$$I_2(x') \leqslant C(1+|x'|)^{-2(1+\varepsilon)}\,.$$

To evaluate $I_3(x')$, we use the inequality

$$(1+|x-x'|)^{-1} \leqslant (1+|x|)^{-1}(1+|x'|)\,.$$

(Note that the constant γ is allowed to be positive. See assumption (A_0).) Choosing ρ so that $\gamma<\rho<j$, using (2.4), we see that

$$\begin{aligned}I_3(x') &\leqslant C(1+|x'|)^{2\rho}\int_{B^c(|x'|/2)\cap\Gamma}(1+|x|)^{2(\gamma-\rho)}(1+|x-x'|^2)^{-j+\rho}dx\\ &\leqslant C(1+|x'|)^{2\gamma}\,\mathrm{meas}\,(B^c(|x'|/2)\cap\Gamma)\\ &\leqslant C(1+|x'|)^{2(\gamma-\theta)} \qquad (\text{using } (2.2))\,.\end{aligned}$$

Putting $\tilde{\varepsilon}=\min\{(2j-n-2|\gamma|)/2, 1+\varepsilon, \theta-\gamma\}-1$, we have proven (3.1). Q.E.D.

Proof of Lemma 2.1. Using Lemma 3.1, we see that for $u\in\mathcal{S}(\boldsymbol{R}^n)$

$$\begin{aligned}||A(X, D)u|| &= ||A(X, D)\Lambda^{-M}\Lambda^{M}u|| \\ &\leqslant C||\Lambda^{M}u|| = C||u||_{M}\,,\end{aligned}$$

which gives (2.6). Q.E.D.

Corollary 3.2. *Let (A_0) be satisfied. Then,*

$$||A\Lambda^{-N}u||^2\leqslant C\int(1+|x|)^{-2(1+\tilde{\varepsilon})}|u(x)|^2dx\,,\qquad u\in L^2(\boldsymbol{R}^n)\,. \tag{3.7}$$

For the proof it suffices to note the definition of A (see (2.7) and (2.8)).

Proof of Theorem 1 (i). We shall prove that

$$\int_{-\infty}^{\infty}||Ae^{-itH_0}u||dt<\infty \tag{3.8}$$

for any $u\in\mathcal{S}(\boldsymbol{R}^n)$ with $\hat{u}\in C_0^\infty(\boldsymbol{R}^n\backslash\sum)$. Here $\sum$ denotes the set of critical points of $p(\xi)$,

$$\sum=\{\xi\in\boldsymbol{R}^n\,|\,p'(\xi)=0\}\,.$$

Since, by Proposition I.3.2, $\mathcal{F}C_0^\infty(\boldsymbol{R}^n\backslash\sum)$ is dense in $\mathcal{H}_{\mathrm{ac}}(H_0)$ and since

$$(H-H_0)e^{-itH_0}u=Ae^{-itH_0}u\,,\qquad u\in\mathcal{S}(\boldsymbol{R}^n)\,,$$

(3.8) will prove conclusion (i). (See Theoıem XI.4, p. 20 of [9].)
Let $u\in\mathcal{F}C_0^\infty(\boldsymbol{R}^n\backslash\sum)$ and choose $\delta>0$ so that

$$|p'(\xi)|>3\delta\qquad\text{if }\xi\in\operatorname{supp}\hat{u}\,. \tag{3.9}$$

Then,

$$||Ae^{-itH_0}u||\leqslant||A\Lambda^{-N}(1-F_{\delta|t|})e^{-itH_0}\Lambda^{N}u||+||A\Lambda^{-N}||\,||F_{\delta|t|}e^{-itH_0}\Lambda^{N}u||\,, \tag{3.10}$$

so it suffices to prove that

$$\int_{-\infty}^{\infty}||A\Lambda^{-N}(1-F_{\delta|t|})e^{-itH_0}\Lambda^{N}u||dt<\infty\,, \tag{3.11}$$

and

$$\int_{-\infty}^{\infty}||F_{\delta|t|}e^{-itH_0}\Lambda^{N}u||dt<\infty\,. \tag{3.12}$$

By Corollary 3.2 the first term on the right hand side of (3.10) is dominated by

$$\begin{aligned}&\Big\{\int_{|x|>\delta|t|}(1+|x|)^{-2(1+\tilde{\varepsilon})}|(e^{-itH_0}\Lambda^{N}u)(x)|^2dx\Big\}^{1/2}\\ &\leqslant(1+\delta|t|)^{-1-\tilde{\varepsilon}}||u||_{N}\,.\end{aligned}$$

This yields (3.11). To prove (3.12), we apply the method of stationary phase to the integral

$$\int e^{i(\langle x,\xi\rangle-tp(\xi))}(1+|\xi|^2)^{N/2}\hat{u}(\xi)d\xi\,.$$

By (3.9)

$$|x-tp'(\xi)|/(|x|+|t|)\geqslant\delta/(1+2\delta)\qquad\text{when } \xi\in\operatorname{supp}\hat{u},\ |x|\leqslant 2\delta|t|.$$

Hence Lemma I.I (see Appendix of [12]) shows that for every ν

$$|(F_{\delta|t|}e^{-itH_0}\Lambda^N u)(x)|\leqslant C_\nu(1+|x|+|t|)^{-\nu}\,.$$

This implies (3.12). Q.E.D.

4. Some lemmas

In this section we prove several lemmas which will be used later on.

Lemma 4.1. *Let (H) be fulfilled. Then the operator $F_r(H+i)^{-1}$ is a compact operator for every $r>0$.*

Proof. Since $\mathscr{D}(H)\subset H_s(\boldsymbol{R}^n)$, we see that $\Lambda^s(H+i)^{-1}$ is a closed operator with domain $L^2(\boldsymbol{R}^n)$. By the closed graph theorem it follows that $\Lambda^s(H+i)^{-1}$ is a bounded operator. Noting that every bounded sequence in $H_s(\boldsymbol{R}^n)$ has a subsequence convergent in L^2_{loc}, we see that $F_r\Lambda^{-s}$ is a compact operator in $L^2(\boldsymbol{R}^n)$. Writing

$$F_r(H+i)^{-1}=F_r\Lambda^{-s}\Lambda^s(H+i))^{-1}\,,$$

the result follows. Q.E.D.

REMARK. By Theorem XI.115 of [9], the above lemma implies that for any $r>0$ and any $u\in\mathscr{H}_c(H)$

$$\lim_{T\to\infty}(2T)^{-1}\int_{-T}^{T}||F_re^{-itH}u||^2dt=0\,.$$

Lemma 4.2. *Let (H) be fulfilled. Then the operator $\Lambda^sE(B)$ is a bounded operator for every bounded Borel set B of the real line.*

Proof. First note that if B is a bounded Borel set, then $(H+i)E(B)$ is a bounded operator. It was observed before that $\Lambda^s(H+i)^{-1}$ is a bounded operator. Writing

$$\Lambda^sE(B)=\Lambda^s(H+i)^{-1}(H+i)E(B)\,,$$

we obtain the result. Q.E.D.

Lemma 4.3. *Let (P_0), (A_0), (A_1), (H) be fulfilled and let $\Phi\in C_\infty(\boldsymbol{R})$. Then for all $a, b\in\boldsymbol{R}$*

$$\|\{\Phi(H)-\Phi(H_0)\}(1-F_r)E([a, b])\|\to 0$$

as $r\to\infty$.

Proof. We first claim that for non-real z

$$\|\{(H-z)^{-1}-(H_0-z)^{-1}\}\Lambda^{-N}(1-F_r)\|\to 0 \tag{4.1}$$

as $r\to\infty$. Since $H\Lambda^{-N}=(H_0+A)\Lambda^{-N}$, we see that

$$\begin{aligned} &\|\{(H-z)^{-1}-(H_0-z)^{-1}\}\Lambda^{-N}(1-F_r)\| \\ &\leqslant\|(H-z)^{-1}A\Lambda^{-N}[(1-F_r), (H_0-z)^{-1}]\|+ \\ &\quad+\|(H-z)^{-1}\|\,\|A\Lambda^{-N}(1-F_r)\|\,\|(H_0-z)^{-1}\|\,. \end{aligned} \tag{4.2}$$

The first term on the right side of (4.2) goes to zero by Lemma 3 in Section 2 of [11]. The second goes to zero by Corollary 3.2. Thus we have shown (4.1).

Now, by the arguments in the proof of Lemma 4 in Section 2 of [11], (4.1) implies that

$$\|\{\Phi(H)-\Phi(H_0)\}\Lambda^{-N}(1-F_r)\|\to 0 \tag{4.3}$$

as $r\to\infty$. Since, by Lemma 4.2, $\Lambda^s E([a, b])$ is a bounded operator, the desired result follows from (4.3) in just the same way as in the proof of Lemma 5 in Section 2 of [11]. Q.E.D.

Let χ_α be the characteristic function of the unit cube centered at $\alpha\in\boldsymbol{Z}^n$. Choose $f\in\mathcal{S}(\boldsymbol{R}^n)$ so that

$$\int f(x)dx=1 \tag{4.4}$$

and let

$$f_\alpha=f*\chi_\alpha\,. \tag{4.5}$$

Then

$$\sum_\alpha f_\alpha(x)=1\,. \tag{4.6}$$

Since $f\in\mathcal{S}(\boldsymbol{R}^n)$, it follows that $f_\alpha\in\mathcal{S}(\boldsymbol{R}^n)$ and that

$$\sup_\alpha\int(1+|x-\alpha|)^{2\nu}|f_\alpha(x)|^2dx<+\infty \tag{4.7}$$

for each ν.

Lemma 4.4. *Let $f\in\mathcal{S}(\boldsymbol{R}^n)$ and let f_α be given by* (4.5). *For each α, let $g_\alpha\in C^{2\nu}(\boldsymbol{R}^n)$ be given with $\sup|g_\alpha|_{2\nu}<+\infty$. Here*

$$|g|_{2\nu}=\sum_{|\mu|\leqslant 2\nu}\sup_{\xi}|(\partial/\partial\xi)^\mu g(\xi)|.$$

Then

$$\sup_\alpha\|(1+|x-\alpha|^2)^\nu\overline{\mathcal{F}}g_\alpha\mathcal{F}(f_\alpha u)\|\leqslant C\|u\|,\qquad u\in L^2(\boldsymbol{R}^n)\tag{4.8}$$

where C is a constant depending only on ν, f and $\sup_\alpha|g_\alpha|_{2\nu}$.

Proof. First we note that

$$(1+|x-\alpha|^2)^\nu\overline{\mathcal{F}}g_\alpha\mathcal{F}(f_\alpha u)=\overline{\mathcal{F}}((1+|D_\xi+\alpha|^2)^\nu(g_\alpha\mathcal{F}(f_\alpha u)))\tag{4.9}$$

where $D_\xi=-i\partial/\partial\xi$. By Leibniz' formula it follows that for any multi-index μ with $|\mu|\leqslant 2\nu$

$$(D_\xi+\alpha)^\mu(g_\alpha\mathcal{F}(f_\alpha u))=\sum_{\tilde{\mu}\leqslant\mu}\binom{\mu}{\tilde{\mu}}D_\xi^{\mu-\tilde{\mu}}g_\alpha(D_\xi+\alpha)^{\tilde{\mu}}(\mathcal{F}(f_\alpha u)).\tag{4.10}$$

Since $f_\alpha\in\mathcal{S}(\boldsymbol{R}^n)$, it follows that

$$(D_\xi+\alpha)^{\tilde{\mu}}(\mathcal{F}(f_\alpha u))=(2\pi)^{-n}((D_\xi+\alpha)^{\tilde{\mu}}\hat{f}_\alpha)*\hat{u}.$$

Moreover, it follows by Young's inequality that $(D_\xi+\alpha)^{\tilde{\mu}}(\mathcal{F}(f_\alpha u))\in L^2(\boldsymbol{R}^n)$ and that

$$\|(D_\xi+\alpha)^{\tilde{\mu}}(\mathcal{F}(f_\alpha u))\|\leqslant(2\pi)^{-n}\|(D_\xi+\alpha)^{\tilde{\mu}}\hat{f}_\alpha\|_{L^1}\|u\|.\tag{4.11}$$

Hence, combining (4.10) and (4.11), we get

$$\|(D_\xi+\alpha)^\mu(g_\alpha\mathcal{F}(f_\alpha u))\|\leqslant C(\sup_\alpha|g_\alpha|_{2\nu})(\sum_{|\mu|\leqslant 2\nu}\|(D_\xi+\alpha)^\mu\hat{f}_\alpha\|_{L^1})\|u\|.\tag{4.12}$$

Using the fact that $\hat{f}_\alpha=\hat{f}\hat{\chi}_\alpha$, we see that

$$\sup_\alpha\|(D_\xi+\alpha)^\mu\hat{f}_\alpha\|_{L^1}<+\infty\tag{4.13}$$

for each μ. Thus the estimate (4.8) follows from (4.9), (4.12) and (4.13). Q.E.D.

Lemma 4.5. *Let $f\in\mathcal{S}(\boldsymbol{R}^n)$ be a non-negative function satisfying condition* (4.4). *Let f_α be given by* (4.5). *For ecah α, let $g_\alpha\in C^{2n}(\boldsymbol{R}^n)$ be given with $\sup_\alpha\|(1-\Delta)^n g_\alpha\|_{L^2}<+\infty$. For $u\in L^2(\boldsymbol{R}^n)$, define*

$$Tu=\sum_\alpha\overline{\mathcal{F}}g_\alpha\mathcal{F}(f_\alpha u).$$

Then

$$\|Tu\|\leqslant C(\sup_\alpha\|(1-\Delta)^n g_\alpha\|_{L^2})\|u\|$$

where C is a constant depending only on n. Moreover,

$$s-\lim_{j\to\infty}\sum_{|\alpha|\leqslant j}\bar{\mathcal{F}}g_\alpha\mathcal{F}(f_\alpha u)=Tu \qquad in\ L^2(\boldsymbol{R}^n)\,.$$

This was proved by Simon [11, Section 2].

5. Enss decomposition principle

In the present section we construct the so-called decomposition operators $D_j^\pm$, D_j^0 mentioned in the introduction, with the aid of the lemmas in the preceding section.

Theorem 2. *Let (P_0), (P_1), (A_0), (A_1) and (H) be fulfilled. Let $[a, b]$ be an interval such that $[a, b]$ is disjoint from $\bar{C}_v$. Then there exist three sequences of bounded operators $\{D_j^\pm\}$ and $\{D_j^0\}$ on $L^2(\boldsymbol{R}^n)$ with the following properties:*

(i) $E([a, b])=(D_j^++D_j^-+D_j^0)E([a, b])$ *for all* j.

(ii) $\|(W_\pm-1)D_j^\pm\|\to 0$ *as* $j\to\infty$.

(iii) $\sup_j\|D_j^\pm\|<+\infty$.

(iv) *There exists a positive number δ depending only on a, b such that for all ν and j the estimates*

$$\|F_{\delta(j\pm t)}e^{-itH_0}D_j^\pm\|\leqslant C_{\nu,a,b}(1+j\pm t)^{-\nu}\,, \qquad t\gtrless 0\,,$$

are valid for some constant $C_{\nu,a,b}$.

(v) *Let $\{u_j\}$ be a bounded sequence in the range of $E([a, b])$ with $s-\lim F_ju_j=0$. Then $s-\lim D_j^0u_j=0$.*

Proof. (following ideas in [11].) Choose a', $b'\in\boldsymbol{R}$ so that $[a, b]\subset(a', b')\subset[a', b']$ is disjoint from $\bar{C}_v$. Let Φ be a function in $C_0^\infty(\boldsymbol{R})$ such that $0\leqslant\Phi\leqslant 1$ and

$$\Phi(\lambda)=\begin{cases}1 & \lambda\in[a, b]\\ 0 & \lambda\notin[a', b']\,.\end{cases} \tag{5.1}$$

Put $K=p^{-1}([a', b'])$. Since K is a compact set disjoint from Σ(see the proof of Theorem 1 (i) for the notation Σ), we can find, a bounded open set Ω and $d>0$, so that

$$K+B_d\subset\Omega\subset\bar{\Omega}\subset\boldsymbol{R}^n\backslash\Sigma\,. \tag{5.2}$$

Here $B_d=\{\xi\in\boldsymbol{R}^n\,|\,|\xi|\leqslant d\}$. Let f be a non-negative function in $\mathcal{S}(\boldsymbol{R}^n)$, with $\operatorname{supp}\hat{f}\subset B_d$, verifying (4.4) and let f_α be given by (4.5).

Now, let

$$D_j^0=\Phi(H)-\Phi(H_0)+\sum_{|\alpha|\leqslant j/2}f_\alpha(x)\Phi(H_0)\,. \tag{5.3}$$

Then conclusion (v) holds. In fact, let $\{u_j\}$ be a sequence as in (v). Using $E([a, b])u_j=u_j$ we see that

$$||\{\Phi(H)-\Phi(H_0)\}u_j|| \leqslant ||\Phi(H)-\Phi(H_0)||\,||F_j u_j||+ + ||\{\Phi(H)-\Phi(H_0)\}(1-F_j)E([a, b])u_j|| . \tag{5.4}$$

By the hypothesis of (v), the first term on the right hand side of (5.4) goes to zero as $j\to\infty$. By Lemma 4.3 the second term goes to zero as $j\to\infty$. Hence the left side of (5.4) goes to zero as $j\to\infty$. Since $\Phi(H)u_j=u_j$, this implies that

$$||F_j\Phi(H_0)u_j|| \to 0 \tag{5.5}$$

as $j\to\infty$. In view of (4.7) it follows from (5.5) that

$$||(\sum_{|\alpha|\leqslant j/2} f_\alpha)\Phi(H_0)u_j|| \to 0$$

which proves conclusion (v).

Next, we construct the operators $D_j^\pm$. Noting (5.2) we can find positive numbers r_1, r_2 such that

$$\{v\in \boldsymbol{R}^n\,|\,v=p'(\xi) \text{ for some } \xi\in\Omega\}\subset\{v\in \boldsymbol{R}^n\,|\,r_1<|v|<r_2\} .$$

Choose two functions $G^\pm$ in $C_0^\infty(\boldsymbol{R}^n)$ so that

(1) $G^+(v)+G^-(v)=1$ if $r_1<|v|<r_2$.

(2) $G^\pm(v)=0$ if $r_1<|v|<r_2$ and the angle between v and $(\mp 1, 0, \cdots, 0)$ is smaller than 45°.

Let ω be a function in $C_0^\infty(\boldsymbol{R}^n)$ such that

$$\omega(\xi)=1 \qquad \text{if } \xi\in\Omega .$$

For $\alpha\in \boldsymbol{Z}^n$, let R_α be a rotation taking α to $(|\alpha|, 0, \cdots, 0)$ and let

$$g_\alpha^\pm(\xi)=\omega(\xi)G^\pm(R_\alpha p'(\xi)) .$$

Then $\{g_\alpha^\pm\}$ obey the hypotheses of Lemmas 4.4 and 4.5. We define

$$D_j^\pm=(\sum_{|\alpha|>j/2}\bar{\mathcal{F}} g_\alpha^\pm \mathcal{F} f_\alpha)\Phi(H_0) .$$

By Lemma 4.5, $D_j^\pm$ are bounded operators on $L^2(\boldsymbol{R}^n)$, and

$$||D_j^\pm||\leqslant C\sup_\alpha ||(1-\Delta)^n g_\alpha^\pm||_{L^2} .$$

Thus, conclusion (iii) follows.

We are in a position to prove conclusion (i). Let $u\in L^2(\boldsymbol{R}^n)$. Note that $(\mathcal{F}(\Phi(H_0)u))(\xi)=\Phi(p(\xi))\hat{u}(\xi)$. It follows from (5.1) that $\operatorname{supp}\Phi\circ p$ is contained in K. Since

$$(\mathcal{F}(f_\alpha\Phi(H_0)u))(\xi)=(2\pi)^{-n}(\hat{f}_\alpha * \mathcal{F}(\Phi(H_0)u))(\xi) \tag{5.6}$$

and since $\operatorname{supp}\hat{f}_\alpha\subset B_d$, we have

$$\operatorname{supp} \mathcal{F}(f_\alpha \Phi(H_0)u) \subset \Omega . \tag{5.7}$$

Using (4.6) and the fact that $g_\alpha^+(\xi)+g_\alpha^-(\xi)=1$ on Ω, we see that

$$\begin{aligned}(D_j^+ + D_j^- + D_j^0)u &= \sum_{|\alpha|>j/2} \bar{\mathcal{F}}(g_\alpha^+ + g_\alpha^-)\mathcal{F}(f_\alpha \Phi(H_0)u) + \\ &\quad + \{\Phi(H)-\Phi(H_0)\}u + \sum_{|\alpha|\leqslant j/2} f_\alpha \Phi(H_0)u \\ &= \Phi(H)u .\end{aligned} \tag{5.8}$$

Since $\Phi(H)E([a, b])=E([a, b])$, (5.8) implies conclusion (i).

We shall now prove conclusion (iv) for D_j^-. The proof for D_j^+ is similar. Let $u \in L^2(\boldsymbol{R}^n)$ and put

$$u_\alpha(x) = (\bar{\mathcal{F}} g_\alpha^- \mathcal{F}(f_\alpha \Phi(H_0)u))(x+\alpha) .$$

Then one easily sees that

$$\hat{u}_\alpha(\xi) = (2\pi)^{-n} e^{i\langle \alpha, \xi\rangle} g_\alpha^-(\xi)(\hat{f}_\alpha * ((\Phi\circ p)\hat{u}))(\xi) . \tag{5.9}$$

It follows from (5.9) that $\hat{u}_\alpha \in C_0^\infty(\boldsymbol{R}^n)$, and that $\operatorname{supp} \hat{u}_\alpha \subset \Omega$. Therefore, we have

$$(e^{-itH_0}\bar{\mathcal{F}} g_\alpha^- \mathcal{F}(f_\alpha \Phi(H_0)u))(x) = \int e^{i(\langle x-\alpha, \xi\rangle - tp(\xi))} \hat{u}_\alpha(\xi) d\xi . \tag{5.10}$$

Now we apply the method of stationary phase to the integral in (5.10). Note that each derivative of the function

$$\xi \to (\langle x-\alpha, \xi\rangle - tp(\xi))/(|x-\alpha|+|t|)$$

is bounded on Ω, uniformly in x, α and t. Choose $\delta>0$ so that $2\delta \leqslant r_1/(6(r_1+1))$. Then simple computations show that

$$(x-\alpha-tp'(\xi))/(|x-\alpha|+|t|) \geqslant r_1/(2(r_1+3))$$

when $|\alpha|>j/2$, $|x|\leqslant 2\delta(j-t)$, $t<0$ and $\xi \in \operatorname{supp} \hat{u}_\alpha$. Hence Lemma I.I shows that for every ν

$$\begin{aligned}&|(e^{-itH_0}\bar{\mathcal{F}} g_\alpha^- \mathcal{F}(f_\alpha \Phi(H_0)u))(x)| \\ &\leqslant C(1+|x-\alpha|+|t|)^{-\nu} \sum_{|\mu|\leqslant \nu} \int |D^\mu \hat{u}_\alpha(\xi)| d\xi\end{aligned} \tag{5.11}$$

if $|\alpha|>j/2$, $|x|\leqslant 2\delta(j-t)$ and $t<0$. (See also Lemma 1 in Section 2 of [11] for the method of stationary phase.) Here C is a constant depending only on ν, r_1 and Ω. By Schwarz' inequality

$$\int |D^\mu \hat{u}_\alpha(\xi)| d\xi \leqslant (\operatorname{meas}(\Omega))^{1/2} \Big(\int |D^\mu \hat{u}_\alpha(\xi)|^2 d\xi \Big)^{1/2} .$$

Since, by (5.9),

$$D^{\mu}\hat{u}_{\alpha}=\mathcal{F}((\alpha-x)^{\mu}\bar{\mathcal{F}}g_{\alpha}^{-}\mathcal{F}(f_{\alpha}\Phi(H_0)u)),$$

Parseval's formula and Lemma 4.4 show that

$$\sum_{|\mu|\leqslant\nu}\int|D^{\mu}\hat{u}_{\alpha}(\xi)|\,d\xi\leqslant C\|u\| \tag{5.12}$$

where C is a constant depending only on ν, Ω and G^{-}. We note that

$$|x-\alpha|\geqslant(2/3)|\alpha|-(1/6)|t| \tag{5.13}$$

when $|\alpha|>j/2$, $|x|\leqslant2\delta(j+|t|)$. Thus, combining (5.11), (5.12) and (5.13), it follows that for every ν the left hand side of (5.11) is bounded by

$$C_{\nu,a,b}(1+|\alpha|+|t|)^{-\nu}\|u\|$$

where $C_{\nu,a,b}$ is a constant depending only on a, b and ν. Since

$$\|F_{\delta(j-t)}e^{-itH_0}D_j^{-}u\|\leqslant\sum_{|\alpha|>j/2}\|F_{\delta(j-t)}e^{-itH_0}\bar{\mathcal{F}}g_{\alpha}^{-}\mathcal{F}(f_{\alpha}\Phi(H_0)u)\|,$$

this yields conclusion (iv).

Finally, to prove conclusion (ii) (for D_j^{-}), we set

$$h_{\alpha}^{-}(\xi)=(1+|\xi|^2)^{N/2}g_{\alpha}^{-}(\xi).$$

Then $\{h_{\alpha}^{-}\}$ also obeys the hypotheses of Lemmas 4.4 and 4.5. Moreover, by the closed graph theorem it follows that

$$\Lambda^{N}D_j^{-}=(\sum_{|\alpha|>j/2}\bar{\mathcal{F}}h_{\alpha}^{-}\mathcal{F}f_{\alpha})\Phi(H_0).$$

By Lemma 4.5, we have

$$\|\Lambda^{N}D_j^{-}\|\leqslant C\sup_{\alpha}\|(1-\Delta)^{n}h_{\alpha}^{-}\|_{L^2} \tag{5.14}$$

where C is some constant. Repeating the same arguments used in the proof of conclusion (iv), we see that for every ν

$$\|F_{\delta(j-t)}e^{-itH_0}\Lambda^{N}D_j^{-}\|\leqslant C(1+j-t)^{-\nu},\qquad t<0. \tag{5.15}$$

Here δ is the same constant as before and C is a constant depending only on a, b and ν. Now, let $u\in L^2(\boldsymbol{R}^n)$. Since $D_j^{-}u\in\mathcal{H}_{\mathrm{ac}}(H_0)\cap\mathcal{D}(H_0)$ and since $e^{-itH_0}D_j^{-}u\in\mathcal{D}(H)$, we have

$$\begin{aligned}\|(W_{-}-1)D_j^{-}u\|&\leqslant\int_{-\infty}^{0}\|Ae^{-itH_0}D_j^{-}u\|\,dt \\ &\leqslant\int_{-\infty}^{0}\|A\Lambda^{-N}F_{\delta(j-t)}e^{-itH_0}\Lambda^{N}D_j^{-}u\|dt+\end{aligned} \tag{5.16}$$

$$+\int_{-\infty}^{0}||A\Lambda^{-N}(1-F_{\delta(j-t)})e^{-itH_0}\Lambda^N D_j^- u||dt$$
$$\equiv I_j+\tilde{I}_j\,.$$

By Lemma 2.1 and (5.15), I_j is bounded by

$$C||A\Lambda^{-N}||\int_{-\infty}^{0}(1+j-t)^{-?}dt||u||\,.$$

By Lemma 3.1, $\tilde{I}_j$ is bounded by

$$C||\Lambda^N D_j^-||\int_{-\infty}^{0}(1+\delta(j-t))^{-1-\tilde{\varepsilon}}dt||u||\,.$$

Combining (5.14), (5.16) and these estimates for I_j and $\tilde{I}_j$, conclusion (ii) follows. Q.E.D.

6. Completeness of wave operators

We now turn to the proof of the other conclusions of Theorem 1. The proof depends heavily upon Theorem 2.

Proof of Theorem 1(ii)–(iv). We prove (ii) for W_+. The proof for W_- is similar.

It suffices to show that if $u\in\mathcal{H}_c(H)\ominus\mathcal{R}(W_+)$ then

$$E([a,b])u=0 \tag{6.1}$$

for any interval $[a,b]$ disjoint from $\bar{C}_v$ (here $\mathcal{R}(W_+)$ denotes the range of W_+). For then, it would follow, by hypothesis (P_1), that the range of the function $e(\lambda)=(E(\lambda)u,u)$ is at most countable. Noting that $e(\lambda)$ is continuous on $\boldsymbol{R}$, it would follow that $e(\lambda)$ is a constant function on $\boldsymbol{R}$. Letting $\lambda\to\pm\infty$, we shall find that $||u||=0$, which implies that $\mathcal{R}(W_+)=\mathcal{H}_c(H)$.

Now, to prove (6.1) we note that $E([a,b])u\in\mathcal{H}_c(H)\ominus\mathcal{R}(W_+)$. By Lemma 4.1 and the remark following it, we have

$$\lim_{T\to\infty}T^{-1}\int_0^T||F_je^{-itH}E([a,b])u||^2dt=0$$

for $j=1,2,\cdots$. In particular, we can find $\{t_j\}$ inductively with $t_j\geqslant\max(j,t_{j-1})$ so that $||F_je^{-it_jH}E([a,b])u||\leqslant 1/j$. Applying Theorem 2(i), we decompose $||E([a,b])u||^2$ as a sum of three terms.

$$\alpha_j^+=(e^{-it_jH}E([a,b])u,\ D_j^+e^{-it_jH}E([a,b])u)\,,$$
$$\alpha_j^-=(e^{-it_jH}E([a,b])u,\ D_j^-e^{-it_jH}E([a,b])u)\,,$$
$$\alpha_j^0=(e^{-it_jH}E([a,b])u,\ D_j^0e^{-it_jH}E([a,b])u)\,.$$

Since, by Theorem 2(v), $\|D_j^0 e^{-it_jH}E([a, b])u\| \to 0$ as $j \to \infty$, α_j^0 tends to zero as $j \to \infty$. Using the fact that $e^{-itH}E([a, b])u$ is orthogonal to the range of W_+, we have

$$\begin{aligned}(6.2)\qquad \alpha_j^+ &\leqslant |(e^{-it_jH}E([a, b])u, (1-W_+)D_j^+ e^{-it_jH}E([a, b])u)| \\ &\leqslant \|(1-W_+)D_j^+\| \, \|E([a, b])u\|^2 .\end{aligned}$$

Combining (6.2) and Theorem 2(ii), it follows that $\alpha_j^+ \to 0$ as $j \to \infty$. To show that $\alpha_j^- \to 0$ as $j \to \infty$, we write

$$\begin{aligned}(6.3)\qquad \alpha_j^- \leqslant\ & |(W_-^* E([a, b])u, e^{it_jH_0}D_j^- e^{-it_jH}E([a, b])u)| + \\ &+ |(e^{-it_jH}E([a, b])u, (1-W_-)D_j^- e^{-it_jH}E([a, b])u)| .\end{aligned}$$

By Theorem 2(ii), the second term on the right side of (6.3) goes to zero as $j \to \infty$. The first term is bounded by

$$\begin{aligned}& |(W_-^* E([a, b])u, F_{\delta j} e^{it_jH_0}D_j^- e^{-it_jH}E([a, b]u)| + \\ &+ |((1-F_{\delta j})W_-^* E([a, b])u, e^{it_jH_0}D_j^- e^{-it_jH}E([a, b])u)| \\ &= \beta_j + \gamma_j .\end{aligned}$$

It follows immediately from Theorem 2(iv) that β_j goes to zero as $j \to \infty$. Obviously, it follows (by Theorem 2(iii)) that γ_j goes to zero as $j \to \infty$. Hence we have shown that $E([a, b])u = 0$. This completes the proof of (ii).

Next, we prove the first statement of (iii). The other half of (iii) is proved similarly.

Suppose, to get a contradiction, that there exists an orthonormal family $\{u_j\}$ with $Hu_j = \lambda_j u_j$ and $\lambda_j \to \lambda_\infty \notin \bar{C}_v$. By throwing out finitely many u_j's we can suppose that each $\lambda_j \in [a, b]$, an interval disjoint from $\bar{C}_v$. Moreover, noting the fact that $u_j \to 0$ weakly, using Lemma 4.1, we have that

$$\|F_r u_j\| = |\lambda_j + i| \, \|F_r(H+i)^{-1}u_j\| \to 0 ,$$

so that by passing to a subsequence we can suppose that $s-\lim F_j u_j = 0$. Thus Theorem 2 is applicable and we obtain

$$\begin{aligned}(6.4)\qquad (u_j, u_j) \leqslant\ & \|(1-W_-)D_j^-\| + \|(1-W_+)D_j^+\| + \|D_j^0 u_j\| + \\ &+ |(W_- D_j^- u_j, u_j)| + |(W_+ D_j^+ u_j, u_j)| .\end{aligned}$$

Noting that each u_j is orthogonal to the range of $W_\pm$, it follows from Theorem 2(ii), (v) that $(u_j, u_j) \to 0$ as $j \to \infty$. This contradicts the fact that $(u_j, u_j) = 1$ for $j = 1, 2, \cdots$.

Finally, (iv) follows readily from (i)–(iii). Q.E.D.

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Department of Mathematics
Kyoto University
Kyoto 606
Japan