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# SEMISIMPLE NORMAL SUBGROUPS OF TRANSITIVE RIEMANNIAN ISOMETRY GROUPS

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**1. Introduction.** In this paper we prove the following:

**Theorem.** *Suppose the connected Lie group $A$ is a product $A=GL$ of a connected subgroup $G$ and a compact subgroup $L$. Let $H$ be a connected semisimple normal subgroup of $G$. Then*
(a) *if $H$ is of noncompact type, $H$ is normal in $A$;*
(b) *if $H$ is compact, then $H$ is contained in a compact semisimple normal subgroup of $A$.*

Here $H$ "of noncompact type" means all simple connected normal subgroups of $H$ are noncompact.

This theorem is related to the problem of describing the group of all isometries of a connected homogeneous Riemannian manifold $M$ in terms of a given transitive connected subgroup $G$. Indeed if $A$ is the connected component of the identity in the full isometry group of $M$, then $A=GL$ where $L$, the isotropy subgroup of $A$ at a point of $M$, is compact.

Part (a) of the theorem generalizes and provides a new proof of a result of [1] in which the normality of $G$ in $A$ is established when $G$ itself is semisimple of noncompact type. Following the proof of the theorem, we will note a sufficient condition for equality of the noncompact parts of Levi factors of $G$ and $A$, generalizing a further result of [1].

**2.** Recall that all maximal compact subgroups of a connected Lie group $A$ are conjugate under an inner automorphism of $A$. If $A=GL$ with $L$ compact and if $U$ is a maximal compact subgroup of $A$, then a conjugate of $L$ lies in $U$. It is then easily verified that $A=GU$. Thus we are free to replace $L$ by any convenient maximal compact subgroup of $A$.

A maximal connected semisimple subgroup $A_{ss}$ of $A$ will as usual be called a Levi factor of $A$. Being semisimple, $A_{ss}$ is a product $A_{ss}=A_{nc}A_c$ of connected normal semisimple subgroups $A_{nc}$ and $A_c$ of noncompact and compact type, respectively. $A_{nc}$ and $A_c$ will be called the noncompact and compact

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parts of $A_{ss}$. Similar notation will be used for the corresponding Lie algebras.

Proof of the theorem. Choose Levi factors $G_{ss}$ and $A_{ss}$ of $G$ and $A$ with $H\subset G_{ss}\subset A_{ss}$. Denote by $\mathfrak{a}$, $\mathfrak{g}$, $\mathfrak{a}_{ss}$, $\mathfrak{g}_{ss}$, and $\mathfrak{h}$ the Lie algebras of $A$, $G$, $A_{ss}$, $G_{ss}$, and $H$, respectively, and by $\mathfrak{a}_r$ and $\mathfrak{g}_r$ the radicals of $\mathfrak{a}$ and $\mathfrak{g}$. As above we write $\mathfrak{a}_{ss}=\mathfrak{a}_{nc}+\mathfrak{a}_c$ and $\mathfrak{g}_{ss}=\mathfrak{g}_{nc}+\mathfrak{g}_c$. Let $\pi_{nc}\colon \mathfrak{a}\to\mathfrak{a}_{nc}$ and $\pi_c\colon \mathfrak{a}\to\mathfrak{a}_c$ be the projections relative to the vector space direct sum $\mathfrak{a}=\mathfrak{a}_{nc}+\mathfrak{a}_c+\mathfrak{a}_r$. Note that $\pi_c(\mathfrak{g}_{nc})=\{0\}$ since $\mathfrak{a}_c$ contains no noncompact semisimple subalgebras, so $\mathfrak{g}_{nc}\subset\mathfrak{a}_{nc}$.

Let $\mathfrak{g}_{nc}=\mathfrak{k}+\mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$, i.e., $\mathfrak{k}$ is a maximal compactly imbedded subalgebra of $\mathfrak{g}_{nc}$, $[\mathfrak{k},\mathfrak{p}]=\mathfrak{p}$ and $[\mathfrak{p},\mathfrak{p}]=\mathfrak{k}$. By a theorem of Mostow (see [2], pp. 277 and 569), $\mathfrak{a}_{nc}$ has a Cartan decomposition $\mathfrak{a}_{nc}=\mathfrak{k}'+\mathfrak{p}'$ with

$$\mathfrak{k}+\pi_{nc}(\mathfrak{g}_c)\subset\mathfrak{k}' \quad\text{and}\quad \mathfrak{p}\subset\mathfrak{p}' . \tag{1}$$

Note that $\mathfrak{k}'+\mathfrak{a}_c$ is a maximal compact subalgebra of $\mathfrak{a}_{ss}$. Hence, letting $\mathfrak{u}$ be any maximal compactly imbedded subalgebra of $\mathfrak{a}$ containing $\mathfrak{k}'+\mathfrak{a}_c$, it follows easily that

$$\mathfrak{u}=(\mathfrak{k}'+\mathfrak{a}_c)+(\mathfrak{u}\cap\mathfrak{a}_r) . \tag{2}$$

$\mathfrak{u}\cap\mathfrak{a}_r$ is a solvable ideal of $\mathfrak{u}$ and hence is central in $\mathfrak{u}$. Thus

$$[\mathfrak{u},\mathfrak{u}]\subset\mathfrak{a}_{ss} . \tag{3}$$

The connected subgroup $U$ of $A$ with Lie algebra $\mathfrak{u}$ contains a maximal compact subgroup of $A$, so $A=GU$ and

$$\mathfrak{a}=\mathfrak{g}+\mathfrak{u} . \tag{4}$$

We first show that $[\mathfrak{h},\mathfrak{a}_r]=\{0\}$. We may assume that $\mathfrak{h}$ is simple. If $\mathfrak{h}$ is of noncompact type, then $\mathfrak{h}$ is a $\mathfrak{g}_{nc}$-ideal and therefore has a Cartan decomposition $\mathfrak{h}=\mathfrak{k}_0+\mathfrak{p}_0$ with $\mathfrak{k}_0=\mathfrak{k}\cap\mathfrak{h}$ and $\mathfrak{p}_0=\mathfrak{p}\cap\mathfrak{h}$. If $\mathfrak{h}$ is compact, then $\mathfrak{h}\subset\mathfrak{g}_c\subset\mathfrak{u}$ by (1) and (2). Thus in either case, $\mathfrak{h}\cap\mathfrak{u}\neq\{0\}$. In view of (3) and (4) we have

$$[\mathfrak{h}\cap\mathfrak{u},\mathfrak{a}]\subset[\mathfrak{h},\mathfrak{g}]+[\mathfrak{u},\mathfrak{u}]\subset\mathfrak{h}+(\mathfrak{u}\cap\mathfrak{a}_{ss})\subset\mathfrak{a}_{ss} . \tag{5}$$

Hence $[\mathfrak{h}\cap\mathfrak{u},\mathfrak{a}_r]=\{0\}$. Since the annihilator in $\mathfrak{g}$ of $\mathfrak{a}_r$ is a $\mathfrak{g}$-ideal, it follows that

$$[\mathfrak{h},\mathfrak{a}_r]=\{0\} . \tag{6}$$

Suppose now that $\mathfrak{h}$ is of noncompact type and write $\mathfrak{h}=\mathfrak{k}_0+\mathfrak{p}_0$ as in the paragraph above. By (5),

$$[\mathfrak{k}_0,\mathfrak{a}_{nc}]\subset(\mathfrak{h}+\mathfrak{u})\cap\mathfrak{a}_{nc}=\mathfrak{k}'+\mathfrak{h}=\mathfrak{k}'+\mathfrak{p}_0 .$$

In particular,

$$[\mathfrak{k}_0,\mathfrak{p}']\subset(\mathfrak{k}'+\mathfrak{p}_0)\cap\mathfrak{p}'=\mathfrak{p}_0 . \tag{7}$$

$[\mathfrak{u}, \mathfrak{p}_0]=[\mathfrak{k}', \mathfrak{p}_0]$ by (2) and (6), so (4) implies

$$[\mathfrak{a}_{nc}, \mathfrak{p}_0]\subset[\mathfrak{g}, \mathfrak{p}_0]+[\mathfrak{k}', \mathfrak{p}_0]\subset\mathfrak{h}+\mathfrak{p}' .$$

Therefore

$$[\mathfrak{p}', \mathfrak{p}_0]\subset(\mathfrak{h}+\mathfrak{p}')\cap\mathfrak{k}' = \mathfrak{k}_0 . \tag{8}$$

(7) and (8) together yield $[\mathfrak{p}', \mathfrak{h}]\subset\mathfrak{h}$. Since $\mathfrak{a}_{nc}=[\mathfrak{p}', \mathfrak{p}']+\mathfrak{p}'$, $\mathfrak{h}$ is an $\mathfrak{a}_{nc}$-ideal. (i) now follows from (6).

Next suppose $H$ is compact. Then $\mathfrak{h}\subset\mathfrak{u}\cap\mathfrak{a}_{ss}$ and $[\mathfrak{h}, \mathfrak{a}]\subset\mathfrak{u}$ by (5). Noting that $[\mathfrak{u}\cap\mathfrak{a}_{ss}, \mathfrak{p}']=[\mathfrak{k}', \mathfrak{p}']=\mathfrak{p}'$, we have $[\mathfrak{h}, \mathfrak{p}']\subset\mathfrak{u}\cap\mathfrak{p}'=\{0\}$. Hence $[\mathfrak{h}, \mathfrak{a}_{nc}]=\{0\}$ and $\mathfrak{h}\subset\mathfrak{a}_c$. Let $\mathfrak{h}'$ be the minimal $\mathfrak{a}_c$-ideal containing $\mathfrak{h}$. By (6), $[\mathfrak{h}', \mathfrak{a}_r]=\{0\}$ so $\mathfrak{h}'$ is an $\mathfrak{a}$-ideal. The corresponding connected subgroup $H'$ of $A$ is a compact semisimple normal subgroup containing $H$.

REMARK. Examples are easily constructed indicating that part (b) of the theorem cannot in general be strengthened. Even when $G$ itself is a semisimple, compact, simply transitive group of isometries of a Riemannian manifold M. Ozeki has shown in [3] that $G$ need not be normal in the full connected isometry group $A$ of $M$.

**Corollary.** *Given Lie groups $A=GL$ with $A$ and $G$ connected and $L$ compact, suppose that the noncompact part $G_{nc}$ of a Levi factor of $G$ is normal in $G$. If no homomorphic image of the radical of $G$ is isomorphic to a transitive group of isometries of a Riemannian symmetric space of the noncompact type, then $G_{nc}$ is the noncompact part of every Levi factor of $A$ (and is normal in $A$).*

Proof. By the theorem, $G_{nc}$ is normal in $A$. Since all Levi factors of a connected Lie group are conjugate, $G_{nc}$ lies in every Levi factor of $G$ and $A$. Let $G_{ss}=G_{nc}G_c$ and $A_{ss}=A_{nc}A_c$ be any Levi factors and let $G_r$ and $A_r$ denote the radicals of $G$ and $A$. Let

$$\pi\colon\ A\to A/(G_{nc}A_cA_r)$$

be the projection. Note that $\pi(A)$ is semisimple of noncompact type and is trivial if and only if $G_{nc}=A_{nc}$. Modding out the discrete center if necessary, we assume $\pi(A)$ has finite center. Now

$$\pi(A) = \pi(G)\pi(L) = \pi(G_c)\pi(G_r)\pi(L) .$$

As noted previously, we may replace $\pi(L)$ by a maximal compact subgroup $U$ of $\pi(A)$ containing $\pi(G_c)$, so that $\pi(A)=\pi(G_r)U$. Under any left-invariant Riemannian metric, $\pi(A)/U$ is a symmetric space of the noncompact type (see [2], pp. 252–253) on which $\pi(G_r)$ acts transitively by isometries. The corollary follows.

## References

[1] C. Gordon: *Riemannian isometry groups containing transitive reductive subgroups,* Math. Ann. **248** (1980), 185–192.

[2] S. Helgason: Differential geometry, Lie groups, and symmetric spaces, New York, Academic Press, 1978.

[3] H. Ozeki: *On a transitive transformation group of a compact group manifold,* Osaka J. Math. **14** (1977), 519–531.

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