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MULTI-SOLITON SOLUTIONS AND QUASI-PERIODIC SOLUTIONS OF NONLINEAR EQUATIONS OF SINE-GORDON TYPE

ETSURO DATE

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In this paper we are concerned mainly with the sine-Gordon equation

$$u_{\xi\eta}+\sin u = 0, \quad u = u(\xi, \eta) \tag{0.1}$$

and the equation of Pohlmeyer-Lund-Regge [18], [24]

$$u_{\xi\eta}-\frac{v_\xi v_\eta \sin(u/2)}{2\cos^3(u/2)}+\sin u = 0\,, \quad u = u(\xi, \eta)\,, \quad v = v(\xi, \eta)\,, \tag{0.2}$$
$$v_{\xi\eta}+\frac{u_\xi v_\eta+u_\eta v_\xi}{\sin u} = 0\,.$$

In part I of this paper we construct multi-soliton solutions of these equations and some related nonlinear equations. In part II, we construct quasi-periodic solutions by using the abelian integrals. We also discuss the following relation between these equations (0.1), (0.2) from a view-point of the theory of algebraic curves: if v=const. in (0.2), then (0.2) reduces to (0.1). We show that this reduction corresponds to fixed point free involutions of hyperelliptic curves.

We explain briefly the background of this work. Multi-soliton solutions are a class of exact solutions characteristic of nonlinear equations solvable by the inverse scattering method. Several methods of constructing multi-solition solutions are given (see, for example, [22], [29]). A typical class of nonlinear differential equations solvable by the inverse scattering method is the class of nonlinear differential equations for $(M\times M)$-matrix-valued functions $u_j(x, y, t)$, $0\leq j\leq m-1$, $v_k(x, y, t)$, $0\leq k\leq n-1$ which admit the so-called Zakharov-Shabat representations

$$[\textstyle\sum_{j=0}^{m} u_j D^j-\partial/\partial y,\ \sum_{j=0}^{n} v_j D^j-\partial/\partial t],\ D = \partial/\partial x \tag{0.3}$$

where u_m, v_n are non-singular constant diagonal matrices [29]. The Korteweg-de Vries (KdV) equation, the Boussinesq equation, the Kadomtsev-Petviashvili equation and the nonlinear Schrödinger equation are examples of equations in this class. Hereafter we call this class the Zakharov-Shabat systems. The class of nonlinear equations equivalent to the Lax representations

$$\frac{\partial L}{\partial t} = [L, M], \quad L = \sum_{j=0}^{m} u_j(x, t)D^j, \quad M = \sum_{j=0}^{n} v_j(x, t)D^j$$

is a subclass of the Zakharov-Shabat systems.

Periodic and quasi-periodic solutions are studied as a periodic analogue of multi-soliton solutions (see, for example, Dubrobin-Matveev-Novikov [11]). In these studies connection with the theory of Riemann surfaces (algebraic curves) was found via spectral theories of linear operators used in the inverse scattering method. Kricheber [16] extended this connection with the theory of Riemann surfaces to the Zakharov-Shabat systems without using the spectral theory explicitly and gave an unified view-point for quasi-periodic solutions of the Zakharov-Shabat systems. This connection between the theory of quasi-periodic solutions of the Zakharov-Shabat systems and the theory of Riemann surfaces is further formulated in algebro-geometric languages (see, for example, Drinfeld [10], Manin [20], Mumford [23]).

The crucial role in this connection is played by functions $\Phi(x, y, t, p)$ which as functions of (x, y, t) are simultaneous solutions of two linear operators in the Zakharov-Shabat representation (0.3) and as functions on Riemann surfaces have essential singularities at prescribed M points. The forms of singularities depend on the orders of linear operators in (0.3). These functions Φ are constructed by using the theory of abelian integrals. Such a construction goes back to Baker [3] and Akhiezer [2].

This connection, on the one hand, suggests a direct method of constructing multi-soliton solutions of the Zakharov-Shabat systems. Namely, multi-soliton solutions are obtained by applying the above construction to rational algebraic curves with double points. Such an algebro-geometric method is given by Kricheber [16] and Manin [20] (see also [7]).

The equations (0.1), (0.2) are not included in the Zakharov-Shabat systems. The equation (0.1) is the integrability condition of the following pair of linear differential equations

$$\text{(0.4)} \qquad \begin{aligned} &i\Phi_\xi + \frac{u_\xi}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\Phi + \frac{\lambda}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\Phi = 0, \\ &i\Phi_\eta + \frac{1}{2\lambda}\begin{pmatrix}0 & \exp(iu)\\ \exp(-iu) & 0\end{pmatrix}\Phi = 0 \end{aligned}$$

where λ is a parameter ([1], [31]) and the equation (0.2) is the integrability conditions of the following pair of linear differential equations

$$\text{(0.5)} \qquad \begin{aligned} &i\Phi_\xi + \begin{pmatrix}0 & a^*\\ a & 0\end{pmatrix}\Phi + \frac{\lambda}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\Phi = 0, \\ &i\Phi_\eta + \frac{1}{2\lambda}\begin{pmatrix}\cos u & -\exp(-i\omega)\sin u\\ -\exp(i\omega)\sin u & -\cos u\end{pmatrix}\Phi = 0 \end{aligned}$$

where

$$(0.6)\qquad a = i(\exp(i\omega)\sin u)_\xi/2\cos u\,,\quad \omega_\xi = v_\xi \cos u/2\cos^2(u/2)\,,\quad \omega_\eta = v_\eta/2\cos^2(u/2)\,,$$

λ is a parameter and a^* denotes the complex-conjugate of a ([18], [24]). The main difference from the Zakharov-Shabat systems consists in the point that coefficients of the linear equations (0.4), (0.5) depend on a parameter rationally (For the Zakharov-Shabat systems, the parameter appears linearly as a spectral parameter in the case of the Lax representations.). This makes the procedure of the inverse scattering method complicated. Quasi-periodic solutions of (0.1) are discussed by Kozel-Kotlyarov [15], Its [21] and Cherednik [4,5]. In these results it is shown that quasi-periodic solutions of (0.1) correspond to hyperelliptic curves of the forms

$$(0.7)\qquad w^2 + az\prod_{j=1}^{2g}(z-z_j) = 0$$

where a is a constant. In [15], it is shown that a parameter λ in (0.4) is related to the meromorphic function z on the Riemann surfaces of these curves (0.7) by the relation $\lambda = z^{1/2}$ by using the pair of linear differential equations (0.4). In [21], simultaneous solutions of linear differential equations of the form (0.4) are constructed by using abelian integrals on the Riemann surfaces of curves (0.7). In this construction simultaneous solutions are two-valued on these surfaces and the parameter λ is related to the meromorphic function z by the realtion $\lambda = z^{1/2}$.

In part I of this paper we generalize the method of Kricheber and Manin to equations (0.1), (0.2) and some related nonlinear equations and give explicit formulae of multi-soliton solutions by this method.

In part II, we construct quasi-periodic solutions of (0.1), (0.2) by combining a method similar to that of Kricheber [16] and fixed point free involutions of hyperelliptic curves. We also give an explanation of the appearance of two-valued functions in [15], [21].

The contents of this paper is as follows. In section 1 we formulate the method of Kricheber and Manin for constructing multi-soliton solutions of the Zakharov-Shabat systems as given in [7]. Next in section 2 we explain a generalization of this method to a class of nonlinear differential equations which is proposed by Zakharov-Mikhailov [28] and Zakharov-Shabat [30]. This class consists of nonlinear differential equations of the integrability conditions of pairs of linear differential equations of the following forms

$$(0.8)\qquad \Phi_\xi = U(\xi, \eta, \lambda)\Phi\,,\quad \Phi_\eta = V(\xi, \eta, \lambda)\Phi\,,\quad \Phi = \Phi(\xi, \eta, \lambda)$$

where Φ, U, V are $(M\times M)$-matrix-valued functions and U, V are rational functions of a parameter λ whose poles are independent of (ξ, η). The equations (0.1), (0.2) are examples of equations in this class. We construct multi-soliton solutions by constructing functions $\Phi(\xi, \eta, \lambda)$ which turn out to be simultaneous solutions of equations of the forms (0.8) by simple characterizations as functions of λ. By this method we construct multi-soliton solutions of (0.1), (0.2) and the equation of the classical massive Thirring model

$$(0.9)\qquad \begin{aligned} iu_\eta+2v+2|v|^2u &= 0 \\ iv_\xi+2u+2|u|^2v &= 0 \end{aligned}$$

in section 3. Multi-soliton solutions of the equation of the Toda lattice, which is a typical example of nonlinear differential-difference equations solvable by the inverse scattering method, are also constructed by this method in section 3. In section 4, quasi-periodic solutions of the class of nonlinear differential equations of the integrability conditions of pairs of linear differential equations

$$\Phi_\xi = (\textstyle\sum_{j=0}^m \lambda^j M_j(\xi, \eta))\Phi\,, \quad \Phi_\eta = (\textstyle\sum_{j=0}^n \lambda^{-j} N_j(\xi, \eta))\Phi\,, \quad \Phi = \Phi(\xi, \eta, \lambda)\,,$$

where Φ, M_i, N_j are (2×2)-matrices and λ is a parameter, by using the theory of abelian integrals on hyperelliptic curves. As a particular case of this construction, we obtain quasi-periodic solutions of (0.2) in section 5.

Quasi-periodic solutions of (0.1) are constructed in Section 6 by introducing fixed point free involutions of hyperelliptic curves. We also explain the appearance of two-valued functions on Riemann surfaces in [15], [21] from our viewpoint, at the same time discussing a reduction of the equation (0.2) to the equation (0.1). Finally in section 8 we discuss the condition on Riemann surfaces to make our solutions real-valued. For that purpose we employ the concept of symmetric Riemann surfaces, which is introduced by Klein as the Riemann surfaces that correspond to real algebraic curves and developed by Weichold [26].

After the completion of the present work, a paper of Cherednik's [6] was published, in which quasi-periodic solutions of equations expressed as the integrability conditions of linear equations like (0.8) are considered, but the reductions by involutions in this paper nor the concept of symmetric Riemann surfaces are not discussed there.

Some of results in this paper are announced in [7], [8], [9].

Throughout this paper for a matrix c, c^* denotes the complex-conjugate matrix of c.

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Part I. A direct method of constructing multi-soliton solutions

1. The Zakharov-Shabat systems

In this section we review the method of Kricheber and Manin for constructing multi-soliton solutions of the Zakharov-Shabat systems in a manner given in [7].

We consider $(M\times M)$-matrix valued functions $F(x, y, t, \lambda)$ of the following forms

$$F(x, y, t, \lambda) = (\textstyle\sum_{j=0}^{N} \lambda^j F_j(x, y, t)) \exp (x\lambda P+yQ(\lambda)+tR(\lambda)) \tag{1.1}$$

where N is an arbitrary positive integer, $\lambda\in \boldsymbol{C}$, P is a non-singular constant diagonal matrix with entries p_j and $Q(\lambda)$, $R(\lambda)$ are diagonal matrices with entries $q_j(\lambda)=\sum_{k=0}^{m} q_{jk}\lambda^k$, $q_{jk}\in \boldsymbol{C}$, $r_j(\lambda)=\sum_{k=0}^{n} r_{jk}\lambda^k$, $r_{jk}\in \boldsymbol{C}$, respectively.

Let $\alpha_1, \cdots, \alpha_N$, $\beta_1, \cdots, \beta_N$ be mutually distinct complex numbers such that all possible expressions $\sum_{k=1}^{M}\sum_{j=1}^{N} \nu_{jk}p_{jk}$, $\nu_{jk}\in \{\alpha_j, \beta_j\}$, $j=1, \cdots, N$, $p_{jk}\in \{p_1, \cdots, p_M\}$ are mutually distinct and $C_1, \cdots, C_M$, $C_j=(c_{j,ab})$ be arbitrary constant matrices.

Proposition 1.1. *There exists a unique function* $\Phi(x, y, t, \lambda)$ *of the form* (1.1) *that satisfies the following conditions*

(1.2) $F_N = I$.

(1.3) $\Phi(x, y, t, \alpha_j) = \Phi(x, y, t, \beta_j)C_j, \quad j = 1, \cdots, N$.

Proof. By condition (1.2), we can denote as

$$\Phi(x, y, t,)\ \lambda = (\lambda^N I+\textstyle\sum_{j=0}^{N-1} \lambda^j\Phi_j(x, y, t)) \exp (x\lambda P+yQ(\lambda)+tR(\lambda))$$

where I is the identity matrix. The conditions (1.3) are equivalent to the following system of linear equations for unknowns $\Phi_k(x, y, t)=(\Phi_{k,ab}(x, y, t))$, $a, b=1, \cdots, M, k=0, \cdots, N-1$:

$$\begin{aligned}\textstyle\sum_{k=0}^{N-1} \Phi_k\{\alpha_j^k \exp (x\alpha_jP+yQ(\alpha_j)+tR(\alpha_j))-\beta_j^k \exp (x\beta_jP+yQ(\beta_j)+tR(\beta_j))C_j\}\\ = -\alpha_j^N \exp (x\alpha_jP+yQ(\alpha_j)+tR(\alpha_j))+\beta_j^N \exp (x\beta_jP+yQ(\beta_j)+tR(\beta_j))C_j\,,\\ j = 1, \cdots, N\,.\end{aligned}$$

This system splits into M systems of linear equations for $\Phi_{k,ab}(x, y, t)$, $k=0, \cdots, N-1$, $b=1, \cdots, M$:

$$\begin{aligned}\textstyle\sum_{k=0}^{N-1}\sum_{b=1}^{M} \Phi_{k,ab}\{\alpha_j^k e_b(\alpha_j)\delta_{bc}-\beta_j^k e_b(\beta_j)c_{j,bc}\}\\ = -\alpha_j^N e_a(\alpha_j)\delta_{ab}+\beta_j^N e_a(\beta_j)c_{j,ab}\,, \quad j=1, \cdots, N,\ c=1, \cdots, M\end{aligned} \tag{1.4}$$

where $e_a(\lambda)= \exp (x\lambda p_a+yq_a(\lambda)+tr_a(\lambda))$. The coefficient matrices of these systems which are labeled by $a=1, \cdots, M$ are the same. The determinant of the coefficient matrix is a linear combination of

$$\textstyle\prod_{k=1}^{N}\prod_{j=1}^{N} e_{j_k}(\nu_{j_k})\,, \quad \nu_{j_k}\in \{\alpha_j, \beta_j\}\ , \quad j_k = 1, \cdots, M\,.$$

By our assumptions on α_j, β_j, these functions of x are linearly independent. Consequently the determinant does not vanish identically as a function of x.

Therefore the coefficients $\Phi_j(x, y, t)$, $k=0, \cdots, N-1$ are uniquely determined. Q.E.D.

REMARK 1.2. i) The conditions (1.2), (1.3) are suggested by the method of Kricheber and Manin.

ii) For our arguments, we need only the fact that the coefficient matrix of the system (1.4) is non-singular.

Next we derive a pair of linear differential equations which the function $\Phi(x, y, t, \lambda)$ satisfies.

Proposition 1.3. *The function $\Phi(x, y, t, \lambda)$ satisfies the following pair of linear differential equations*

$$\sum_{j=0}^{m} u_j(x, y, t)D^j\Phi = (\partial/\partial y)\Phi\,, \quad \sum_{j=0}^{n} v_j(x, y, t)D^j\Phi = (\partial/\partial t)\Phi$$

where u_j (resp. v_j) are determined by the equations

$$\sum_{j=0}^{m} u_j \sum_{k=l}^{j} {}_jC_k D^{j-k}\xi_{k-l} = \sum_{j=l}^{m} \xi_{j-l}Q_j\,, \quad l=0, \cdots, m\,,$$
$$(resp.\ \sum_{j=0}^{n} v_j \sum_{k=l}^{j} {}_jC_k D^{j-k}\xi_{k-l} = \sum_{j=l}^{n} \xi_{j-l}R_j\,, \quad l=0, \cdots, n)$$

$\xi_j=\Phi_{N-j}$ and Q_j, R_j are constant diagonal matrices of order M with entries q_{kj}, r_{kj}, respectively.

Proof. First we note that the system of linear equations

$$(1.5)\qquad \sum_{j=0}^{m} u_j \sum_{k=l}^{j} {}_jC_k D^{j-k}\xi_{k-l} = \sum_{j=l}^{m} \xi_{j-l}Q_j\,, \quad l=0, \cdots, m$$

is uniquely solvable. For the coefficient matrix of this system is similar to a triangular matrix whose diagonal entries are 1. Consider the function

$$F(x, y, t, \lambda) = \sum_{j=0}^{m} u_j D^j\Phi-(\partial/\partial y)\Phi\,.$$

Since functions $u_j(x, y, t)$ are determined by the equations (1.5), the function $F(x, y, t, \lambda)$ has the form

$$F(x, y, t, \lambda) = (\sum_{j=0}^{N-1} \lambda^j F_j(x, y, t)) \exp(x\lambda P+yQ(\lambda)+tR(\lambda))\,.$$

This function satisfies the relations

$$F(x, y, t, \alpha_j) = F(x, y, t, \beta_j)C_j\,, \quad j=1, \cdots, N\,.$$

By the argument in the proof of Prop. 1.1. and the fact that in this case the coefficient of λ^N is zero, we see that F_j satisfy the system of linear equations (1.4) in which the right hand side is set equal to zero. Therefore we have $F_j=0$. Q.E.D.

Theorem 1.4. $[\sum_{j=0}^{m} u_j D^j-\partial/\partial y, \sum_{j=0}^{n} v_j D^j-\partial/\partial t]=0$, *that is, we have a Zakharov-Shabat representation.*

Proof. By Prof. 1.3., we have

$$[\sum_{j=0}^{m} u_j D^j - \partial/\partial y,\ \sum_{j=0}^{n} v_j D^j - \partial/\partial t]\Phi = 0\,,$$

The operator on the left hand side is a linear ordinary differential operator with respect to x and does not contain a parameter λ. The kernel of this operator contains one parameter family $\Phi(x, y, t, \lambda)$ and consequently infinite dimensional. Therefore this operator must be identically zero. Q.E.D.

The Lax representations are drived as follows.

Corollary 1.5. *If $Q(\alpha_j)=Q(\beta_j)$, $j=1, \cdots, N$, the functions u_j, v_k are independent of y and we have*

$$[\sum_{j=0}^{m} u_j D^j,\ \sum_{j=0}^{n} v_j D^j - \partial/\partial t] = 0$$

Proof. If $Q(\alpha_j)=Q(\beta_j)$, we can cancel $q_b(\alpha_j)=q_b(\beta_j)$ on the both hand sides of (1.4) and the resulting system is independent of y. Thus Φ_k, and consequently u_j, are functions independent of y. Q.E.D.

Corollary 1.6. *If $Q(\alpha_j)=Q(\beta_j)$, $R(\alpha_j)=R(\beta_j)$, $j=1, \cdots, N$, the functions u_j, v_k are independent of y, t and*

$$[\sum_{j=0}^{m} u_j D^j,\ \sum_{j=0}^{n} v_j D^j] = 0$$

holds.

REMARK 1.7. For scalar cases ($M=1$), the assumptions on α_j, β_j are simplified. We only require that α_j, β_j are mutually distinct. The coefficient matrix of the system (1.4) is the wronskian of functions (of x)

$$f_j(x) = \exp(\alpha_j x + yQ(\alpha_j) + tR(\alpha_j)) - c_j \exp(\beta_j x + yQ(\beta_j) + tR(\beta_j))\,,$$
$$j=1, \cdots, N$$

where we put $p=1$ without loss of generality. This wronskian does not vanish identically, because we have the following.

Proposition 1.8. *If the wronskian of analytic functions $F_j(x)$, $j=1, \cdots, n$, of x vanishes identically, then functions F_j are linearly dependent.*

Proof. There exist a natural number $m (1 \leq m \leq n-1)$ and $j_1, \cdots, j_m (<n)$ such that the wronskian of $F_{j_1}, \cdots, F_{j_m}$ does not vanish at some point $x=x_0$ and for all $i_1, \cdots, i_l$ ($l \geq m+1$) the wronskians of $F_{i_1}, \cdots, F_{i_l}$ vanish identically. By renumbering the indices, we put $j_1=1, \cdots, j_m=m$. The functions $F_1, \cdots, F_m$ form a fundamental system of solutions of the linear ordinary differential equation $L(F)=w(F, F_1, \cdots, F_m)=0$ in a neighborhood U of x_0 where $w(F, F_1, \cdots, F_m)$ denotes the wronskian of functions $F, F_1, \cdots, F_m$. By the choice of m, we have $L(F_k)=0$, $k \geq m+1$. Therefore $F_k (k>m)$ are linear combinations of

$F_1, \cdots, F_m$ in U. By analyticity, these linear relations hold in the whole domain of definition of F_j. Q.E.D.

Corresponding to various choice of P, $Q(\lambda)$, $R(\lambda)$, we obtain various Zakharov-Shabat, or Lax, representations. We give an example.

EXAMPLE i) For $M=1$, $Q(\lambda)=\lambda^2$, $R(\lambda)=\lambda^3+b\lambda$, $b\in \boldsymbol{C}$, we have

$$[D^2+u-\partial/\partial y,\ D^3+(3u/2+b)D+v-\partial/\partial t]=0$$

where

$$u=-2(\partial/\partial x)\Phi_{N-1}\,,\quad v=(\partial/\partial x)(3(\Phi_{N-1})^2/2-3(\partial/\partial x)\Phi_{N-1}-3\Phi_{N-2})\,.$$

This operator equation is equivalent to the following system of nonlinear equations for u, v,

$$3u_y=4v_y-3u_{xx}\,,$$
$$v_y-u_t=v_{xx}-u_{xxx}-3uu_x/2-bu_x\,.$$

By elminating v, we have the Kadomtsev-Petviashvili equation

$$3u_{yy}/4+(-u_t+bu_x+u_{xxx}/4+3uu_x/2)_x=0\,.$$

In other words, we have a solution $u=-2(\partial/\partial x)\Phi_{N-1}$ of the Kadomtsev-Petviashvili equation by the above construction for the choice $Q(\lambda)=\lambda^2$, $R(\lambda)=\lambda^3+b\lambda$.

ii) The Korteweg-de Vries equation is derived as follows. In the construction in i) we put $\beta_j=-\alpha_j$, $j=1,\cdots,N$ and $b=0$, then $Q(\alpha_j)=Q(\beta_j)$, $j=1,\cdots,N$. By Cor. 1.5., we conclude that $u(x,t)=-2(\partial/\partial x)\Phi_{N-1}(x,t)$ is a solution of the KdV equation

$$u_t-3uu_x/2-u_{xxx}=0\,.$$

Next we show that solutions constructed in this way are the same as N-soliton solutions obtained by the inverse scattering method. We show this for the Kadomtsev-Petviashvili equation.

By Cramer's formula and $u=-2(\partial/\partial x)\Phi_{N-1}$, we have

$$u=2(\partial^2/\partial x^2)\log w(f_1,\cdots,f_N)$$

where $f_j(x,y,t)=\exp(\alpha_jx+\alpha_j^2y+(\alpha_j^3+b\alpha_j)t)-c_j\exp(\beta_jx+\beta_j^2y+(\beta_j^3+b\beta_j)t)$, $j=1,\cdots,N$ and $w(f_1,\cdots,f_N)$ is the wronskian of functions $f_1,\cdots,f_N$ of x. By direct calculations we have

$$\begin{aligned}w(f_1\cdots,f_N)&=\textstyle\prod_{j=1}^N e(\alpha_j)\det[\alpha_j^{k-1}-c_j\beta_j^{k-1}e(\beta_j)e(\alpha_j)^{-1}]\\&=\textstyle\prod_{j=1}^N e(\alpha_j)(\det(\alpha_j^{k-1}))[\det\{\alpha_j^{k-1}-c_j\beta_j^{k-1}e(\beta_j)e(\alpha_j)^{-1}\}]\times\\&\qquad\times[\det\{(\alpha_j^{k-1})^{-1}\}]\\&=\prod_{j=1}^N e(\alpha_j)\prod_{N\geq a>b\geq 1}(\alpha_a-\alpha_b)\det\left[\delta_{jk}-\frac{\prod_{l=1;\neq k}^N(\beta_j-\alpha_l)}{\prod_{l=1;\neq k}^N(\alpha_k-\alpha_l)}e(\beta_j)e(\alpha_j)^{-1}\right]\end{aligned}$$

where $e(\lambda)=\exp(\lambda x+\lambda^2 y+(\lambda^3+b)t))$. Introducing notations $g(\lambda)=\prod_{j=1}^{N}(\lambda-\alpha_j)$, $\dot{g}=dg/d\lambda$, we can rewrite the above expression as

$$=\prod_{j=1}^{N} e(\alpha_j)\prod_{N\geq a>b\geq 1}(\alpha_a-\alpha_b)\det[\delta_{jk}-\frac{c_j}{\beta_j-\alpha_k}\frac{g(\beta_j)}{\dot{g}(\alpha_k)}e(\beta_j)e(\alpha_j)^{-1}].$$

Thus we have

$$u(x, y, t)=2\frac{\partial^2}{\partial x^2}\log\det[\delta_{jk}-\frac{c_j}{\beta_j-\alpha_k}\frac{g(\beta_j)}{\dot{g}(\alpha_k)}\exp\{(\beta_j-\alpha_j)x+(\beta_j^2-\alpha_j^2)y+(\beta_j^3-\alpha_j^3+b\beta_j-b\alpha_j)t\}].$$

From this expression, we see easily that our solutions are identical with N-soliton solutions.

2. A generalization of the method in Section 1

In this section we explain a generalization of the method described in Section 1. to a class of nonlinear equations of the integrability conditions of pairs of linear differential equations (0.8). To simplify the arguments, we assume that functions U, V do not have poles at ∞.

We aim to construct $(M\times M)$-matrix-valued functions $\Phi(\xi, \eta, \lambda)$ which turn out to be simultaneous solutions of pairs of linear differential equations of the following forms

$$\Phi_\xi=(\sum_{j=1}^{m}\sum_{k=0}^{r_j}\frac{1}{(\lambda-a_j)^k}M_{jk}(\xi, \eta))\Phi,$$

$$\Phi_\eta=(\sum_{j=1}^{m}\sum_{k=0}^{r_j}\frac{1}{(\lambda-a_j)^k}N_{jk}(\xi, \eta))\Phi$$

where $m, r_1, \cdots, r_m$ are fixed positive integers and $a_1, \cdots, a_m$ are fixed mutually distinct complex numbers.

We consider $(M\times M)$-matrix valued functions of the following forms

$$F(\xi, \eta, \lambda)=(F_0+\sum_{j=1}^{N}\frac{1}{(\lambda-\lambda_j)}F(\xi, \eta))\exp(\sum_{j=1}^{m}\sum_{k=0}^{r_j}\frac{1}{(\lambda-a_j)^k}f_{jk}(\xi, \eta)) \tag{2.1}$$

where N is an arbitrary positive integer, $\lambda_1, \cdots, \lambda_N$ are arbitrary mutually distinct complex numbers and $f_{jk}(\xi, \eta)$ are $(M\times M)$-matrix-valued smooth functions of (ξ, η). We intend to single the function $\Phi(\xi, \eta, \lambda)$ out from functions of the form (2.1) by the conditions

$$F_0=I$$

$$\Phi(\xi, \eta, \mu_j)c_j=0, \quad j=1, \cdots, MN$$

where $\mu_1, \cdots, \mu_{MN}$ are mutually distinct complex numbers and $c_1, \cdots, c_{MN}$ are contant vectors. These conditions are equivalent to a system of linear equations for unknown coefficients $\Phi_j(\xi,\eta), j=1,\cdots,N$ of

$$\Phi(\xi,\eta,\lambda)=(I+\textstyle\sum_{j=1}^{N}\frac{1}{\lambda-\lambda_j}\Phi_j(\xi,\eta))\exp(\textstyle\sum_{j=1}^{m}\sum_{k=0}^{r_j}\frac{1}{(\lambda-a_j)^k}f_{jk}(\xi,\eta)).$$

We assume that the coefficient matrix of the system is non-singular as a function of ξ(or η). This is an assumption on μ_k, c_k for fixed λ_j, f_{jk}, a_k. Though to express this assumption explicitly in terms of μ_k, c_k is difficult in general, we can write down this assumption explicitly for each special case which we discuss later.

If this assumption is satisfied, then the function $\Phi(\xi,\eta,\lambda)$ is uniquely determined.

Further by an argument similar to that in Section 1, we can show that the function $\Phi(\xi,\eta,\lambda)$ satisfies the pair of linear differential equations of the form

$$\Phi_\xi=(\textstyle\sum_{j=1}^{m}\sum_{k=0}^{r_j}\frac{1}{(\lambda-a_j)^k}M_{jk}(\xi,\eta))\Phi,$$
(2.2)
$$\Phi_\eta=(\textstyle\sum_{j=1}^{m}\sum_{k=0}^{r_j}\frac{1}{(\lambda-a_j)^k}N_{jk}(\xi,\eta))\Phi$$

where $M_{jk}(\xi,\eta)$, $N_{jk}(\xi,\eta)$ are rational functions of elements of $(\partial/\partial\xi)f_{jk}(\xi,\eta)$, $(\partial/\partial\eta)f_{jk}(\xi,\eta)$, $\Phi_j(\xi,\eta)$.

In this way we can construct solutions of nonlinear equations of the integrability conditions of pairs of linear differential equations of the form (2.2).

3. Construction of multi-soliton solutions

In this section we construct multi-soliton solutions of (0.1), (0.2) (0.9) and the equation of the Toda lattice by the method explained in Section 2.

3.1 The sine-Gordon equation.

Let N be an arbitrary positive integer $\alpha_1.\cdots,\alpha_N$ be mutually distinct complex numbers such that $\alpha_j\neq-\alpha_k$, $j,k=1,\cdots,N$ and $c_1,\cdots,c_N$ be arbitrary complex numbers.

We consider functions $\Phi_n(\xi,\eta,\lambda)$, $n=1,2$ of the following forms

(3.1) $$\Phi_n(\xi,\eta,\lambda)=(\lambda^N+\textstyle\sum_{j=0}^{N-1}\phi_{nj}(\xi,\eta)\lambda^j)\exp(2^{-1}i(\xi\lambda+\eta\lambda^{-1})).$$

Lemma 3.1. *There exist unique functions* $\Phi_{nj}(\xi,\eta,\lambda)$, $n=1,2$ *of the form* (3.1) *that satisfy the conditions*

(3.2) $$\Phi_n(\xi,\eta,\alpha_j)=(-1)^{n-1}c_j\Phi_n(\xi,\eta,\alpha_j),\quad j=1,\cdots,N,n=1,2.$$

Proof. Conditions (3.2) are equivalent to the following systems of linear equations for unknowns $\phi_{nj}(\xi,\eta)$, $j=1,\cdots,N$, labeled by $n=1,2$:

$$\begin{aligned}(3.3)_n \quad & \sum_{k=0}^{N-1} \{\alpha_j^k \exp(2^{-1}i(\alpha_j\xi+\alpha_j^{-1}\eta))+(-1)^n c_j(-\alpha_j)^k \exp(-2^{-1}i(\alpha_j\xi \\ & \qquad +\alpha_j^{-1}\eta))\}\phi_{nk}(\xi,\eta) \\ & = -\alpha_j^N \exp(2^{-1}i(\alpha_j\xi+\alpha_j^{-1}\eta)+(-1)^n c_j(-\alpha_j)^N \exp(-2^{-1}i(\alpha_j\xi+\alpha_j^{-1}\eta)), \\ & \qquad j=1,\cdots,N.\end{aligned}$$

The determinants of the coefficient matrices of these systems are constant multiples of the wronskians of functions (of ξ)

$$f_{nj}(\xi) = \exp(2^{-1}i(\alpha_j\xi+\alpha_j^{-1}\eta))+(-1)^n c_j \exp(-2^{-1}i(\alpha_j\xi+\alpha_j^{-1}\eta)), \quad j=1,\cdots,N.$$

By Prof. 1.8. and assumptions on α_j, these wronskians do not vanish identically Q.E.D.

REMARK 3.2. We can put the above argument to fit in with the argument in Section 2 by defining (2×2)-matrix-valued function $\Phi(\xi,\eta,\lambda)$ by

$$\Phi(\xi,\eta,\lambda) = \begin{pmatrix}\Phi_1(\xi,\eta,\lambda) & \Phi_1(\xi,\eta,-\lambda)\\ \Phi_2(\xi,\eta,\lambda) & -\Phi_2(\xi,\eta,-\lambda)\end{pmatrix}$$

Then the conditions (3.2) are written as

$$\Phi(\xi,\eta,\alpha_j)^t(1,c_j)=0,\quad \Phi(\xi,\eta,-\alpha_j)^t(c_j,1)=0, \quad j=1,\cdots,N.$$

As in Section 1, we can show that the function $\Phi(\xi,\eta,\lambda)$ satisfies the linear differential equations (0.4) in which the coefficients are given by

$$(3.4) \qquad u_\xi = \phi_{1,N-1}-\phi_{2\,N-1}, \quad \exp(iu) = \phi_{1,0}/\phi_{2,0}.$$

In this way, we have

Theorem 3.3. *The function* $u=-i\log(\phi_{10}/\phi_{20})$ *is a solution of the sine-Gordon equation* (0.1).

Next we show that solutions we have constructed are identical with N-soliton solutions of (0.1) obtained by the inverse scattering method. We denote the coefficient matrices of the systems $(3.3)_n$ by A_n. Using the Cramer's formula, we have

$$(\det A_1)\phi_{10} = (-1)^N(\textstyle\prod_{j=1}^N \alpha_j)\det A_2, \quad (\det A_2)\phi_{20} = (-1)^N(\textstyle\prod_{j=1}^N \alpha_j)\det A_1$$

and consequently

$$\phi_{10}/\phi_{20} = (\det A_2/\det A_1)^2.$$

As in Section 1, we can rewrite $\det A_n$ as

$$\begin{aligned}\det A_n &= \exp\{\textstyle\sum_{j=1}^N 2^{-1}i(\alpha_j\xi+\alpha_j^{-1}\eta)\}\textstyle\prod_{N\geq a>b\geq 1}(\alpha_a-\alpha_b)\times \\ &\times \det\left[\delta_{jk}+(-1)^{n+1}\frac{c_j}{\alpha_j+\alpha_k}\frac{g(-\alpha_j)}{\dot g(\alpha_k)}\exp\{-2^{-1}i\xi(\alpha_j+\alpha_k)-2^{-1}i\eta(\alpha_j^{-1}+\alpha_k^{-1})\}\right].\end{aligned}$$

Using these expressions, we can identify our solutions with N-soliton solutions obtained by the inverse scattering method.

Our solutions are complex-valued in general. Real-valued solutions are obtained in the following way. We choose $\alpha_1, \cdots, \alpha_N, c_1, \cdots, c_N$ so that for a suitable permutation σ of $\{1, \cdots, N\}$, $\alpha_j^* = -\alpha_{\sigma(j)}$, $c_j^* = -c_{\sigma(j)}$, $j=1, \cdots, N$ hold. Then using $(3.3)_n$, we have $\phi_{1j}^* = (-1)^{N+j}\phi_{2j}$. In particular, we have $\phi_{20} = (-1)^N \phi_{10}^*$. In view of (3.4), we have a real-valued solution of the sine-Gordon equation.

3.2. The equation of Pohlmeyer-Lund-Regge.

In this case we consider (2×2)-matrix-valued functions $\Phi(\xi, \eta, \lambda) = (\Phi_{jk}(\xi, \eta, \lambda))$ of the following forms

$$
\begin{aligned}
&\Phi_{11}(\xi, \eta, \lambda) = (\lambda^N + \textstyle\sum_{j=0}^{N-1} \phi_{1j}(\xi, \eta)\lambda^j) \exp(2^{-1}i(\lambda\xi + \lambda^{-1}\eta))\,, \\
(3.5) \qquad &\Phi_{21}(\xi, \eta, \lambda) = (\textstyle\sum_{j=0}^{N-1} \phi_{2j}(\xi, \eta)\lambda^j) \exp(2^{-1}i(\lambda\xi + \lambda^{-1}\eta))\,, \\
&\Phi_{12}(\xi, \eta, \lambda) = -\Phi_{21}(\xi, \eta, \lambda^*)^*, \quad \Phi_{22}(\xi, \eta, \lambda) = \Phi_{11}(\xi, \eta, \lambda^*)^*
\end{aligned}
$$

where N is an arbitrary positive integer and * denotes the complex-conjugation. The choice of the form of $\Phi(\xi, \eta, \lambda)$ is based on the following observation: if ${}^t(\Phi_1(\xi, \eta, \lambda), \Phi_2(\xi, \eta, \lambda))$ is a solution of the equation with real u, v, then ${}^t(-\Phi_2(\xi, \eta, \lambda^*)^*, \Phi_1(\xi, \eta, \lambda^*)^*)$ is also a solution of (0.5).

Let $\alpha_1, \cdots, \alpha_N$ be mutually distinct complex numbers such that for all j $\mathrm{Im}\,\alpha_j$ have the same signature and c_j be arbitrary complex numbers.

Lemma 3.4. *There exists a unique function $\Phi(\xi, \eta, \lambda)$ of the form* (3.5) *that satisfies the conditions.*

$$
(3.6) \qquad \Phi(\xi, \eta, \alpha_j)\,{}^t(1, c_j) = 0\,, \quad j=1, \cdots, N\,.
$$

Proof. The conditions (3.6) are equivalent to the following system of linear equations for unknowns $\phi_{nj}(\xi, \eta)$, $n=1, 2, j=0, \cdots, N-1$:

$$
\begin{aligned}
&\alpha_j^N \exp(2^{-1}(\alpha_j\xi + \alpha_j^{-1}\eta) + \textstyle\sum_{k=0}^{N-1} \alpha_j^k \exp(2^{-1}i(\alpha_j\xi + \alpha_j^{-1}\eta))\phi_{1k}(\xi, \eta) \\
&\qquad\qquad = -c_j \textstyle\sum_{k=0}^{N-1} \alpha_j^k \exp(-2^{-1}i(\alpha_j\xi + \alpha_j^{-1}\eta))\phi_{2k}(\xi, \eta)^* \\
&\qquad \textstyle\sum_{k=0}^{N-1} \alpha_j^k \exp(2^{-1}i(\alpha_j\xi + \alpha_j^{-1}\eta))\phi_{2k}(\xi, \eta) \\
&= c_j\, \alpha_j^N \exp(-2^{-1}(\alpha_j\xi + \alpha_j^{-1}\eta)) + c_j \textstyle\sum_{k=0}^{N-1} \alpha_j^k \exp\{-2^{-1}i(\alpha_j\xi + \alpha_j^{-1}\eta))\phi_{1k}(\xi, \eta)\}^* \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j=1, \cdots, N\,.
\end{aligned}
$$

In matrix notations this system is written as

$$
\begin{aligned}
&\begin{bmatrix} EA & CE^{-1}A \\ C^*E^{*-1}A^* & -E^*A^* \end{bmatrix} {}^t(\phi_{10}, \cdots, \phi_{1,N-1}, \phi_{20}^*, \cdots, \phi_{2,N-1}^*) \\
&= -{}^t(\alpha_1^N e(\alpha_1), \cdots, \alpha_N^N e(\alpha_N), c_1^*(\alpha_1^*)^N e(\alpha_1^*), \cdots, c_N^*(\alpha_N^*)^N e(\alpha_N^*))
\end{aligned}
$$

where A is the $(N\times N)$-matrix with (j, k)-elements α_j^{k-1}, E, C are diagonal matrices of order N with entries $e(\alpha_j)$, c_j, respectively and $e(\lambda)=\exp(2^{-1}(\lambda\xi+\lambda^{-1}\eta))$. The coefficient matrix of this system is similar to

$$\begin{bmatrix} E+CE^{-1}AA^{*^{-1}}E^{*^{-2}}C^*A^*A^{-1} & CE^{-1}AA^{*^{-1}} \\ 0 & -E^* \end{bmatrix}.$$

We show that the matrix $E+CE^{-1}AA^{*^{-1}}E^{*^{-2}}C^*A^*A^{-1}$ is non-singular. First we have

$$E+CE^{-1}AA^{*^{-1}}E^{*^{-2}}C^*A^*A^{-1}=E[I+(CE^{-2}AA^{*^{-1}})(CE^{-2}AA^{*^{-1}})^*]$$

By direct calculations, we have

$$AA^{*^{-1}}=GBH \tag{3.7}$$

where B is the $(N\times N)$-matrix with (j,k)-elements $(\alpha_j-\alpha_k^*)^{-1}$ and G, H are diagonal matrices of order N with entries $\prod_{l=1}^N(\alpha_j-\alpha_l^*)$, $\prod_{l=1;\neq j}^N(\alpha_j^*-\alpha_l^*)$ respectively. On the other hand by using Lagrange's interpolation formula, we have

$$(BG^*H)^{-1}=B^*GH^*. \tag{3.8}$$

Using these relations (3.7), (3.8), we have

$$E+CE^{-1}AA^{*^{-1}}E^{*^{-2}}C^*A^*A^{-1}=E\{GBG^*+CGE^{-2}B(CGE^{-2})^*\}HB^*H^*$$

The matrix $\pm i(GBG^*+CGE^{-2}B(CGE^{-2})^*)$ is the Gram matrix of functions of x

$$f_j(x)={}^t(1,\ c_j\exp(-i(\alpha_j\xi+\alpha_j^{-1}\eta)))\exp(\pm\alpha_jx),\quad j=1,\cdots,N\quad\text{on}\quad[0,\infty).$$

Therefore under our assumption on α_j, this Gram matrix is non-singular and consequently our coefficient matrix is non-singular. Q.E.D.

By an argument similar to that in Section 1, we see that the function $\Phi(\xi,\eta,\lambda)$ satisfies the following linear differential equations

$$\begin{aligned} &i\Phi_\xi+\begin{pmatrix} 0 & \phi_{2,N-1}^* \\ \phi_{2,N-1} & 0 \end{pmatrix}\Phi+\frac{\lambda}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\Phi=0, \\ &i\Phi_\eta+\frac{1}{2\lambda(|\phi_{10}|^2+|\phi_{20}|^2)}\begin{pmatrix} |\phi_{10}|^2-|\phi_{20}|^2 & 2\phi_{10}\phi_{20}^* \\ 2\phi_{10}^*\phi_{20} & -|\phi_{10}|^2+|\phi_{20}|^2 \end{pmatrix}\Phi=0. \end{aligned} \tag{3.9}$$

Comparing (0.5) with (3.9), we put

$$\begin{aligned} \cos u &= (|\phi_{10}|^2-|\phi_{20}|^2)/(|\phi_{10}|^2+|\phi_{20}|^2), \\ \exp(i\omega)\sin u &= -2\phi_{10}^*\phi_{20}/(|\phi_{10}|^2+|\phi_{20}|^2). \end{aligned} \tag{3.10}$$

Then we have

$$\exp(2i\omega)=\phi_{10}^*\phi_{20}/\phi_{10}\phi_{20}^*,$$

On the other hand, by comparing the coefficients of $\exp(2^{-1}i(\lambda\xi+\lambda^{-1}\eta))$ of (3.9), we see that the relations

$$(3.11)\qquad i(\partial/\partial\xi)\phi_{10} = -\phi^*_{2,N-1},\ \phi_{20}\,, \quad i(\partial/\partial\xi)\phi_{20} = -\phi_{2,N-1}\phi_{10}$$

hold. Using these relations, we have

$$(3.12)\qquad \omega_\xi = 2^{-1}(|\phi_{10}|^2-|\phi_{20}|^2)\{(\phi_{10}\phi^*_{20})^{-1}\phi^*_{2,N-1}+(\phi^*_{10}\phi_{20})^{-1}\phi_{2,N-1}\}$$

Combining (3.10) and (3.12), we have

$$2\omega_\xi \cos^2(u/2)/\cos u = \phi_{2,N-1}\phi_{10}/\phi_{20}+\phi^*_{2,N-1}\phi^*_{10}/\phi^*_{20}\,.$$

Again using (3.11), we have

$$2\omega_\xi \cos^2(u/2)/\cos u = i(\partial/\partial\xi)\log(-\phi^*_{20}/\phi_{20})\,.$$

Similarly we have

$$2\omega_\eta \cos^2(u/2) = i(\partial/\partial\eta)\log(-\phi^*_{20}/\phi_{20})\,.$$

In view of (0.6), we have

Theorem 3.5. *The pair of functions*

$$u = \arccos((|\phi_{10}|^2-|\phi_{20}|^2)/(|\phi_{10}|^2+|\phi_{20}|^2)), \quad v = 2\arg(\phi_{20})+v_0, \quad v_0\in\boldsymbol{R},$$

is a solution of (0.2).

REMARK 3.6. For $N=1$, $\alpha_1=d\exp(i\delta)$, $c_1=r\exp(i\gamma)$, we have

$$u = 2\arcsin[\sin\delta/\cosh\{(d\xi-d^{-1}\eta)\sin\delta+\log r\}]\,,$$
$$v = -(d\xi+d^{-1}\eta)+v_0\,.$$

In view of the relations

$$\theta = 2^{-1}u\,, \quad \lambda_\xi = 2^{-1}v_\xi\tan^2(u/2)\,, \quad \lambda_\eta = -2^{-1}v_\eta\tan^2(u/2)$$

where θ, λ are variables used in Lund [18], our solution is the same as the one-soliton solution of (0.2) given by Lund [18]

Restricting the choice of α_j, c_j so that for a suitable permutation σ of $\{1,\cdots,N\}$ the relations

$$\alpha^*_j = -\alpha_{\sigma(j)}\,, \quad c^*_j = -c_{\sigma(j)}\,, \quad j=1,\cdots,N$$

hold, we have

$$\phi^*_{1j} = \phi_{1j}\,, \quad \phi^*_{2j} = \phi_{2j}\,.$$

That is, we have a solution of the sine-Gordon equation

3.3. The equation of the massive Thirring model.

This equation (0.9) is the integrability conditions of the linear differential equations

$$(3.13)\qquad \begin{aligned} &i\Phi_\xi+2\lambda\begin{bmatrix}0 & a\\ a^* & 0\end{bmatrix}\Phi+\lambda^2\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\Phi=0\,,\\ &i\Phi_\eta+2|v|^2\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\Phi+\frac{2}{\lambda}\begin{bmatrix}0 & b\\ b^* & 0\end{bmatrix}\Phi+\frac{1}{\lambda^2}\begin{bmatrix}-1 & 0\\ 1 & 0\end{bmatrix}\Phi=0\,,\end{aligned}$$

where λ is a parameter and $a=\exp\left(\int_\xi^\infty |u|^2 d\xi\right)u$, $b=\exp\left(\int_\xi^\infty |u|^2 d\xi\right)v$ ([14], [17]).

In this case, we consider (2×2)-matrix valued functions $\Phi(\xi,\eta,\lambda)=(\Phi_{jk}(\xi,\eta,\lambda))$,

$$(3.14)\qquad \begin{aligned} &\Phi_{12}(\xi,\eta,\lambda)=(\textstyle\sum_{j=1}^N \phi_{1j}(\xi,\eta)\lambda^{2j-1})\exp(i(\lambda^2\xi+\lambda^{-2}\eta))\,,\\ &\Phi_{22}(\xi,\eta,\lambda)=(1+\textstyle\sum_{j=1}^N \phi_{2j}(\xi,\eta)\lambda^{2j})\exp(i(\lambda^2\xi+\lambda^{-2}\eta))\,,\\ &\Phi_{11}(\xi,\eta,\lambda)=\Phi_{22}(\xi,\eta,\lambda^*)^*\,,\quad \Phi_{21}(\xi,\eta,\lambda)=-\Phi_{12}(\xi,\eta,\lambda^*)^*\end{aligned}$$

where N is an arbitrary positive integer.

Let $\alpha_1,\cdots,\alpha_N$ be mutually distinct complex numbers such that for all j, $\operatorname{Im}\alpha_j$ have the same signature and $c_1,\cdots,c_N$ be arbitrary complex numbers.

As in the preceding subsection 3.2., we have

Lemma 3.7. *There exists a unique function $\Phi(\xi,\eta,\lambda)$ of the form* (3.14) *that satisfies the conditions*

$$\Phi(\xi,\eta,\alpha_j)^t(1,c_j)=0\,,\quad j=1,\cdots,N\,.$$

Further the function $\Phi(\xi,\eta,\lambda)$ satisfies the following linear differential equations

$$(3.15)\qquad \begin{aligned} &i\Phi_\xi+2\lambda\begin{bmatrix}0 & \phi_{1N}/\phi_{2N}\\ \phi_{1N}^*/\phi_{2N}^* & 0\end{bmatrix}\Phi+\lambda^2\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\Phi=0\,,\\ &i\Phi_\eta+2|\phi_{11}|^2\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\Phi+\frac{2}{\lambda}\begin{bmatrix}0 & \phi_{11}\\ \phi_{11}^* & 0\end{bmatrix}\Phi+\frac{1}{\lambda^2}\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\Phi=0\,.\end{aligned}$$

Comparing (3.13) with (3.15), we put

$$(3.16)\qquad \exp\left(2i\int_\xi^\infty |u^2|\,d\xi\right)=\phi_{1N}/\phi_{2N}\,,\quad \exp\left(2i\int_\xi^\infty |u|^2 d\xi\right)v=\phi_{11}\,.$$

On the other hand, by comparing the coefficients of λ^{2N} in (3.15), we have

$$(3.17)\qquad (\partial/\partial\xi)\phi_{2N}=2i|\phi_{1N}|^2/\phi_{2N}^*\,.$$

Using (3.16) and (3.17), we conclude

Theorem 3.8. *The pair of functions*

$$u(\xi, \eta) = \phi_{2N}(\xi, \eta)/\phi_{1N}(\infty, \eta), \quad v(\xi, \eta) = \phi_{1N}(\xi, \eta)\phi_{21}(\xi, \eta)/\phi_{1N}(\infty, \eta)$$

is a solution of the equation of the massive Thirring model (0.9).

REMARK. 3.9. For $N=1$, $\alpha_1=d\exp(i\delta)$, $c_1=r\exp(i\gamma)$, we have

$$\begin{aligned} u &= -id\sin(2\delta)\exp\{-2i(d^2\xi+d^2\eta)\cos(2\delta)-i\gamma\} \\ &\qquad \times \operatorname{sech}\{2(d^2\xi-d^{-2}\eta)\sin(2\delta)-\log r+i\delta\}, \\ v &= id^{-1}\sin(2\delta)\exp\{-2i(d^2\xi+d^{-2}\eta)\cos 2\delta-i\gamma\} \\ &\qquad \times \operatorname{sech}\{2(d^2\xi-d^{-2}\eta)\cos 2\delta-\log r-i\delta\}, \end{aligned}$$

which is the same as the one-soliton solution of (0.9) given by Kuznetsov-Mikhailov [17] and Kaup-Newell [14].

3.4. The equation of the Toda lattice.

The equation of the Tada lattice is the following:

$$(d/dt)Q_n = P_n, \quad (d/dt)P_n = \exp(Q_{n-1}-Q_n) - \exp(Q_n-Q_{n+1}), \quad n\in \boldsymbol{Z}$$

or

$$(d/dt)a_n = 2a_n(b_{n+1}-b_n), \quad (d/dt)b_n = 2a_n(a_n-a_{n-1}) \tag{3.18}$$

where

$$a_n = 4^{-1}\exp\{4^{-1}(Q_{n-1}-Q_n)\}, \quad b_n = -2^{-1}P_{n-1}.$$

This equation is the compatibility conditions of the following linear equations

$$\begin{aligned} & L\Phi = (\lambda+\lambda^{-1})\Phi, \quad M = (d/dt)\Phi, \quad \Phi = \{\Phi_n\} \\ & (L\Phi)_n = \Phi_{n+1}+b_n\Phi_n+a_{n-1}\Phi_{n-1}, \\ & (M\Phi)_n = \Phi_{n+1}+b_n\Phi_n-a_{n-1}\Phi_{n-1} \end{aligned} \tag{3.19}$$

where λ is a parameter ([13], [19]).

In this case we consider sequences of functions $\Phi_n(t, \lambda)$, $n\in\boldsymbol{Z}$, $\lambda\in\boldsymbol{C}$ of the following form:

$$\Phi_n(t, \lambda) = \lambda^n(\lambda^N+\textstyle\sum_{j=0}^{N-1}\phi_{nj}(t)\lambda^j)\exp(t(\lambda-\lambda^{-1})) \tag{3.20}$$

where N is an arbitrary positive integer.

Let $\alpha_1, \cdots, \alpha_N$ be mutually distinct complex numbers such that $\alpha_j \neq \alpha_k^{-1}$, $j, k=1, \cdots, N$ and $c_1, \cdots, c_N$ be arbitrary complex numbers.

Lemma. 3.10. *For each $n\in\boldsymbol{Z}$, there exists a unique function $\Phi_n(t, \lambda)$ of the form* (3.20) *that satisfies the conditions*

(3.21) $$\Phi_n(t, \alpha_j) = c_j \Phi_n(t, \alpha_j^{-1}), \quad j=1, \cdots, N.$$

Proof. The conditions (3.21) are equivalent to the following system of linear equations for unknowns $\phi_{nj}(t), j=0, \cdots, N-1$;

$$\begin{aligned}\textstyle\sum_{k=0}^{N-1} \{\alpha_j^{n+k} \exp(t(\alpha_j-\alpha_j^{-1})) - c_j\alpha_j^{-n-k} \exp(-t(\alpha_j-\alpha_j^{-1}))\}\phi_{nk}(t) \\ = -\alpha_j^{N+n} \exp(t(\alpha_j-\alpha_j^{-1})) + c_j\alpha_j^{-N-n} \exp(-t(\alpha_j-\alpha_j^{-1})), \\ j=1, \cdots, N.\end{aligned}$$

The determinant of the coefficient matrix of this system is a linear combination of $\exp\{\sum_{j=1}^{N} \pm t(\alpha_j-\alpha_j^{-1})\}$ and the coefficients of $\exp\{\sum_{j=1}^{N} t(\alpha_i-\alpha_j^{-1})\}$ is not zero. Therefore the determinant does not vanish identically as a function of t.

Q.E.D.

By an analogous argument as in Section 1, we see that the sequence of functions $\{\Phi_n(t)\}$ satisfies the linear equations (3.19) with coefficients

$$a_n = \phi_{n+1,0}/\phi_{n,0}, \quad b_n = \phi_{n,N-1} - \phi_{n+1,N-1}.$$

Thus we have

Theorem 3.11 *The sequence of functions*

$$a_n = \phi_{n+1,0}/\phi_{n,0}, \quad b_n = \phi_{n,N-1} - \phi_{n+1,N-1}$$

is a solution of the equation of the Toda lattice (3.18).

We put

$$D_n(t, x) = \det[f_n(t, x), \cdots, f_{n+N-1}(t, x)]$$

where $f_j(t, x) = {}^t(f_{1k}(t, x), \cdots, f_{Nk}(t, x))$,

$$f_{jk}(t, x) = \alpha_j^k \exp(t(\alpha_j-\alpha_j^{-1}) + \alpha_j x) - c_j\alpha_j^{-k} \exp(-t(\alpha_j-\alpha_j^{-1}) + \alpha_j^{-1}x).$$

Then by Cramer's formula, we have

$$\begin{aligned}\phi_{n,0} &= (-1)^{N-1} D_{n+1}(t, 0)/D_n(t, 0) \\ \phi_{n,N-1} &= \det([f_n, \cdots, f_{n+N-2}, f_{n+N}](t, 0))/D_n(t, 0) = ((\partial/\partial x) \log D_n)(t, 0).\end{aligned}$$

On the other hand by direct calculations, we have

$$\begin{aligned}D_n(t, x) &= \exp(\textstyle\sum_{j=1}^{N} t(\alpha_j-\alpha_j^{-1}) + x\alpha_j) \prod_{N\ge a>b\ge 1} (\alpha_a-\alpha_b) E_n(t, x) \\ E_n(t, x) &= \det[\delta_{jk} - \frac{c_j\alpha_j^{-2n}}{\alpha_j^{-1}-\alpha_k} \frac{g(\alpha_j^{-1})}{\dot{g}(\alpha_k)} \exp(-t(\alpha_j+\alpha_k-\alpha_j^{-1}-\alpha_k^{-1}) \\ &\quad -\frac{x}{2}(\alpha_j+\alpha_k-\alpha_j^{-1}-\alpha_k^{-1}))]\end{aligned}$$

where $g(\lambda) = \prod_{j=1}^{N} (\lambda-\alpha_j)$, $\dot{g} = dg/d\lambda$.

Thus we have

$$a_n=(E_nE_{n+2}/E_{n+1}^2)(t,0)\,,\quad b_n=((\partial/\partial_x)\log(E_n/E_{n+1}))(t,0)\,.$$

In this way, we see that our solutions are identical with N-soliton solutions obtained by the inverse scattering method.

Real-valued solutions are obtained by restricting the choices of α_j, c_j so that for a suitable permutation σ of $\{1,\cdots,N\}$ the relations

$$\alpha_j^*=\alpha_{\sigma(j)}\,,\quad c_j^*=c_{\sigma(j)}\qquad j=1,\cdots,N$$

hold.

Part II. Quasi-periodic solutions

4. Construction of quasi-periodic solutions

In this section we construct quasi-periodic solutions of a class of nonlinear differential equations of the integrability conditions of pairs of linear differential equations

$$\Phi_\xi=(\textstyle\sum_{j=0}^m\lambda^jM_j(\xi,\eta))\Phi\,,$$
$$\Phi_\eta=(\textstyle\sum_{j=0}^n\lambda^{-j}N_j(\xi,\eta))\Phi$$

where Φ, M_j, N_j are (2×2)-matrix valued functions and λ is a parameter, by modifying the method of Kricheber for the Zakharov-Shabat systems [16]. A generalization of our method to the class of equations proposed by Zakharov-Mikhailov [28] and Zakharov-Shabat [30] is straightforeward.

4.1. Construction of $\Phi(\xi,\eta,p)$.

Let R be the Riemann surface of hyperelliptic curve $\mu^2+\alpha\prod_{j=1}^{2g+2}(\lambda-\lambda_j)=0$, $\alpha=$constant, $\lambda_j\neq\lambda_k(j\neq k)$, $\lambda_j\neq0$ of genus g. Denote by p_j(resp. q_j) the points on R whose projections on Riemann sphere CP^1 by λ are ∞(resp. 0). As local parameters around p_j(resp. q_j), we take λ^{-1}(resp. λ). Let $\delta=d_1+\cdots+d_{g+1}(d_j\in R)$ be an effective divisor on R such that $l(\delta-p_j)=1, j=1, 2$, where for a divisor δ' on R $l(\delta')$ denotes the dimension of the vector space $L(\delta')$ of meromorphic functions for which $(f)+\delta'$ are effective divisors, (f)=the divisor defined by f. Further let $f_j(\xi,\lambda)=\sum_{k=0}^m f_{jk}(\xi)\lambda^k$ and $g_j(\eta,\lambda)=\sum_{k=0}^n g_{jk}(\eta)\lambda^{-k}$, $j=1, 2$ be smooth functions of ξ, η with $f_j(0,\lambda)=g_j(0,\lambda)=0$.

First, we have

Theorem 4.1. *For given δ, f_j, g_j, there exist unique functions $\Phi_j(\xi,\eta,p)$, $j=1, 2$ on R with the following properties, parametrized by $(\xi,\eta)\in U$ where U is a neighborhood of $0\in\boldsymbol{R}^2$ depending on δ and f_j, g_j.*

i) *Φ_j are meromorphic on $R-\{p_1,p_2,q_1,q_2\}$ and whose pole divisors are δ,*

ii) *around p_k(resp. q_k), $\Phi_j\exp(-f_k)$ (resp. $\Phi_j\exp(-g_k)$) are holomorphic and*

$(\Phi_j\, exp\,(-f_k))(p_k)=\delta_{jk}$.

Proof. On R we take a canonical homology basis a_j, b_j, $1\leq j\leq g$ and let $\omega_j\, 1\leq j\leq g$ be the normalized basis of abelian differentials of the first kind on R; $\int_{a_j}\omega_k=\delta_{jk}$. For distinct points $p, q\in R$ let ω_{pq} be the normalized differential of the third kind that has single poles at p, q with residues 1, -1 respectively. Further let ω_{f_j}(resp. ω_{g_j}) be the normalized defferential of the second kind with poles only at p_j(resp. q_j) of the forms $(\partial/\partial z)f_j(\xi, z)dz$ $(z=\lambda^{-1})$ (resp. $(\partial/\partial\lambda)g_j(\eta, \lambda)d\lambda$).

At first we assume that the functions Φ_j with the above properties exist. Then $d\log\Phi_j$ are abelian differentials on R. The location of their poles are as follows: at $d_k(1\leq k\leq g+1)$ poles of first order with residues -1, at $p_k(k=1, 2)$ poles of the forms $\{(\partial/\partial z)f_k(\xi, z)+(1-\delta_{jk})z^{-1}\}dz$ $(z=\lambda^{-1})$, at $q_k(k=1, 2)$ poles of the forms $\{(\partial/\partial\lambda)g_k(\eta, \lambda)\}d\lambda$, at zeros $p_{jk}(\xi, \eta)$ $(1\leq l\leq g)$ of Φ_j poles of first order with residues 1, and there are no other poles. Therefore $d\log\Phi_j$ are written as

$$d\log\Phi_j=\textstyle\sum_{l=1}^{2}(\omega_{f_l}+\omega_{g_l})+\sum_{l=1}^{g}\omega_{p_{jl}(\xi,\eta),d_l}+\delta_{j2}\omega_{p_1d_{g+1}} \\ +\delta_{j1}\omega_{p_2d_{g+1}}+\textstyle\sum_{l=1}^{g}c_{jl}\omega_l \tag{4.1}$$

with $c_{jl}\in \boldsymbol{C}$. Since Φ_j are single-valued functions on R, we must have

$$\int_{a_k}d\log\Phi_j=2\pi i m_{jk}\,,\quad \int_{b_k}d\log\Phi_j=2\pi i n_{jk}\,,\quad k=1,\cdots,g$$

with m_{jk}, $n_{jk}\in\boldsymbol{Z}$. From the first relations we have $c_{jl}=2\pi i m_{jl}$. From the second relations and the reciprocity law for differentials of the first and the third kind, we have

$$2\pi i n_{jk}=\textstyle\sum_{l=1}^{2}\int_{b_k}(\omega_{f_l}+\omega_{g_l})+2\pi i\sum_{l=1}^{g}\int_{d_l}^{p_{jl}(\xi,\eta)}\omega_k+2\pi i\delta_{j2}\int_{d_{g+1}}^{p_1}\omega_k \\ +2\pi i\delta_{j1}\int_{d_{g+1}}^{p_2}\omega_k+2\pi i\textstyle\sum_{l=1}^{g}m_{jl}\tau_{lk}$$

where $\tau_{jk}=\int_{b_j}\omega_k$. Thus the divisor $p_{j1}(\xi, \eta)+\cdots+p_{jg}(\xi, \eta)$ formed by the zeroes of Φ_j are the solutions of the following Jacobi's inversion problem on R:

$$(\textstyle\sum_{l=1}^{g}\int_{p_0}^{p_{jl}(\xi,\eta)}\omega_1,\cdots,\sum_{l=1}^{g}\int_{p_0}^{p_{jl}(\xi,\eta)}\omega_g)$$

$$\equiv(F_1+\textstyle\sum_{l=1}^{g+1}\int_{p_0}^{d_l}\omega_1-\delta_{j2}\int_{p_0}^{p_1}\omega_1-\delta_{j1}\int_{p_0}^{p_2}\omega_1,\cdots,F_g+\sum_{l=1}^{g+1}\int_{p_0}^{d_l}\omega_g \\ -\delta_{j2}\int_{p_0}^{p_1}\omega_g-\delta_{j1}\int_{p_0}^{p_2}\omega_g)\qquad(\text{mod.}\ \Gamma) \tag{4.2}$$

where $F_k=-(2\pi i)^{-1}\sum_{l=1}^{2}\int_{b_k}(\omega_{f_l}+\omega_{g_l})$, p_0 is a fixed point on R and Γ is the lattice in $\boldsymbol{C}^g$ generated by the columns of period matrix (I_g, τ), $\tau=(\tau_{jk})$.

Now we proceed to the construction of Φ_j. Since under our assumptions on δ, f_j, g_j the Jacobi's inversion problems (4.2) are uniquely solvable for (ξ, η) in a neighborhood of $0\in\boldsymbol{R}^2$, we determine the divisors $p_{j1}(\xi, \eta)+\cdots+p_{jg}(\xi, \eta)$, $j=1, 2$ by solving (4.2). Next define abelian differentials ψ_j by the right hand sides of (4.1) with $p_{jl}(\xi, \eta)$ determined as solutions of (4.2), then the functions $\exp\left(\int_{p_0}^{p}\psi_j\right)\Big/\exp\left(\int_{p}^{p_j}\psi_j\right)$ have the properties i), ii). Q.E.D.

Next we express the functions Φ_j in terms of theta functions and abelian integrals on R. First by our assumption on δ there exist unique functions ϕ_1(resp. ϕ_2) that belong to $L(\delta-p_2)$ (resp. $L(\delta-p_1)$) and $\phi_j(p_j)=1$. We write $(\phi_1)=\delta_1+p_2-\delta$, and $(\phi_2)=\delta_2+p_1-\delta$, then δ_1 and δ_2 are effective general divisors of degree g. We define the mapping $w\colon R\to J(R)=\boldsymbol{C}^g/\Gamma$ (=the Jacobian variety of R) by $w(p)=(w_1(p), \cdots, w_g(p))$, $w_j(p)=\int_{p_0}^{p}\omega_j$ $p\in R$ and extend this mapping to the divisor group linearly. We denote this mapping by the same notation w. By using this notation the above Jacobi's inversion problems are written as

$$w(p_{j1}(\xi, \eta)+\cdots+p_{jg}(\xi, \eta)) = F+w(\delta_j)\,, \quad j=1, 2$$

where $F=(F_1, \cdots, F_g)$. Next we define

$$w_{p_j}=\lim_{p\to p_j}\int_{p_0}^{p}(\omega_{f_j}-f_j dz)\,, \quad w_{g_j}=\lim_{p\to p_j}\int_{p_0}^{p}(\omega_{g_j}-g_j d\lambda)\,.$$

On the other hand, for effective general divisors of degree g $\delta_1=r_1+\cdots+r_g$, $\delta_2=t_1+\cdots+t_g$ on R, we have

$$\sum_{j=1}^{g}\omega_{r_j t_j}=d\log\frac{\theta(w(p)-w(\delta_1)-K)}{\theta(w(p)-w(\delta_2)-K)}$$

where $\theta(u)$, $u\in\boldsymbol{C}^g$ is the Riemann theta function on R defined by

$$\theta(u)=\sum_{m\in\boldsymbol{Z}^g}\exp(2\pi i u^t m+\pi i m\tau^t m)$$

and K is the Riemann's constant vector

$$K=(K_1, \cdots, K_g)\,, \quad K_j=2^{-1}\tau_{jj}+\int_{p_0}^{r_j}\omega_j-\sum_{k=1}^{g}\int_{a_k}w_j\omega_k \tag{4.3}$$

where r_j are the starting points of b_j. Using this fact, we have the following expressions of the functions Φ_j:

$$\begin{aligned}\Phi_j(\xi, \eta, p) &= \exp\left\{\sum_{k=1}^{2}\int_{p_0}^{p}(\omega_{f_k}-w_{p_k})+\sum_{k=1}^{2}\int_{p_0}^{p}(\omega_{g_k}-w_{g_k})\right\}\phi_j(p)\\ &\times\frac{\theta(w(p)-F-w(\delta_j)-K)\theta(w(p_j)-w(\delta_j)-K)}{\theta(w(p)-w(\delta_j)-K)\theta(w(p_j)-F-w(\delta_j)-K)}\,, \quad j=1, 2\,.\end{aligned} \tag{4.4}$$

4.2. Derivation of linear differential equations.

Here we derive a pair of linear differential equations with respect to ξ and η which the function $\Phi={}^t(\Phi_1, \Phi_2)$ satisfies.

Let the expansions of Φ_j around p_k(resp. q_k) be

$$\Phi_j = \exp(f_k(\xi, \lambda))(\textstyle\sum_{l=0}^{\infty} \alpha_{jk,l}(\xi, \eta)\lambda^{-l}), \quad \alpha_{jk,0} = \delta_{jk}$$

(resp. $\Phi_j=\exp(g_k(\eta, \lambda))(\sum_{l=0}^{\infty} \beta_{jk,l}(\xi, \eta)\lambda^{l})$). Then the expansions of $(\partial/\partial\xi)\Phi_j$ around p_k are

$$\begin{aligned}(\partial/\partial\xi)\Phi_j &= \exp(f_k)\{(\textstyle\sum_{l=0}^{m} (\partial/\partial\xi)f_{kl}\lambda^{l})(\sum_{l=0}^{\infty} \alpha_{jk,l}\lambda^{-l})+\sum_{l=0}^{\infty} (\partial/\partial\xi)\alpha_{jk,l}\lambda^{-l}\} \\ &= \exp(f_k)[\textstyle\sum_{l=-m}^{0} \sum_{s=-l}^{m} ((\partial/\partial\xi)f_{ks})\alpha_{jk,l+s}\lambda^{-l} \\ &\quad +\textstyle\sum_{l=1}^{\infty} \{\sum_{s=0}^{m} ((\partial/\partial\xi)f_{ks})\alpha_{jk,l+s}+(\partial/\partial\xi)\alpha_{jk,l}\}\lambda^{-l}]\end{aligned}$$

We want to determine the functions $m_{jk,l}(\xi, \eta)$, $j, k=1, 2$, $l=0, \cdots, m$ so that the expansions of $(\partial/\partial\xi)\Phi_j-\sum_{k=1}^{2} (\sum_{l=0}^{m} \lambda^l m_{jk,l})\Phi_k$ around p_k have the forms $\exp(f_k)(\sum_{l=1}^{\infty} h_{jk,l}(\xi, \eta)\lambda^{-l})$. These requirements are equivalent to the following system of linear equations for unknowns $m_{jk,l}(\xi, \eta)$:

$$\textstyle\sum_{s=l}^{m} \sum_{p=1}^{2} m_{jp,s}\alpha_{pk,s-l} = \sum_{s=l}^{m} ((\partial/\partial\xi)f_{ks})\alpha_{jk,s-l}, \quad l=0, \cdots, m, \quad j, k=1, 2$$

or in matrix notation

$$\textstyle\sum_{s=l}^{m} M_s\alpha_{s-l} = \sum_{s=l}^{\xi} \alpha_{s-l}((\partial/\partial\xi)f_s), \quad l=0, \cdots, m \tag{4.5}$$

where $M_s=(m_{jk,s})$, $\alpha_s=(\alpha_{jk,s})$ and f_s are the diagonal matrices with entries f_{1s}, f_{2s}. Matrices M_s are uniquely determined in decreasing order of s from this system, since $\alpha_{jk,0}=\delta_{jk}$. Consider the functions

$$\{(\partial/\partial\xi)\Phi_j-(\textstyle\sum_{k=1}^{2} \sum_{l=0}^{m} \lambda^l m_{jk,l})\Phi_k\}/\Phi_j .$$

These functions belong to $L(p_{j1}(\xi, \eta)+\cdots+p_{jg}(\xi, \eta))$ and vanish at p_j. Since the divisors $p_{j1}(\xi, \eta)+\cdots+p_{jg}(\xi, \eta)$ are general for $(\xi, \eta)\in U$, the functions

$$(\partial/\partial\xi)\Phi_j-\textstyle\sum_{k=1}^{2} \sum_{l=0}^{m} \lambda^l m_{jk,l}\Phi_k$$

are identically zero for $p\in R$, $(\xi, \eta)\in U$.

Next we consider $(\partial/\partial\eta)\Phi_j$.

Proposition 4.2. *The matrix $(\beta_{jk,0})$ is non-singular.*

Proof. Suppose the contrary. Then there exists a number $c(\xi, \eta)$ such that

$$\beta_{2j,0} = c\beta_{1j,0}, \quad j=1, 2 .$$

Therefore the function $\Phi_1-c\Phi_2$ has zero at q_1 and q_2. Consider the function

$$(\Phi_1-\Phi_2)/\lambda\Phi_1 .$$

This function belongs to $L(p_{11}(\xi, \eta)+\cdots+p_{1g}(\xi, \eta)-p_1)$. Since the divisor $p_{11}(\xi, \eta)+\cdots+p_{1g}(\xi, \eta)$ is general, we must have

$$\Phi_1 = c\Phi_2 ,$$

which is a contradiction. Q.E.D.

The system of linear equations corresponding to (4.5) is

$$\sum_{s=l}^{n} N_s\beta_{s-l} = \sum_{s=l}^{n} \beta_{s-l}((\partial/\partial\eta)g_s) , \quad l=0, \cdots, n$$

where $N_s=(n_{jk,s})$, $\beta_s=(\beta\beta_{jk,s})$ and g_s are the diagonal matrices with entries g_{s1}, g_{2s}. Since by Prop. 4.2, the matrix β_0 is non-singular, this system is also uniquely solvable.

Summarizing, we have

Theorem 4.3. *There exist unique functions $m_{jk,l}(\xi, \eta)$, $n_{jk,l}(\xi, \eta)$ independent of $p\in R$ such that the equations*

$$\Phi_\xi = (\textstyle\sum_{l=0}^{m} \lambda^l M)\Phi , \quad \Phi_\eta = (\textstyle\sum_{l=0}^{n} \lambda^{-l} N_l)\Phi$$

hold for $p\in R$, $(\xi, \eta)\in U$ where $M_l=(m_{jk,l})$, $N_l=(n_{jk,l})$.

5. The equation of Pohlmeyer-Lund-Regge

In this section we construct quasi-periodic solutions of the equation of the system of Pohlmeyer-Lund-Regge by applying the result in the preceding section.

We construct the function $\Phi(\xi, \eta, p)$ by putting $f_1(\xi, \lambda)=2^{-1}i\lambda\xi$, $f_2(\xi, \lambda)=-2^{-1}i\lambda\xi$, $g_1(\eta, \lambda)=2^{-1}\lambda^{-1}i\eta$, $g_2(\eta, \lambda)=-2^{-1}\lambda^{-1}i\eta$. Then this function $\Phi(\xi, \eta, p)$ satisfies the following pair of linear differential equations

$$\text{(5.1)} \quad \begin{aligned} &i\Phi_\xi+\begin{bmatrix}0 & -\alpha_{12}\\ \alpha_{21} & 0\end{bmatrix}\Phi+2^{-1}\lambda\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\Phi = 0 \\ &i\Phi_\eta+\frac{1}{2\lambda(\beta_{11}\beta_{22}-\beta_{12}\beta_{21})}\begin{bmatrix}\beta_{11}\beta_{22}+\beta_{12}\beta_{21} & -2\beta_{11}\beta_{12}\\ \beta_{21}\beta_{22} & -\beta_{11}\beta_{22}-\beta_{12}\beta_{21}\end{bmatrix}\Phi = 0 \end{aligned}$$

where $\alpha_{jk}=\alpha_{ik,1}$, $\beta_{jk}=\beta_{jk,0}$.

Comparing (5.1) with (0.5), we put

$$\text{(5.2)} \quad \begin{aligned} \cos u &= (\beta_{11}\beta_{22}+\beta_{12}\beta_{21})/(\beta_{11}\beta_{22}-\beta_{12}\beta_{21}) , \\ \exp(i\omega)\sin u &= -2\beta_{21}\beta_{22}/(\beta_{11}\beta_{22}-\beta_{12}\beta_{21}) , \\ \exp(-i\omega)\sin u &= 2\beta_{11}\beta_{12}/(\beta_{11}\beta_{22}-\beta_{12}\beta_{21}) . \end{aligned}$$

Thus we have

$$\exp(2i\omega) = -\beta_{21}\beta_{22}/\beta_{11}\beta_{12} .$$

On the other hand, by comparing the constant terms of (5.1) at q_k we see that the relations

$$(5.3)\qquad i(\partial/\partial\xi)\beta_{1k}=\alpha_{12}\beta_{2k}\,,\quad i(\partial/\partial\xi)\beta_{2k}=-\alpha_{21}\beta_{1k}$$

hold. Using these relations we have

$$(5.4)\qquad \omega_\xi=(\beta_{11}\beta_{22}+\beta_{12}\beta_{21})(\alpha_{21}\beta_{11}\beta_{12}+\alpha_{12}\beta_{21}\beta_{22})/2\beta_{11}\beta_{12}\beta_{21}\beta_{22}\,.$$

Combining (5.2) and (5.4), we have

$$2\omega_\xi\cos^2(u/2)/\cos u=\alpha_{21}\beta_{11}/\beta_{21}+\alpha_{12}\beta_{22}/\beta_{12}\,.$$

Again using (5.3), we have

$$2\omega_\xi\cos^2(u/2)/\cos u=i(\partial/\partial\xi)\log(\beta_{12}/\beta_{21})\,.$$

Similarly we have

$$2\omega_\eta\cos^2(u/2)=i(\partial/\partial\eta)\log(\beta_{12}/\beta_{21})\,.$$

In view of (0.6), we have

Theorem 5.1. *The pair of functions*

$$(5.5)\qquad \begin{aligned} u&=\arccos\{(\beta_{11}\beta_{22}+\beta_{12}\beta_{21})/(\beta_{11}\beta_{22}-\beta_{12}\beta_{21})\}\,,\\ v&=i\log(\beta_{12}/\beta_{21})+v_0\,,\quad v_0\in\boldsymbol{C}\end{aligned}$$

is a solution of (0.2).

REMARK. These solutions are expressed by Riemann theta functions, in view of (4.4).

6. The sine-Gordon equation and fixed point free involutions

In this section we construct quasi-periodic solutions of the sine-Gordon equation (0.1) by introducing fixed point free involutions of hyperelliptic curves.

First we describe the actions of fixed point free involutions of compact Riemann surfaces on one-dimensional homology groups and period matrices, following Rauch-Farkus [25] and Fay [12].

Let R_1 be a compact Riemann surface of genus g_1 with a fixed point free involution T. Let R be the quotient of R_1 by T. Then by the Riemann-Hurwitz formula, we have $g_1=2g-1$ where g is the genus of R.

Proposition 6.1. *There exists a canonical basis a_j,b_j, $1\le j\le g_1$ of $H_1(R,\boldsymbol{Z})$ with the following property*

$$Ta_1=a_1\,,\quad Tb_1=b_1\,,\quad Ta_j=a_{j+g-1}\,,\quad Tb_j=b_{j+g-1}\,,\qquad j=2,\cdots,g\,.$$

Let ω_j $1\le j\le g_1$ be the normalized basis of abelian differentials of the first

kind with respect to the above basis a_j, b_j of $H_1(R_1, \boldsymbol{Z})$; $\int_{a_j}\omega_k=\delta_{jk}$.
Then we have

Proposition 6.2. $T^*\omega_1=\omega_1$, $T^*\omega_j=\omega_{j+g-1}$, $j=2, \cdots, g$. *where* $T^*\omega$ *denotes the pullback of* ω *by* T.

Therefore we have the followiug relations for $\tau_{jk}=\int_{b_j}\omega_k$.

Proposition 6.3. $\tau_{j+g-1,k}=\tau_{j,k+g-1}$, $\tau_{j+g-1,k+g-1}=\tau_{jk}$

$$\tau_{1,j+g-1}=\tau_{1,j}\,, \qquad j, k=2, \cdots, g\,.$$

The involution T acts on the Jacobian variety of R_1 and this action is extended to the universal covering space $\boldsymbol{C}^{g_1}$ of $J(R_1)$;

$$T:\ (u_1, u_2, \cdots, u_g, u_{g+1}, \cdots, u_{2g-1}) \to (u_1, u_{g+1}, \cdots, u_{2g-1}, u_2, \cdots, u_g)\,.$$

Defining the theta function associated to $\tau=(\tau_{jk})$ by

$$\theta(u)=\sum_{m\in Z^{g_1}}\exp(2\pi i m^t u+\pi i m\tau^t m)\,, \quad u=(u_1, \cdots, u_{g_1})\in \boldsymbol{C}^{g_1}\,,$$

we have the following "symmetry" of $\theta(u)$;

$$\theta(Tu)=\theta(u)\,. \tag{6.1}$$

Now we turn to the construction of quasi-periodic solutions.

Let R_1 be the Riemann surface of the hyperelliptic curve $\mu^2+\alpha\prod_{i=1}^{2g+2}(\lambda-\lambda_j)\times(\lambda+\lambda_j)=0$, $\alpha=$const., $\lambda_j\neq\lambda_k(j\neq k)$, $\lambda_j\neq 0$ of genus $g_1=2g-1$. This curve admits a fixed point free involution T: $(\lambda, \mu)\to(-\lambda, -\mu)$. We take a canonical basis of $H_1(R_1, \boldsymbol{Z})$ with the property stated in Prop. 6.1.. Let ω_{p_j}(resp. ω_{q_j}), $j=1, 2$ be the normalized abelian differentials of the second kind that have poles only at p_j(resp. q_j) of the forms $2^{-1}z^{-2}dz$, $z=\lambda^{-1}$, (resp. $2^{-1}\lambda^{-2}d\lambda$). Then the differentials ω_{f_j}, ω_{g_j} in Section 4 for $f_1(\xi, \lambda)=2^{-1}i\xi\lambda$, $f_2(\xi, \lambda)=-2^{-1}i\xi\lambda$, $g_1(\eta, \lambda)=2^{-1}\lambda^{-1}i\eta$, $g_2(\eta, \lambda)=-2^{-1}\lambda^{-1}i\eta$ are expressed as

$$\omega_{f_1}=-i\xi\omega_{p_1}\,, \quad \omega_{f_2}=i\xi\omega_{p_2}\,, \quad \omega_{g_1}=-i\eta\omega_{q_1}\,, \quad \omega_{g_2}=i\eta\omega_{q_2}\,.$$

Lemma. 6.4. $\omega_{p_2}=-T^*\omega_{p_1}$, $\omega_{q_2}=-T^*\omega_{q_1}$.

Proof. Since $Tp_2=p_1$, $T^*\omega_{p_1}$ is a normalized differential of the second kind which has poles only at p_2 of the form $-2^{-1}z^{-2}dz$, $z=\lambda^{-1}$. By the uniqueness of normalized differentials of prescribed poles, we have $T^*\omega_{p_1}=-\omega_{p_2}$.

Q.E.D.

By putting $U_{jk}=(2\pi i)^{-1}\int_{b_k}\omega_{p_j}$, $V_{jk}=(2\pi i)^{-1}\int_{b_k}\omega_{q_j}$, $j=1, 2$, $k=1, \cdots, 2g-1$

$U_j=(U_{j1}, \cdots, U_{j,2g-1})$, $V_j=(V_{j1}, \cdots, V_{j,2g-1})$, the vector F in (4.4) is expressed as

$$F = i\xi(U_1 - U_2) + i\eta(V_1 - V_2)\,.$$

Lemma 6.5. $U_2 = -TU_1$, $V_2 = -TV_1$.

Proof. By the above Lemma 6.5. and Prop. 6.1., we have

$$U_{2k} = (2\pi i)^{-1}\int_{b_k}\omega_{p_2} = -(2\pi i)^{-1}\int_{b_k} T^*\omega_{p_1} = -(2\pi i)^{-1}\int_{Tb_k}\omega_{p_1}$$

$$= \begin{cases} -(2\pi i)^{-1}\int_{b_k}\omega_{p_1}\,, & k=1 \\ -(2\pi i)^{-1}\int_{b_{k+g-1}}\omega_{p_1}\,, & k=2,\cdots,g \\ -(2\pi i)^{-1}\int_{b_{k-g+1}}\omega_{p_1}\,, & k=g+1,\cdots,2g-1 \end{cases} = -(TU_1)_k\,.$$

Q.E.D.

Denote by w_{p_0} w_{Tp_0}, the mapping defined in Section 4 with base points p_0, Tp_0 respectively. Further let K_{p_0}, K_{Tp_0} be the Riemann's constant vector (4.3) with base points p_0, Tp_0 respectively. Then by a similar calculation, we have

Lemma 6.6. $TK_{p_0} = K_{Tp_0}$.

We construct functions $\Phi_j(\xi, \eta, p)$, $j=1,2$ as in the preceding section by choosing δ such that $T\delta=\delta$. Since $Tp_1 = p_2$, we have $\phi_2(p) = \phi_1(Tp)$. Using this fact and Lemmas 6.4., 6.5., we have the following expressions for the functions $\Phi_j(\xi, \eta, p)$:

$$\Phi_1(\xi, \eta, p) = \exp\{i\xi\int_{p_0}^{p}(\omega_{p_1} + T^*\omega_{p_1} - w_{p_1} + w_{p_2}) + i\eta\int_{p_0}^{p}(\omega_{q_1} + T^*\omega_{q_1} - w_{q_1} + w_{q_2})\}$$

$$\times\phi_1(p)$$

$$\times\frac{\theta(w_{p_0} - i\xi(U_1 + TU_1) - i\eta(V_1 + TV_1) - w_{p_0}(\delta_1) - K_{p_0})\theta(w_{p_0}(p_1) - w_{p_0}(\delta_1) - K_{p_0})}{\theta(w_{p_0}(p) - w_{p_0}(\delta_1) - K_{p_0})\theta(W_{p_0}(p_1) - i\xi(U_1 + TU_1) - i\eta(V_1 + TV_1) - w_{p_0}(\delta_1) - K_{p_0})},$$

$$\Phi_2(\xi, \eta, p) = \exp\{i\xi\int_{Tp_0}^{p}(\omega_{p_1} + T^*\omega_{p_1} - w_{p_1} + w_{p_2}) + i\eta\int_{Tp_0}^{p}(\omega_{q_1} + T^*\omega_{q_1} - w_{q_1} + w_{q_2})\}$$

$$\times\phi(Tp)\frac{\theta(w_{Tp_0}(p) - i\xi(U_1 + TU_1) - i\eta(V_1 + TV_1) - w_{Tp_0}(T\delta_1) - K_{Tp_0})}{\theta(w_{Tp_0}(p) - w_{Tp_0}(T\delta_1) - K_{Tp_0})}$$

$$\times\frac{\theta(w_{Tp_0}(p_2) - w_{Tp_0}(T\delta_1) - K_{Tp_0})}{\theta(w_{Tp_0}(p_2) - i\xi(U_1 + TU_1) - i\eta(v_1 + TV_1) - w_{Tp_0}(T\delta_1) - K_{Tp_0})}.$$

Using these expressions, (6.1) and Lemma 6.6., we have the following relations

$$\Phi_1(\xi, \eta, Tp) = \Phi_2(\xi, \eta, p)\,,$$

$$\alpha_{12} = -\alpha_{21}\,,\quad \beta_{11} = \beta_{22}\,,\quad \beta_{12} = \beta_{21}\,.$$

Therefore we have v=const. in (5.5), that is, we have a solution of the sine-Gordon equation.

In order to recover the linear differential equations (0.4), we put

$$\Psi = {}^t(\Psi_1, \Psi_2)\,, \quad \Psi_1 = \Phi_1 + \Phi_2\,, \quad \Psi_2 = \Phi_1 - \Phi_2\,.$$

Then the function Ψ satisfies the following linear differential equations

$$i\Psi_\xi + \begin{pmatrix} \alpha_{21} & 0 \\ 0 & -\alpha_{21} \end{pmatrix}\Psi + \frac{\lambda}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\Psi = 0\,,$$

$$i\Psi_\eta + \frac{1}{2\lambda}\begin{pmatrix} 0 & c \\ c^{-1} & 0 \end{pmatrix}\Psi = 0\,, \quad c = (\beta_{11}+\beta_{12})/(\beta_{11}-\beta_{12})\,.$$

On the other hand the quotient of the Riemann surface R_1 by T is the Riemann surface of the hyperelliptic curve $w^2 + \alpha z \prod_{j=1}^{2g}(z-\lambda_j^2)=0$ and the projection $R_1 \to R$ is given by $(\lambda, \mu) \to (z, w) = (\lambda^2, \lambda\mu)$. Since the function Ψ_1(resp. Ψ_2) is invariant (resp. anti-invariant) under T, Ψ_1(resp. Ψ_2) is single-valued (resp. two-valued) on R. This fact together with the fact that λ is two-valued on R explain the appearance of two-valued functions in [15], [21].

7. Real-valued solutions and symmetric Riemann surfaces

In this section we construct real-valued quasi-periodic solutions of the equations (0.1), (0.2) by using the theory of symmetric Riemann surfaces introduced by Klein and developped by Weichold [26]. At first we describe some of results in [26].

We call a pair (R, σ) a symmetric Riemann surface when R is a compact Riemann surface and σ is an anti-holomorphic involution on R.

For a symmetric Riemann surface (R, σ), let R_0 be the fixed point set of R by σ. As for the set $R-R_0$, we have

Proposition 7.1. *Either the set $R-R_0$ is connected (in this case the quotient of R by σ is non-orientable) or it consists of exactly two connected components (in this case the quotient of R by σ is orientable).*

For the latter case we assign the invariant $\varepsilon=+$ and for the former case $\varepsilon=-$. Further let r be the number of connected components of R_0. In this way we assign for each symmetric Riemann surface the triple (g, r, ε) where g is the genus of R.

We call a symmetric Riemann surface (R, σ) of type (g, r, ε) when the triple assigned to (R, σ) is (g, r, ε).

Proposition 7.2. *The range of r is as follows:*
i) *for type $(g, r, +)$, $g-r+1$=even and $1 \le r \le g+1$,*
ii) *for type $(g, r, -)$, $0 \le r \le g$.*

The action of σ on $H_1(R, \boldsymbol{Z})$ is described as follows.

Proposition 7.3. *There exists a canonical basis a_j, b_j, $1\leq j\leq g$ of $H_1(R, \boldsymbol{Z})$ with the following property.*

i) *for type $(g, r, +)$*

$\sigma a_j=-a_j\,,\quad j=1, \cdots, g,$

$\sigma b_j=-b_j\,,\quad j=1, \cdots, r-1,$

$\sigma b_{r+2j}=b_{r+2j}+a_{r+2j+1}\,,\quad j=0, \cdots, (g-r+1)/2-1,$

$\sigma b_{r+2j-1}=b_{r+2j-1}+a_{r+2j-2}\,,\quad j=1, \cdots, (g-r+1)/2,$

where $b_j(j=1, \cdots, r-1)$ are connected components of R_0,

ii) *for type $(g, r, -)$ $(r>0)$*

$\sigma a_j=-a_j\,,\quad j=1, \cdots, g,$

$\sigma b_j=b_j\,,\quad j=1, \cdots, r-1,$

$\sigma b_{r+j-1}=b_{r+j-1}+a_{r+j-1}\,,\quad j=1, \cdots, g-r+1.$

where $b_j(j=1, \cdots, r-1)$ are connected components of R_0,

iii) *for type $(g, 0, -)$*

$\sigma a_j=a_j\,,\quad j=1, \cdots, g$

$\sigma b_j=-b_j+\sum_{k=1\,;\,\neq j}^{g} a_k\,,\quad j=1, \cdots, g$

where a_j have no real points (that is, without points $\sigma p=p$).

Let $\omega_1, \cdots, \omega_g$ be the normalized basis of abelian differentials of the first kind with respect to the above basis a_j, b_j. Then we have

Proposition 7.4.

i) *For types $(g, r, +)$, $(g, r, -)$ $(r>0)$*

$$(\sigma^*\omega_j)^* = -\omega_j\,,\quad j=1, \cdots, g\,,$$

ii) *for type $(g, 0, -)$*

$$(\sigma^*\omega_j)^* = \omega_j\,,\quad j=1, \cdots, g\,.$$

where $\sigma^*\omega$ denotes the pull back of ω by σ.

Let $\tau_{jk}=\int_{b_j}\omega_k$ and $\tau=(\tau_{jk})$, then for each type $Re\,\tau$ is given by the following.

Proposition 7.5.

i) *For type $(g, r, +)$*

$$2Re\,\tau_{jk}=\begin{cases}1, & (j, k), (k, j)=(r-1+2l, r-2+2l)\,,\quad l=1, \cdots, (g-r+1)/2,\\ 0, & (j, k)=otherwise.\end{cases}$$

ii) *for type $(g, r, -)$ $(r>0)$*

$$2Re\,\tau_{jk}=\begin{cases}1, & (j, k)=(l, l)\,,\quad l=r, \cdots, g,\\ 0, & (j, k)=otheruise\end{cases}$$

iii) *for type* $(g, 0, -)$

$$2Re\,\tau_{jk}=\begin{cases}1, & (j,k)\neq(l,l),\quad l=1,\cdots,g,\\ 0, & (j,k)=otherwise.\end{cases}$$

By Prop. 7.5., the theta function $\theta(u)$ associated to τ satisfies the relation

$$\theta(u)^* = \theta(u^*) \quad u\in \boldsymbol{C}^g \tag{7.1}$$

for symmetric Riemann surfaces of types $(g, r, +)$, $(g, 0, -)$.

On the other hand for symmetric Riemann surfaces of type $(g, 0, -)$, Witt [27] proved the following.

Theorem 7.6. *On symmetric Riemann surfaces of type* $(g, 0, -)$ *there exist meromorphic functions* f *with the property*

$$ff^\sigma = -1$$

where $f^\sigma(p)=(f(\sigma p))^*$.

Since we need functions f in Theorem 7.6. with additional properties in the construction of quasi-periodic solutions, we reproduce the proof of this theorem.

First we rephrase the above Prop. 7.3. as follows.

Proposition 7.7. *For symmetric Riemann surfaces of type* $(g, 0, -)$ *there exists a canonical basis* c_j, d_j, $1\leq j\leq g$ *of* $H_1(R, \boldsymbol{Z})$ *with the following property.*

i) *for* $g=even$.

$$\sigma c_j = c_j\,,\quad j=1,\cdots,g$$

$$\sigma d_{2j} = d_{2j}-c_{2j-1}\,,\quad \sigma d_{2j-1} = d_{2j-1}-c_{2j}\,,\quad j=1,\cdots,g/2$$

ii) *for* $g=odd$.

$$\sigma c_j = c_j\,,\quad j=1,\cdots,g$$

$$\sigma d_1 = d_1\,,\quad \sigma d_{2j} = d_{2j}-c_{2j+1}\,,\quad \sigma d_{2j+1} = d_{2j+1}-c_{2j}\,,\quad j=1,\cdots,(g-1)/2\,.$$

This is shown by using the fact that the matrix representations of the action of σ on $H_1(R, \boldsymbol{Z})$ given in Prop. 7.3. and Prop. 7.7. are equivalent by an element of $Sp(2g, \boldsymbol{Z})$.

Let ω'_j, $1\leq j\leq g$ be the normalized basis of abelian differential with respect to the above basis c_j, d_j of $H_1(R, \boldsymbol{Z})$, then we have

$$\sigma^*\omega'_j = \omega'_j\,,\quad 1\leq j\leq g\,,$$

that is, ω'_j are real abelian differentials of the first kind. Putting

$$\tau'_{jk} = \int_{d_j}\omega'_k\,,\quad \tau' = (\tau'_{jk})\,,$$

we have

Proposition 7.8.

i) *for* g=*even*

$$2Re\,\tau'_{jk} = \begin{cases} 1, & (j,k),(k,j)=(2l-1,2l), \quad l=1,\cdots,g/2. \\ 0, & \text{otherwise,} \end{cases}$$

ii) *for* g=*odd*

$$2Re\,\tau'_{jk} = \begin{cases} 1, & (j,k),(k,j)=(2l,2l+1), \quad l=1,\cdots,(g-1)/2, \\ 0, & \text{otherwise.} \end{cases}$$

Namely the columns of the period matrix (I_g, τ') (=a basis of periods of real differentials of the first kind) of a symmetric Riemann surface of type $(g, 0, -)$ has the following form

$$\begin{matrix} e_1, \cdots, e_g \\ f_1, \cdots, f_t, f_{t+1}+2^{-1}e_{t+1}, \cdots, f_g+2^{-1}e_g\,, \end{matrix} \qquad t = \begin{cases} 0, & g=\text{even} \\ 1, & g=\text{odd} \end{cases} \tag{7.2}$$

where e_j are real vector and f_j are purely imaginary vector.

Now we proceed to the proof of theorem.

Lemma 7.9. *If* $ff^{\sigma}=a=\text{constant}>0$, *then there exists a meromorphic function* g *which satisfies the relation*

$$(f) = (g) - \sigma(g)\,.$$

Proof. If f=const., then we put $g=1$. In case of $f\neq$const., we put $g=f+a^{1/2}$. Then

$$f(f^{\sigma}+a^{1/2}) = ff^{\sigma}+fa^{1/2} = a^{1/2}(f+a^{1/2})\,. \qquad \text{Q.E.D.}$$

Proof of Theorem 7.6. Let q be an arbitrary point on R. We denote

$$w'(q-\sigma q) = A+iB$$

where w' is defined as in Section 4. by ω'_j. Since ω'_j are real differentials, we have

$$w'(\sigma q - q) = A - iB\,.$$

and consequently, we see that $2A$ is a period.

Case i) g=even. By (7.2), A is congruent to a imaginary vector. Therefore we have

$$w(q-\sigma q) = iC = \text{purely imaginary.}$$

Determine an effective divosor δ of odd degree $>g$ by solving the following Jacobi's inversion problem

$$w'(\delta-(\deg\delta)q) = -2^{-1}(\deg\delta)w'(q-\sigma q) = -2^{-1}i(\deg\delta)C\,,$$

then we have

$$\begin{aligned} w'(\delta-\sigma\delta) &= w'(\delta-\deg \delta)q)+w'((\deg \delta)\sigma q-\sigma\delta)+(\deg \delta)w'(q-\sigma q) \\ &= -2^{-1}i(\deg \delta)C-2^{-1}i(\deg \delta)C+i(\deg \delta)C = 0\,. \end{aligned}$$

By Abel's theorem there exists a meromorphic funtion f with the property

$$(f) = \delta-\sigma\delta\,,$$

then we have

$$ff^{\sigma} = \text{constant.}$$

Suppose that there exists a meromorphic function g with the property

$$\delta-\sigma\delta = (g)-\sigma(g)\,.$$

In that case we must have

$$\delta = \delta'+\sigma\delta'+(g)$$

for a suitable divisor δ'. By counting the degrees of the both hand sides, we have a contradiction. Therefore by Lemma 7.9., we conclude that $ff^{\sigma}<0$.

Case ii) g=odd.

Determine a divosor δ of even degree $>g$ by solving the following Jacobi's inversion problem

$$w'(\delta-(\deg \delta)q) = 2^{-1}f_1-2^{-1}i(\deg \delta)B$$

where f_1 is the purely imaginary vector in (7.2). Since deg δ=even, we have

$$\begin{aligned} w'(\delta-\sigma\delta) &= w'(\delta-(\deg \delta)q)+w'((\deg \delta)\sigma q-\sigma\delta)+(\deg \delta)w'(q-\sigma q) \\ &= 2^{-1}f_1-2^{-1}i(\deg \delta)B-(-2^{-1}f_1+2^{-1}i(\deg \delta)B)+(\deg \delta)A+i(\deg \delta)B \\ &= 0\,. \end{aligned}$$

Accordingly by Abel's theorem, there exists a meromorphic function f with the property

$$\delta-\sigma\delta = (f)\,.$$

We have

$$ff^{\sigma} = \text{constant.}$$

Suppose that there exists a meromorphic function with the property

$$\delta-\sigma\delta = (g)-\sigma(g)\,.$$

Then we must have

$$\delta = \delta'+\sigma\delta'+(g)$$

for a suitable divisor δ'. By Abel's theorem, we have

$$\begin{aligned} 2^{-1}f_1-2^{-1}i(\deg\delta)B &= w'(\delta-(\deg\delta)q) \\ &= w'(\delta'-2^{-1}(\deg\delta)q)+w'(\sigma\delta'-2^{-1}(\deg\delta)\sigma q)+2^{-1}(\deg\delta)w'(\sigma q-q) \\ &= 2Re\, w'(\delta'-2^{-1}(\deg\delta)q)+2^{-1}(\deg\delta)A-2^{-1}i(\deg\delta)B\,. \end{aligned}$$

This implies that f_1 is congruent to a real vector, which is a contradiction. Again by Lemma 7.9., we conclude that $ff^{\sigma}<0$. Q.E.D.

REMARK. In particular, we can take $\deg\delta=g+1$.

After these preliminaries, we construct real-valued quasi-periodic solutions. Let R be the Riemann surface of the hyperelliptic curve

$$\mu^2+\Pi_{j=1}^{g+1}(\lambda-\lambda_j)(\lambda-\lambda_j^*)=0\,,\quad \lambda_j\neq\lambda_k(j\neq k)\,,\quad \lambda_j\neq\lambda_k^*\,,\quad \lambda_j\neq 0\,.$$

This Riemann surface admits an anti-holomorphic involution $\sigma\colon(\lambda,\mu)\to(\lambda^*,\mu^*)$. This symmetric Riemann surface (R,σ) is of type $(g, 0, -)$. We take a canonical basis a_j, b_j, $1\leq j\leq g$ of $H_1(R, \boldsymbol{Z})$ with the property in Prop. 7.3.. Let ω_{p_j}, ω_{q_j}, U_j, V_j be as in Section 5. Then we have

Lemma 7.10. $\omega_{p_2}=(\sigma^*\omega_{p_1})^*$, $\omega_{q_2}=(\sigma^*\omega_{q_1})^*$.

Proof. Since $\sigma(p_1)=p_2$, $(\sigma^*\omega_{p_1})^*$ is an abelian differential of the second kind with pole only at p_2 of the form $2^{-1}z^{-2}dz$, $z=\lambda^{-1}$. Further by Prop 7.3., we have

$$\int_{a_k}(\sigma^*\omega_{p_1})^*=\Big(\int_{\sigma a_k}\omega_{p_1}\Big)^*=\Big(\int_{a_k}\omega_{p_1}\Big)^*=0\,,\quad k=1,\cdots,g\,,$$

that is, $(\sigma^*\omega_{p_1})^*$ is also a normalized differential. By the uniqueness of the normalized differential of the second kind with prescribed poles, we have $\omega_{p_2}=(\sigma^*\omega_{p_1})^*$. Q.E.D.

Lemma. 7.11. $U_2=U_1^*$, $V_2=V_1^*$.

Proof. By Prop. 7.3. and the above Lemma 7.10, we have

$$\begin{aligned} U_{2k} &= (2\pi i)^{-1}\int_{b_k}\omega_{p_2}=(2\pi i)^{-1}\Big(\int_{b_k}\sigma^*\omega_{p_1}\Big)^*=(2\pi i)^{-1}\Big(\int_{\sigma b_k}\omega_{p_1}\Big)^* \\ &= -(2\pi i)^{-1}\Big(\int_{b_k}\omega_{p_1}\Big)^*=\Big[(2\pi i)^{-1}\Big(\int_{b_k}\omega_{p_1}\Big)\Big]^*=U_{1k}{}^*\,. \end{aligned}$$ Q.E.D.

Denote by w_{p_0}, $w_{\sigma p_0}$, K_{p_0}, $K_{\sigma p_0}$ the mapping defined in Section 4. and the Riemann's constant vector (4.3) with base points p_0, σp_0 respectively. Then by a similar calculation, we have

Lemma 7.12. $K_{p_0}^*=K_{\sigma p_0}$.

Let δ be the pole divosir of a meromorphic function f on R with the property in Theorem 7.6 of degree $g+1$. By examining the proof of Theorem

7.6., we can choose δ such that $l(\delta - p_j)=1, j=1,2$.

Then we have the following expressions of the functions $\Phi_j(\xi, \eta, p)$:

$$\Phi_1(\xi, \eta, p) = \exp [i\xi \int_{p_0}^{p} \{\omega_{p_1} - (\sigma^* \omega_{p_1})^* - w_{p_1} + w_{p_1}^*\} + i\eta \int_{p_0}^{p} \{\omega_{q_1} - (\sigma^* \omega_{q_1})^* - w_{q_1} + w_{q_1}^*\}]$$
$$\times \frac{(f(p)-f(p_2))\theta(w_{p_0}(p) - i\xi(U_1 - U_1^*) - i\eta(V_1 - V_1^*) - w_{p_0}(\delta_1) - K_{p_0})}{(f(p_1)-f(p_2))\theta(w_{p_0}(p) - w_{p_0}(\delta_1) - K_{p_0})}$$
$$\times \frac{\theta(w_{p_0}(p_1) - w_{p_0}(\delta_1) - K_{p_0})}{\theta(w_{p_0}(p_1) - i\xi(U_1 - U_1^*) - i\eta(V_1 - V_1^*) - w_{p_0}(\delta_1) - K_{p_0})},$$

$$\Phi_2(\xi, \eta, p) = \exp [i\xi \int_{\sigma p_0}^{p} \{\omega_{p_1} - (\sigma^* \omega_{p_1})^* - w_{p_1} + w_{p_1}^*\} + i\eta \int_{\sigma p_0}^{p} \{\omega_{q_1} - (\sigma^* \omega_{q_1})^* - w_{q_1} + w_{q_1}^*\}]$$
$$\times \frac{(f(p)-f(p_1))\theta(w_{\sigma p_0}(p) - i\xi(U_1 - U_1^*) - i\eta(V_1 - V_1^*) - w_{\sigma p_0}(\sigma\delta_1) - K_{\sigma p_0})}{(f(p_2)-f(p_1))\theta(w_{\sigma p_0}(p) - w_{\sigma p_0}(\sigma\delta_1) - K_{\sigma p_0})}$$
$$\times \frac{\theta(w_{\sigma p_0}(p_2) - w_{\sigma p_0}(\sigma\delta_1) - K\sigma_{p_0})}{\theta(w_{\sigma p_0}(p_2) - i\xi(U_1 - U_1^*) - i\eta(V_1 - V_1^*) - w_{\sigma p_0}(\sigma\delta_1) - K_{\sigma p_0})}.$$

By using these expressions, (7.1) and Lemma 7.12,. we see that the coefficients β_{jk} have the following forms:

$$\beta_{11}(\xi, \eta) = (f(p_1)-f(p_2))^{-1}(f(q_1)-f(p_2))a(\xi, \eta),$$
$$\beta_{12}(\xi, \eta) = (f(p_1)-f(p_2))^{-1}(f(q_2)-f(p_2))b(\xi, \eta),$$
$$\beta_{21}(\xi, \eta) = (f(p_2)-f(p_1))^{-1}(f(q_1)-f(p_1))b(\xi, \eta)^*,$$
$$\beta_{22}(\xi, \eta) = (f(p_2)-f(p_1))^{-1}(f(q_2)-f(p_1))a(\xi, \eta)^*.$$

On the other hand by Th. 7.6., we have

$$f(p_1)(f(p_2))^* = f(q_1)(f(q_2))^* = -1.$$

Using these relations, we conclude that the inequality

$$-1 \leq (\beta_{11}\beta_{22} + \beta_{12}\beta_{21})/(\beta_{11}\beta_{22} - \beta_{12}\beta_{21}) \leq 1$$

holds and that the function β_{12}/β_{21} has the form

$$\beta_{12}/\beta_{21} = r(b(\xi, \eta)^*)^{-1} b(\xi, \eta)$$

with a constant r. Therefore the pair of function

$$u = \arccos \{(\beta_{11}\beta_{22} + \beta_{12}\beta_{21})/(\beta_{11}\beta_{22} - \beta_{22}\beta_{21})\}$$
$$v = i \log (\beta_{12}/\beta_{21}) + v_0$$

is a real-valued solution of (0.2) with a suitable constant v_0.

In the same way real-valued solutions of the sine-Gordon equation are obtained by starting with the hyperelliptic curve $\mu^2+\prod_{j=1}^{g}(\lambda-\lambda_j)(\lambda-\lambda_j^*)\times(\lambda+\lambda_j)(\lambda+\lambda_j^*)=0$, $\lambda_j^2\neq\lambda_k^2$, $j\neq k$, $\lambda_j^2\neq(\lambda_k^*)^2$, $\lambda_j\neq 0$, which admits a fixed point free involution $T:(\lambda,\mu)\to(-\lambda,-\mu)$ and an anti-holomorphic involution $\sigma:(\lambda,\mu)\to(\lambda^*,\mu^*)$. Since T and σ commute, the constructions in Section 6 and the present section are compatible.

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