

ON QUASICONFORMAL MAPPINGS COMPATIBLE WITH A FUCHSIAN GROUP

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1. Introduction. Let U be the upper half-plane and let $\hat{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$ be the extended real line. The conformal automorphisms of U form the real Möbius group $SL'(2, \mathbf{R})$, that is, $SL(2, \mathbf{R})$ modulo its center. A discrete subgroup G of $SL'(2, \mathbf{R})$ is called a Fuchsian group; the trivial group $G = \{1\}$ is included, where 1 means the identity transformation of $SL'(2, \mathbf{R})$. It is well-known that a Fuchsian group G acts discontinuously on U and that the limit set $\Lambda(G)$ is a closed subset of $\hat{\mathbf{R}}$, which is invariant under G .

We denote by $L_\infty(U)$ the Banach space consisting of measurable functions μ on U with finite L_∞ norm $\|\mu\|$. Let f be a sense-preserving homeomorphism of U . Such a mapping f is said to be quasiconformal if it has generalized derivatives $f_{\bar{z}}$ and f_z which satisfy the Beltrami equation

$$(1.1) \quad f_{\bar{z}} = \mu f_z$$

for some $\mu \in L_\infty(U)$ with $\|\mu\| < 1$. As is known, a quasiconformal automorphism f of U is extensible to a homeomorphism of $U \cup \hat{\mathbf{R}}$. By abuse of language, we introduce no new name for the extended mapping. For every $\mu \in L_\infty(U)$ with $\|\mu\| < 1$, we denote by f_μ the homeomorphic solution of the equation (1.1) which leaves the points 0, 1, ∞ fixed; it is well-known that f_μ exists and that f_μ is uniquely determined by μ (see [9]).

Let $h: \hat{\mathbf{R}} \rightarrow \hat{\mathbf{R}}$ be a sense-preserving homeomorphism such that $h(\infty) = \infty$. Following [9], we say that h is quasisymmetric if there exists a constant $\rho \geq 1$ such that

$$(1.2) \quad 1/\rho \leq (h(x+t) - h(x)) / (h(x) - h(x-t)) \leq \rho$$

for $x \in \mathbf{R}$ and $t > 0$. In this case, we denote by ρ_h the infimum of the constants ρ which satisfy (1.2).

Let G be a Fuchsian group. We define $H(G)$ as the set of all the quasisymmetric functions h which satisfy the following conditions: for every $\gamma \in G$, there exists $\gamma^* \in SL'(2, \mathbf{R})$ such that

$$(1.3) \quad h \circ \gamma \circ h^{-1} = \gamma^*|_{\hat{\mathbf{R}}},$$

and

(1.4) h leaves the points $0, 1, \infty$ fixed,

where $\gamma^*|_{\hat{\mathbf{R}}}$ means the restriction of γ^* to $\hat{\mathbf{R}}$. We define $\Sigma(G)$ as the family of all the closed subsets σ of $\hat{\mathbf{R}}$, which are invariant under G and satisfy

$$(1.5) \quad \sigma \supset \Lambda(G) \cup \{0, 1, \infty\}.$$

For $h \in H(G)$ and $\sigma \in \Sigma(G)$, we define $F(G, h, \sigma)$ as the set of all the quasiconformal automorphisms f of U satisfying

$$(1.6) \quad fGf^{-1} \subset SL'(2, \mathbf{R}),$$

and

$$(1.7) \quad f|_{\sigma} = h|_{\sigma}.$$

In view of (1.4), (1.5) and (1.7), we see that each $f \in F(G, h, \sigma)$ is of the form $f = f_{\mu}$ for some $\mu \in L_{\infty}(U)$ with $\|\mu\| < 1$. Moreover, for $\mu \in L_{\infty}(U)$ with $\|\mu\| < 1$, we can easily check that $f = f_{\mu}$ satisfies (1.6) if and only if μ satisfies the condition

$$\mu(\gamma(z))\overline{\gamma'(z)}/\gamma'(z) = \mu(z) \quad \text{for every } \gamma \in G.$$

In the case where $F(G, h, \sigma)$ is not empty, we put

$$(1.8) \quad k(G, h, \sigma) = \inf \|\mu\|,$$

where the infimum is taken over all $f_{\mu} \in F(G, h, \sigma)$. By means of a normal family argument of quasiconformal mappings, we can check that there exists some $f_{\nu} \in F(G, h, \sigma)$ with $\|\nu\| = k(G, h, \sigma)$ (see [9]). Such a mapping f_{ν} is said to be extremal in the class $F(G, h, \sigma)$.

Let G_1 be a Fuchsian group and let G_2 be a subgroup of G_1 . Then G_2 is, of course, a Fuchsian group. It follows from the definitions that $H(G_1) \subset H(G_2)$, $\Sigma(G_1) \subset \Sigma(G_2)$ and that

$$(1.9) \quad F(G_1, h, \sigma) \subset F(G_2, h, \sigma)$$

for $h \in H(G_1)$ and $\sigma \in \Sigma(G_1)$. It is quite interesting to know how far informations about $F(G_i, h, \sigma)$ for $i=1, 2$, have an effect on each other. In this paper we shall be concerned with this kind of problems and prove some theorems.

2. Teichmüller space theory and $F(G, h, \sigma)$. In this section we translate some known results about Teichmüller space theory into the language of our formulation.

At first, the following result follows easily from the definitions.

Lemma 1. *Let G be a Fuchsian group, $h \in H(G)$ and $\sigma \in \Sigma(G)$. Suppose that $F(G, h, \sigma)$ is not empty. Let γ be an arbitrarily chosen and fixed element of G . Then $f_\mu \circ \gamma \circ f_\mu^{-1}$ is identical with γ^* for every $f_\mu \in F(G, h, \sigma)$, where $\gamma^* \in SL'(2, \mathbf{R})$ is determined by the condition (1.3) and does not depend on the choice of f_μ . Further, $k(G, h, \sigma) = 0$ if and only if $1 \in F(G, h, \sigma)$.*

Let L be the lower half-plane. For every Fuchsian group G , we denote by $B(L, G)$ the Banach space consisting of all the holomorphic functions ϕ in L , which satisfy

$$\phi(\gamma(z))\gamma'(z)^2 = \phi(z) \quad \text{for every } \gamma \in G, \text{ and } \sup |(z - \bar{z})^2 \phi(z)| < \infty.$$

If we define $T(G)$ as the set of those $h \in H(G)$ such that $F(G, h, \hat{\mathbf{R}})$ is not empty, then $T(G)$ is the so-called (unreduced) Teichmüller space for the Fuchsian group G . From now on, the trivial group $\{1\}$ is simply denoted by 1 , provided that there is no confusion. It is well-known in Teichmüller space theory that there exists a canonical injection $\Psi: H(1) \rightarrow B(L, 1)$. Earle [4] includes the result:

$$(2.1) \quad \Psi(H(G)) = \Psi(H(1)) \cap B(L, G)$$

for every Fuchsian group G . Kra [6] proved that

$$(2.2) \quad \Psi(T(G)) = \Psi(H(1)) \cap B(L, G)$$

for every finitely generated Fuchsian group G . Furthermore, it is obvious that

$$(2.3) \quad F(G, h, \hat{\mathbf{R}}) \subset F(G, h, \sigma) \quad \text{for every } \sigma \in \Sigma(G).$$

According to (2.1), (2.2) and (2.3), we have the following lemma.

Lemma 2. *Let G be a finitely generated Fuchsian group. Then $F(G, h, \sigma)$ is not empty for every $h \in H(G)$ and every $\sigma \in \Sigma(G)$.*

REMARK 1. In the case where $G=1$ and $\sigma = \hat{\mathbf{R}}$, it is well-known that Ahlfors and Beurling [1] includes the following result: Suppose that $h \in H(1)$. Then $F(1, h, \hat{\mathbf{R}})$ is not empty, and $k(1, h, \hat{\mathbf{R}}) \leq \rho_h^2$.

Now we state the following Lemma 3, of which we shall not make use in this paper. It is, however, a noteworthy information with respect to arbitrary, not necessarily finitely generated, Fuchsian groups. The argument developed in the proof of Theorem 2 in Lehto [8] and (2.3) ensure the validity of our Lemma 3.

Lemma 3. *Let G be a Fuchsian group. Then $F(G, h, \sigma)$ is not empty for every $\sigma \in \Sigma(G)$ and every $h \in H(G)$ such that $\rho_h < \sqrt{2}$.*

3. Extremal quasiconformal mappings. Let G be a Fuchsian group

and $\sigma \in \Sigma(G)$. Let ϕ be a holomorphic quadratic differential for the configuration (U, G) , that is, a holomorphic function in U satisfying

$$\phi(\gamma(z))\gamma'(z)^2 = \phi(z) \quad \text{for every } \gamma \in G.$$

We denote by $A(G, \sigma)$ the space consisting of all the holomorphic quadratic differentials ϕ for (U, G) , which are continuously extensible to $\mathbf{R} \setminus \sigma$ and real on $\mathbf{R} \setminus \sigma$, and satisfy

$$\|\phi\|_G \equiv \iint_{U/G} |\phi(z)| dx dy < \infty.$$

The space $A(G, \sigma)$ is a real Banach space with norm $\|\cdot\|_G$. We denote by $A(G, \sigma)_1$ the set of those $\phi \in A(G, \sigma)$ with $\|\phi\|_G = 1$. As is well-known, $\dim_{\mathbf{R}} A(G, \sigma)$ is finite if and only if G is finitely generated and the set $(\sigma \setminus \Lambda(G))/G$ is finite.

The fundamental inequality, referred to in the title of Bers [3], plays an important role to characterize extremal mappings. In his paper [3], Bers proved the most generalized form of the inequality (see Theorem 2 in [3]). It has several very important applications (see [5] and [11]). Here we state slightly modified forms of some results in Gardiner [5] as the following Lemmas 4 and 5. The corresponding results in [5] say that the following Lemmas 4 and 5 hold whenever G has no elliptic elements. But the proofs of them in [5] are based on Theorem 2 in [3] and are still effectively valid, even if G may have elliptic elements.

Lemma 4. *Suppose that $f_\mu \in F(G, h, \sigma)$. Put $K = (1+k)/(1-k)$, where $k = k(G, h, \sigma)$. Then, for every ϕ in $A(G, \sigma)_1$,*

$$(3.1) \quad 1/K \leq \iint_{U/G} |\phi| \frac{|1 - \mu\phi| |\phi|^2}{1 - |\mu|^2} dx dy.$$

Lemma 5. *Suppose that $f_\mu \in F(G, h, \sigma)$, where*

$$(3.2) \quad \mu = k|\phi|/\phi$$

for a constant k such that $0 \leq k < 1$ and some $\phi \in A(G, \sigma)_1$. Then f_μ is unique extremal in the class $F(G, h, \sigma)$.

A quasiconformal mapping $f_\mu \in F(G, h, \sigma)$, with μ of the above form (3.2), is said to be a Teichmüller mapping with finite norm in the class $F(G, h, \sigma)$.

Lemma 6. *Suppose that $f_\mu \in F(G, h, \sigma)$. Then f_μ is extremal in the class $F(G, h, \sigma)$ if and only if*

$$(3.3) \quad \|\mu\| = \sup \left| \operatorname{Re} \iint_{U/G} \mu \phi dx dy \right|,$$

where the supremum is taken over all $\phi \in A(G, \sigma)_1$, and $\text{Re } A$ denotes the real part of A .

Proof. The necessity is due to Bers [2]. The sufficiency follows as a by-product of the fundamental inequality. In fact, we can prove the sufficiency by making use of Lemma 4 and, for the sake of completeness, we shall give the proof.

We assume that μ satisfies (3.3). For ϕ in $A(G, \sigma)_1$, we have $-\phi \in A(G, \sigma)_1$. Thus we see that there exists a sequence $\phi_n \in A(G, \sigma)_1$, $n=1, 2, \dots$, such that

$$(3.4) \quad \|\mu\| = \lim_{n \rightarrow \infty} \text{Re} \iint_{U/G} \mu \phi_n dx dy .$$

Put $K=(1+k)/(1-k)$, where $k=k(G, h, \sigma)$. Then, by Lemma 4,

$$\begin{aligned} 1/K &\leq \iint_{U/G} \frac{|\phi_n| |1-\mu\phi_n| |\phi_n|^2}{1-|\mu|^2} dx dy \\ &\leq ((1+\|\mu\|^2)/(1-\|\mu\|^2)) - (2 \text{Re} \iint_{U/G} \mu \phi_n dx dy)/(1-\|\mu\|^2) . \end{aligned}$$

By letting n tend to ∞ in the above inequality, in view of (3.4), we have

$$(3.5) \quad 1/K \leq (1-\|\mu\|)/(1+\|\mu\|) .$$

It follows from (1.8) and (3.5) that $\|\mu\|=k$. This means that f_μ is extremal in the class $F(G, h, \sigma)$.

Let $f_\mu \in F(G, h, \sigma)$ be extremal. Then we have seen that μ satisfies (3.3). In this case, we can easily check that f_μ is a Teichmüller mapping with finite norm if and only if the supremum in the right-hand side of (3.3) is attained for some $\phi_0 \in A(G, \sigma)_1$ (see [10]). In particular, this happens, provided that $\dim_{\mathbb{R}} A(G, \sigma)$ is finite. Thus, noting Lemma 2, we have the following result.

Lemma 7. *Let G be a finitely generated Fuchsian group and $\sigma \in \Sigma(G)$. Suppose that the set $(\sigma \backslash \Lambda(G))/G$ is finite. Then, for every $h \in H(G)$, $F(G, h, \sigma)$ is not empty and contains a Teichmüller mapping with finite norm.*

Let G be a Fuchsian group and let Φ be a holomorphic function in U . The Poincaré series $\Theta_G \Phi$ of Φ is defined by

$$(3.6) \quad (\Theta_G \Phi)(z) = \sum_{\gamma \in G} \Phi(\gamma(z)) \gamma'(z)^2 .$$

Bers [3] proved the fundamental inequality by an application of Poincaré series. The following lemma is implicitly established in [3] on the way of the proof of the inequality.

Lemma 8. *Let G be a Fuchsian group and $\sigma \in \Sigma(G)$. Then, for every Φ in*

$A(1, \sigma)$, the series $\Theta_G \Phi$, defined by (3.6), converges absolutely and uniformly on compact subsets of U . And, further, the mapping Θ_G gives a real linear continuous mapping of $A(1, \sigma)$ onto $A(G, \sigma)$, and the operator norm of Θ_G is not greater than 1.

REMARK 2. In the case $\sigma = \hat{R}$, the assertion is well-known (see the presentation in Kra [7]).

4. A Fuchsian group G_1 and its subgroup G_2 . Let G_1 be a Fuchsian group and let G_2 be a subgroup of G_1 . Let $h \in H(G_1)$ and $\sigma \in \Sigma(G_1)$ be given. Suppose that $F(G_1, h, \sigma)$ is not empty. Then, owing to (1.9), we see that $k(G_1, h, \sigma) \geq k(G_2, h, \sigma)$. Now we prove the following theorem.

Theorem 1. *Let $G_1 \supseteq G_2$ be Fuchsian groups such that the index $[G_1 : G_2]$ of G_2 in G_1 is finite. Let $h \in H(G_1)$ and $\sigma \in \Sigma(G_1)$ be given. Suppose that $F(G_2, h, \sigma)$ is not empty. Then*

(1) $F(G_1, h, \sigma)$ is not empty,

and

(2) $k(G_1, h, \sigma) = k(G_2, h, \sigma)$.

Proof. By our assumption, the index $n \equiv [G_1 : G_2]$ is finite. In this case, it clearly follows that $A(G_1, \sigma) \subset A(G_2, \sigma)$ and that, for every ϕ in $A(G_1, \sigma)$,

$$(4.1) \quad \|\phi\|_{G_1} = \|\phi\|_{G_2}/n.$$

First we show that, if $F(G_1, h, \sigma)$ is not empty, then $k(G_1, h, \sigma) = k(G_2, h, \sigma)$. Let f_μ be extremal in the class $F(G_1, h, \sigma)$. Then, by Lemma 6, there exists a sequence $\phi_j \in A(G_1, \sigma)_1, j=1, 2, \dots$, such that

$$(4.2) \quad k(G_1, h, \sigma) = \|\mu\| = \lim_{j \rightarrow \infty} \left| \operatorname{Re} \iint_{U/G_1} \mu \phi_j dx dy \right|.$$

But we can rewrite (4.2) as follows:

$$(4.3) \quad \|\mu\| = \lim_{j \rightarrow \infty} \left| \operatorname{Re} \iint_{U/G_2} \mu \phi_j / n dx dy \right|.$$

Since $\phi_j/n \in A(G_2, \sigma)_1$, we see by (1.9), (4.3) and Lemma 6 that f_μ is extremal in the class $F(G_2, h, \sigma)$. This means $k(G_1, h, \sigma) = k(G_2, h, \sigma)$.

Now we show (1). Since the index $n = [G_1 : G_2]$ is finite, if G_2 is a finite group, then so is G_1 . In this case, (1) clearly follows by Lemma 2. Thus it remains to prove (1) in the case where G_2 is an infinite group. Since every Fuchsian group consists of a countable number of elements, we may assume that $G_2 = \{1, \gamma_1, \gamma_2, \dots\}$. For every $j=1, 2, \dots$, let $G_{2,j}$ be the group generated by $\{\gamma_1, \gamma_2, \dots, \gamma_j\}$. Then

$$G_{2,1} \subset G_{2,2} \subset \dots, \text{ and } G_2 = G_{2,1} \cup G_{2,2} \cup \dots.$$

Choose coset representatives $\delta_0=1, \delta_1, \dots, \delta_{n-1}$ in G_1 such that

$$G_1 = G_2 + G_2\delta_1 + \dots + G_2\delta_{n-1}.$$

For every $j=1, 2, \dots$, let $G_{1,j}$ be the group generated by $G_{2,j}$ and $\{\delta_1, \delta_2, \dots, \delta_{n-1}\}$, and put $G_{2,j}^* = G_{1,j} \cap G_2$. Then, as is well-known in group theory, it follows that

$$(4.4) \quad [G_{1,j} : G_{2,j}^*] \leq [G_1 : G_2] < \infty \quad \text{for } j = 1, 2, \dots.$$

Since every $G_{1,j}$ is finitely generated, every $F(G_{1,j}, h, \sigma)$ is not empty by Lemma 2. Let f_{μ_j} be extremal in the class $F(G_{1,j}, h, \sigma)$ for $j=1, 2, \dots$. Then, as was proved in the first half of the proof, we see by (4.4) that

$$(4.5) \quad \|\mu_j\| = k(G_{2,j}^*, h, \sigma) \quad \text{for } j = 1, 2, \dots.$$

On the other hand, because $F(G_2, h, \sigma) \subset F(G_{2,j}^*, h, \sigma)$ for $j=1, 2, \dots$, we have

$$(4.6) \quad k \equiv k(G_2, h, \sigma) \geq k(G_{2,j}^*, h, \sigma) \quad \text{for } j = 1, 2, \dots.$$

According to (4.5) and (4.6), we see that the family $\{f_{\mu_j}\}, j=1, 2, \dots$, forms a subfamily of a closed normal family of K -quasiconformal automorphisms of U , where $K=(1+k)/(1-k)$ (see [9]). Taking a subsequence if necessary, we may assume that the sequence $f_{\mu_j}, j=1, 2, \dots$, converges to a quasiconformal automorphism f_μ of U . Noting Lemma 1, we can easily check that f_μ belongs to $F(G_1, h, \sigma)$. This completes the proof of the theorem.

REMARK 3. Let G be a Fuchsian group, $h \in H(G)$ and $\sigma \in \Sigma(G)$. If G is an infinite group, then we have an enumeration $1, \gamma_1, \gamma_2, \dots$, of all the elements of G . In this case, for every $j=1, 2, \dots$, let G_j be the group generated by $\{\gamma_1, \gamma_2, \dots, \gamma_j\}$. Since every topological subspace of \hat{R} is separable, if the set $\sigma \setminus (\Lambda(G) \cup \{0, 1, \infty\})$ is infinite, then there exists a countable dense subset $\{a_1, a_2, \dots, a_j, \dots\}$ of $\sigma \setminus (\Lambda(G) \cup \{0, 1, \infty\})$. In this case, for every $j=1, 2, \dots$, let σ_j be the smallest subset of σ , which belongs to $\Sigma(G)$ and which contains $\{a_1, a_2, \dots, a_j\}$. Then, for every $j=1, 2, \dots$, we can easily check that an arbitrary element of $\sigma_j \setminus \Lambda(G)$ is of the form $\gamma(x)$ for some $\gamma \in G$, where $x=0, 1, \infty$ or $a_i, 1 \leq i \leq j$. In particular, for every $j=1, 2, \dots$, the set $(\sigma_j \setminus \Lambda(G))/G$ is finite. Under the above assumptions and notations, suppose that $F(G, h, \sigma)$ is not empty. Then, as in the latter part of the proof of Theorem 1, we can check the following results by a normal family argument and Lemma 1:

$$(a) \quad k(G, h, \sigma) = \lim_{j \rightarrow \infty} k(G_j, h, \sigma)$$

whenever G is an infinite group, and

$$(b) \quad k(G, h, \sigma) = \lim_{j \rightarrow \infty} k(G, h, \sigma_j)$$

whenever the set $\sigma \setminus (\Lambda(G) \cup \{0, 1, \infty\})$ is infinite.

Now we shall state one more lemma and prove some theorems.

Lemma 9. *Let $G_1 \cong G_2$ be Fuchsian groups and $\sigma \in \Sigma(G_1)$. Let Φ be an arbitrarily chosen and fixed element of $A(1, \sigma)$. Put $\phi_1 = \Theta_{G_1}\Phi$ and $\phi_2 = \Theta_{G_2}\Phi$. Suppose that $\phi_2 \neq 0$. Then*

$$\|\phi_1\|_{G_1} \leq \|\phi_2\|_{G_2} \leq \|\Phi\|_1.$$

Furthermore, the three following conditions are equivalent to each other:

$$(1) \quad \|\phi_1\|_{G_1} = \|\phi_2\|_{G_2},$$

$$(2) \quad \phi_2 \in A(G_1, \sigma),$$

and

$$(3) \quad n \equiv [G_1: G_2] < \infty, \quad \text{and} \quad \phi_1 = n\phi_2.$$

Proof. Choose coset representatives $\delta_0 = 1, \delta_1, \delta_2, \dots$, in G_1 , which end with δ_{n-1} if $n = [G_1: G_2] < \infty$, such that

$$G_1 = G_2 + G_2\delta_1 + G_2\delta_2 + \dots.$$

Let ω be a fundamental region in U with respect to G_1 , that is, ω is an open subset of U such that the measure of the boundary $\partial\omega$ of ω in U vanishes, $\omega \cap \gamma(\omega)$ is empty for $1 \neq \gamma \in G_1$, and such that each $z \in U$ is contained in some $\gamma(\omega \cup \partial\omega)$, $\gamma \in G_1$. Then $\omega \cup \delta_1(\omega) \cup \delta_2(\omega) \cup \dots$ is a fundamental region in U with respect to G_2 . We put $m = n - 1 (\geq 1)$ if $n = [G_1: G_2] < \infty$, otherwise $m = \infty$. Since the series $\Theta_{G_1}\Phi$ converges absolutely and uniformly on compact subsets of U , we see that

$$(4.7) \quad \begin{aligned} \phi_1(z) &= \sum_{i=0}^m \left(\sum_{\gamma \in G_2} \Phi(\gamma \circ \delta_i(z)) \gamma'(\delta_i(z))^2 \right) \delta_i'(z)^2 \\ &= \sum_{i=0}^m \phi_2(\delta_i(z)) \delta_i'(z)^2. \end{aligned}$$

Thus

$$(4.8) \quad \begin{aligned} \iint_{\omega} |\phi_1(z)| dx dy &\leq \sum_{i=0}^m \iint_{\omega} |\phi_2(\delta_i(z)) \delta_i'(z)^2| dx dy \\ &= \sum_{i=0}^m \iint_{\delta_i(\omega)} |\phi_2(z)| dx dy \\ &= \iint_{\omega \cup \delta_1(\omega) \cup \dots} |\phi_2(z)| dx dy \\ &= \iint_{U/G_2} |\phi_2(z)| dx dy. \end{aligned}$$

Similarly

$$\iint_{U/G_2} |\phi_2| dx dy \leq \iint_U |\Phi| dx dy .$$

It remains to prove the latter part of the lemma. Assume that (4.8) holds with equality. Then, for every $i=1, 2, \dots$,

$$\begin{aligned} (4.9) \quad & \iint_{\omega} |\phi_2(z) + \phi_2(\delta_i(z))\delta'_i(z)^2| dx dy \\ & = \iint_{\omega} (|\phi_2(z)| + |\phi_2(\delta_i(z))\delta'_i(z)^2|) dx dy . \end{aligned}$$

By (4.9), we see that, for every $i=1, 2, \dots$,

$$(4.10) \quad \phi_2(z)\overline{\phi_2(\delta_i(z))\delta'_i(z)^2} \geq 0 \quad \text{for } z \in \omega .$$

We can easily check that, for every $i=1, 2, \dots$, (4.10) happens if and only if there exists a constant $c_i \geq 0$ such that

$$\phi_2(\delta_i(z))\delta'_i(z)^2 = c_i\phi_2(z) \quad \text{for } z \in \omega, \text{ hence for } z \in U .$$

As $\phi_1 \neq \infty$, we see by (4.7) that $c \equiv 1 + \sum_{i=1}^m c_i$ is finite and that $\phi_1(z) = c\phi_2(z)$. Thus (1) implies (2).

Now we note that, if the index $[G_1 : G_2]$ is infinite, then $A(G_1, \sigma) \cap A(G_2, \sigma) = \{0\}$. Noting this, we see by (4.7) that (2) implies (3). Finally, by (4.1), (3) clearly implies (1). The proof of our lemma is complete.

Theorem 2. *Let $G_1 \cong G_2$ be Fuchsian groups and $\sigma \in \Sigma(G_1)$. Let ν be a measurable function in U such that $\nu = k|\phi_2|/|\phi_2|$, where k is a constant such that $0 < k < 1$ and $\phi_2 \in A(G_2, \sigma)_1$. Then ϕ_2 belongs to $A(G_1, \sigma)$ if and only if $f_\nu G_1 f_\nu^{-1} \subset SL'(2, \mathbf{R})$. Furthermore, in the case where $[G_1 : G_2] < \infty$ and $\phi_2 \notin A(G_1, \sigma)$, $f_\nu \mathbf{R}$ does not belong to $H(G_1)$.*

Proof. If ϕ_2 belongs to $A(G_1, \sigma)$, then

$$\nu(\gamma(z))\overline{\gamma'(z)}/\gamma'(z) = \nu(z) \quad \text{for every } \gamma \in G_1 .$$

So f_ν is compatible with G_1 , that is, $f_\nu G_1 f_\nu^{-1} \subset SL'(2, \mathbf{R})$.

Now we show the converse. Assume that f_ν is compatible with G_1 . By Lemma 8, there exists some $\Phi \in A(1, \sigma)$ such that $\phi_2 = \Theta_{G_2}\Phi$. Put $\phi_1 = \Theta_{G_1}\Phi$. Then we can easily check that

$$(4.11) \quad \|\nu\| = \iint_{U/G_2} \nu\phi_2 dx dy = \iint_{U/G_1} \nu\phi_1 dx dy .$$

By Lemma 9, we have $\|\phi_1\|_{G_1} \leq 1$. But (4.11) does not hold whenever $\|\phi_1\|_{G_1} < 1$. This implies $\|\phi_1\|_{G_1} = 1$. Hence we see by Lemma 9 that $\phi_2 \in A(G_1, \sigma)$.

It remains to prove the latter part of the theorem. Assume that $[G_1:G_2]<\infty$ and that $\phi_2 \in A(G_1, \sigma)$. Put $h=f_\nu|_k$. Clearly f_ν is contained in $F(G_2, h, \sigma)$. Now we shall prove that $h \in H(G_1)$. Assume the contrary. Then h belongs to $H(G_1)$, and thus $F(G_1, h, \sigma)$ is not empty by (1) in Theorem 1. Let f_μ be extremal in the class $F(G_1, h, \sigma)$. By (2) in Theorem 1, we see that f_μ is extremal in the class $F(G_2, h, \sigma)$, too. On the other hand, by Lemma 5, f_ν is unique extremal in the class $F(G_2, h, \sigma)$. Thus f_μ is identical with f_ν . Therefore we see that f_ν is compatible with G_1 , and that $\phi_2 \in A(G_1, \sigma)$. This contradicts our assumption. This shows that $h \in H(G_1)$.

As immediate corollaries, we deduce the following results.

Corollary 2.1. *Let G_1 be a Fuchsian group which has a subgroup G_2 such that $[G_1:G_2]=\infty$ and let $\sigma \in \Sigma(G_1)$. Suppose that $\nu=k|\phi_2|/\phi_2$, where k is a constant such that $0 < k < 1$ and $\phi_2 \in A(G_2, \sigma)$. Then f_ν is not compatible with G_1 .*

Corollary 2.2. *Let G_1 be a Fuchsian group which has a finitely generated subgroup G_2 such that $[G_1:G_2]=\infty$. For a given $\sigma \in \Sigma(G_1)$ and a given $h \in H(G_1)$ with $h|_\sigma \neq 1|_\sigma$, suppose that $F(G_1, h, \sigma)$ is not empty. Then*

$$k(G_1, h, \sigma) > k(G_2, h, \sigma^*)$$

for every $\sigma^* \in \Sigma(G_2)$ with $\sigma^* \subset \sigma$, such that the set $(\sigma^* \setminus \Lambda(G_2))/G_2$ is finite.

Proof. Assume the contrary. Then $k(G_1, h, \sigma) = k(G_2, h, \sigma^*)$ for some $\sigma^* \in \Sigma(G_2)$, where $\sigma^* \subset \sigma$ and the set $(\sigma^* \setminus \Lambda(G_2))/G_2$ is finite. Let f_μ be extremal in the class $F(G_1, h, \sigma)$. As $h|_\sigma \neq 1|_\sigma$, f_μ is not conformal by Lemma 1. By our assumption, f_μ is extremal in the class $F(G_2, h, \sigma^*)$, too. Thus, owing to Lemma 5 and Lemma 7, we see that f_μ is a Teichmüller mapping with finite norm in the class $F(G_2, h, \sigma^*)$, and that f_μ is not conformal. Hence, by Corollary 2.1, f_μ does not belong to $F(G_1, h, \sigma)$. This is absurd. This contradiction proves Corollary 2.2.

Let G be a Fuchsian group and $\sigma \in \Sigma(G)$ and let f be a quasiconformal automorphism of U . We say that f is compatible with G on σ if f satisfies the following property: for every $\gamma \in G$, there exists $\gamma^* \in SL'(2, \mathbf{R})$ such that

$$(4.12) \quad f \circ \gamma(x) = \gamma^* \circ f(x) \quad \text{for all } x \in \sigma .$$

Theorem 3. *Let G_1 be a Fuchsian group which has a normal subgroup G_2 such that $[G_1:G_2]=\infty$. For a given $\sigma \in \Sigma(G_1)$ and a given $h \in H(G_1)$ with $h|_\sigma \neq 1|_\sigma$, suppose that $F(G_2, h, \sigma)$ is not empty. Then $F(G_2, h, \sigma)$ admits no Teichmüller mappings with finite norm.*

We shall prove the following slightly stronger form of Theorem 3.

Theorem 3'. *Let G_1 be a Fuchsian group which has a normal subgroup G_2 such that $[G_1: G_2]=\infty$. Suppose that $\sigma \in \Sigma(G_1)$ and that $\nu=k|\phi|/\phi$, where k is a constant such that $0 < k < 1$ and $\phi \in A(G_2, \sigma)_1$. Then f_ν is not compatible with G_1 on σ . In particular, $f_{\nu \hat{k}}$ does not belong to $H(G_1)$.*

Proof. Assume the contrary. Then f_ν is compatible with G_1 on σ . Let γ be an arbitrary element of G_1 . Then there exists $\gamma^* \in SL'(2, \mathbf{R})$ which satisfies (4.12) for $f=f_\nu$. We put $g=\gamma^{*-1} \circ f_\nu \circ \gamma$ and $h=g_{1 \hat{k}}$. Since G_2 is a normal subgroup of G_1 , we can easily check that h belongs to $H(G_2)$ and that both g and f_ν belong to $F(G_2, h, \sigma)$. By Lemma 5, we see that f_ν is unique extremal in the class $F(G_2, h, \sigma)$. On the other hand, the norm of the complex dilatation of g is equal to $|\nu|$, and thus g is also extremal in the class $F(G_2, h, \sigma)$. Therefore g is identical with f_ν . In other words, it holds that

$$(4.13) \quad f_\nu \circ \gamma(z) = \gamma^* \circ f_\nu(z) \quad \text{for } z \in U.$$

Because the choice of $\gamma \in G_1$ is arbitrary, we see by (4.13) that f_ν is compatible with G_1 . But it contradicts Corollary 2.1. This contradiction proves the theorem.

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