Suyama, Y.
Osaka J. Math.
19 (1982), 287-309

HOLOMORPHIC ISOMETRIC IMBEDDING INTO $Q_m(C)$

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(Received September 8, 1980)

0. Introduction

In this paper, we shall study the existence and rigidity problems of a holomorphic isometric imbedding of a Kaehler manifold into a complex quadric $\boldsymbol{Q}_m(\boldsymbol{C})$.

A systematic study of the holomorphic isometric imbeddings of Kaehler manifolds with analytic metrics was done by E. Calabi [2]. He considered the so-called diastatic function of a Kaehler manifold and showed that this function plays an important role in study of the holomorphic isometric imbedding [see §1 in this paper]. Especially giving an explicit representation of the diastatic function of a simply connected complete Kaehler manifold with constant holomorphic sectional curvature, he found a necessary and sufficient condition on a Kaehler manifold $\boldsymbol{M}$ in order that a holomorphic isometric imbedding of $\boldsymbol{M}$ into this space exists. And then he proved the rigidity theorem for such an imbedding.

We shall give here an explicit representation of the diastatic function of $\boldsymbol{Q}_m(\boldsymbol{C})$. By making use of this function, we shall make a special coordinate system in $\boldsymbol{Q}_m(\boldsymbol{C})$ around each point [§2]. These representation and coordinate system are the core of this paper [§3 and §4].

The complex quadric $\boldsymbol{Q}_m(\boldsymbol{C})$ is a complex hypersurface in the projective space $\boldsymbol{P}_{m+1}(\boldsymbol{C})$ defined by

$$(z^0)^2+(z^1)^2+\cdots+(z^{m+1})^2 = 0$$

with respect to the homogeneous coordinate system $(z^0, \cdots, z^{m+1})$ of $\boldsymbol{P}_{m+1}(\boldsymbol{C})$. As a Kaehler metric on $\boldsymbol{Q}_m(\boldsymbol{C})$, we take the metric induced from that on $\boldsymbol{P}_{m+1}(\boldsymbol{C})$, which is the Fubini-Study metric with constant holomorphic sectional curvature 4. The complex quadric $\boldsymbol{Q}_m(\boldsymbol{C})$ has the group of holomorphic isometric transformations, which acts on $\boldsymbol{Q}_m(\boldsymbol{C})$ transitively.

We shall show that $\boldsymbol{P}_n(\boldsymbol{C})$ is holomorphically and isometrically imbedded into $\boldsymbol{Q}_l(\boldsymbol{C})$ only for $l \geqq 2n$ [§3, Ex. 3 and Th. 1]. This implies that, for a Kaehler manifold $\boldsymbol{M}$, the existence problem of a holomorphic isometric imbedding of $\boldsymbol{M}$ into $\boldsymbol{Q}_m(\boldsymbol{C})$ is equivalent to that of such an imbedding into $\boldsymbol{P}_n(\boldsymbol{C})$.

Partially supported by Grant-in-Aid for Scientific Research.

The way to imbed $P_n(C)$ into $Q_l(C)$ ($l \geqq 2n$) holomorphically and isometrically is essentially only one [§3, Th. 1]. Although $Q_m(C)$ can be imbedded holomorphically and isometrically into $Q_l(C)$ only for $l \geqq m$, yet the way to imbed $Q_m(C)$ into $Q_l(C)$ ($l \geqq m$) is not always only one. That is, the set $\{$holomorphic isometric imbeddings h of $Q_m(C)$ into $Q_l(C)\}/\approx$ consists of only one element for $m \leqq l < 2(m+1)$, but it corresponds bijectively to the closed interval $[0, 1]$ for $l \geqq 2(m+1)$, where $h \approx h'$ implies that h and h' are essentially equivalent [§4, Th.4 and Ex.6]. This fact is different from the case of the holomorphic isometric imbedding into $P_n(C)$.

We assume that a Kaehler manifold M can be imbedded holomorphically and isometrically into $Q_m(C)$ for some natural number m. Then we shall give a way to find the natural number $m_0 = \mathrm{Mim.}\{m$: there exists a holomorphic isometric imbedding h of M into $Q_m(C)\}$ [§4, Th. 2 and Cor. 2]. Furthermore, we shall give a necessary and sufficient condition on M and a natural number l in order that there exists the following closed domain D around 0 in C^k:

(1) If $D \ni w$, then $\exp(\sqrt{-1}\,\theta)w \in D$ for $\theta \in [0, 2\pi)$.

(2) The set $D/\sim$ corresponds bjiectively to the set $\{$holomorphic isometric imbeddings h of M into $Q_l(C)\}/\approx$, where $w \sim w'$ implies that $w = \exp(\sqrt{-1}\,\theta)w'$ for some $\theta \in [0, 2\pi)$ and $h \approx h'$ implies that h and h' are essentially equivalent [§4, Th. 4].

Under some additional condition, there exists a small domain $D' \subset D$ such that the correspondence $D' \ni x \mapsto h(x) \in \{$holomorphic isometric imbeddings of M into $Q_l(C)\}$ varies continuously with respect to x, where the class $[x]$ corresponds to the class $[h(x)]$ under the correspondence of (2) [§4, Rem. 5, Ex. 6 and Ex. 7].

Before we finish this introduction, we add some remarks. P. Griffiths considered the rigidity problem for non-degenerate holomorphic curves in the complex Grassmannian $G(n, 2n)$ of n-planes in C^{2n} (with the canonical metric g) with interest in variation of Hodge structure. The Grassmannian $G(n, 2n)$ has a group G of holomorphic isometric transformations, which acts on $G(n, 2n)$ transitively. Let f and $\tilde{f}$ be two non-degenerate holomorphic mappings of a Riemann surface M into $G(n, 2n)$. Then, he proved the following fact. If, for each $x \in M$, there exists $h_x \in G$ depending on x such that f and $h_x\tilde{f}$ agree up to order 2 at x (then, $f^*g = \tilde{f}^*g$ on M holds), there exists a fixed $h \in G$ such that $f = h\tilde{f}$ [4, p. 806]. M.L. Green also studied the rigidity problem for a holomorphic isometric mapping into a Kaehler manifold [3]. In his paper, he showed that there exists a case where the way to imbed a Kaehler manifold into an algebraic variety in $P_n(C)$ holomorphically and isometrically is not essentially only one [3, Ex. 6].

The author expresses his hearty thanks to Professor H. Ozeki for his many valuable advices.

1. Calabi's results

In this section, we summarize some of Calabi's results on the holomorphic isometric imbedding [2] for our later use.

(a) diastatic function and canonical coordinate system

Let M be a k-dimensional Kaehler manifold with an analytic metric g. Then, a diastatic function $D_M(p, q)$ is defined on some neighborhood of the diagonal set $\{(p, p); p \in M\}$ of the product space $M \times M$. The function $D_M(p, q)$ has the following properties:

(1) The function $D_M(p, q)$ is uniquely determined by the Kaehler metric g.

(2) The function $D_M(p, q)$ is symmetric in p and q, and $D_M(p, p)=0$.

(3) The function $D_M(p, q)$ is real valued and analytic on the domain.

(4) Let $(z^1, \cdots, z^k)$ be a coordinate system in M and U its coordinate neighborhood. Let $g=\sum_{\alpha,\beta} g_{\alpha\bar{\beta}}(z, \bar{z})dz^\alpha d\bar{z}^\beta$ on U, where $z=(z^1, \cdots, z^k)$ and $\bar{z}=(\bar{z}^1, \cdots, \bar{z}^k)$. Then we have

$$\frac{\partial^2 D_M(p, q)}{\partial z^\alpha(p)\partial \overline{z^\beta(p)}} = g_{\alpha\bar{\beta}}(z(p), \overline{z(p)})$$

and

$$\frac{\partial^2 D_M(p, q)}{\partial z^\alpha(q)\partial \overline{z^\beta(q)}} = g_{\alpha\bar{\beta}}(z(q), \overline{z(q)})\,.$$

DEFINITION. Let p be a point in a Kaehler manifold M with an analytic metric. A coordinate system $(z^1, \cdots, z^k)$ in M around p is called a *canonical coordinate system* if it satisfies:

(1) $z^i(p)=0 \qquad (1 \leqq i \leqq k)$.

(2) For a k-tuple $\boldsymbol{\alpha}=(\alpha_1, \cdots, \alpha_k)$ of non-negative integers, we put $z^{\boldsymbol{\alpha}}=(z^1)^{\alpha_1}\cdots(z^k)^{\alpha_k}$, $\bar{z}^{\boldsymbol{\alpha}}=(\bar{z}^1)^{\alpha_1}\cdots(\bar{z}^k)^{\alpha_k}$ and $|\boldsymbol{\alpha}|=\alpha_1+\cdots+\alpha_k$. Then, the Taylor expansion of the diastatic function $D_M(p, q)$ at $q=p$ has a form

$$D_M(p, q) = \sum_{i=1}^k |z^i|^2(q)+\Phi_{2,2}(z(q), \overline{z(q)})\,,$$

where

$$\Phi_{2,2}(z, \bar{z}) = \sum_{\alpha,\beta} a_{\alpha\beta} z^{\boldsymbol{\alpha}} \bar{z}^{\boldsymbol{\beta}} \qquad (|\boldsymbol{\alpha}| \geqq 2 \text{ and } |\boldsymbol{\beta}| \geqq 2)\,.$$

Proposition A. *Let M be a Kaehler manifold with an analytic metric. Then each point p in M has a canonical coordinate system around its point. Furthermore, the canonical coordinate systems around any point p in M are unique up to a unitary transformation.*

EXAMPLE 1. Let b be a positive real number and $P_n(C, b)$ a complex projective space with constant holomorphic sectional curvature $4b$. Let $(z^0, \cdots, z^n)$ be the homogeneous coordinate system of $P_n(C, b)$ and π the natural projection of $C^{n+1}-\{0\}$ onto $P_n(C, b)$ [6, p. 169]. For the point $\pi(1, 0, \cdots, 0)=p$, a canonical coordinate system $(x^1, \cdots, x^n)$ around p and the diastatic function $D_{P_n(b)}(p, q)$ in the coordinate neighborhood are given by

$$(1.1)\qquad \begin{cases} \sqrt{b}\, x^i = z^i/z^0 \qquad (z^0 \neq 0) \\ D_{P_n(b)}(p, q) = \dfrac{1}{b} \log\left[1 + b \sum_{i=1}^n |x^i|^2(q)\right], \end{cases}$$

respectively. In this case, the diastatic function is analytically extended to the product space $P_n(C, b) \times P_n(C, b)$, and given by

$$D_{P_n(b)}(p, q) = \frac{1}{b} \log\left[\frac{(\sum_{i=0}^n |z^i|^2(p))(\sum_{i=0}^n |z^i|^2(q))}{|\sum_{i=0}^n z^i(p)\overline{z^i(q)}|^2}\right].$$

(b) holomorphic isometric imbedding

Proposition B. *Let N be a Kaehler manifold with an analytic metric and M a complex submanifold in N. Then, the diastatic function $D_N(p, q)$ of N, restricted to pairs of points in M, is the same as the diastatic function $D_M(p, q)$ obtained from the metric induced on M.*

REMARK 1. (1) Let M and N be Kaehler manifolds with analytic metrics, and f a holomorphic mapping of M into N. Then $D_N(f(p), f(q)) = D_M(p, q)$ holds on some neighborhood of the diagonal set $\{(p, p); p \in M\}$, if and only if f is an isometric immersion.

(2) If there exists a holomorphic isometric immersion of a Kaehler manifold M into $P_n(C, b)$, then M has the diastatic function extended analytically to $M \times M$.

In fact, the statement (1) is derived from the property (4) of the diastatic function and Proposition B. The statement (2) is derived from Example 1 and Proposition B.

(c) holomorphic isometric imbedding into $P_n(C, b)$

Theorem C. (1) *Let M be a simply connected (or for ecah point p of M the maximal analytic extension of the diastatic function $D_M(p, q)$ in q is single valued) Kaehler manifold with an analytic metric. Suppose that there exist an connected open set V in M and a holomorphic isometric immersion f of V into $P_n(C, b)$. Then the mapping f is holomorphically and isometrically extended to M.*

(2) *Suppose that a Kaehler manifold M is holomorphically and isometrically immersed into a complex projective space with holomorphic sectional curvature $4b$. Then the dimension n of $P_n(C, b)$, in which the image of M by a holomorphic isometric mapping is full, is determined by the manifold M and its metric g. (Therefore, for a natural number m, which is smaller than above dimension n, no holomorphic isometric immersion of M into $P_m(C, b)$ exists.)*

(3) *Suppose that $\boldsymbol{f}$ and $\boldsymbol{g}$ are two holomorphic isometric immersions of a Kaehler manifold $\boldsymbol{M}$ into $\boldsymbol{P}_n(\boldsymbol{C}, b)$. Then there exists an element A in the group of holomorphic isometric transformations of $\boldsymbol{P}_n(\boldsymbol{C}, b)$ satisfying $\boldsymbol{f}(z)=A\boldsymbol{g}(z)$ for z of $\boldsymbol{M}$.*

(4) *A holomorphic isometric immersion $\boldsymbol{f}$ of a Kaehler manifold $\boldsymbol{M}$ into $\boldsymbol{P}_n(\boldsymbol{C}, b)$ is an imbedding, if and only if*

$$\boldsymbol{D}_M(p, q) = 0 \qquad \textit{only for } p = q\,.$$

2. Good canonical coordinate system and diastatic function of $\boldsymbol{Q}_m(\boldsymbol{C})$

Let π be the natural projection of $\boldsymbol{C}^{m+2}-\{0\}$ onto $\boldsymbol{P}_{m+1}(\boldsymbol{C})$. Then a complex quadric $\boldsymbol{Q}_m(\boldsymbol{C})$ is a complex hypersurface in $\boldsymbol{P}_{m+1}(\boldsymbol{C})$ defined by the equation

$$(2.1) \qquad (z^0)^2+(z^1)^2+\cdots+(z^{m+1})^2=0\,,$$

where $(z^0, \cdots, z^{m+1})$ denotes the homorgeneous coordinate system of $\boldsymbol{P}_{m+1}(\boldsymbol{C})$. As its Kaehler metric we take the metric induced from that on $\boldsymbol{P}_{m+1}(\boldsymbol{C}, 1)$. We denote by ι_1 the imbedding of $\boldsymbol{Q}_m(\boldsymbol{C})$ into $\boldsymbol{P}_{m+1}(\boldsymbol{C}, 1)$ given by the equation (2.1).

There exists a special canonical coordinate system in $\boldsymbol{Q}_m(\boldsymbol{C})$ around each point.

DEFINITION. Let $p\in\boldsymbol{Q}_m(\boldsymbol{C})$. A canonical coordinate system $(y^1, \cdots, y^m)$ in $\boldsymbol{Q}_m(\boldsymbol{C})$ around p is called a *good canonical coordinate system,* if the diastatic function $\boldsymbol{D}_{Q_m}(p, q)$ is given by

$$(2.2) \qquad \boldsymbol{D}_{Q_m}(p, q) = \log\Big[1+\textstyle\sum_{i=1}^m |y^i|^2(q)+\frac{1}{4}\big|\sum_{i=1}^m (y^i)^2(q)\big|^2\Big]$$

with respect to the coordinate system $(y^1, \cdots, y^m)$. A canonical coordinate system $(y^0, y^1, \cdots, y^m)$ in $\boldsymbol{P}_{m+1}(\boldsymbol{C}, 1)$ around $\iota_1(p)$ is called a *canonical coordinate system associated with $\boldsymbol{Q}_m(\boldsymbol{C})$*, if it satisfies the following conditions:

(1) The system $(y^1, \cdots, y^m)$, restricted to $\boldsymbol{Q}_m(\boldsymbol{C})$, becomes a good canonical coordinate system in $\boldsymbol{Q}_m(\boldsymbol{C})$ around p.

(2) The complex quasric $\boldsymbol{Q}_m(\boldsymbol{C})$ is defined by the equation

$$2y^0=\sqrt{-1}\,[(y^1)^2+\cdots+(y^m)^2]$$

in its coordinate neighborhood.

NOTATION 1. In the equation (2.2) we put

$$\textstyle\sum_{i=1}^m (y^i)^2(q) = \langle \boldsymbol{y}, \bar{\boldsymbol{y}}\rangle(q)\,, \quad \sum_{i=1}^m |y^i|^2(q) = |\boldsymbol{y}|^2(q)$$

using hermitian inner product $\langle\cdot, \cdot\rangle$ and norm $|\cdot|$ in $\boldsymbol{C}^m$. Then (2.2) is expressed as

$$(2.2)' \qquad D_{Q_m}(p, q) = \log\left[1 + |\boldsymbol{y}|^2(q) + \frac{1}{4}|\langle \boldsymbol{y}, \bar{\boldsymbol{y}}\rangle|^2\right].$$

Proposition 1. *There exists a canonical coordinate system in $\boldsymbol{P}_{m+1}(\boldsymbol{C}, 1)$ associated with $\boldsymbol{Q}_m(\boldsymbol{C})$ around each point of $\iota_1(\boldsymbol{Q}_m(\boldsymbol{C}))$. Furthermore, the good canonical coordinate systems around any point p in $\boldsymbol{Q}_m(\boldsymbol{C})$ are unique up to a transformation of $\{exp(\sqrt{-1})\theta\} \times \boldsymbol{O}(m)$, where $0 \leqq \theta < 2\pi$ and $\boldsymbol{O}(m)$ denotes the real orthogonal group of degree m.*

Proof. Let $(z^0, \cdots, z^{m+1})$ be the homogeneous coordinate system of $\boldsymbol{P}_{m+1}(\boldsymbol{C}, 1)$. The group $\boldsymbol{O}(m+2)$ acts as $\boldsymbol{z} \to A\boldsymbol{z}$ on $\boldsymbol{Q}_m(\boldsymbol{C})$ transitively, where $A \in \boldsymbol{O}(m+2)$ and $\boldsymbol{z} = {}^t[z^0, \cdots, z^{m+1}]$. And it acts on $\boldsymbol{Q}_m(\boldsymbol{C})$ holomorphically and isometrically. We put $A(\pi(\boldsymbol{z})) = \pi(A(\boldsymbol{z}))$ for $A \in \boldsymbol{O}(m+2)$ and $\boldsymbol{z} \in \boldsymbol{C}^{m+2} - \{0\}$.

(1) Let $p \in \iota_1(\boldsymbol{Q}_m(\boldsymbol{C}))$. Suppose that there exists a canonical coordinate system $(y^0, \cdots, y^m)$ in $\boldsymbol{P}_{m+1}(\boldsymbol{C}, 1)$ associated with $\boldsymbol{Q}_m(\boldsymbol{C})$ around p. For a transformation $A \in \boldsymbol{O}(m+2)$ we put

$$s^i(A(q)) = y^i(q) \qquad (0 \leqq i \leqq m).$$

Then the coordinate system $(s^0, \cdots, s^m)$ around $A(p)$ is a canonical coordinate system associated with $\boldsymbol{Q}_m(\boldsymbol{C})$. In fact, since the diastatic function $D_{Q_m}(p, q)$ is invariant by a holomorphic isometric transformation, we have

$$\begin{aligned} D_{Q_m}(A(p), A(q)) &= D_{Q_m}(p, q) \\ &= \log\left[1 + |\boldsymbol{y}|^2(q) + \frac{1}{4}|\langle \boldsymbol{y}, \bar{\boldsymbol{y}}\rangle|^2(q)\right] \\ &= \log\left[1 + |\boldsymbol{s}|^2(A(q)) + \frac{1}{4}|\langle \boldsymbol{s}, \bar{\boldsymbol{s}}\rangle|^2(A(q))\right] \end{aligned}$$

for $q \in \iota_1(\boldsymbol{Q}_m(\boldsymbol{C}))$. And we have

$$\begin{aligned} 2s^0(A(q)) &= 2y^0(q) = \sqrt{-1}\langle \boldsymbol{y}, \bar{\boldsymbol{y}}\rangle(q) \\ &= \sqrt{-1}\langle \boldsymbol{s}, \bar{\boldsymbol{s}}\rangle(A(q)) \qquad \text{for } q \in \iota_1(\boldsymbol{Q}_m(\boldsymbol{C})). \end{aligned}$$

(2) Let π be the natural projection of $\boldsymbol{C}^{m+2} - \{0\}$ onto $\boldsymbol{P}_{m+1}(\boldsymbol{C}, 1)$. Let Let $\pi(1, \sqrt{-1}, 0, \cdots, 0) = p_0$ and

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}}, & \sqrt{\frac{-1}{2}} & & & \\ \sqrt{\frac{-1}{2}}, & \frac{1}{\sqrt{2}} & & 0 & \\ & & 1 & & \\ & 0 & & \ddots & \\ & & & & 1 \end{pmatrix}$$

a unitary matrix of degree $m+2$. We take a canonical coordinate system $(y^0, \cdots, y^m)$ in $\boldsymbol{P}_{m+1}(\boldsymbol{C}, 1)$ around p_0 defined by

$$(2.3)\qquad y^i(U(q)) = (z^{i+1}/z^0)(q) \qquad (z^0 \neq 0) \quad \text{for } 0 \leqq i \leqq m .$$

Then this is a canonical coordinate system associated with $\boldsymbol{Q}_m(\boldsymbol{C})$. In fact, we have

$$U\boldsymbol{z} = U\begin{pmatrix} z^0 \\ z^1 \\ z^2 \\ \vdots \\ z^{m+1} \end{pmatrix} = \begin{pmatrix} (z^0+\sqrt{-1}z^1)/\sqrt{2} \\ (\sqrt{-1}z^0+z^1)/\sqrt{2} \\ z^2 \\ \vdots \\ z^{m+1} \end{pmatrix}.$$

If $U\boldsymbol{z} \in \iota_1(\boldsymbol{Q}_m(\boldsymbol{C}))$, we have

$$2\sqrt{-1}z^0z^1+(z^2)^2+\cdots+(z^{m+1})^2 = 0 .$$

Therefore we have

$$2y^0 = \sqrt{-1}\langle \boldsymbol{y}, \boldsymbol{y}\rangle \qquad \text{for } (y^0, \cdots, y^m) \in \iota_1(\boldsymbol{Q}_m(\boldsymbol{C})) .$$

(3) Let $(y^1, \cdots, y^m)$ and $(s^1, \cdots, s^m)$ be two good canonical coordinate systems around p of $\boldsymbol{Q}_m(\boldsymbol{C})$. By Proposition A, there exists a unitary matrix B satisfying $B\boldsymbol{y}=\boldsymbol{s}$, where $\boldsymbol{y}={}^t[y^1, \cdots, y^m]$ and $\boldsymbol{s}={}^t[s^1, \cdots, s^m]$. From $|\boldsymbol{y}|^2(q)=|\boldsymbol{s}|^2(q)$ and

$$\log\left[1+|\boldsymbol{y}|^2(q)+\frac{1}{4}|\langle \boldsymbol{y}, \bar{\boldsymbol{y}}\rangle|^2(q)\right]$$
$$= \boldsymbol{D}_{Q_m}(p, q) = \log\left[1+|\boldsymbol{s}|^2(q)+\frac{1}{4}|\langle \boldsymbol{s}, \bar{\boldsymbol{s}}\rangle|^2(q)\right],$$

we have the equality $|\langle \boldsymbol{y}, \bar{\boldsymbol{y}}\rangle|^2(q)=|\langle \boldsymbol{s}, \bar{\boldsymbol{s}}\rangle|^2(q)$. Putting $B=[b^i_j]$ $(1\leqq i, j\leqq m)$, we have

$$\textstyle\sum_{k,l,s,t}(\sum_i b^i_k b^i_l)(\sum_j \overline{b^j_s}\overline{b^j_t})y^k y^l \overline{y^s}\overline{y^t} = \sum_{i,j}(y^i)^2(\overline{y^j})^2 .$$

Therefore, we have $\sum_i b^i_k b^i_l = \exp[2\sqrt{-1}\theta]\delta_{kl}$ ($\delta_{kl}=1$ for $k=l$ and 0 for $k\neq l$) for some constant θ. Since $\exp[-\sqrt{-1}\theta]B$ is also unitary, we have only to show that a unitary transformation B satisfying $\langle B\boldsymbol{y}, \overline{B\boldsymbol{y}}\rangle=\langle \boldsymbol{y}, \bar{\boldsymbol{y}}\rangle$ for any $\boldsymbol{y}$ of $\boldsymbol{C}^m$ is a real orthogonal transformation. But this is easily seen. Q.E.D.

Proposition 2. *Let p be a point of $\boldsymbol{Q}_m(\boldsymbol{C})$. The holomorphic isometric transformation group of $\boldsymbol{Q}_m(\boldsymbol{C})$, which fixes the point p, is given by $\{exp[\sqrt{-1}\theta\}\times \boldsymbol{O}(m)$ with respect to a good canonical coordinate $(y^1, \cdots, y^m)$ in $\boldsymbol{Q}_m(\boldsymbol{C})$ around p, where $0\leqq\theta<2\pi$.*

Proof. Let U be an element of the group stated in Proposition. Let $(y^1, \cdots, y^m)$ be a good canonical coordinate system around p. We have

$$
(2.4)\quad
\begin{aligned}
&|\boldsymbol{y}|^2(q)+\frac{1}{4}|\langle \boldsymbol{y}, \bar{\boldsymbol{y}}\rangle|^2(q) = \exp \boldsymbol{D}_{Q_m}(p, q)-1 \\
&= \exp \boldsymbol{D}_{Q_m}(p, U(q))-1 = |\boldsymbol{y}|^2(U(q))+\frac{1}{4}|\langle \boldsymbol{y}, \bar{\boldsymbol{y}}\rangle|^2(U(q)).
\end{aligned}
$$

Suppose that the Taylor expansion of each $y^i(U(q))$ $(1\leqq i\leqq m)$ at $q=p$ is given by

$$
(2.5)\qquad y^i(U(q)) = \sum_j a^i_j y^j(q)+\sum_{k,l} b^i_{kl} y^k(q) y^l(q)+\Phi_3(y(q)),
$$

where $b^i_{kl}=b^i_{lk}$ and the degree of each term of $\Phi_3(y)$ is at least 3 with respect to the variables (y^i). Looking at the terms of degree 2 in the equation (2.4), it follows that the matrix $[a^i_j]$ $(1\leqq i, j\leqq m)$ is unitary. From the terms of degree 3 in the equation (2.4), we have

$$
\sum_{j,k,l}(\sum_i a^i_j \overline{b^i_{kl}}) y^j \overline{y^k}\,\overline{y^l} = 0 .
$$

Since each vector $\boldsymbol{b}_{kl}=[b^1_{kl}, \cdots, b^m_{kl}]$ is orthogonal to all vectors $\boldsymbol{a}_j=[a^1_j, \cdots, a^m_j]$ $(1\leqq j\leqq m)$, we have $\boldsymbol{b}_{kl}=0$. Furthermore, from the terms of degree 4, we have

$$
\sum_{k,l,s,t}(\sum_i a^i_k a^i_l)(\sum_j \overline{a^j_s}\,\overline{a^j_t}) y^k y^l \overline{y^s}\,\overline{y^t} = |\langle y, \bar{y}\rangle|^2 .
$$

Therefore, we have $[a^i_j]\in\{\exp[\sqrt{-1}\,\theta]\}\times \boldsymbol{O}(m)$. Similarly, we can prove that all terms of degree higher than 2 in the equation (2.5) vanish. Q.E.D.

Proposition 3. *Let $\boldsymbol{M}$ be a simply connected (or for each point p of $\boldsymbol{M}$ the maximal analytic extension of the diastatic function $\boldsymbol{D}_M(p, q)$ in q is single valued) Kaehler manifold with an analytic metric. Suppose that there exist a connected open set V in $\boldsymbol{M}$ and a holomorphic isometric immersion $\boldsymbol{f}$ of V into $\boldsymbol{Q}_m(\boldsymbol{C})$. Then the mapping $\boldsymbol{f}$ is holomorphically and isometrically extended to $\boldsymbol{M}$.*

Proof. Suppose that a holomorphic mapping $\boldsymbol{h}$ of $\boldsymbol{M}$ into $\boldsymbol{P}_{m+1}(\boldsymbol{C})$ satisfies the following condition: There exists an open set $\boldsymbol{U}$ in $\boldsymbol{M}$ and the image $\boldsymbol{h}(\boldsymbol{U})$ is contained in $\boldsymbol{Q}_m(\boldsymbol{C})$. Then, by the theorem of identity, the image $\boldsymbol{h}(\boldsymbol{M})$ of the mapping $\boldsymbol{h}$ is contined in $\boldsymbol{Q}_m(\boldsymbol{C})$. From this fact and (1) of Theorem C, we have Proposition 3. Q.E.D.

The following example is well-known. We show it as an application of diastatic functions of $\boldsymbol{Q}_m(\boldsymbol{C})$ and $\boldsymbol{P}_n(\boldsymbol{C}, b)$.

EXAMPLE 2. The projective space $\boldsymbol{P}_1(\boldsymbol{C}, 1/2)$ is isomorphic to the complex quadric $\boldsymbol{Q}_1(\boldsymbol{C})$ holomorphically and isometrically.

In fact, let p be a point in $\boldsymbol{P}_1(\boldsymbol{C}, 1/2)$ and x a canonical coordinate system around p. The diastatic function of $\boldsymbol{P}_1(\boldsymbol{C}, 1/2)$ is given by

$$\begin{aligned} \boldsymbol{D}_{\boldsymbol{P}_1(1/2)}(p, q) &= 2\log\left[1+\frac{1}{2}|x|^2(q)\right] \\ &= \log\left[1+|x|^2(q)+\frac{1}{4}|x|^4(q)\right] \end{aligned}$$

in the coordinate neighborhood. This implies that the correspondence of the canonical coordinate in $\boldsymbol{P}_1(\boldsymbol{C}, 1/2)$ around p to a good canonical coordinate in $\boldsymbol{Q}_1(\boldsymbol{C})$ around some point is a holomorphic isometric isomorphism between both coordinate neighborhoods. From this fact, Proposition 3 and (4) of Theorem C, we see Example 2.

3. Existence problem of holomorphic isometric imbedding into $\boldsymbol{Q}_m(\boldsymbol{C})$

From now on $\boldsymbol{P}_n(\boldsymbol{C}, 1)$ is denoted simply by $\boldsymbol{P}_n(\boldsymbol{C})$. At first, we give some examples of holomorpic isometric imbeddings into $\boldsymbol{Q}_m(\boldsymbol{C})$. These examples are fundamental for our study.

EXAMPLE 3. Let n and m be natural numbers with $m \geqq 2n$. The projective space $\boldsymbol{P}_n(\boldsymbol{C})$ is imbedded into $\boldsymbol{Q}_m(\boldsymbol{C})$.

In fact, let π: $\boldsymbol{C}^{n+1}-\{0\} \to \boldsymbol{P}_n(\boldsymbol{C})$ and π': $\boldsymbol{C}^{m+2}-\{0\} \to \boldsymbol{P}_{m+1}(C)$ be the natural projections. Put $\pi(1, 0, \cdots, 0)=p$ and $\pi'(1, \sqrt{-1}, 0, \cdots, 0)=p_0$. Let $(x^1, \cdots, x^n)$ be the canonical coordinate system in $\boldsymbol{P}_n(\boldsymbol{C})$ around p given by the equation (1.1) replaced b by 1 and $(y^0, \cdots, y^m)$ the canonical coordinate system in $\boldsymbol{P}_{m+1}(\boldsymbol{C})$ associated with $\boldsymbol{Q}_m(\boldsymbol{C})$ around p_0 given by the equation (2.3). We define a holomorphic mapping $\boldsymbol{g}$ of the coordinate neighborhood in $\boldsymbol{P}_n(\boldsymbol{C})$ into $\boldsymbol{P}_{m+1}(\boldsymbol{C})$ by

$$(3.1) \qquad \boldsymbol{g}(\boldsymbol{x}) = \frac{1}{\sqrt{2}}(0, x^1, \sqrt{-1}x^1, x^2, \sqrt{-1}x^2, \cdots, x^n, \sqrt{-1}x^n, 0, \cdots, 0),$$

where $\boldsymbol{x}=(x^1, \cdots, x^n)$. Then we have $\boldsymbol{g}(x) \in \boldsymbol{Q}_m(\boldsymbol{C})$ and

$$\boldsymbol{D}_{\boldsymbol{Q}_m}(0, \boldsymbol{g}(\boldsymbol{x})) = \log[1+|\boldsymbol{x}|^2] = \boldsymbol{D}_{\boldsymbol{P}_n}(0, \boldsymbol{x}).$$

This implies that $\boldsymbol{g}$ is a holomorphic isometric mapping of the coordinate neighborhood around p into $\boldsymbol{Q}_m(\boldsymbol{C})$. Therefore, $\boldsymbol{g}$ is holomorphically and isometrically extended to the mapping of $\boldsymbol{P}_n(\boldsymbol{C})$ into $\boldsymbol{Q}_m(\boldsymbol{C})$ from Proposition 3. Then this mapping $\boldsymbol{g}$ is an imbedding by (4) of Theorem C.

Let $(z^0, \cdots, z^n)$ and $(w^0, \cdots, w^{m+1})$ be the homogeneous coordinate systems of $\boldsymbol{P}_n(\boldsymbol{C})$ and $\boldsymbol{P}_{m+1}(\boldsymbol{C})$ respectively. Then the global expression of $\boldsymbol{g}$ is given by

$$(3.1)' \quad \boldsymbol{g}\colon (z^0, \cdots, z^n) \mapsto (z^0, \sqrt{-1}z^0, z^1, \sqrt{-1}z^1, \cdots, z^n, \sqrt{-1}z^n, 0, \cdots, 0).$$

EXAMPLE 4. For $l \geqq 2(m+1)$, $\boldsymbol{Q}_m(\boldsymbol{C})$ is holomorphically and isometrically imbedded into $\boldsymbol{Q}_l(\boldsymbol{C})$ by the composition of the natural imbedding ι_1: $\boldsymbol{Q}_m(\boldsymbol{C}) \to \boldsymbol{P}_{m+1}(\boldsymbol{C})$ and the imbedding $\boldsymbol{g}$ of Example 3.

EXAMPLE 5. For $l \geqq m$, $\boldsymbol{Q}_m(\boldsymbol{C})$ is naturally imbedded into $\boldsymbol{Q}_l(\boldsymbol{C})$. The image is characterized by $z^{m+2} = \cdots = z^{l+1} = 0$, where $(z^0, \cdots, z^{l+1})$ is the homogeneous coordinate system of $\boldsymbol{P}_{l+1}(\boldsymbol{C})$. This imbedding is denoted by ι_2.

REMARK 2. Let $l \geqq 2(m+1)$ and consider Examples 4 and 5. Then, there is no holomorphic isometric transformation A of $\boldsymbol{Q}_l(C)$ satisfying $\boldsymbol{g}\iota_1 = A\iota_2$.

In fact, let π: $\boldsymbol{C}^{m+2} - \{0\} \to \boldsymbol{P}_{m+1}(\boldsymbol{C})$ and $\tilde{\pi}$: $\boldsymbol{C}^{l+2} - \{0\} \to \boldsymbol{P}_{l+1}(\boldsymbol{C})$ be the natural projections. Put $\pi(1, 0, \cdots, 0) = p$, $\pi(1, \sqrt{-1}, 0, \cdots, 0) = p_0$ and $\tilde{\pi}(1, \sqrt{-1}, 0, \cdots, 0) = \tilde{p}_0$. Let $(x^1, \cdots, x^{m+1})$ be the canonical coordinate system in $\boldsymbol{P}_{m+1}(\boldsymbol{C})$ around p given by the equation (1.1) and $(y^0, \cdots, y^m)$ the canonical coordinate system in $\boldsymbol{P}_{m+1}(\boldsymbol{C})$ around p_0 associated with $\boldsymbol{Q}_m(\boldsymbol{C})$ given by the equation (2.3). Then, there exists a holomorphic isometric transformation T of $\boldsymbol{P}_{m+1}(\boldsymbol{C})$ such that $T(p_0) = p$ and $T((y^0, \cdots, y^m)) = (x^1, \cdots, x^{m+1})$, by (2) in Proof of Proposition 1. Thus we have $\boldsymbol{g}T\iota_1(p_0) = \tilde{p}_0 = \iota_2(p_0)$. So, we take the canonical coordinate system $(w^0, \cdots, w^l)$ in $\boldsymbol{P}_{l+1}(\boldsymbol{C})$ around $\tilde{p}_0$ associated with $\boldsymbol{Q}_l(\boldsymbol{C})$ defined by the equation (2.3). Then, two mappings $\boldsymbol{g}T\iota_1$ and ι_2: $\boldsymbol{Q}_m(\boldsymbol{C}) \to \boldsymbol{P}_{l+1}(\boldsymbol{C})$ are locally given by

$$\begin{aligned} \boldsymbol{g}T\iota_1 \colon (y^1, \cdots, y^m) \mapsto \frac{1}{\sqrt{2}}\Big(0, \frac{\sqrt{-1}}{2}[(y^1)^2 + \cdots + (y^m)^2], \\ \frac{-1}{2}[(y^1)^2 + \cdots + (y^m)^2], y^1, \sqrt{-1}y^1, \cdots, y^m, \sqrt{-1}y^m, 0, \cdots, 0\Big) \end{aligned}$$

and

$$\iota_2 \colon (y^1, \cdots, y^m) \mapsto \Big(\frac{\sqrt{-1}}{2}[(y^1)^2 + \cdots + (y^m)^2], y^1, \cdots, y^m, 0, \cdots, 0\Big)$$

respectively. By Proposition 2, there is no holomorphic isometric transformation A of $\boldsymbol{Q}_l(\boldsymbol{C})$ such that $\boldsymbol{g}T\iota_1 = A\iota_2$. Furthermore, there is a holomophic isometric transformation $\tilde{T}$ of $\boldsymbol{Q}_l(\boldsymbol{C})$ such that $\tilde{T}\boldsymbol{g} = \boldsymbol{g}T$ [c.f. the following Theorem 1]. This shows Remark 2.

Theorem 1. (*a*) *In case $m < 2n$, there is no holomorphic isometric immersion of $\boldsymbol{P}_n(\boldsymbol{C})$ into $\boldsymbol{Q}_m(\boldsymbol{C})$.*

(*b*) *In case $m \geqq 2n$, for any holomorphic isometric immersion $\boldsymbol{f}$ of $\boldsymbol{P}_n(\boldsymbol{C})$ into $\boldsymbol{Q}_m(\boldsymbol{C})$ there is an element A in the group of holomorphic isometric transformations of $\boldsymbol{Q}_m(\boldsymbol{C})$ satisfying $A\boldsymbol{f} = \boldsymbol{g}$ on $\boldsymbol{P}_n(\boldsymbol{C})$, where $\boldsymbol{g}$ is the mapping given in Example 3. Especially, $\boldsymbol{f}$ is an imbedding.*

Proof. We use the same notations as in Proof of Example 3. Let $\boldsymbol{f}$ be a holomorphic isometric immersion of $\boldsymbol{P}_n(\boldsymbol{C})$ into $\boldsymbol{Q}_m(\boldsymbol{C})$. We can assume that $\boldsymbol{f}(p)=p_0$, since the group of holomorphic isometric transformations of $\boldsymbol{Q}_m(\boldsymbol{C})$ acts on $\boldsymbol{Q}_m(\boldsymbol{C})$ transitively. Suppose that $\boldsymbol{f}$ is locally given by

$$\boldsymbol{f}(\boldsymbol{x}) = (f^0(\boldsymbol{x}), \boldsymbol{f}_1(\boldsymbol{x})) = (f^0(\boldsymbol{x}), f^1(\boldsymbol{x}), \cdots, f^m(\boldsymbol{x}))\,.$$

(1) We shall show that $m\geqq 2n$ and each $f^i(\boldsymbol{x})$ $(1\leqq i\leqq m)$ consists of only terms of degree 1. Let the Taylor expansion of $f^i(1\leqq i\leqq m)$ at $x=0$ be given by

$$f^i(\boldsymbol{x}) = \textstyle\sum_j a^i_j x^j + \sum_{k,l} a^i_{kl} x^k x^l + \Phi_3(\boldsymbol{x})\,, \tag{3.2}$$

where $a^i_{kl}=a^i_{lk}$ and the degree of each term of $\Phi_3(\boldsymbol{x})$ is at least 3 with respect to the variables (x^i). Since $\boldsymbol{f}$ is holomorphic and isometric, we have

$$\begin{aligned} |\boldsymbol{x}|^2 &= \exp \boldsymbol{D}_{\boldsymbol{P}_n}(0, \boldsymbol{x})-1 = \exp \boldsymbol{D}_{\boldsymbol{Q}_m}(\boldsymbol{f}(0), \boldsymbol{f}(\boldsymbol{x}))-1 \\ &= |\boldsymbol{f}_1(\boldsymbol{x})|^2+\frac{1}{4}|\langle \boldsymbol{f}_1(\boldsymbol{x}), \overline{\boldsymbol{f}_1(\boldsymbol{x})}\rangle|^2. \end{aligned} \tag{3.3}$$

Let $\boldsymbol{a}_i=[a^1_i, \cdots, a^m_i]$ $(1\leqq i\leqq n)$ and $\boldsymbol{a}_{kl}=[a^1_{kl}, \cdots, a^m_{kl}]$ $(1\leqq k, l\leqq n)$. Then we have

$$|\boldsymbol{f}_1(\boldsymbol{x})|^2 = \textstyle\sum_{k,l}\langle \boldsymbol{a}_k, \boldsymbol{a}_l\rangle x^k \bar{x}^l + \sum_{k\,l,s,t}\langle \boldsymbol{a}_{kl}, \boldsymbol{a}_{st}\rangle x^k x^l \bar{x}^s \bar{x}^t + \Phi_5(\boldsymbol{x})\,, \tag{3.4}$$

where the degree of each term of $\Phi_5(\boldsymbol{x})$ is at least 5 with respect to the variables $(x^i, \bar{x}^i)$. And we have

$$|\langle \boldsymbol{f}_1(\boldsymbol{x}), \overline{\boldsymbol{f}_1(\boldsymbol{x})}\rangle|^2 = \textstyle\sum_{k,l,s,t}\langle \boldsymbol{a}_k, \bar{\boldsymbol{a}}_l\rangle\langle \bar{\boldsymbol{a}}_s, \boldsymbol{a}_t\rangle x^k x^l \bar{x}^s \bar{x}^t + \Phi'_5(\boldsymbol{x})\,, \tag{3.5}$$

where the degree of each term of $\Phi'_5(\boldsymbol{x})$ is at least 5 with respect to the variables $(x^i, \bar{x}^i)$. The equations (3.3), (3.4) and (3.5) imply that

$$\langle \boldsymbol{a}_i, \boldsymbol{a}_j\rangle = \delta_{ij} \qquad (1\leqq i, j\leqq n) \tag{3.6}$$

and

$$\langle \boldsymbol{a}_{ij}, \boldsymbol{a}_{kl}\rangle+\frac{1}{4}\langle \boldsymbol{a}_i, \bar{\boldsymbol{a}}_j\rangle\langle \bar{\boldsymbol{a}}_k, \boldsymbol{a}_l\rangle = 0 \qquad (1\leqq i, j, k, l\leqq n)\,. \tag{3.7}$$

In the equation (3.7), putting $i=k$ and $j=l$, we have

$$a^k_{ij} = 0 \qquad (1\leqq k\leqq m,\ 1\leqq i, j\leqq n) \tag{3.8}$$

and

$$\langle \boldsymbol{a}_i, \bar{\boldsymbol{a}}_j\rangle = 0 \qquad (1\leqq i, j\leqq n)\,. \tag{3.9}$$

The equation (3.8) implies that each function $f^i(\boldsymbol{x})$ $(1\leqq i\leqq m)$ does not contain terms of degree 2. And the equations (3.6) and (3.9) imply that the system $\{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n, \bar{\boldsymbol{a}}_1, \cdots, \bar{\boldsymbol{a}}_n\}$ of vectors is an orthonormal system of $\boldsymbol{C}^m$. This shows $m\geqq 2n$.

Since each function $f^i(x)$ $(1 \leqq i \leqq m)$ does not contain terms of degree 2 and $\langle a_i, \bar{a}_j \rangle = 0$, the right hand side of the equation (3.5) consists of terms of degree at least 8 with respect to the variables $(x^i, \bar{x}^i)$. Therefore, from the equation (3.3), the right hand side of the equation (3.4) does not contain terms of degree 6. Therefore, each function $f^i(x)$ $(1 \leqq i \leqq m)$ does not contain terms of degree 3 in the same way as we get the equations (3.8) and (3.9) from the equation (3.7). Similarly, we can prove that each function $f^i(x)$ $(1 \leqq i \leqq m)$ consists of only terms of degree 1. Then the equation (3.9) implies $f^0(x) = 0$ from $f(x) \in Q_m(C)$.

(2) Put $a_i = (b_i + \sqrt{-1} c_i)/\sqrt{2}$ $(1 \leqq i \leqq n)$, where b_i and c_i are real vectors of R^m. From the equations (3.6) and (3.9) the system $\{b_1, \cdots, b_n, c_1, \cdots, c_n\}$ of real vectors is an orthonormal system of R^m. And we have, by the equation (3.2),

$$f_1(x) = \frac{1}{\sqrt{2}} [b_1 + \sqrt{-1}\, c_1, \cdots, b_n + \sqrt{-1}\, c_n] x$$

where $x = {}^t[x^1, \cdots, x^n]$. By extending $b_1, c_1, \cdots, b_n, c_n$ to an ordered orthonormal basis $b_1, c_1, \cdots, b_n, c_n, d_{2n+1}, \cdots, d_m$ in R^m, we have $A \in O(m)$ with these column vectors. Then we have

$$f_1(x) = \frac{1}{\sqrt{2}} A \begin{pmatrix} \frac{1}{\sqrt{-1}} & & & \\ & \frac{1}{\sqrt{-1}} & & 0 \\ 0 & & \ddots & \\ & & & \frac{1}{\sqrt{-1}} \\ & 0 & & \end{pmatrix} x = A g(x) .$$

Thus, we have Theorem 4 by the theorem of identity. Q.E.D.

Corollary 1. *Let M be a Kaehler manifold. Then, the existence problem of a holomorphic isometric immersion (resp. imbedding) of M into $Q_m(C)$ is equivalent to that of such an immersion (resp. imbedding) of M into $P_n(C)$.*

REMARK 3. (1) Let M be a Kaehler manifold. E. Calabi gave a necessary and sufficient condition in order that M is holomorphically and isometrically immersed into $P_n(C)$. This condition is stated in terms of the diastatic function of M [2].

(2) In the last section, we shall show a result which contains Theorem 1 [c.f. Cor. of Th. 3].

4. Rigidity problem of holomorphic isometric imbedding into $Q_m(C)$

We adopt the following:

NOTATION 2. Let $(z^0, \cdots, z^n)$ be the homogeneous coordinate system of $P_n(C)$ and π the natural projection of $C^{n+1}-\{0\}$ onto $P_n(C)$. Put $\pi(1, \sqrt{-1}, 0, \cdots, 0)=p_0$. A quadric in $P_n(C)$ through the point p_0 is a hypersurface (we admit a case where it has singular set) in $P_n(C)$ defined by the equation

$$(4.1) \qquad \sum_{i,j=0}^{n} c_{ij} z^i z^j = 0\,,$$

and then the coefficients satisfy $(c_{00}, \cdots, c_{nn}) \neq (0, \cdots, 0)$, $c_{ij}=c_{ji}$ and $c_{00}-c_{11}+2\sqrt{-1}\,c_{01}=0$. For the quadric (4.1), we put

$$(4.2) \qquad \begin{cases} b_{i-1,j-1} = c_{ij} \qquad (2 \leqq i, j \leqq n)\,, \ b_{00} = 2\sqrt{-1}\,c_{01} \\ b_{0j-1} = [-c_{0j}+\sqrt{-1}c_{1j}]/\sqrt{2} \qquad (2 \leqq j \leqq n) \\ a_0 = [c_{00}+c_{11}]/2\,, \quad a_{j-1} = [c_{1j}-\sqrt{-1}\,c_{0j}]/\sqrt{2} \qquad (2 \leqq j \leqq n)\,. \end{cases}$$

Taking the canonical coordinate system $(y^0, \cdots, y^{n-1})$ in $P_n(C)$ around p_0 associated with $Q_{n-1}(C)$ given by the equation (2.3), the quadric (4.1) is represented as

$$(4.3) \qquad 2\sum_{i=0}^{n-1} a_i y^i = \sqrt{-1} \sum_{i,j=0}^{n-1} b_{ij} y^i y^j$$

in the coordinate neighborhood, using constants (a_i, b_{ij}) defined by the equation (4.2).

For the quadric (4.1) and each $x \in C$, we define a real symmetric matrix $A(x)=[a_{ij}(x)]$ $(0 \leqq i, j \leqq 2n-1)$ of degree $2n$ with the following components:

$$(4.4) \qquad \begin{cases} 2a_{2i,2j}(x) = \delta_{ij} + \text{Real part}(xb_{ij} - |x|^2 a_i \bar{a}_j) \\ 2a_{2i+1,2j+1}(x) = \delta_{ij} - \text{Real part}(xb_{ij} + |x|^2 a_i \bar{a}_j) \qquad (0 \leqq i, j \leqq n-1) \\ 2a_{2i+1,2j}(x) = \text{Imagenary part}(xb_{ij} - |x|^2 a_i \bar{a}_j)\,. \end{cases}$$

In the above notation, the matrix $A(x)$ is positive definite at $x=0$. Since $A(x)$ is continuous with respect to $x \in C$, each eigenvalue of $A(x)$ is continuous with respect to $x \in C$ [5, p. 107, Th. 5.1]. Therefore, there exists an open set D at $x=0$ in C such that $A(x)$ is positive definite for $x \in D$.

In the following, we shall state our several results together. These results are related each other and their proofs will be given later.

Theorem 2. *We use same notations as Notation 2. Let M be a Kaehler manifold. Suppose that there exists a holomorphic isometric full immersion (resp. imbedding) f of M into $P_n(C)$ through the point p_0. Then, the following two conditions (a) and (b) are equivalent for fixed natural number $(n-1 \leqq)m < 2n$.*

(*a*) *There exists a holomorphic isometric immersion* (*resp. imbedding*) $\boldsymbol{h}$ *of* $\boldsymbol{M}$ *into* $\boldsymbol{Q}_m(\boldsymbol{C})$.

(*b*) *The image* $\boldsymbol{f}(\boldsymbol{M})$ *is contained in a quadric* $\sum_{i,j=0}^{n} c_{ij} z^i z^j = 0$ *in* $\boldsymbol{P}_n(\boldsymbol{C})$ *through the point* p_0 *satisfying thefolllowing condition*: *there exists a complex number* x_0 *such that the matrix* $A(x_0)$ *defined by the quadric is positive semi-definite of rank at most* m.

REMARK 4. If there exists a holomorphic isometric immersion $\boldsymbol{f}$ of $\boldsymbol{M}$ into $\boldsymbol{P}_n(\boldsymbol{C})$, we can get a holomorphic isometric immersion $\tilde{\boldsymbol{f}}$ of $\boldsymbol{M}$ into $\boldsymbol{P}_n(\boldsymbol{C})$ through the point p_0. In fact, the group of holomorphic isometric transformations of $\boldsymbol{P}_n(\boldsymbol{C})$ acts on $\boldsymbol{P}_n(\boldsymbol{C})$ transitively.

Theorem 3. *Let* $\boldsymbol{M}$ *and* $\boldsymbol{f}$ *be the same as in Theorem* 2. *Suppose that the image* $\boldsymbol{f}(\boldsymbol{M})$ *is contained in a quadric in* $\boldsymbol{P}_n(\boldsymbol{C})$ *through the point* p_0. *Then, we have a positive real number* r *satisfying the following* (*a*) *and* (*b*):

(*a*) *There exists a holomorphic isometric immersion* (*resp. imbedding*) $\boldsymbol{h}(y)$ *of* $\boldsymbol{M}$ *into* $\boldsymbol{Q}_l(\boldsymbol{C})$ $(l \geqq 2n)$ *for* $y \in [0, r]$. *And each mapping* $\boldsymbol{h}(y)$ *for* $y \in [0, r]$ *is essentially different from each other.*

(*b*) *There exists a natural number* m $(<2n)$ *such that* $\boldsymbol{h}(r)$ *determined by* (*a*) *is a holomorphic isometric immersion* (*resp. imbedding*) *of* $\boldsymbol{M}$ *into* $\boldsymbol{Q}_m(\boldsymbol{C})$.

Corollary 2. *Let* $\boldsymbol{M}$ *be a Kaehler manifold. Suppose that there exists a holomorphic isometric full immersion* (*resp. imbedding*) $\boldsymbol{f}$ *of* $\boldsymbol{M}$ *into* $\boldsymbol{P}_n(\boldsymbol{C})$ *such that the image* $\boldsymbol{f}(\boldsymbol{M})$ *is not contained in any quadric in* $\boldsymbol{P}_n(\boldsymbol{C})$. *Then, there exists a holomorphic isometric immersion* (*resp. imbedding*) $\boldsymbol{h}$ *of* $\boldsymbol{M}$ *into* $\boldsymbol{Q}_l(\boldsymbol{C})$ *if and only if* $l \geqq 2n$. *Furthermore, such an immersion* (*resp. imbedding*) *is essentially only one.*

Theorem 4. *We use the same notations as in Notation* 2. *Let* $\boldsymbol{M}$ *be a Kaehler manifold and* $\boldsymbol{f}$ *a holomorphic isometric full immersion* (*resp. imbedding*) *of* $\boldsymbol{M}$ *into* $\boldsymbol{P}_n(\boldsymbol{C})$ *through the point* p_0. *Then, the following conditions* (*a*) *and* (*b*) *are equivalent:*

(*a*) *For a quadric* $\sum_{i,j=0}^{n} c_{ij} z^i z^j = 0$ *in* $\boldsymbol{P}_n(\boldsymbol{C})$, *put*

$$\begin{cases} \mathfrak{A}[c_{ij}] = \{\boldsymbol{z} \in \boldsymbol{P}_n(\boldsymbol{C}) \colon \sum_{i,j=0}^{n} c_{ij} z^i z^j = 0\}\,, \\ c'_{ij} = 2c_{ij} \qquad (0 \leqq i < j \leqq n) \text{ and } c'_{ii} = c_{ii} \ (0 \leqq i \leqq n)\,. \end{cases}$$

Then, the dimension of the linear space $\{(c'_{ij})_{0 \leqq i \leqq j \leqq n} \in \boldsymbol{C}^{(n+1)(n+2)/2} \colon \boldsymbol{f}(\boldsymbol{M}) \subset \mathfrak{A}[c_{ij}]\} \cup \{0\}$ *is* k.

(*b*) *There exists a closed domain* $\boldsymbol{D}$ *around* 0 *in* $\boldsymbol{C}^k$ *such that, if* $\boldsymbol{D} \ni \boldsymbol{w}$, *then* $exp(\sqrt{-1}\,\theta)\boldsymbol{w} \in \boldsymbol{D}$ *for* $\theta \in [0, 2\pi)$. *And, for* $l \geqq 2n$, *the set* $\boldsymbol{D}/\!\sim$ *corresponds bijectively to the set* {*holomorphic isometric immersions* (*resp. imbeddings*) $\boldsymbol{h}$ *of* $\boldsymbol{M}$ *into* $\boldsymbol{Q}_l(\boldsymbol{C})$}$/\!\approx$, *where* $\boldsymbol{w} \sim \boldsymbol{w}'$ *implies* $\boldsymbol{w}' = exp(\sqrt{-1}\,\theta)\boldsymbol{w}$ *for some* $\theta \in [0, 2\pi)$ *and* $\boldsymbol{h} \approx \boldsymbol{h}'$ *implies that* $\boldsymbol{h}$ *and* $\boldsymbol{h}'$ *are essentially equivalent.*

Proof of Theorem 2. Let $(z^0, \cdots, z^{m+1})$ be the homogeneous coordinate system of $\boldsymbol{P}_{m+1}(\boldsymbol{C})$. We consider as $\boldsymbol{P}_n(\boldsymbol{C})\subset\boldsymbol{P}_{m+1}(\boldsymbol{C})$ (i.e., $\boldsymbol{P}_n(\boldsymbol{C})$ is identified with the submanifold of $\boldsymbol{P}_{m+1}(\boldsymbol{C})$ defined by $z^{n+1}=\cdots=z^{m+1}=0$). Therefore, we also denote by π the natural projection of $\boldsymbol{C}^{m+2}-\{0\}$ onto $\boldsymbol{P}_{m+1}(\boldsymbol{C})$ and $\pi(1, \sqrt{-1}, 0, \cdots, 0)=p_0$. Then we take the canonical coordinate system $(y^0, \cdots, y^m)$ in $\boldsymbol{P}_{m+1}(\boldsymbol{C})$ around p_0 associated with $\boldsymbol{Q}_m(\boldsymbol{C})$ given by the equation (2.3).

(1) Suppose that there exists a holomorphic isometric mapping $\boldsymbol{h}$ of $\boldsymbol{M}$ into $\boldsymbol{Q}_m(\boldsymbol{C})$ $(n-1\leqq m<2n)$. Let $\boldsymbol{h}(z_0)=\boldsymbol{f}(z_0)=p_0$ for some point $z_0\in\boldsymbol{M}$. Let $\boldsymbol{f}$ and $\boldsymbol{h}$ be locally defined by

$$(4.5)\qquad \begin{cases} \boldsymbol{f}(z)=(f^0(z), \cdots, f^{n-1}(z), 0, \cdots, 0) \\ \boldsymbol{h}(z)=(h^0(z), h^1(z), \cdots, h^m(z)) \end{cases} \qquad \text{for } z\in \boldsymbol{V},$$

respectively, where $\boldsymbol{V}$ is some open set around z_0.

From (3) of Theorem C, we can take a unitary matrix $C=[c_j^i]$ $(0\leqq i, j\leqq m)$ of degree $m+1$ satisfying $\boldsymbol{h}(z)=C\boldsymbol{f}(z)$ for $z\in \boldsymbol{V}$. Put $C=[\boldsymbol{c}_0, \cdots, \boldsymbol{c}_m]$, where each $\boldsymbol{c}_i$ is a column vector. Then $\boldsymbol{h}(z)\in\boldsymbol{Q}_m(\boldsymbol{C})$ for $z\in \boldsymbol{V}$, if and only if

$$(4.6)\quad 2\sum_{i=0}^{n-1} c_i^0 f^i(z)=\sqrt{-1}\sum_{i,j=0}^{n-1}\{\langle \boldsymbol{c}_i, \bar{\boldsymbol{c}}_j\rangle - c_i^0 c_j^0\} f^i(z) f^j(z) \qquad \text{for } \boldsymbol{V}\in z\,.$$

Put

$$\begin{cases} \langle \boldsymbol{c}_i, \bar{\boldsymbol{c}}_j\rangle - c_i^0 c_j^0 = b_{ij} & (0\leqq i, j\leqq n-1) \\ c_i^0 = a_i & (0\leqq i\leqq n-1)\,. \end{cases}$$

The equation (4.6) implies that $\boldsymbol{f}(\boldsymbol{V})$ is contained in a quadric $2\sum_{i=0}^{n-1} a_i y^i = \sqrt{-1}\sum_{i,j=0}^{n-1} b_{ij} y^i y^j$. Putting $\boldsymbol{c}_i=\boldsymbol{c}_i'+a_i\boldsymbol{e}^0$, where $\boldsymbol{e}^0={}^t[1, 0, \cdots, 0]$, we have

$$(4.7)\qquad \begin{cases} \langle \boldsymbol{c}_i', \bar{\boldsymbol{c}}_j'\rangle = b_{ij} \\ \langle \boldsymbol{c}_i', \boldsymbol{c}_j'\rangle = \delta_{ij} - a_i\bar{a}_j\,. \end{cases} \qquad (0\leqq i, j\leqq n-1)$$

Therefore, putting $\boldsymbol{c}_j'=\boldsymbol{d}_{2j}+\sqrt{-1}\,\boldsymbol{d}_{2j+1}$ $(0\leqq j\leqq n-1)$, where each $\boldsymbol{d}_j$ is a real vector in $\boldsymbol{R}^m$, the equation (4.7) implies

$$\begin{aligned} 2\langle \boldsymbol{d}_{2i}, \boldsymbol{d}_{2j}\rangle &= \delta_{ij}+\text{Real part}(b_{ij}-a_i\bar{a}_j) \\ 2\langle \boldsymbol{d}_{2i+1}, \boldsymbol{d}_{2j+1}\rangle &= \delta_{ij}-\text{Real part}(b_{ij}+a_i\bar{a}_j) \\ 2\langle \boldsymbol{d}_{2i+1}, \boldsymbol{d}_{2j}\rangle &= \text{Imagenary part}(b_{ij}-a_i\bar{a}_j)\,. \end{aligned}$$

Since this shows

$$A(1)=\begin{pmatrix} {}^t\boldsymbol{d}_0 \\ {}^t\boldsymbol{d}_1 \\ \vdots \\ {}^t\boldsymbol{d}_{2n-1} \end{pmatrix} [\boldsymbol{d}_0, \boldsymbol{d}_1, \cdots, \boldsymbol{d}_{2n-1}]\,,$$

the matrix $A(1)$ is positive semi-definite of rank at most m.

(2) Suppose that the mapping $\boldsymbol{f}$ of $\boldsymbol{M}$ into $\boldsymbol{P}_n(\boldsymbol{C})$ satisfies (b) of this theorem. Therefore, from the equation (4.3), we have

$$2\sum_{i=0}^{n-1} a_i f^i(z) = \sqrt{-1}\sum_{i,j=0}^{n-1} b_{ij} f^i(z) f^j(z) \qquad \text{for } z \in \boldsymbol{V}$$

using the first equation of (4.5) and, for the quadric $2\sum_{i=0}^{n-1} a_i y^i = \sqrt{-1}\sum_{i,j=0}^{n-1} b_{ij} y^i y^j$, there exists a complex number x_0 such that $A(x_0)$ is positive semi-definit of rank at most m. Therefore, there exists an orthogonal matrix $E \in \boldsymbol{O}(2n)$ satisfying

$$(4.8) \qquad EA(x_0)^t E = \begin{pmatrix} \lambda_1 & & & & & \\ & \lambda_2 & & & 0 & \\ & & \ddots & & & \\ & 0 & & \lambda_m & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{pmatrix} \qquad (\lambda_i \geqq 0).$$

We define a matrix D of degree $2n$ by

$$(4.9) \qquad D = \begin{pmatrix} \sqrt{\lambda_1} & & & & & \\ & \ddots & & 0 & & \\ & & \ddots & & & \\ 0 & & & \sqrt{\lambda_m} & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{pmatrix} E = \begin{pmatrix} \boldsymbol{d}_0, \boldsymbol{d}_1, \cdots, \boldsymbol{d}_{2n-1} \\ \\ 0 \\ \\ \end{pmatrix},$$

where each $\boldsymbol{d}_i$ $(0 \leqq i \leqq 2n-1)$ is a real vector of $\boldsymbol{R}^m$. The equations (4.8) and (4.9) imply

$$A(x_0) = {}^t DD, \text{ i.e., } a_{ij}(x_0) = \langle \boldsymbol{d}_i, \boldsymbol{d}_j \rangle \qquad (0 \leqq i, j \leqq 2n-1).$$

Therefore, putting $\boldsymbol{e}^0 = {}^t[1, 0, \cdots, 0]$, $\boldsymbol{d}_i = {}^t[0, {}^t\boldsymbol{d}_i]$ $(0 \leqq i \leqq 2n-1)$ and $\boldsymbol{c}_i = \boldsymbol{d}_{2i} + \sqrt{-1}\,\boldsymbol{d}_{2i+1} + x_0 a_i \boldsymbol{e}^0$ $(0 \leqq i \leqq n-1)$, we have

$$(4.10) \qquad \begin{cases} \langle \boldsymbol{c}_i, \boldsymbol{c}_j \rangle = \delta_{ij} \\ \langle \boldsymbol{c}_i, \bar{\boldsymbol{c}}_j \rangle = x_0 b_{ij} + x_0^2 a_i a_j . \end{cases} \qquad (0 \leqq i, j \leqq n-1)$$

This shows

$$2\sum_{i=0}^{n-1} x_0 a_i f^i(z) = \sqrt{-1}\sum_{i,j=0}^{n-1} \{\langle \boldsymbol{c}_i, \bar{\boldsymbol{c}}_j \rangle - x_0^2 a_i a_j\} f^i(z) f^j(z)$$

for $z \in \boldsymbol{V}$. Therefore, extending vectors $\boldsymbol{c}_0, \cdots, \boldsymbol{c}_{n-1}$ to an ordered orthonormal basis $\boldsymbol{c}_0, \cdots, \boldsymbol{c}_m$ of $\boldsymbol{C}^{m+1}$, we take the unitary matrix $C = [\boldsymbol{c}_0, \cdots, \boldsymbol{c}_m]$ with these column vectors. Put $\boldsymbol{h}(z)=C\boldsymbol{f}(z)$ for $z\in \boldsymbol{V}$. Then $\boldsymbol{h}(z)$ is a holomorphic isometric mapping of $\boldsymbol{V}$ into $\boldsymbol{Q}_m(\boldsymbol{C})$. And it satisfies $\boldsymbol{h}(z_0)=p_0$.

(3) Proposition 3 still holds good although we replace $\boldsymbol{Q}_m(\boldsymbol{C})$ by any quadric in $\boldsymbol{P}_n(\boldsymbol{C})$. Therefore, the above statements (1) and (2) imply Theorem 2 from Proposition 3. Q.E.D.

Proof of Theorem 3. (1) Let $(z^0, \cdots, z^n)$ be the homogeneous coordinate system of $\boldsymbol{P}_n(\boldsymbol{C})$. Suppose, for a quadric

$$(4.11)\qquad \textstyle\sum_{i,j=0}^{n} c_{ij}z_i z^j = 0 \qquad (c_{ij}=c_{ji} \text{ and } c_{00}-c_{11}+2\sqrt{-1}\,c_{01}=0)\,,$$

$A(w)$ is positive definite for some $w\in\boldsymbol{C}$. Then the manifold $\boldsymbol{M}$, whose image $\boldsymbol{f}(\boldsymbol{M})$ is contained in the quadric, can be holomorphically and isometrically mapped into $\boldsymbol{Q}_l(\boldsymbol{C})$ $(l\geqq 2n)$ in the same way as Proof of Theorem 2. Therefore, from the statement after Notation 2, there exists an open set $\boldsymbol{D}$ at $x=0$ in $\boldsymbol{C}$ such that, for each $w\in\boldsymbol{D}$, there exists a holomorphic isometric mapping $\boldsymbol{h}(w)$ of $\boldsymbol{M}$ into $\boldsymbol{Q}_l(\boldsymbol{C})$ $(l\geqq 2n)$.

(2) Let $\boldsymbol{P}_n(\boldsymbol{C})\subset\boldsymbol{P}_{m+1}(\boldsymbol{C})$ and let π be the natural projection of $\boldsymbol{C}^{m+2}-\{0\}$ onto $\boldsymbol{P}_{m+1}(\boldsymbol{C})$. We take the good cannonical coordinate system $(y^1, \cdots, y^m)$ in $\boldsymbol{Q}_m(\boldsymbol{C})$ around p_0.

We shall show that, if $A(x_0)$ is positive semi-definite of rank m, so is $A(\exp(\sqrt{-1}\,\theta)x_0)$ for $\theta\in[0, 2\pi)$ and that, for two holomorphic isometric mappings $\boldsymbol{h}(x_0)$ and $\boldsymbol{h}(\exp\sqrt{-1}\,\theta)x_0)$ of $\boldsymbol{M}$ into $\boldsymbol{Q}_m(\boldsymbol{C})$ defined by (1) of this proof (or Theorem 2), there exists a holomorphic isometric transformation B of $\boldsymbol{Q}_m(\boldsymbol{C})$ such that $B\boldsymbol{h}(x_0)=\boldsymbol{h}(\exp(\sqrt{-1}\,\theta)x_0)$ on $\boldsymbol{M}$. From (2) in Proof of Theorem 2 (we admit a case of $m=2n$), the mapping $\boldsymbol{h}(x_0)$ is locally given by

$$(4.12)\qquad \boldsymbol{h}(x_0)(z)=\begin{pmatrix} y^1(z)\\ y^2(z)\\ \vdots\\ \vdots\\ y^m(z)\end{pmatrix}=D\begin{pmatrix} f^0(z)\\ \sqrt{-1}f^0(z)\\ \vdots\\ f^{n-1}(z)\\ \sqrt{-1}f^{n-1}(z)\end{pmatrix}\qquad \text{for } z\in\boldsymbol{V}.$$

We remark that this mapping $\boldsymbol{h}(x_0)$ determined by $A(x_0)$ is essentially only one, i.e., two mappings $\boldsymbol{h}(x_0)$ and $\tilde{\boldsymbol{h}}(x_0)$ are determined by $A(x_0)$, then there exists an orthogonal matrix O of degree m such that $O\boldsymbol{h}(x_0)=\tilde{\boldsymbol{h}}(x_0)$. Now we shall consider about $A(\exp(\sqrt{-1}\theta)x_0)$. From (1) in Proof of Theorem 2, we have

$$A(\exp(\sqrt{-1}\theta)x_0)=\begin{pmatrix} {}^tB(\theta) & & \\ & \ddots & \\ & & {}^tB(\theta)\end{pmatrix} {}^tDD \begin{pmatrix} B(\theta) & & \\ & \ddots & \\ & & B(\theta)\end{pmatrix},$$

where

$$B(\theta)=\begin{bmatrix}\cos\left(\dfrac{\theta}{2}\right), & \sin\left(\dfrac{\theta}{2}\right)\\ -\sin\left(\dfrac{\theta}{2}\right), & \cos\left(\dfrac{\theta}{2}\right)\end{bmatrix}.$$

Therefore, from (2) in Proof of Theorem 2, we have

$$\text{(4.13)}\quad \boldsymbol{h}(\exp(\sqrt{-1}\theta)x_0)(z)=D\begin{pmatrix}B(\theta) & & & \\ & \ddots & & \\ & & \ddots & \\ & & & B(\theta)\end{pmatrix}\begin{pmatrix}f^0(z)\\ \sqrt{-1}f^0(z)\\ \vdots\\ f^{n-1}(z)\\ \sqrt{-1}f^{n-1}(z)\end{pmatrix}\quad \text{for } z\in \boldsymbol{V}.$$

Thus we have $\exp(\sqrt{-1}\,\theta/2)\boldsymbol{h}(x_0)(z)=\boldsymbol{h}(\exp(\sqrt{-1}\theta)x_0)(z)$ for $z\in \boldsymbol{V}$.

(3) Suppose that $A(1)$ is positive definite for the quadric (4.11). Using constants of the equation (4.2), we define the following symmetric matricies of degree $2n$:

$$I_{2n}=[\delta_{ij}]\,,\quad E=[e_{ij}] \text{ and } F=[f_{ij}]\,,$$

where

$$\begin{cases} 2e_{2i,2j}=\text{Real part}(b_{ij})\\ 2e_{2i+1,2j+1}=-\text{Real part}(b_{ij}) & (0\leqq i,j\leqq n-1)\\ 2e_{2i+1,2j}=\text{Imagenary part}(b_{ij})\end{cases}$$

and

$$\begin{cases} 2f_{2i,2j}=\text{Real part}(a_i\bar{a}_j)\\ 2f_{2i+1,2j+1}=\text{Real part}(a_i\bar{a}_j) & (0\leqq i,j\leqq n-1)\\ 2f_{2i+1,2j}=\text{Imagenary part}(a_i\bar{a}_j)\,.\end{cases}$$

For a real number y, we have

$$\text{(4.14)}\qquad A(y)=I_{2n}+yE-y^2F\,.$$

First assume some $a_i\neq 0$. Since the matrix F is a real representation of the complex matrix $(1/2)[a_i\bar{a}_j]$ $(0\leqq i,j\leqq n-1)$, F is a positive semidefinite of rank 2 (two eigenvalues are equal). As the real number y tends to ∞, $y^{-2}A(y)$ tends to $-F$. Since each eigenvalue is continuous with respect to y, there exists a real number r such that $A(r)$ is positive semi-definite of rank lower than $2n$. If every a_i $(0\leqq i\leqq n-1)$ is equal to zero, we have only to do the above argument for the term yE in the equation (4.14).

(4) We shall show that each mapping $\boldsymbol{h}(y)$ for $y\in[0, r]$ is essentially different from each other. First, assume some $a_i\neq 0$. Then, we can see this fact from (2) in Proof of Theorem 2. Therefore, we assume all a_i $(i=0,\cdots,n-1)$ is zero. We can see this fact from the equations (4.9), (4.12) and (4.14). Q.E.D.

Proof of Corollary 2. We use the same notations as in Proof of Theorem 2. Suppose that there exists a holomorphic isometric mapping $\boldsymbol{h}$: $\boldsymbol{M}\rightarrow\boldsymbol{Q}_m(\boldsymbol{C})$ $(m\geqq 2n)$ and $\boldsymbol{h}(z_0)=\boldsymbol{f}(z_0)=p_0$. Then, by the same way as (1) in Proof of Theorem 2 we have the following two cases. One is the case where $\boldsymbol{f}(\boldsymbol{M})$ is contained in a quadric in $\boldsymbol{P}_n(\boldsymbol{C})$ such that $A(1)$ is positive semi-definite and $\operatorname{rank} A(1)\leqq 2n$. The other is the case where $\boldsymbol{f}(\boldsymbol{M})$ is not contained in any quadric in $\boldsymbol{P}_n(\boldsymbol{C})$ (this is the case satisfying $c_i^0=0$ and $\langle c_i, \bar{c}_j\rangle-c_i^0c_j^0=0$ for $0\leqq i\leqq n-1$, $0\leqq j\leqq n-1$ in the equation (4.6). In the latter case, there exists $T\in O(m+2)$ such that $\boldsymbol{h}=T\boldsymbol{g}\boldsymbol{f}$, where $\boldsymbol{g}$ is given in Example 3. And the former case does not occur by the assumption. Q.E.D.

Proof of Theorem 4. (a)$\rightarrow$(b).

(1) For a quadric $\sum_{i,j=0}^n c_{ij}z^iz^j=0$ through the point p_0, we take a real symmetric matrix $A(x)$ defined by Notation 2. Suppose that $A(r)$ is a positive semi-definite of $\operatorname{rank} A(r)<2n$ for a positive real number r. Then, if $r'>r$, $A(r')$ is not positive semi-definite. In fact, let $\boldsymbol{e}$ be a real vector of $\boldsymbol{R}^{2n}$ such that $A(r)\boldsymbol{e}=0$. By the equation (4.14), we have

$$A(r)\boldsymbol{e}=\boldsymbol{e}+rE\boldsymbol{e}-r^2F\boldsymbol{e}=0$$

and

$$A(r')\boldsymbol{e}=\boldsymbol{e}+r'E\boldsymbol{e}-(r')^2F\boldsymbol{e}\,.$$

Then we have $F\boldsymbol{e}=\lambda\boldsymbol{e}_1$, where $\lambda\geqq 0$ and $\boldsymbol{e}_1$ is the projection of $\boldsymbol{e}$ to some real two dimensional space. Therefore, we have

$$\langle A(r')\boldsymbol{e}, \boldsymbol{e}\rangle=\frac{1}{r}(r-r')|\boldsymbol{e}|^2+(r-r')r'\lambda|\boldsymbol{e}_1|^2<0\,.$$

(2) We take a basis of the linear space $\{(c'_{ij})_{0\leqq i\leqq j\leqq n}\in\boldsymbol{C}^{(n+1)(n+2)/2}:\ \boldsymbol{f}(\boldsymbol{M})\subset\mathfrak{A}[c_{ij}]\}\cup\{0\}$ and denote it by $(c'^{l}_{ji})_{0\leqq i\leqq j\leqq n}$ $(1\leqq l\leqq k)$. Suppose $\mathfrak{A}[\sum_{l=1}^k x_lc_{ij}^l]=\mathfrak{A}[\sum_{l=1}^k y_lc_{ij}^l]$. Then, there exists a complex number w such that, for $(z^0, \cdots, z^n)\in\boldsymbol{C}^{n+1}$,

$$w\sum_{l=1}^k x_l\left(\sum_{i,j=0}^n c_{ij}^lz^iz^j\right)=\sum_{l=1}^k y_l\left(\sum_{i,j=0}^n c_{ij}^lz^iz^j\right).$$

Since a mapping $\mathcal{S}$: $\boldsymbol{C}^{n+1}\rightarrow\boldsymbol{C}^{(n+1)(n+2)/2}$, given by the equation $\mathcal{S}(z^0, \cdots, z^n)=(z^iz^j)_{0\leqq i\leqq j\leqq n}$, is full, we have $\sum_{l=1}^k(wx_l-y_l)c'^{l}_{ij}=0$ $(0\leqq i\leqq j\leqq n)$. Therefore, we have $w(x_1, \cdots, x_k)=(y_1, \cdots, y_k)$.

(3) Take k-tuples $(x_1, \cdots, x_k)$ and $(y_1, \cdots, y_k)$ of complex numbers such that $w(x_1, \cdots, x_k)\neq(y_1, \cdots, y_k)$ for any $w\in\boldsymbol{C}$. For a quadric $\sum_{l=1}^k x_l(\sum_{i,j=0}^n c_{ij}^lz^iz^j)=0$, we take a real symmetric matrix $A(x)$ of degree $2n$ defined by Notation 2. We denote simply $A(1)$ by A. Similarly, we put $\tilde{A}(1)=\tilde{A}$, where $\tilde{A}(x)$ is a symmetric matrix determined by a quadric $\sum_{l=1}^k y_l(\sum_{i,j=0}^n c_{ij}^lz^iz^j)=0$. Suppose that A and $\tilde{A}$ are positive semi-definite. Then we have, for any $\theta\in[0, 4\pi)$,

$$\tilde{A} \neq \begin{pmatrix} {}^tB(\theta) & & \\ & \ddots & \\ & & {}^tB(\theta) \end{pmatrix} A \begin{pmatrix} B(\theta) & & \\ & \ddots & \\ & & B(\theta) \end{pmatrix}, \tag{4.15}$$

where

$$B(\theta) = \begin{pmatrix} \cos\left(\dfrac{\theta}{2}\right), & \sin\left(\dfrac{\theta}{2}\right) \\ -\sin\left(\dfrac{\theta}{2}\right), & \cos\left(\dfrac{\theta}{2}\right) \end{pmatrix}.$$

In fact, suppose that, for some $\theta_0 \in [0, 4\pi)$, the equality holds in the equation (4.15). Take the canonical coordinate system $(y^0, \cdots, y^{n-1})$ in $\boldsymbol{P}_n(\boldsymbol{C})$ around p_0 given by the equation (2.3). Let the quadrics $\sum_{l=1}^k x_l(\sum_{i,j=0}^n c_{ij}^l z^i z^j) = 0$, $\sum_{l=1}^k y_l(\sum_{i,j=0}^n c_{ij}^l z^i z^j) = 0$ and the mapping $\boldsymbol{f}$ be represented as, in the coordinate neighborhood,

$$\begin{cases} 2\sum_{i=0}^{n-1} a_i y^i = \sqrt{-1}\sum_{i,j=0}^{n-1} b_{ij} y^i y^j, \\ 2\sum_{i=0}^{n-1} \tilde{a}_i y^i = \sqrt{-1}\sum_{i,j=0}^{n-1} \tilde{b}_{ij} y^i y^j \quad \text{and} \\ \boldsymbol{f}(z) = (f^0(z), \cdots, f^{n-1}(z)) \qquad \text{for } z \in \boldsymbol{V} \end{cases}$$

respectively, where $\boldsymbol{V}$ is an open set in $\boldsymbol{M}$. Then, from (2) in Proof of Theorem 2, we have $\tilde{b}_{ij} = \exp(\sqrt{-1}\,\theta_0) b_{ij}$ $(0 \leqq i, j \leqq n-1)$. This shows $\sum_{i=0}^{n-1}[\exp(\sqrt{-1}\,\theta_0)a_i - \tilde{a}_i] f^i(z) = 0$ for $z \in \boldsymbol{V}$. Since $\boldsymbol{f}$ is a full mapping, we have $\exp(\sqrt{-1}\,\theta_0)a_i = \tilde{a}_i$ $(0 \leqq i \leqq n-1)$. Therefore, two quadrics $\sum_{l=1}^k x_l \times (\sum_{i,j=0}^n c_{ij}^l z^i z^j) = 0$ and $\sum_{l=1}^k y_l(\sum_{i,j=0}^n c_{ij}^l z^i z^j) = 0$ are equal. This is a contradiction.

(4) We use same notations as in above (3). Let $\boldsymbol{h}$ and $\tilde{\boldsymbol{h}}$ be holomorphic isometric mappings of $\boldsymbol{M}$ into $\boldsymbol{Q}_l(\boldsymbol{C})$ $(l \geqq 2n)$ determined by A and $\tilde{A}$ respectively. Then, $\boldsymbol{h}$ and $\tilde{\boldsymbol{h}}$ are essentially different from each other. In fact, from (2) in Proof of Theorem 2 (or (2) in Proof of Theorem 3), $\boldsymbol{h}$ and $\tilde{\boldsymbol{h}}$ are locally defined by

$$\boldsymbol{h}(z) = D_1 \begin{pmatrix} f^0(z) \\ \sqrt{-1} f^0(z) \\ \vdots \\ f^{n-1}(z) \\ \sqrt{-1} f^{n-1}(z) \end{pmatrix} \quad \text{and} \quad \tilde{\boldsymbol{h}}(z) = D_2 \begin{pmatrix} f^0(z) \\ \sqrt{-1} f^0(z) \\ \vdots \\ f^{n-1}(z) \\ \sqrt{-1} f^{n-1}(z) \end{pmatrix} \qquad \text{for } z \in \boldsymbol{V}.$$

respectively, where $A = {}^tD_1D_1$ and $\tilde{A} = {}^tD_2D_2$. Suppose that there exists a holomorphic isometric transformation T of $\boldsymbol{Q}_l(\boldsymbol{C})$ such that $T\boldsymbol{h} = \tilde{\boldsymbol{h}}$ on $\boldsymbol{M}$. Since $\boldsymbol{f}$ is a full mapping into $\boldsymbol{P}_n(\boldsymbol{C})$, the system $(f^0(z), \cdots, f^{n-1}(z))$ for $z \in \boldsymbol{V}$ is linearly indegendent over complex numbers. Therefore, there exist $\theta \in [0, 4\pi)$ and $O \in \boldsymbol{O}(m)$ such that $OD_2B(\theta) = D_2$. Then we have

$${}^tB(\theta)AB(\theta) = {}^tB(\theta){}^tD_1D_1B(\theta) = {}^tB(\theta){}^tD_1{}^tOOD_1B(\theta) = {}^tD_2D_2 = \tilde{A}.$$

This is a contradiction.

(5) For the quadric $\sum_{l=1}^{k} x_l(\sum_{i,j=0}^{n} c_{ij}^{l} z^i z^j)=0$, we take a real symmetric matrix $A(x)$ of degree $2n$ defined by Notation 2. Then we can represent $A(x)$ as

$$A(x) = A(x, x_1, \cdots, x_k) = (xx_1, xx_2, \cdots, xx_k)\,.$$

for $x \in \boldsymbol{C}$ and $(x_1, \cdots, x_k) \in \boldsymbol{C}^k$. Since the matrix $A(w_1, \cdots, w_k)$ is continuous with respect to $(w_1, \cdots, w_k) \in \boldsymbol{C}^k$, each eigenvalue of it is continuous with respect to $(w_1, \cdots, w_k) \in \boldsymbol{C}^k$. Therefore, we have (b) from the above arguments and Proof of Theorem 3.

(b)$\rightarrow$(a). We can show this from Corollary 2, Theorem 3 and the proof of (a)$\rightarrow$(b). Q.E.D.

REMARK 5. In (5) of Proof of Theorem 4, if $A(w_1, \cdots, w_k)$ is positive definite and each eigenvalue of it is simple at $(w_1, \cdots, w_k)=(a_1, \cdots, a_k)$, then there exist some open set $\boldsymbol{D}'$ around the point and a holomorphic isometric mapping $\boldsymbol{h}(\boldsymbol{x})$ of $\boldsymbol{M}$ into $\boldsymbol{Q}_l(\boldsymbol{C})$ for each point $\boldsymbol{x} \in \boldsymbol{D}'$. This correspondence $\boldsymbol{D}' \ni \boldsymbol{x} \mapsto \boldsymbol{h}(\boldsymbol{x})$ gives the correspondence in (b) of Theorem 4 and varies continuously with respect to $\boldsymbol{x}$.

In fact, Remark 5 is derived from the fact that each eigenvector of $A(w_1, \cdots, w_l)$ varies continuously around $(a_1, \cdots, a_k)$ [5, p. 110, 3].

Finally, we shall give two examples for Theorem 4.

EXAMPLE 6. In the case of $\boldsymbol{Q}_m(\boldsymbol{C})$, we have, for $l \geqq 2(m+1)$, the closed interval $[0, 1]$ corresponds bijectively to the set {holomorphic isometric imbeddings of $\boldsymbol{Q}_m(\boldsymbol{C})$ into $\boldsymbol{Q}_l(\boldsymbol{C})\}/\approx$. Especially, putting $[0, 1] \ni y \mapsto I(y)$ with respect to above equality, then $I(1)$ is the class of holomorphic isometric transformations of $\boldsymbol{Q}_m(\boldsymbol{C})$. And there exists a holcmorphic isometric imbedding $\boldsymbol{h}(y)$ of $\boldsymbol{Q}_m(\boldsymbol{C})$ into $\boldsymbol{Q}_l(\boldsymbol{C})$ for each $y \in [0, 1]$ such that $\boldsymbol{h}(y)$ varies continuously with respect to y, where the equivalent class $[\boldsymbol{h}(y)]$ denotes $I(y)$.

In fact, the natural imbedding ι_1: $\boldsymbol{Q}_m(\boldsymbol{C}) \rightarrow \boldsymbol{P}_{m+1}(\boldsymbol{C})$ is full. For a canonical coordinate system $(y^0, \cdots, y^m)$ in $\boldsymbol{P}_{m+1}(\boldsymbol{C})$ around $\pi(1, \sqrt{-1}, 0, \cdots, 0)$ associated with $\boldsymbol{Q}_m(\boldsymbol{C})$, $\boldsymbol{Q}_m(\boldsymbol{C})$ is defined by the equation

$$2y^0 = \sqrt{-1}[(y^1)^2 + \cdots + (y^m)^2]\,.$$

Thus we have, for $y \in \boldsymbol{R}$,

$$A(y) = \frac{1}{2}\begin{pmatrix} 1-y^2, & & & & \\ & 1-y^2, & & 0 & \\ & & B & & \\ & 0 & & \ddots & \\ & & & & B \end{pmatrix},$$

where

$$B=\begin{bmatrix}1+y, & 0\\ 0, & 1-y\end{bmatrix}.$$

EXAMPLE 7. Consider the following quadric $\boldsymbol{M}$ in $\boldsymbol{P}_2(\boldsymbol{C})$:

$$\boldsymbol{M}:\ \frac{1}{2}(z^0)^2+\frac{1}{2}(z^1)^2+\frac{3}{2\sqrt{2}}z^1z^2+\frac{\sqrt{-1}}{2\sqrt{2}}z^0z^2=0\,.$$

This is a non-singular surface. As a metric of $\boldsymbol{M}$ we take the metric induced from that on $\boldsymbol{P}_2(\boldsymbol{C})$. Then we have, for $l\geqq 4$, the closed interval $[0, -1/4+\sqrt{33}/4]$ corresponds bijetively to the set {holomorphic isometric imbeddings of $\boldsymbol{M}$ into $\boldsymbol{Q}_l(\boldsymbol{C})\}/\approx$. Especially, $I(-1/4+\sqrt{33}/4)$ is the class of holomorphic isometric imbeddings of $\boldsymbol{M}$ into $\boldsymbol{Q}_3(\boldsymbol{C})$. And there exists a holomorphic isometric imbedding $\boldsymbol{h}(y)$ for each $y\in[0, 1/4+\sqrt{33}/4]$ such that $h(y)$ varies continuously with respect to y, where the equivalent class $[h(y)]$ denotes $I(y)$.

In fact, let (x^0, x^1) be the canonical coordinate system in $\boldsymbol{P}_2(\boldsymbol{C})$ around $\pi(1, \sqrt{-1}, 0)$ given by the equation (2.3). Then $\boldsymbol{M}$ is represented as

$$\frac{1}{2}x^0+\frac{1}{2}x^1=\frac{\sqrt{-1}}{4}x^0x^1$$

in the coordinate neighborhood. The matrix $A(y)$ for $y\in\boldsymbol{R}$ is given by

$$A(y)=\begin{pmatrix}\frac{1}{2}-\frac{y^2}{8}, & 0, & \frac{y}{8}(1-y), & 0\\ 0, & \frac{1}{2}-\frac{y^2}{8}, & 0, & -\frac{y}{8}(1+y)\\ \frac{y}{8}(1-y), & 0, & \frac{1}{2}-\frac{y^2}{8}, & 0\\ 0, & -\frac{y}{8}(1+y), & 0, & \frac{1}{2}-\frac{y^2}{8}\end{pmatrix}.$$

The matrix $A(y)$ has the following eigenvalues and the correspondent eigenvectors:

$$\frac{1}{2}+\frac{y}{8},\qquad {}^t\!\left[0,\ \frac{1}{\sqrt{2}},\ 0,\ -\frac{1}{\sqrt{2}}\right],$$

$$\frac{1}{2}-\frac{y}{8},\qquad {}^t\!\left[\frac{1}{\sqrt{2}},\ 0,\ -\frac{1}{\sqrt{2}},\ 0\right],$$

$$\frac{1}{2}-\frac{y}{8}-\frac{y^2}{4},\qquad {}^t\!\left[0,\ \frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right],$$

$$\frac{1}{2}+\frac{y}{8}-\frac{y^2}{4},\ {}^t\left[\frac{1}{\sqrt{2}},\quad 0\ ,\quad \frac{1}{\sqrt{2}},\quad 0\quad\right].$$

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