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ON THE JACOBI DIFFERENTIAL OPERATORS ASSOCIATED TO MINIMAL ISOMETRIC IMMERSIONS OF SYMMETRIC SPACES INTO SPHERES II

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Contents

Introduction

This is a continuation of our first paper [6]. In this paper we shall study the linear mappings S_σ in our paper [6].

In section 6 we study subrepresentations of a representation of a compact Lie group G with respect to its closed subgroup K. We introduce certain constants associated to a representation of G which describe some intertwining homomorphisms. And under certain conditions, called (P_1) and (P_2), we prove some properties of these constants, which play important roles for the study of the linear mappings S_σ.

In section 7 we consider an orthogonal symmetric Lie algebra $(\mathfrak{g}, \sigma)$, and study a 3-dimensional subalgebra of $\mathfrak{g}$ as well as its representation induced from that of $\mathfrak{g}$. The results in section 6 and 7 will be used in the later computations.

In section 8 we study minimally imbedded symmetric R-spaces into spheres. It is shown that in these cases the Jacobi differential operator $\tilde{S}$ reduces to Casimir operators (Proposition 8.3.1).

In section 9 we recall some basic results on representations of the special orthogonal group $SO(n+1)$, and study in detail certain representations of the group $SO(n+1)$. It is shown that the properties (P_1) and (P_2) are satisfied in the cases where immersed manifolds are spheres.

In the forthcoming paper III, applying the results in sections 6, 7 and 9, we shall study on the spectra of the Jacobi differential operator $\tilde{S}$ for the minimally immersed spheres.

We shall denote by [I] our first paper [6] for short, and retain the definitions and notation in [I].

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6. Subrepresentations

6.1. Let H be a Lie group. Let $\rho: H\to GL(V)$ and $\sigma: H\to GL(W)$ be complex representations of H. We denote by V^* (resp. by W^*) the dual space of V(resp. of W). The dual spaces V^* and W^* are H-modules by the contragradient representations ρ^*: $H\to GL(V^*)$ and σ^*: $H\to GL(W^*)$. For a linear mapping f: $V\to W$, we define a subspace V^*_f of V^* by $V^*_f={}^tf(W^*)$, where ${}^tf$ is the transposed mapping of f. Let $\mathrm{Hom}(V, W)$ be the vector space of all linear mappings of V to W. We identify $\mathrm{Hom}(V, W)$ with $V^*\otimes W$ in a canonical manner.

The following two lemmas are obtained easily.

Lemma 6.1.1. *If a linear mapping f: $V\to W$ is an H-homomorphism, so is ${}^tf$: $W^*\to V^*$.*

Lemma 6.12. (1) *A linear mapping f: $V\to W$ is an H-homomorphism, if and only if $(\rho^*\otimes\sigma)(x)f=f$ for every $x\in G$.*

(2) *If W is an irreducible H-module and if f: $V\to W$ is a non-trivial H-homomorphism, then ${}^tf$: $W^*\to V^*_f$ is an H-isomorphism.*

6.2. In the rest of this section we assume the followings. Let G be a compact connected Lie group and K a closed subgroup of G. The Lie algebra $\mathfrak{g}$, the inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$, and the subspaces $\mathfrak{k}$ and $\mathfrak{p}$ of $\mathfrak{g}$ are the same as in subsection 2.1 of [I]. We also denote by $\langle\ ,\ \rangle$ the Hermitian inner product on $\mathfrak{g}^C$, which is the extension of the inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$. Then $\mathfrak{g}^C$ is a unitary G-module via the adjoint action of G. We denote by $(\ ,\ )$ the symmetric bilinear form on $\mathfrak{g}^C$, which is the $\boldsymbol{C}$-bilinear extension of the inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$. In this paper we will not distinguish G-modules and representations of G.

Let χ: $G\to GL(W)$ be a unitary representation of G (not necessarily irreducible), and $W=W_1+W_2+\cdots+W_h$ a direct sum decomposition into irreducible components as K-modules. If any pair W_i, W_j of the components with $i\neq j$ are not K-isomorphic, we say that χ has the *property* (P_1).

Let U and V be complex vector spaces. We define a equivalence relation $\sim$ in $\mathrm{Hom}(U,V)$ as follows: For $f, g\in\mathrm{Hom}(U,V)$, $f\sim g$, if there exists a complex number c such that $|c|=1$ and that $g=cf$. We denote by $[f]$ the equivalence class to which f belongs, and by $[\mathrm{Hom}(U,V)]$ the set of all $\sim$-equivalence classes of $\mathrm{Hom}(U,V)$.

Let V' and W' be irreducible unitary K-modules and $\rho: G\to GL(W)$ a unitary representation of G with the property (P_1). We define $\rho_{V',W'}\in[\mathrm{Hom}(\mathfrak{g}^C\otimes V', W')]$ as follows:

(1) The case where both V' and W' are contained in the representation $\rho_{|K}$, the restriction of ρ to K(More precisely each one is isomorphic to a K-submodule of W, regarding W as a K-module via $\rho_{|K}$): Let $W=W_1+W_2+\cdots+W_h$ be a direct sum decomposition into irreducible components as K-modules. Let f_i: $V'\to W_i$ and f_j: $W'\to W_j$ be unitary K-isomorphisms. We define a linear mapping f: $\mathfrak{g}^C\otimes V'\to W'$ by

$$f(X\otimes v)=f_j^{-1}((d\rho(X)f_i(v))^j)\quad \text{for } X\in\mathfrak{g}^C \text{ and } v\in V',$$

where $(d\rho(X)f_i(v))^j$ denotes the W_j-component of $d\rho(X)f_i(v)$ with respect to the above direct sum decomposition of W. We define $\rho_{V',W'}$ by $\rho_{V',W'}=[f]$.

(2) Otherwise: We define $\rho_{V',W'}$ by $\rho_{V',W'}=[0]$.

REMARK 6.2.1. Suppose that both V' and W' are contained in the representation $\rho_{|K}$. Then for every $f'\in\rho_{V',W'}$ there exist unitary K-isomorphisms f'_i: $V'\to W_i$ and f'_j: $W'\to W_j$ such that $f'(X\otimes v)=f'^{-1}_j((d\rho(X)f'_i(v))^j)$ for $X\in\mathfrak{g}^C$ and $v\in V'$. The above fact is evident by the definition of $\rho_{V',W'}$.

Lemma 6.2.1. *Let ϕ: $K\to GL(V')$ and ψ: $K\to GL(W')$ be irreducible unitary representations, and ρ: $G\to GL(W)$ be a unitary representation of G with the property (P_1). Then we have for every $f\in\rho_{V',W'}$*

$$f\circ(Ad\otimes\phi)(k)=\psi(k)\circ f\quad \text{for } k\in K.$$

Proof. (1) The case where both V' and W' are contained in the representation $\rho_{|K}$: Let $W=W_1+W_2+\cdots+W_h$ be a direct sum decomposition into irreducible components as K-modules. By Remark 6.2.1 there exist unitary K-isomorphisms f_i: $V'\to W_i$ and f_j: $W'\to W_j$ such that $f(X\otimes v)=f_j^{-1}((d\rho(X)f_i(v))^j)$ for $X\in\mathfrak{g}^C$ and $v\in V'$. Hence

$$\begin{aligned}
f((Ad\otimes\phi)(k)(X\otimes v)) &= f_j^{-1}(\{d\rho(Ad(k)X)f_i(\phi(k)v)\}^j)\\
&= f_j^{-1}(\{(\rho(k)d\rho(X)\rho(k^{-1}))(\rho(k)f_i(v))\}^j)\\
&= f_j^{-1}(\rho(k)\{d\rho(X)f_i(v)\}^j)\\
&= \psi(k)f_j^{-1}(\{d\rho(X)f_i(v)\}^j)\\
&= \psi(k)f(X\otimes v).
\end{aligned}$$

(2) Otherwise: Since $f=0$, the statement is evident. Q.E.D.

We denote by $\mathfrak{k}^C$(resp. by $\mathfrak{p}^C$) the complex subspace of $\mathfrak{g}^C$ generated by

$\mathfrak{k}$(resp. by $\mathfrak{p}$). These spaces $\mathfrak{k}^C$ and $\mathfrak{p}^C$ are unitary K-submodules of $\mathfrak{g}^C$ with the adjoint K-action.

Lemma 6.2.2. *Under the assumptions of the above lemma, for every $f\in\rho_{V',W'}$ the linear mappings $f\colon \mathfrak{k}^C\otimes V'\to W'$ and $f\colon \mathfrak{p}^C\otimes V'\to W'$ induced from f are K-homomorphisms.*

Proof. We have the lemma by the proof of the above lemma. Q.E.D.

Let V' and W' be irreducible unitary K-modules and $\rho\colon G\to GL(W)$ a unitary representation of G with the property (P_1). We define $\rho_{V',W'}(\mathfrak{k})$, $\rho_{V'W'}(\mathfrak{p})\in[\mathrm{Hom}(\mathfrak{g}^C\otimes V',W')]$ as follows: Take an element $f\in\rho_{V',W'}$. Let $f_{\mathfrak{k}}$ and $f_{\mathfrak{p}}$ be the linear mappings of $\mathfrak{g}^C\otimes V'$ to W' defined by

$$f_{\mathfrak{k}|\mathfrak{k}^C\otimes V'}=f_{|\mathfrak{k}^C\otimes V'}\,,\qquad f_{\mathfrak{k}|\mathfrak{p}^C\otimes V'}=0\,,$$
$$f_{\mathfrak{p}|\mathfrak{k}^C\otimes V'}=0\,,\qquad f_{\mathfrak{p}|\mathfrak{p}^C\otimes V'}=f_{|\mathfrak{p}^C\otimes V'}\,.$$

We define $\rho_{V',W'}(\mathfrak{k})$ (resp. $\rho_{V',W'}(\mathfrak{p})$) by $\rho_{V',W'}(\mathfrak{k})=[f_{\mathfrak{k}}]$ (resp. by $\rho_{V',W'}(\mathfrak{p})=[f_{\mathfrak{p}}]$). Then we have

Lemma 6.2.3. (1) *If V' is not K-isomorphic to W', we have*

$$\rho_{V',W'}=\rho_{V',W'}(\mathfrak{p})\,.$$

(2) *Under the same assumption of* (1), *if moreover the K-module $\mathfrak{p}^C\otimes V'$ does not contain W', then we have*

$$\rho_{V',W'}=[0]\,.$$

(3) *If the K-module $\mathfrak{p}^C\otimes V'$ does not contain V', we have*

$$\rho_{V',V'}=\rho_{V',V'}(\mathfrak{k})\,.$$

Proof. It is sufficient to prove the statements when both V' and W' are contained in the representation $\rho_{|K}$. Let $W=W_1+W_2+\cdots+W_h$ be a direct sum decomposition into irreducible components as K-modules. Let $f_i\colon V'\to W_i$ and $f_j\colon W'\to W_j$ be unitary K-isomorphisms. We define $f\in\rho_{V',W'}$ by

$$f(X\otimes v)=f_j^{-1}((d\rho(X)f_i(v))^j)\quad\text{for } X\in\mathfrak{g}^C \text{ and } v\in V'\,.$$

(1) If $X\in\mathfrak{k}^C$, then $d\rho(X)f_i(v)$ is contained in W_i for $v\in V'$. Hence

$$f(X\otimes v)=f_j^{-1}((d\rho(X)f_i(v))^j)=0\quad\text{for } X\in\mathfrak{k}^C \text{ and } v\in V'\,.$$

Therefore we have $f=f_{\mathfrak{p}}$, and hence we have $\rho_{V',W'}=\rho_{V',W'}(\mathfrak{p})$.

(2) It follows from Lemma 6.2.2 and Schur's lemma that $f(\mathfrak{p}^C\otimes V')=\{0\}$. Therefore we have $f=0$, and obtain the assertion.

(3) It follows from Lemma 6.2.2 and Schur's lemma that $f(\mathfrak{p}^C\otimes V')=\{0\}$.

Therefore we have $f=f_{\mathfrak{f}}$. Q.E.D.

Let V' and W' be irreducible unitary K-modules. The Hermitian inner products on $\mathfrak{g}^C$, V' and W' induce a natural Hermitian inner product on Hom $(\mathfrak{g}^C\otimes V',W')$, identifying $\mathrm{Hom}(\mathfrak{g}^C\otimes V',W')$ with $(\mathfrak{g}^C)^*\otimes(V')^*\otimes W'$. We denote by $\langle\ ,\ \rangle$ this Hermitian inner product on $\mathrm{Hom}(\mathfrak{g}^C\otimes V',W')$. For $[f], [g]\in [\mathrm{Hom}(\mathfrak{g}^C\otimes V',W')]$, we define a real number $([f],[g])$ by

$$([f], [g]) = |\langle f, g\rangle| . \tag{6.2.1}$$

6.3. Throughout this subsection we assume the followings. Let $\rho: G\to GL(V)$ and $\chi: G\to GL(W)$ be unitary representations of G with the property (P_1). We define a linear mapping $L(\chi, \rho)$ of $W\otimes V$ as in subsection 5.2 of [I]. Then we have

$$L(\chi, \rho)((W\otimes V)_0)\subset(W\otimes V)_0 , \tag{6.3.1}$$

where $(W\otimes V)_0=\{u\in W\otimes V;\ (\chi\otimes\rho)(k)u=u \text{ for } k\in K\}$. We decompose V and W into a vector space direct sum with the following properties:

(a) $V=V_1+V_2+\cdots+V_k$, $W=W_1+W_2+\cdots+W_m$,

here each V_h(resp. W_j) is an irreducible K-module.

(b) There exists a non-negative integer $d(d\leqq k)$ with the following two properties:

(1) If $h\leqq d$, V_h is K-isomorphic to W_h.

(2) If $d<h$, W_h is not K-isomorphic to any V_j.

For $h\leqq d$, let a_h: $V_h\to W_h$ be a unitary K-isomorphism. We choose an orthonormal basis $\{v_{h;1}, \cdots, v_{h;n(h)}\}$ (resp. $\{w_{j;1}, \cdots, w_{j;p(j)}\}$) of V_h(resp. of W_j) such that $a_h(v_{h;\alpha})=w_{h;\alpha}(h\leqq d)$. Let $\{w_{j;1}{}^*, \cdots, w_{j;p(j)}{}^*\}$ be the basis of $W_j{}^*$ dual to $\{w_{j;1}, \cdots, w_{j;p(j)}\}$. In the followings we assume that $d\geqq 1$. Then

$$\{\omega_h = \sum_{\alpha=1}^{n(h)} w_{h;\alpha}{}^*\otimes v_{h;\alpha};\ h = 1, 2, \cdots, d\}$$

is a basis of $(W^*\otimes V)_0$. By (6.3.1) we define complex numbers $c(\chi^*, \rho)^j{}_h$ by

$$L(\chi^*, \rho)\omega_h = \sum_{j=1}^{d} c(\chi^*, \rho)^j{}_h\omega_j . \tag{6.3.2}$$

In particular if $\chi=\rho$, we choose the identity mapping of V_h as a_h, $h=1, 2, \cdots, k$. Then $\{\omega_h=\sum_{\alpha=1}^{n(h)} v_{h;\alpha}{}^*\otimes v_{h;\alpha};\ h=1, 2, \cdots, k\}$ is independent of the choice of an orthonormal basis of V_h. We denote $c(\rho^*, \rho)^j{}_h$ by $c(\rho)^j{}_h$. Let $\{E_1, E_2, \cdots, E_{n+p}\}$ be an orthonormal basis of $\mathfrak{g}$. Put

$$d\rho(E_i)v_{h;\alpha} = \sum_{j=1}^{k}\sum_{\beta=1}^{n(j)} A_{i,h\alpha}^{j\beta} v_{j;\beta} \tag{6.3.3}$$

and

$$d\chi(E_i)w_{s;\gamma}=\sum_{t=1}^{m}\sum_{\delta=1}^{p(t)}B_{i,}{}_{s\gamma}^{t\delta}w_{t;\delta}\,. \tag{6.3.4}$$

We have

Lemma 6.3.1. *The following equality holds for* $h=1,2,\cdots,d$:

$$\sum_{i=1}^{n+p}\sum_{\alpha=1}^{n(h)}A_{i,}{}_{h\alpha}^{j\beta}\bar{B}_{i,}{}_{h\alpha}^{t\delta}=\begin{cases}c(\chi^*,\rho)^j{}_h, & \text{if } j\leqq d,\ t=j \text{ and } \delta=\beta,\\ 0, & \text{otherwise,}\end{cases} \tag{6.3.5}$$

where $\bar{B}_{i,}{}_{h\alpha}^{t\delta}$ *is the complex conjugate of* $B_{i,}{}_{h\alpha}^{t\delta}$. *Therefore each* $c(\rho)^j{}_h$ *is a non-negative real number.*

Proof. By the definitions of $L(\chi^*,\rho)$ and ω_h, we have

$$L(\chi^*,\rho)\omega_h=(\sum_{i=1}^{n+p}d\chi^*(E_i)\otimes d\rho(E_i))\,(\sum_{\alpha=1}^{n(h)}w_{h;\alpha}{}^*\otimes v_{h;\alpha})$$

$$=\sum_{j=1}^{k}\sum_{\beta=1}^{n(j)}\sum_{t=1}^{m}\sum_{\delta=1}^{p(t)}(\sum_{i=1}^{n+p}\sum_{\alpha=1}^{n(h)}A_{i,}{}_{h\alpha}^{j\beta}\bar{B}_{i,}{}_{h\alpha}^{t\delta})w_{t;\delta}{}^*\otimes v_{j;\beta}\,.$$

Comparing the above equality with the right hand side of (6.3.2), we obtain (6.3.5). Q.E.D.

Lemma 6.3.2. We have

$$c(\chi^*,\rho)^j{}_h\ dim\ V_j=\bar{c}(\chi^*,\rho)^h{}_j\ dim\ V_h \qquad for\ h,j=1,2,\cdots,d,$$

where $\bar{c}(\chi^*,\rho)^h{}_j$ *is the complex conjugate of* $c(\chi^*,\rho)^h{}_j$.

Proof. By (6.3.5) we have

$$c(\chi^*,\rho)^j{}_h\dim V_j=\sum_{\beta=1}^{n(j)}(\sum_{i=1}^{n+p}\sum_{\alpha=1}^{n(h)}A_{i,}{}_{h\alpha}^{j\beta}\bar{B}_{i,}{}_{h\alpha}^{j\beta})\,.$$

Since ρ and χ are unitary representations, we have

$$c(\chi^*,\rho)^j{}_h\dim V_j=\sum_{\beta=1}^{n(j)}(\sum_{i=1}^{n+p}\sum_{\alpha=1}^{n(h)}\bar{A}_{i,}{}_{j\beta}^{h\alpha}B_{i,}{}_{j\beta}^{h\alpha})$$

$$=\sum_{\alpha=1}^{n(h)}(\sum_{i=1}^{n+p}\sum_{\beta=1}^{n(j)}\bar{A}_{i,}{}_{j\beta}^{h\alpha}B_{i,}{}_{j\beta}^{h\alpha})$$

$$=\bar{c}(\chi^*,\rho)^h{}_j\dim V_h\,.$$

Q.E.D.

Let $\{F_1,F_2,\cdots,F_n\}$ and $\{F'_1,F'_2,\cdots,F'_n\}$ be bases of $\mathfrak{p}^C$ with the property $(F_i,F'_j)=\delta_{ij}$, $i,j=1,2,\cdots,n$. Let $\{F_{n+1},F_{n+2},\cdots,F_{n+p}\}$ and $\{F'_{n+1},F'_{n+2},\cdots,F'_{n+p}\}$ be bases of $\mathfrak{k}^C$ with the property $(F_i,F'_j)=\delta_{ij}$, $i,j=n+1,n+2,\cdots,n+p$. We define linear mappings $L(\chi,\rho)_{\mathfrak{k}}$ and $L(\chi,\rho)_{\mathfrak{p}}$ of $W\otimes V$ as follows:

$$L(\chi,\rho)_{\mathfrak{k}}=\sum_{i=n+1}^{n+p}d\chi(F_i)\otimes d\rho(F'_i)\,,$$

$$L(\chi,\rho)_{\mathfrak{p}}=\sum_{i=1}^{n}d\chi(F_i)\otimes d\rho(F'_i)\,.$$

Then this linear mapping $L(\chi,\rho)_{\mathfrak{k}}$(resp. $L(\chi,\rho)_{\mathfrak{p}}$) is independent of the choice of bases of $\mathfrak{k}^C$(resp. of $\mathfrak{p}^C$) with the above property by the same reason as for $L(\chi,\rho)$. Since $\mathfrak{p}^C$ is orthogonal to $\mathfrak{k}^C$ with respect to the bilinear form (,) on $\mathfrak{g}^C$, we have

$$L(\chi, \rho) = L(\chi, \rho)_{\mathfrak{k}} + L(\chi, \rho)_{\mathfrak{p}} . \tag{6.3.6}$$

Lemma 6.3.3. *We have for $k \in K$*

$$(\chi\otimes\rho)(k)\circ L(\chi, \rho)_{\mathfrak{k}} = L(\chi, \rho)_{\mathfrak{k}}\circ(\chi\otimes\rho)(k) ,$$
$$(\chi\otimes\rho)(k)\circ L(\chi, \rho)_{\mathfrak{p}} = L(\chi, \rho)_{\mathfrak{p}}\circ(\chi\otimes\rho)(k) .$$

Therefore the linear mappings $L(\chi, \rho)_{\mathfrak{k}}$ and $L(\chi, \rho)_{\mathfrak{p}}$ of $W\otimes V$ leave $(W\otimes V)_0$ invariant.

Proof. We have for $k\in K$

$$\begin{aligned}&(\chi\otimes\rho)(k)\circ L(\chi, \rho)_{\mathfrak{k}}\\ &=\sum_{i=n+1}^{n+p}\{\chi(k)d\chi(F_i)\chi(k^{-1})\}\otimes\{\rho(k)d\rho(F'_i)\rho(k^{-1})\}\circ(\chi\otimes\rho)(k)\\ &=\sum_{i=n+1}^{n+p}d\chi(Ad(k)F_i)\otimes d\rho(Ad(k)F'_i)\circ(\chi\otimes\rho)(k) .\end{aligned}$$

Since the bilinear form (,) on $\mathfrak{g}^C$ is $Ad(G)$-invariant, we have

$$\sum_{i=n+1}^{n+p}d\chi(Ad(k)F_i)\otimes d\rho(Ad(k)F'_i) = L(\chi, \rho)_{\mathfrak{k}} .$$

Thus we obtain the first equality. The second equality is obtained in the same way. Q.E.D.

By the above lemma we define complex numbers $c(\chi^*,\rho;\mathfrak{k})^j{}_h$ and $c(\chi^*,\rho;\mathfrak{p})^j{}_h$ by

$$\begin{cases}L(\chi^*, \rho)_{\mathfrak{k}}\omega_h = \sum_{j=1}^{d} c(\chi^*, \rho; \mathfrak{k})^j{}_h\omega_j ,\\ L(\chi^*, \rho)_{\mathfrak{p}}\omega_h = \sum_{j=1}^{d} c(\chi^*, \rho; \mathfrak{p})^j{}_h\omega_j .\end{cases} \tag{6.3.7}$$

We denote by $c(\rho;\mathfrak{k})^j{}_h$ (resp. by $c(\rho;\mathfrak{p})^j{}_h$) $c(\rho^*, \rho; \mathfrak{k})^j{}_h$ (resp. $c(\rho^*, \rho; \mathfrak{p})^j{}_h$). Let σ_h be the irreducible representation ρ: $K\to GL(V_h)$ induced from ρ, and c_{σ_h} the eigenvalue of the Casimir operator C_{σ_h} of σ_h, $h=1,2,\cdots,k$. Then we have the following lemma.

Lemma 6.3.4.

(1) $c(\chi^*,\rho)^j{}_h=c(\chi^*,\rho;\mathfrak{k})^j{}_h+c(\chi^*,\rho;\mathfrak{p})^j{}_h$ *for j, $h=1,2,\cdots,d$.*

(2) $c(\chi^*,\rho)^j{}_h=c(\chi^*,\rho:\mathfrak{p})^j{}_h$ *for j, $h=1,2,\cdots,d$ with $j\neq h$.*

(3) $c(\rho;\mathfrak{k})^h{}_h=-c_{\sigma_h}$ *for $h=1,2,\cdots,k$.*

Proof. (1) We have the equality by (6.3.2), (6.3.6) and (6.3.7).

(2) Take an orthonormal basis $\{E_1, \cdots, E_{n+p}\}$ of $\mathfrak{g}$ such that $\{E_1, \cdots, E_n\}$ (resp. $\{E_{n+1}, \cdots, E_{n+p}\}$) is an orthonomal basis of $\mathfrak{p}$(resp. of $\mathfrak{k}$). Then for $\beta = 1, 2, \cdots, n(j)$, we have the followings in the same way as in (6.3.5):

$$(6.3.8) \qquad \begin{cases} c(\chi^*, \rho; \mathfrak{k})^j{}_h = \sum_{i=n+1}^{n+p} \sum_{\alpha=1}^{n(h)} A_{i,}{}^{j\beta}_{h\alpha} \bar{B}_{i,}{}^{j\beta}_{h\alpha}, \\ c(\chi^*, \rho; \mathfrak{p})^j{}_h = \sum_{i=1}^{n} \sum_{\alpha=1}^{n(h)} A_{i,}{}^{j\beta}_{h\alpha} \bar{B}_{i,}{}^{j\beta}_{h\alpha}. \end{cases}$$

Therefore we have for $j, h = 1, 2, \cdots, d$ with $j \neq h$

$$c(\chi^*, \rho; \mathfrak{k})^j{}_h = 0.$$

Hence we obtain the desired equality by (1).

(3) Let $\{E_1, \cdots, E_{n+p}\}$ be the orthonormal basis of $\mathfrak{g}$ in the proof of (2). Since $C_{\sigma_h} = \sum_{i=n+1}^{n+p} d\rho(E_i) d\rho(E_i)$ and since σ_h is a unitary representation, we have by (6.3.8)

$$\begin{aligned} C_{\sigma_h} v_{h;\alpha} &= \Big(\sum_{i=n+1}^{n+p} \sum_{\beta=1}^{n(h)} A_{i,}{}^{h\beta}_{h\alpha} A_{i,}{}^{h\alpha}_{h\beta}\Big) v_{h;\alpha} \\ &= -\Big(\sum_{i=n+1}^{n+p} \sum_{\beta=1}^{n(h)} A_{i,}{}^{h\alpha}_{h\beta} \bar{A}_{i,}{}^{h\alpha}_{h\beta}\Big) v_{h;\alpha} \\ &= -c(\rho; \mathfrak{k})^h{}_h v_{h;\alpha}. \end{aligned}$$

Q.E.D.

Let V' and W' be irreducible unitary K-modules. We define a subspace $V'(\rho; W')$ of $(\mathfrak{g}^C)^* \otimes (V')^*$ by $V'(\rho; W') = {}^t f((W')^*)$, taking an element $f \in \rho_{V', W'}$. We denote by $\langle\ ,\ \rangle$ the Hermitian inner product on $(\mathfrak{g}^C)^* \otimes (V')^*$ induced from the Hermitian inner products on $\mathfrak{g}^C$ and V'. Then $(\mathfrak{g}^C)^* \otimes (V')^*$ is a unitary K-module.

Proposition 6.3.5. *Let U', V' and W' be irreducible unitary K-modules. If U' is not K-isomorphic to W', then the subspace $V'(\rho; U')$ is orthogonal to $V'(\chi; W')$.*

Proof. If $\chi_{V', W'} = [0]$ or $\rho_{V', U'} = [0]$, the statement is evident. Suppose that $\chi_{V', W'}$, $\rho_{V', U'} \neq [0]$. Then it follows from Lemma 6.2.1 and (2) of Lemma 6.1.2 that $V'(\chi; W')$ (resp. $V'(\rho; U')$) is K-isomorphic to $(W')^*$(resp. $(U')^*$). Hence the irreducible K-module $V'(\rho; U')$ is not K-isomorphic to the irreducible K-module $V'(\chi; W')$. Therefore we obtain the proposition by the K-invariance of the Hermitian inner product on $(\mathfrak{g}^C)^* \otimes (V')^*$. Q.E.D.

Proposition 6.3.6. *Suppose that an irreducible unitary K-module V'(resp. W') is K-ismorphic to V_1(resp. to V_h). Let $\{w_1, w_2, \cdots, w_{n(h)}\}$ be an orthonormal basis of W' and $\{w_1^*, w_2^*, \cdots, w_{n(h)}^*\}$ its dual basis of $(W')^*$. Then we have for*

$f\in\rho_{V',W'}$

$$\langle {}^tf(w_\alpha^*),\ {}^tf(w_\alpha^*)\rangle = c(\rho)^h{}_1, \quad \alpha=1,2,\cdots,n(h)\,.$$

If $V'(\rho;W')\neq\{0\}$, $\{{}^tf(w_1^*),\ {}^tf(w_2^*),\cdots,{}^tf(w_{n(h)}^*)\}$ *is an orthogonal basis of* $V'(\rho;W')$.

Proof. Let $f_1\colon V'\to V_1$ and $f_h\colon W'\to V_h$ be unitary K-isomorphisms with the following property (Remark 6.2.1):

$$f(X\otimes v)=f_h^{-1}((d\rho(X)f_1(v))^h)\quad\text{for } X\in\mathfrak{g}^C \text{ and } v\in V'\,.$$

Let $\{v_{j\,;\,1},v_{j\,;\,2},\cdots,v_{j\,;\,n(j)}\}$ be an orthonormal basis of $V_j(j=1,2,\cdots,k)$ such that $f_h(w_\alpha)=v_{h\,;\,\alpha}$, $\alpha=1,2,\cdots,n(h)$. Choose the orthonormal basis $\{v_1,v_2,\cdots,v_{n(1)}\}$ of V' such that $f_1(v_\alpha)=v_{1\,;\,\alpha}$, $\alpha=1,2,\cdots,n(1)$, and let $\{v_1^*,v_2^*,\cdots,v_{n(1)}^*\}$ be its dual basis of $(V')^*$. By (6.3.3) we have

$$f(E_i\otimes v_\alpha)=\sum_{\beta=1}^{n(h)}A_{i,\,1\alpha}^{\,h\beta}w_\beta\,.$$

Hence

$${}^tf(w_\beta^*)=\sum_{i=1}^{n+p}\sum_{\alpha=1}^{n(1)}A_{i,\,1\alpha}^{\,h\beta}E_i^*\otimes v_\alpha^*\,,$$

where $\{E_1^*,E_2^*,\cdots,E_{n+p}^*\}$ is the basis of $\mathfrak{g}^*$ dual to the orthonormal basis $\{E_1,E_2,\cdots,E_{n+p}\}$. Therefore we have by (6.3.5)

$$\text{(6.3.9)}\qquad \langle {}^tf(w_\beta^*),\ {}^tf(w_\gamma^*)\rangle=\sum_{i=1}^{n+p}\sum_{\alpha=1}^{n(1)}A_{i,\,1\alpha}^{\,h\beta}\bar{A}_{i,\,1\alpha}^{\,h\gamma} = \begin{cases} c(\rho)^h{}_1, & \text{if } \beta=\gamma\,,\\ 0, & \text{if } \beta\neq\gamma\,.\end{cases}$$

If $V'(\rho;W')\neq\{0\}$, (6.3.9) shows that $\{{}^tf(w_1^*),\ {}^tf(w_2^*),\cdots,{}^tf(w_{n(h)}^*)\}$ is an orthogonal basis of $V'(\rho;W')$. Q.E.D.

Proposition 6.3.7. *Let* V' *and* W' *be irreducible unitary K-modules.*

(a) *The case where both* V' *and* W' *are contained in the representations* $\rho_{|K}$ *and* $\chi_{|K}$: *Suppose that* V_1*(resp.* V_h*) is K-isomorphic to* V'*(resp.* W'*). Then we have*

$$(\rho_{V',W'},\chi_{V',W'})=|c(\chi^*,\rho)^h{}_1|\ \textit{dim}\ W'\,,$$
$$(\rho_{V',W'}(\mathfrak{k}),\chi_{V',W'}(\mathfrak{k}))=|c(\chi^*,\rho;\mathfrak{k})^h{}_1|\ \textit{dim}\ W'\,,$$
$$(\rho_{V',W'}(\mathfrak{p}),\chi_{V',W'}(\mathfrak{p}))=|c(\chi^*,\rho;\mathfrak{p})^h{}_1|\ \textit{dim}\ W'\,.$$

(b) *Otherwise*: *We have*

$$(\rho_{V',W'},\chi_{V',W'})=0\,.$$

Proof. (a) Let $\{E_1,E_2,\cdots,E_{n+p}\}$ be the orthonormal basis of $\mathfrak{g}$ in the proof of Lemma 6.3.4. Let $f_1\colon V'\to V_1$ and $f_h\colon W'\to V_h$ be unitary K-isomorphisms. Put $g_1=a_1\circ f_1$ and $g_h=a_h\circ f_h$. Choose an orthonormal basis $\{v_1,v_2,\cdots,v_{n(1)}\}$ (resp. $\{w_1,w_2,\cdots,w_{n(h)}\}$) of V'(resp. of W') such that $f_1(v_\alpha)=v_{1;\alpha}$(resp. $f_h(w_\beta)=v_{h;\beta}$). We define an element of $f\in\rho_{V',W'}$(resp. $g\in\chi_{V',W'}$) by $f(X\otimes v)=f_h^{-1}((d\rho(X)f_1(v))^h)$ (resp. by $g(X\otimes v)=g_h^{-1}((d\chi(X)g_1(v))^h)$ for $X\in\mathfrak{g}^C$ and $v\in V'$. Then we have by (6.3.3) and (6.3.4)

$$\begin{cases} f=\sum_{\gamma=1}^{n(h)}\sum_{i=1}^{n+p}\sum_{\alpha=1}^{n(1)}A_{i,\,1\alpha}^{h\gamma}E_i^*\otimes v_\alpha^*\otimes w_\gamma\,, \\ g=\sum_{\gamma=1}^{n(h)}\sum_{i=1}^{n+p}\sum_{\alpha=1}^{n(1)}B_{i,\,1\alpha}^{h\gamma}E_i^*\otimes v_\alpha^*\otimes w_\gamma\,. \end{cases}$$

Let $f_{\mathfrak{k}}$, $f_{\mathfrak{p}}$, $g_{\mathfrak{k}}$ and $g_{\mathfrak{p}}$ be those in subsection 6.2. Then we have

$$\begin{cases} f_{\mathfrak{k}}=\sum_{\gamma=1}^{n(h)}\sum_{i=n+1}^{n+p}\sum_{\alpha=1}^{n(1)}A_{i,\,1\alpha}^{h\gamma}E_i^*\otimes v_\alpha^*\otimes w_\gamma\,, \\ f_{\mathfrak{p}}=\sum_{\gamma=1}^{n(h)}\sum_{i=1}^{n}\sum_{\alpha=1}^{n(1)}A_{i,\,1\alpha}^{h\gamma}E_i^*\otimes v_\alpha^*\otimes w_\gamma\,, \end{cases}$$

$$\begin{cases} g_{\mathfrak{k}}=\sum_{\gamma=1}^{n(h)}\sum_{i=n+1}^{n+p}\sum_{\alpha=1}^{n(1)}B_{i,\,1\alpha}^{h\gamma}E_i^*\otimes v_\alpha^*\otimes w_\gamma\,, \\ g_{\mathfrak{p}}=\sum_{\gamma=1}^{n(h)}\sum_{i=1}^{n}\sum_{\alpha=1}^{n(1)}B_{i,\,1\alpha}^{h\gamma}E_i^*\otimes v_\alpha^*\otimes w_\gamma\,. \end{cases}$$

Therefore we have by (6.2.1) and (6.3.5)

$$\begin{aligned}(\rho_{V',W'},\chi_{V',W'}) = |\langle f,g\rangle| &= \Big|\sum_{\gamma=1}^{n(h)}\sum_{i=1}^{n+p}\sum_{\alpha=1}^{n(1)}A_{i,\,1\alpha}^{h\gamma}\bar{B}_{i,\,1\alpha}^{h\gamma}\Big| \\ &= |c(\chi^*,\rho)^h{}_1|\dim W'\,.\end{aligned}$$

Applying (6.3.8), we obtain the other two equalities.

(b) Since $\rho_{V',W'}=[0]$ or $\chi_{V',W'}=[0]$, the statement is evident. Q.E.D.

We denote by C_ρ the Casimir operator of the representation ρ of G. Then we have

$$\rho(k)\circ C_\rho = C_\rho\circ\rho(k) \quad \text{for } k\in K\,.$$

Therefore it follows from the property (P_1) and Schur's lemma that there exist complex numbers $c(\rho)_h$, $h=1,2,\cdots,k$, such that $C_{\rho|V_h}=c(\rho)_h 1_{V_h}$. Here $C_{\rho|V_h}$ is the restriction of C_ρ to V_h. Then we have

Proposition 6.3.8.

$$c(\rho)_h = -\sum_{j=1}^{k} c(\rho)^h{}_j,\quad h=1,2,\cdots,k\,. \tag{6.3.10}$$

Proof. We have by (6.3.3) and (6.3.5)

$$C_\rho(v_{h;\alpha}) = \sum_{i=1}^{n+p} d\rho(E_i)\,(d\rho(E_i)v_{h;\alpha})$$
$$= \sum_{i=1}^{n+p}\sum_{j,s=1}^{k}\sum_{\beta=1}^{n(j)}\sum_{\gamma=1}^{n(s)} A_{i,h\alpha}^{\ j\beta} A_{i,j\beta}^{\ s\gamma} v_{s;\gamma}$$
$$= -\sum_{j,s=1}^{k}\sum_{\gamma=1}^{n(s)}\Big(\sum_{i=1}^{n+p}\sum_{\beta=1}^{n(j)} A_{i,j\beta}^{\ s\gamma}\bar{A}_{i,j\beta}^{\ h\alpha}\Big) v_{s;\gamma}$$
$$= -\sum_{j=1}^{k} c(\rho)^h{}_j v_{h;\alpha}\,.$$

This proves (6.3.10). Q.E.D.

Let $\{E_1, E_2, \cdots, E_q, X_\lambda, X_{-\lambda};\ \lambda\in\Lambda\}$ be a basis of $\mathfrak{g}^C$ with the following property: Put $e_\lambda = \frac{1}{\sqrt{2}}(X_\lambda + X_{-\lambda})$ and $f_\lambda = \frac{\sqrt{-1}}{\sqrt{2}}(X_\lambda - X_{-\lambda})$. Then $\{E_1, E_2, \cdots, E_q, e_\lambda, f_\lambda;\ \lambda\in\Lambda\}$ is an orthonormal basis of $\mathfrak{g}$. Put

$$d\rho(E_i)v_{h;\alpha} = \sum_{j=1}^{k}\sum_{\beta=1}^{n(j)} C_{i,h\alpha}^{\ j\beta} v_{j;\beta}\,,$$
$$d\rho(X_{\pm\lambda})v_{h;\alpha} = \sum_{j=1}^{k}\sum_{\beta=1}^{n(j)} C_{\pm\lambda,h\alpha}^{\ j\beta} v_{j;\beta}\,,$$
$$d\chi(E_i)w_{s;\gamma} = \sum_{t=1}^{m}\sum_{\delta=1}^{p(t)} D_{i,s\gamma}^{\ t\delta} w_{t;\delta}$$

and

$$d\chi(X_{\pm\lambda})w_{s;\gamma} = \sum_{t=1}^{m}\sum_{\delta=1}^{p(t)} D_{\pm\lambda,s\gamma}^{\ t\delta} w_{t;\delta}\,.$$

Then we have the following proposition.

Proposition 6.3.9. (a) *We have for* $h=1,2,\cdots,d$

$$\text{(6.3.11)}\quad \sum_{i=1}^{q}\sum_{\alpha=1}^{n(h)} C_{i,h\alpha}^{\ j\beta}\bar{D}_{i,h\alpha}^{\ t\delta} + \sum_{\lambda\in\Lambda}\sum_{\alpha=1}^{n(h)}\big(C_{\lambda,h\alpha}^{\ j\beta}\bar{D}_{\lambda,h\alpha}^{\ t\delta} + C_{-\lambda,h\alpha}^{\ j\beta}\bar{D}_{-\lambda,h\alpha}^{\ t\delta}\big)$$
$$= \begin{cases} c(\chi^*, \rho)^j{}_h, & \text{if } j\leqq d,\ t=j \text{ and } \delta=\beta\,,\\ 0, & \text{otherwise.}\end{cases}$$

(b) *Suppose that* $\{E_1, E_2, \cdots, E_r, e_\lambda, f_\lambda;\ \lambda\in\Lambda_1\}$ *(resp.* $\{E_{r+1}, E_{r+2}, \cdots, E_q, e_\lambda, f_\lambda;\ \lambda\in\Lambda_2\}$*) is an orthonormal basis of* $\mathfrak{k}$ *(resp. of* $\mathfrak{p}$*). Then we have for* $h, j=1,2,\cdots,d$

$$c(\chi^*, \rho; \mathfrak{k})^j{}_h$$
$$= \sum_{i=1}^{r}\sum_{\alpha=1}^{n(h)} C_{i,h\alpha}^{\ j\beta}\bar{D}_{i,h\alpha}^{\ j\beta} + \sum_{\lambda\in\Lambda_1}\sum_{\alpha=1}^{n(h)}\big(C_{\lambda,h\alpha}^{\ j\beta}\bar{D}_{\lambda,h\alpha}^{\ j\beta} + C_{-\lambda,h\alpha}^{\ j\beta}\bar{D}_{-\lambda,h\alpha}^{\ j\beta}\big)\,,$$
$$c(\chi^*, \rho; \mathfrak{p})^j{}_h$$
$$= \sum_{i=r+1}^{q}\sum_{\alpha=1}^{n(h)} C_{i,h\alpha}^{\ j\beta}\bar{D}_{i,h\alpha}^{\ j\beta} + \sum_{\lambda\in\Lambda_2}\sum_{\alpha=1}^{n(h)}\big(C_{\lambda,h\alpha}^{\ j\beta}\bar{D}_{i,h\alpha}^{\ j\beta} + C_{-\lambda,h\alpha}^{\ j\beta}\bar{D}_{-\lambda,h\alpha}^{\ j\beta}\big)\,.$$

Proof. (a) We have by the definition of $C_{i,h\alpha}^{j\beta}$, $C_{\pm\lambda,h\alpha}^{j\beta}$, $D_{i,h\alpha}^{j\beta}$ and $D_{\pm\lambda,h\alpha}^{j\beta}$

$$L(\chi^*, \rho)\omega_h = \{\sum_{i=1}^{q} d\chi^*(E_i)\otimes d\rho(E_i)+$$

$$\sum_{\lambda\in\Lambda}(d\chi^*(e_\lambda)\otimes d\rho(e_\lambda)+d\chi^*(f_\lambda)\otimes d\rho(f_\lambda))\}(\sum_{\alpha=1}^{n(h)} w_{h;\alpha}{}^*\otimes v_{h;\alpha})$$

$$=\sum_{j=1}^{k}\sum_{t=1}^{m}\sum_{\beta=1}^{n(j)}\sum_{\delta=1}^{p(t)}(\sum_{i=1}^{q}\sum_{\alpha=1}^{n(h)} C_{i,h\alpha}^{j\beta}\bar{D}_{i,h\alpha}^{t\delta}+$$

$$\sum_{\lambda\in\Lambda}\sum_{\alpha=1}^{n(h)}\{\frac{1}{2}(C_{\lambda,h\alpha}^{j\beta}+C_{-\lambda,h\alpha}^{j\beta})(\bar{D}_{\lambda,h\alpha}^{t\delta}+\bar{D}_{\lambda,h\alpha}^{t\delta})+$$

$$\frac{1}{2}(C_{\lambda,h\alpha}^{j}-C_{-\lambda,h\alpha}^{j\beta})(\bar{D}_{\lambda,h\alpha}^{t\delta}-\bar{D}_{-\lambda,h\alpha}^{t\delta})\})w_{t;\delta}{}^*\otimes v_{j;\beta}$$

$$=\sum_{j=1}^{k}\sum_{t=1}^{m}\sum_{\beta=1}^{n(j)}\sum_{\delta=1}^{p(t)}\{\sum_{i=1}^{q}\sum_{\alpha=1}^{n(h)} C_{i,h\alpha}^{j\beta}\bar{D}_{i,h\alpha}^{t\delta}+$$

$$\sum_{\lambda\in\Lambda}\sum_{\alpha=1}^{n(h)}(C_{\lambda,h\alpha}^{j\beta}\bar{D}_{\lambda,h\alpha}^{t\delta}+C_{-\lambda,h\alpha}^{j\beta}\bar{D}_{-\lambda,h\alpha}^{t\delta})\}w_{t;\delta}{}^*\otimes v_{j;\beta}\,.$$

Comparing the above equality with the right hand side of (6.3.2), we obtain (6.3.11).

(b) We obtain the equalities in the similar way to above. Q.E.D.

6.4. We say that the pair (G,K) has the *property* (P_2), if the following condition is satisfied: Let V' be an arbitrary irreducible K-module and $\mathfrak{p}^C\otimes V'=U_1+U_2+\cdots+U_h$ a direct sum decomposition into irreducible components as K-modules. Then any pair U_i, U_j of the components with $i\neq j$ are not K-isomorphic.

Lemma 6.4.1. *Suppose that the pair (G,K) has the property (P_2). Let V' and W' be irreducible unitary K-modules. Then there exists a K-homomorphism f_0: $\mathfrak{p}^C\otimes V'\to W'$ with the following property: For every K-homomorphism f: $\mathfrak{p}^C\otimes V'\to W'$ there exists a complex number $c\in \boldsymbol{C}$ such that $f=cf_0$. Moreover if the K-module $\mathfrak{p}^C\otimes V'$ contains the K-module W', We may choose f_0 in such a way that $\langle f_0, f_0\rangle = dim\ W'$. Here $\langle\ ,\ \rangle$ denotes the Hermitian inner product on $Hom(\mathfrak{p}^C\otimes V',W')$ induced from the Hermitian inner products on $\mathfrak{p}^C$, V' and W'.*

Proof. When the K-module $\mathfrak{p}^C\otimes V'$ does not contain W', any K-homomorphism of $\mathfrak{p}^C\otimes V'$ to W' is trivial. So the first statement is evident. Suppose that the K-module $\mathfrak{p}^C\otimes V'$ contains W'. In the decomposition $\mathfrak{p}^C\otimes V'=U_1+U_2+\cdots+U_h$, we may assume that U_1 is K-isomorphic to W'. Let g: $U_1\to W'$ be a unitary K-isomorphism. Choose an orthonormal basis $\{u_1,u_2,\cdots,u_k\}$ (resp. $\{w_1,w_2,\cdots,w_k\}$) of U_1(resp. of W') such that $g(u_\alpha)=w_\alpha$, $\alpha=1,2,\cdots,k$. Let $\{u_1,\cdots,u_k,u_{k+1},\cdots,u_m\}$ be an orthonormal basis of $\mathfrak{p}^C\otimes V'$ and $\{u_1{}^*,u_2{}^*,\cdots,u_m{}^*\}$ its dual basis of $(\mathfrak{p}^C)^*\otimes(V')^*$. By the property (P_2), for every K-homomorphism

$f: \mathfrak{p}^C \otimes V' \to W'$ there exists a complex number $c \in \boldsymbol{C}$ such that

$$f = c \sum_{\alpha=1}^{k} u_\alpha{}^* \otimes w_\alpha .$$

If we put $f_0 = \sum_{\alpha=1}^{k} u_\alpha{}^* \otimes w_\alpha$, we have $\langle f_0, f_0 \rangle = \dim W'$. Q.E.D.

Suppose that the pair (G,K) has the property (P_2). Let V' and W' be irreducible unitary K-modules. Then we extend f_0 in Lemma 6.4.1 to a K-homomorphism of $\mathfrak{g}^C \otimes V'$ to W', which is also denoted by f_0, by defining as $f_0(\mathfrak{k}^C \otimes V') = \{0\}$.

Proposition 6.4.2. *Suppose that the pair (G, K) has the property (P_2). Let V' and W' be irreducible unitary K-modules.*

(1) *Let $\rho: G \to GL(V)$ be a unitary representation of G with the property (P_1). Then there exists a complex number $c \in \boldsymbol{C}$ such that $\rho_{V',W'}(\mathfrak{p}) = [cf_0]$.*

(2) *Let $\rho: G \to GL(V)$ and $\chi: G \to GL(W)$ be unitary representations with the property (P_1). Suppose that both of the irreducible unitary K-modules V' and W' are contained in the representations $\rho_{|K}$ and $\chi_{|K}$. Let $V = V_1 + V_2 + \cdots + V_k$ and $W = W_1 + W_2 + \cdots + W_m$ be the direct sum decompositions in the begining of subsection* 6.3. *Suppose that V_1(resp. V_h) is K-isomorphic to V'(resp. to W'). By* (1) *above there exists a complex number c(resp. d) such that $\rho_{V',W'}(\mathfrak{p}) = [cf_0]$ (resp. $\chi_{V',W'}(\mathfrak{p}) = [df_0]$). If the K-module $\mathfrak{p}^C \otimes V'$ contains W', then we have*

$$|c(\chi^*, \rho; \mathfrak{p})^h{}_1| = |cd| . \tag{6.4.1}$$

Proof. (1) Take $f \in \rho_{V',W'}$. Then it follows from Lemma 6.2.2 and Lemma 6.4.1 that there exists a complex number c such that $f_{\mathfrak{p}} = cf_0$. Therefore we obtain (1).

(2) We have by Lemma 6.4.1

$$(\rho_{V',W'}(\mathfrak{p}), \chi_{V',W'}(\mathfrak{p})) = |\langle cf_0, df_0 \rangle| = |cd| \dim W' .$$

Therefore we have (6.4.1) by Proposition 6.3.7. Q.E.D.

6.5. In this subsection the assumptions and the notation are the same as in subsection 3.3 of [I]. Moreover we assume that the minimal isometric immersion $F: (M, c\langle\ ,\ \rangle) \to S$ is full and that the unitary representation $\rho: G \to GL(V^C)$ has the property (P_1). Here V^C is the complexification of V. Let $V^C = V_1 + V_2 + \cdots + V_m$ be a direct sum decomposition into irreducible components as K-modules such that $(V^N)^C = V_1 + V_2 \cdots + V_k$ and that $(V^0)^C + (V^T)^C = V_{k+1} + V_{k+2} + \cdots + V_m$. It follows from Lemma 5.2.3 of [I] and the property (P_1) that the operator $\sum_{i=1}^{n+p} \{d\rho(E_i)\,(d\rho(E_i)*)^N\}^N$ leaves every V_h invariant. Therefore by Schur's lemma there exist complex numbers c_h, $h = 1, 2, \cdots, m$, such that

$$\sum_{i=1}^{n+p} \{d\rho(E_i)\,(d\rho(E_i)*)^N\}^N{}_{|V_h} = c_h 1_{V_h}\,,$$

where $\sum_{i=1}^{n+p} \{d\rho(E_i)\,(d\rho(E_i)*)^N\}^N{}_{|V_h}$ denotes the restriction of $\sum_{i=1}^{n+p} \{d\rho(E_i)$ $(d\rho(E_i)*)^N\}^N$ to V_h. Then we have

Proposition 6.5.1.

$$c_h = \begin{cases} -\sum_{j=1}^{k} c(\rho)^h{}_j\,, & \text{if } h = 1,2,\cdots,k\,, \\ 0\,, & \text{if } h = k+1,k+2,\cdots,m\,. \end{cases}$$

Proof. Let $\{v_{j\,;\,1}, v_{j\,;\,2}, \cdots, v_{j\,;\,n(j)}\}$ be an orthonormal basis of V_j and define $A_{i,j\alpha}^{s\beta}$ by (6.3.3). Then we have by (6.3.5)

$$\begin{aligned} c_h v_{h\,;\,\alpha} &= \sum_{i=1}^{n+p} \{d\rho(E_i)\,(d\rho(E_i)v_{h\,;\,\alpha})^N\}^N \\ &= \sum_{i=1}^{n+p}\sum_{s,j=1}^{k}\sum_{\beta=1}^{n(j)}\sum_{\gamma=1}^{n(s)} A_{i,h\alpha}^{j\beta} A_{i,j\beta}^{s\gamma} v_{s\,;\,\gamma} \\ &= -\sum_{s,j=1}^{k}\sum_{\gamma=1}^{n(s)} \Big(\sum_{i=1}^{n+p}\sum_{\beta=1}^{n(j)} A_{i,j\beta}^{s\gamma} \bar{A}_{i,j\beta}^{h\alpha}\Big) v_{s\,;\,\gamma} \\ &= -\sum_{j=1}^{k} c(\rho)^h{}_j v_{h\,;\,\alpha}\,. \end{aligned}$$

It is evident that $c_h=0$ for $h=k+1,k+2,\cdots,m$. Q.E.D.

7. 3-dimensional subalgebras

7.1. In this section we assume the followings. Let G be a compact connected Lie group, K a closed subgroup of G and (G,K) a Riemannian symmetric pair. The Lie algebra $\mathfrak{g}$, and the subspaces $\mathfrak{k}$ and $\mathfrak{p}$ of $\mathfrak{g}$ are the same as in subsection 2.1 of [I]. The Hermitian inner product $\langle\ ,\ \rangle$ and the symmetric bilinear form $(\ ,\ )$ on $\mathfrak{g}^C$, the complexification of $\mathfrak{g}$, are the same as in subsection 6.2. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{k}$. We denote by $\mathfrak{k}^C$(resp. by $\mathfrak{p}^C$ and by $\mathfrak{h}^C$) the complex subspace of $\mathfrak{g}^C$ generated by $\mathfrak{k}$(resp. by $\mathfrak{p}$ and by $\mathfrak{h}$). Let σ be the involutive automorphism of $\mathfrak{g}$ associated to the Riemannian symmetric pair (G,K), and τ the conjugation of $\mathfrak{g}^C$ with respect to $\mathfrak{g}$.

We have

$$(7.1.1) \qquad \langle X, Y\rangle = (X, \tau Y) \qquad \text{for } X, Y\in\mathfrak{g}^C\,.$$

For $\lambda\in\mathfrak{h}^C$ we define a subspace $\mathfrak{p}_\lambda{}^C$ of $\mathfrak{p}^C$ by

$$\mathfrak{p}_\lambda{}^C = \{X\in\mathfrak{p}^C; [H,X] = \sqrt{-1}\,(\lambda, H)X \qquad \text{for } H\in\mathfrak{h}^C\}\,.$$

Put $\mathfrak{S}=\{\lambda\in\mathfrak{h}^C; \mathfrak{p}_\lambda{}^C\neq\{0\}\}$. Then we have

Lemma 7.1.1 (*cf. Araki* [1] *p.* 4). (1) *If* λ *is an element in* $\mathfrak{S}$, *then* λ *is contained in* $\mathfrak{h}$. *Therefore the linear form* $(\lambda, *)$ *is real valued on* $\mathfrak{h}$.

(2) *The subspace* $\mathfrak{p}^C$ *is decomposed into a vector space direct sum in the following way*:

$$\mathfrak{p}^C = \sum_{\lambda \in \mathfrak{S}} \mathfrak{p}_\lambda{}^C .$$

The following lemmas are proved in the similar way to the case of root systems of compact Lie groups.

Lemma 7.1.2. *If* $X \in \mathfrak{p}_\lambda{}^C$ *and* $Y \in \mathfrak{p}_{-\lambda}{}^C$, *then* $[X, Y]$ *is contained in* $\mathfrak{h}^C$.

Lemma 7.1.3. *If* $X \in \mathfrak{p}_\lambda{}^C$ *and* $\lambda \in \mathfrak{S}$, *then* τX *is contained in* $\mathfrak{p}_{-\lambda}{}^C$. *Therefore we have dim* $\mathfrak{p}_\lambda{}^C = $ *dim* $\mathfrak{p}_{-\lambda}{}^C$.

Lemma 7.1.4. *Suppose that* $X \in \mathfrak{p}_\lambda{}^C$ *and* $Y \in \mathfrak{p}_\mu{}^C$. *Then we have* $(X, Y) = 0$, *if* $\lambda + \mu \neq 0$. *In particular the following equality holds*:

$$(X, X) = 0 . \tag{7.1.2}$$

Lemma 7.1.5. *If* $X \in \mathfrak{p}_\lambda{}^C$ *and* $Y \in \mathfrak{p}_{-\lambda}{}^C$, *then we have*

$$[X, Y] = \sqrt{-1}\,(X, Y)\lambda . \tag{7.1.3}$$

7.2. Let $\lambda \in \mathfrak{S}$ with $\lambda \neq 0$. Choose an element $X_\lambda \in \mathfrak{p}_\lambda{}^C$ with the property $(X_\lambda, \tau X_\lambda) = \langle X_\lambda, X_\lambda \rangle = 1$. We define elements e and f of $\mathfrak{g}$ as follows:

$$e = \frac{1}{\sqrt{2}}(X_\lambda + \tau X_\lambda) ,$$

$$f = \frac{\sqrt{-1}}{\sqrt{2}}(X_\lambda - \tau X_\lambda) .$$

It follows from Lemma 7.1.3 and (7.1.2) that

$$\langle e, e \rangle = (e, e) = \frac{1}{2}\{(X_\lambda, X_\lambda) + 2(X_\lambda, \tau X_\lambda) + (\tau X_\lambda, \tau X_\lambda)\}$$
$$= 1 .$$

Similarly we obtain $\langle f, f \rangle = 1$. We have the following lemma.

Lemma 7.2.1.

$$[X_\lambda, \tau X_\lambda] = \sqrt{-1}\,\lambda . \tag{7.2.1}$$

Proof. Applying Lemma 7.1.3 and (7.1.3), we have

$$[X_\lambda, \tau X_\lambda] = \sqrt{-1}\,(X_\lambda, \tau X_\lambda)\lambda = \sqrt{-1}\,\lambda .$$

Q.E.D.

Put $Y_\lambda=\tau X_\lambda$ and $H_\lambda=\sqrt{-1}\,\lambda$. We define a complex subspace $\mathfrak{g}_{X_\lambda}{}^C$ of $\mathfrak{g}^C$ and a real subspace $\mathfrak{g}_{X_\lambda}$ of $\mathfrak{g}$ as follows:

$$\mathfrak{g}_{X_\lambda}{}^C = \{X_\lambda, Y_\lambda, H_\lambda\}_C\,,$$
$$\mathfrak{g}_{X_\lambda} = \{e, f, h\}_R\,,$$

where $h=\sqrt{-1}\,H_\lambda$. Then we have

Lemma 7.2.2. *The subspace $\mathfrak{g}_{X_\lambda}$ is a real Lie subalgebra of $\mathfrak{g}$ and $\mathfrak{g}_{X_\lambda}{}^C$ is a complex Lie subalgebra of $\mathfrak{g}^C$. The Lie subalgebra $\mathfrak{g}_{X_\lambda}$ is a real form of $\mathfrak{g}_{X_\lambda}{}^C$.*

Proof. We have the following equalities by the definitions of X_λ, Y_λ and H_λ:

$$(7.2.2)\qquad \begin{cases}[H_\lambda, X_\lambda] = -(\lambda, \lambda)X_\lambda\,,\\ [H_\lambda, Y_\lambda] = (\lambda, \lambda)Y_\lambda\,,\\ [X_\lambda, Y_\lambda] = H_\lambda\,.\end{cases}$$

These show that $\mathfrak{g}_{X_\lambda}{}^C$ is a Lie subalgebra of $\mathfrak{g}^C$. Applying (7.2.2), we have

$$[h, e] = -(\lambda, \lambda)f,\ \ [h, f] = (\lambda, \lambda)e,\ \ [e, f] = -h\,.$$

By (1) of Lemma 7.1.1, (λ, λ) is a real number. Therefore $\mathfrak{g}_{X_\lambda}$ is a Lie subalgebra of $\mathfrak{g}$. It is evident that $\mathfrak{g}_{X_\lambda}$ is a real form of $\mathfrak{g}_{X_\lambda}{}^C$. Q.E.D.

We denote by $\mathfrak{sl}(2, \boldsymbol{C})$ (resp. by $\mathfrak{su}(2)$) the Lie algebra of the special linear group $SL(2, \boldsymbol{C})$ (resp. the Lie algebra of the special unitary group $SU(2)$). Then $\mathfrak{su}(2)$ is a compact real form of $\mathfrak{sl}(2, \boldsymbol{C})$. We choose a basis $\{X_0, Y_0, H_0\}$ of $\mathfrak{sl}(2, \boldsymbol{C})$ as follows:

$$X_0 = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},\ \ Y_0 = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix},\ \ H_0 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

Then we have

$$(7.2.3)\qquad \begin{cases}[H_0, X_0] = 2X_0\,,\\ [H_0, Y_0] = -2Y_0\,,\\ [X_0, Y_0] = H_0\,.\end{cases}$$

Since $\lambda\in\mathfrak{h}$ and $\lambda\neq 0$, (λ, λ) is strictly positive. Put $k=(\lambda, \lambda)$. We define a linear mapping $\phi\colon \mathfrak{g}_{X_\lambda}{}^C\to\mathfrak{sl}(2, \boldsymbol{C})$ by

$$\phi(H_\lambda) = -\frac{k}{2}H_0,\ \ \phi(X_\lambda) = \sqrt{\frac{k}{2}}X_0,\ \ \phi(Y_\lambda) = -\sqrt{\frac{k}{2}}Y_0\,.$$

Then we have

Lemma 7.2.3. *The linear mapping* $\phi: \mathfrak{g}_{X_\lambda}{}^C \to \mathfrak{sl}(2, \boldsymbol{C})$ *is a Lie algebra isomorphism and maps* $\mathfrak{g}_{X_\lambda}$ *onto* $\mathfrak{su}(2)$.

Proof. Applying (7.2.3), we have

$$\begin{cases} [\phi(H_\lambda), \phi(X_\lambda)] = -(\lambda, \lambda)\phi(X_\lambda), \\ [\phi(H_\lambda), \phi(Y_\lambda)] = (\lambda, \lambda)\phi(Y_\lambda), \\ [\phi(X_\lambda), \phi(Y_\lambda)] = \phi(H_\lambda). \end{cases}$$

Therefore it follows from (7.2.2) that ϕ is a Lie algebra isomorphism. We have

$$\phi(e) = \frac{\sqrt{k}}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ \phi(f) = \frac{\sqrt{k}}{2}\begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \ \phi(h) = -\frac{k}{2}\begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}.$$

Therefore $\phi(e)$, $\phi(f)$ and $\phi(h)$ are contained in $\mathfrak{su}(2)$. Thus the lemma is proved. Q.E.D.

7.3. We define a symmetric bilinear form (,) on $\mathfrak{sl}(2, \boldsymbol{C})$ by

$$(X, Y) = -\frac{1}{2}Tr(XY) \qquad \text{for } X, Y \in \mathfrak{sl}(2, \boldsymbol{C}),$$

where $Tr(XY)$ denotes the trace of the matrix XY. Then this bilinear form (,) is $\mathfrak{sl}(2, \boldsymbol{C})$-invariant and positive definite on $\mathfrak{su}(2)$. Put

$$\mathfrak{t} = \left\{\sqrt{-1}\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}; \ a \in \boldsymbol{R}\right\}.$$

Then $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{su}(2)$. Let $\rho: \mathfrak{su}(2) \to \mathfrak{gl}(V)$ be a complex representation of $\mathfrak{su}(2)$. An element $\psi \in \mathfrak{t}$ is called a *weight* of ρ, if there exists a non-zero vector $v \in V$ such that $\rho(H)v = \sqrt{-1}\,(\psi, H)v$ for all $H \in \mathfrak{t}$. And this vector v is called a *ψ-weight vector* or a *weight vector belonging to* ψ. Put

$$\phi = \sqrt{-1}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Choose a linear order $>$ on $\mathfrak{t}$ such that $\phi > 0$. Suppose that $\rho: \mathfrak{su}(2) \to \mathfrak{gl}(V)$ is an irreducible complex representation. Then the highest weight of ρ is equal to $m\phi$ for some non-negative integer m, the weights of ρ are $\{(m-2i)\phi; i=0, 1, \cdots, m\}$, and $\dim V = m+1$(cf. Serre [8] Chapitre IV).

Let $\rho_m: \mathfrak{sl}(2, \boldsymbol{C}) \to \mathfrak{gl}(U_m)$ be an irreducible representation with the highest weight $m\phi$. Then there exists a basis $\{u_{-m}, u_{-m+2}, \cdots, u_{m-2}, u_m\}$ of U_m with the following properties:

$$(7.3.1)\qquad \begin{cases} \rho_m(H_0)u_{m-2i} = (m-2i)u_{m-2i}\,, \\ \rho_m(X_0)u_{m-2i} = iu_{m+2-2i}\,, \\ \rho_m(Y_0)u_{m-2i} = (m-i)u_{m-2i-2}\,, \qquad i=0,1,\cdots,m\,, \end{cases}$$

where $u_{m+2}=u_{-m-2}=0$. We introduce a Hermitian inner product $\langle\ ,\ \rangle_0$ on U_m such that $\{\sqrt{\frac{1}{i!(m-i)!}}\,u_{m-2i};\ i=0,1,\cdots,m\}$ is an orthonormal basis of U_m. Then $\langle\ ,\ \rangle_0$ is invariant under the action of $\mathfrak{su}(2)$, i.e. $\langle\rho_m(X)u,v\rangle_0+\langle u,\rho_m(X)v\rangle_0=0$ for $u,v\in U_m$ and $X\in\mathfrak{su}(2)$. A Hermitian inner product on U_m which is invariant under the action of $\mathfrak{su}(2)$ is unique up to constant multiple. Then we have

Lemma 7.3.1. *Let $\langle\ ,\ \rangle$ be a Hermitian product on U_m which is invariant under the action of $\mathfrak{su}(2)$. Then there exists an orthonormal basis $\{e_{-m}, e_{-m+2},\cdots, e_{m-2}, e_m\}$ of U_m with the following properties:*

$$(7.3.2)\qquad \begin{cases} \rho_m(H_0)e_i = ie_i\,, \\ \rho_m(X_0)e_i = \dfrac{1}{2}\sqrt{(m-i)(m+i+2)}\,e_{i+2} \\ \rho_m(Y_0)e_i = \dfrac{1}{2}\sqrt{(m+i)(m-i+2)}\,e_{i-2}\,,\quad i=-m,-m+2,\cdots,m-2,m\,. \end{cases}$$

When $m=2m'$ is even, we put $f_i=e_{2i}$, $i=-m',\cdots,-1,0,1,\cdots,m'$. Then we have

$$(7.3.3)\qquad \begin{cases} \rho_{2m'}(H_0)f_i = 2if_i\,, \\ \rho_{2m'}(X_0)f_i = \sqrt{(m'-i)(m'+i+1)}\,f_{i+1}\,, \\ \rho_{2m'}(Y_0)f_i = \sqrt{(m'+i)(m'-i+1)}\,f_{i-1}\,,\quad i=-m',\cdots,0,\cdots,m'\,. \end{cases}$$

Proof. By (7.3.1) we have

$$\rho_m(X_0)\left(\sqrt{\frac{1}{i!(m-i)!}}\,u_{m-2i}\right) = \frac{i}{\sqrt{i!(m-i)!}}\,u_{m+2-2i} = \sqrt{\frac{i(m-i+1)}{(i-1)!(m-i+1)!}}\,u_{m+2-2i}\,,$$

$$\rho_m(Y_0)\left(\sqrt{\frac{1}{i!(m-i)!}}\,u_{m-2i}\right) = \frac{m-i}{\sqrt{i!(m-i)!}}\,u_{m-2i-2} = \sqrt{\frac{(m-i)(i+1)}{(i+1)!(m-i-1)!}}\,u_{m-2i-2}\,.$$

Since $\langle\ ,\ \rangle$ is a constant multiple of $\langle\ ,\ \rangle_0$, we obtain (7.3.2). We have easily (7.3.3) by (7.3.2). Q.E.D.

Let $\rho\colon G\to GL(V)$ be a unitary representation. Then by Lemma 7.2.3 we may consider the differential $d\rho$ of ρ as a representation of $\mathfrak{sl}(2,\boldsymbol{C})$ such that the Hermitian inner product $\langle\ ,\ \rangle$ on V is invariant under the action of $\mathfrak{su}(2)$. Let $|v|$ denote the length $\sqrt{\langle v,v\rangle}$ of a vector $v\in V$. We decompose V into a vector space direct sum in the following way:

$$V=\sum_{m=0}^{s} V^m\,, \tag{7.3.4}$$

where V^m is the subspace of V generated by $\mathfrak{sl}(2,\boldsymbol{C})$-submodules which are $\mathfrak{sl}(2,\boldsymbol{C})$-isomorphic to U_m. Then we have

Lemma 7.3.2. (1) *Let v be an $i\phi$-weight vector with $|v|=1$. If v is contained in V^m, we have*

$$\begin{cases} |d\rho(X_\lambda)v|^2=\dfrac{\langle\lambda,\lambda\rangle}{8}(m-i)(m+i+2)\,, \\ |d\rho(Y_\lambda)v|^2=\dfrac{\langle\lambda,\lambda\rangle}{8}(m+i)(m-i+2)\,. \end{cases} \tag{7.3.5}$$

(1)′ *Let v be a $2i\phi$-weight vector with $|v|=1$. If v is contained in V^{2m}, we have*

$$\begin{cases} |d\rho(X_\lambda)v|^2=\dfrac{\langle\lambda,\lambda\rangle}{2}(m-i)(m+i+1)\,, \\ |d\rho(Y_\lambda)v|^2=\dfrac{\langle\lambda,\lambda\rangle}{2}(m+i)(m-i+1)\,. \end{cases} \tag{7.3.6}$$

Proof. The $\mathfrak{sl}(2,\boldsymbol{C})$-invariant subspace generated by v is $\mathfrak{sl}(2,\boldsymbol{C})$-isomorphic to U_m. Therefore we obtain (7.3.5) by Lemma 7.2.3 and (7.3.2). We have (7.3.6) by (7.3.5) easily. Q.E.D.

8. Symmetric R-spaces

8.1. Let $(\mathfrak{h},\sigma)$ be an orthogonal symmetric Lie algebra of compact type. Put $\mathfrak{h}=\mathfrak{g}+\mathfrak{m}$, where $\mathfrak{g}$(resp. $\mathfrak{m}$) is the 1-eigenspace (resp. -1-eigenspace) of σ. Let Aut$(\mathfrak{h})$ be the group of all automorphisms of $\mathfrak{h}$. Identifying the Lie algebra of Aut$(\mathfrak{h})$ with $\mathfrak{h}$, let G be the connected Lie subgroup of Aut$(\mathfrak{h})$ corresponding to the Lie subalgebra $\mathfrak{g}$ of $\mathfrak{h}$. Then G leaves the subspace $\mathfrak{m}$ invariant. Let $\langle\ ,\ \rangle$ be an Aut$(\mathfrak{h})$-invariant inner product on $\mathfrak{h}$. The Lie group G acts as an isometry group on the Euclidean space $\mathfrak{m}$ with the inner product $\langle\ ,\ \rangle$, the restriction of the inner product $\langle\ ,\ \rangle$ on $\mathfrak{h}$ to $\mathfrak{m}$. Let S be the unit sphere of $\mathfrak{m}$ with center o, the origin of $\mathfrak{m}$. Let H be an element of S and M the orbit of G through H. Denoting by K the stabilizer of H in G, the space M may be identified with the quotient space G/K, which is called an *R-space associated to* $(\mathfrak{g},\sigma)$. If (G,K) is a Riemannian symmetric pair, M is called a *symmetric R-space*. Then the Riemannian submanifold M of S is a Riemannian symmetric space (Takeuchi [9] p. 112).

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{m}$. For $\lambda \in \mathfrak{a}$ we define a subspace $\mathfrak{g}_\lambda$(resp. $\mathfrak{m}_\lambda$) of $\mathfrak{g}$(resp. of $\mathfrak{m}$) as follows:

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g};\ ad(H)^2 X = -\langle \lambda, H\rangle^2 X \quad \text{for any } H \in \mathfrak{a}\}\ ,$$
$$\mathfrak{m}_\lambda = \{X \in \mathfrak{m};\ ad(H)^2 X = -\langle \lambda, H\rangle^2 X \quad \text{for any } H \in \mathfrak{a}\}\ .$$

Then $\mathfrak{g}_{-\lambda} = \mathfrak{g}_\lambda$, $\mathfrak{m}_{-\lambda} = \mathfrak{m}_\lambda$ and $\mathfrak{m}_0 = \mathfrak{a}$. Put $\mathfrak{r} = \{\lambda \in \mathfrak{a};\ \lambda \neq 0,\ \mathfrak{m}_\lambda \neq \{0\}\}$. Then $\mathfrak{r}$ is a root system in $\mathfrak{a}$ (Satake [7] p. 81). This root system $\mathfrak{r}$ is called the restricted root system of $(\mathfrak{h}, \sigma)$. Choose a linear order in $\mathfrak{a}$. Let Δ be the fundamental system of $\mathfrak{r}$ and $\mathfrak{r}^+$ the set of all positive roots in $\mathfrak{r}$. Then we have the following orthogonal decomposition of $\mathfrak{g}$ and $\mathfrak{m}$(*cf.* Helgason [4]):

$$\mathfrak{g} = \mathfrak{g}_0 + \sum_{\lambda \in \mathfrak{r}^+} \mathfrak{g}_\lambda,\quad \mathfrak{m} = \mathfrak{a} + \sum_{\lambda \in \mathfrak{r}^+} \mathfrak{m}_\lambda\ .$$

By virtue of the following lemma we may assume that $H \in S \cap \mathfrak{a}$ and $\langle \lambda, H\rangle \geqq 0$ for any $\lambda \in \mathfrak{r}^+$.

Lemma 8.1.1 (cf. Helgason [4] p. 211, p. 248). *For any $H \in \mathfrak{m}$ there exists an element $x \in G$ such that $xH \in \mathfrak{a}$ and $\langle \lambda, xH\rangle \geqq 0$ for any $\lambda \in \mathfrak{r}^+$.*

We identify the tangent space $T_H(M)$ of M at H with a subspace of $\mathfrak{m}$ in a canonical manner. Then we have $T_H(M) = [\mathfrak{g}, H]$. Put

$$\begin{cases} \mathfrak{r}_1^+ = \{\lambda \in \mathfrak{r}^+;\ \langle \lambda, H\rangle = 0\}\ , \\ \mathfrak{r}_2^+ = \{\lambda \in \mathfrak{r}^+;\ \langle \lambda, H\rangle > 0\}\ . \end{cases}$$

The tangent space $T_H(M)$ and the normal space $N_H(M)$ in S are given by

$$\text{(8.1.1)}\qquad \begin{cases} T_H(M) = \sum_{\lambda \in \mathfrak{r}_2^+} \mathfrak{m}_\lambda\ , \\ N_H(M) = \mathfrak{a}_H + \sum_{\lambda \in \mathfrak{r}_1^+} \mathfrak{m}_\lambda\ , \end{cases}$$

where $\mathfrak{a}_H = \{X \in \mathfrak{a};\ \langle X, H\rangle = 0\}$. Let $\mathfrak{k}$ be the Lie algebra of the stabilizer K of H, and $\mathfrak{p}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$. Then we have

$$\text{(8.1.2)}\qquad \mathfrak{k} = \mathfrak{g}_0 + \sum_{\lambda \in \mathfrak{r}_1^+} \mathfrak{g}_\lambda\ ,\quad \mathfrak{p} = \sum_{\lambda \in \mathfrak{r}_2^+} \mathfrak{g}_\lambda\ .$$

8.2. Put $\Delta_1 = \{\lambda \in \Delta;\ \lambda \in \mathfrak{r}_1^+\}$. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{h}$ containing $\mathfrak{a}$. Let $\mathfrak{h}^C$ be the complexification of $\mathfrak{h}$, and $\mathfrak{t}^C$ the subspace of $\mathfrak{h}^C$ spanned by $\mathfrak{t}$. We denote by $(\ ,\ )$ the symmetric bilinear form on $\mathfrak{h}^C$, which is the $\boldsymbol{C}$-bilinear extension of the inner product $\langle\ ,\ \rangle$ on $\mathfrak{h}$. Let $\tilde{\mathfrak{r}}$ be the root system of $\mathfrak{h}^C$ with respect to $\mathfrak{t}^C$. Recall that an element $\alpha \in \mathfrak{t}^C$ belongs to $\tilde{\mathfrak{r}}$, if $\alpha \neq 0$ and if there exists a non-zero vector $X \in \mathfrak{t}^C$ such that $[H, X] = \sqrt{-1}$ $(\alpha, H)X$ for any $H \in \mathfrak{t}^C$. Then $\mathfrak{t}$ contains the root system $\tilde{\mathfrak{r}}$. We denote by the same letter σ the conjugation of $\mathfrak{h}^C$ with respect to the real form $\mathfrak{g}+\sqrt{-1}$ $\mathfrak{m}$. We choose a σ-order in $\mathfrak{t}$ in the sense of Satake [7] which has the following property: Let $\tilde{\Delta}$ be the fundamental system with respect to this linear order in $\mathfrak{t}$, and let p denote the projection of $\mathfrak{t}$ onto $\mathfrak{a}$. Then $\Delta=p(\tilde{\Delta})-\{0\}$. We also denote by $\tilde{\Delta}$ the Satake diagram of $\tilde{\Delta}$. Put $\tilde{\Delta}_1=p^{-1}(\Delta_1)$. It is known (Takeuchi [9] p. 102) that isomorphic pairs $(\tilde{\Delta}, \tilde{\Delta}_1)$ of Satake diagrams give rise to isomorphic pairs (G,K). Here we say that the pair $(\tilde{\Delta}, \tilde{\Delta}_1)$ is isomorphic to the pair $(\tilde{\Delta}', \tilde{\Delta}'_1)$, if there exists an isomorphism ϕ of $\tilde{\Delta}$ onto $\tilde{\Delta}'$ with $(\tilde{\Delta}_1)=\tilde{\Delta}'_1$, and we say that the pair (G,K) is isomorphic to the pair (G',K'), if there exists an isomorphism f of G onto G' with $f(K)=K'$.

REMARK 8.2.1. Let Δ_1 be a subsystem of Δ, and $(\tilde{\Delta}, \tilde{\Delta}_1)$ the pair of Satake diagram determined by Δ_1. Then there exists a minimal R-space M such that the pair of Satake diagram corresponding to M is isomorphic to $(\tilde{\Delta}, \tilde{\Delta}_1)$ (Nagura [5] p. 210).

8.3. We decompose $(\mathfrak{h}, \sigma)$ into a direct sum of irreducible orthogonal symmetric Lie algebras $(\mathfrak{h}_i, \sigma_i)$:

$$\mathfrak{h} = \mathfrak{h}_1+\mathfrak{h}_2+\cdots+\mathfrak{h}_r\,, \qquad \sigma = \sigma_1+\sigma_2+\cdots+\sigma_r\,.$$

Put $\mathfrak{g}_i=\mathfrak{g}\cap\mathfrak{h}_i$, $\mathfrak{m}_i=\mathfrak{m}\cap\mathfrak{h}_i$ and $\mathfrak{a}_i=\mathfrak{m}_i\cap\mathfrak{a}$. Then $\mathfrak{a}_i$ is a maximal abelian subspace of $\mathfrak{m}_i$. Let M be an R-space associated to $(\mathfrak{h}, \sigma)$. Put

$$\mathfrak{r}_i = \mathfrak{r}\cap\mathfrak{a}_i,\ (\mathfrak{r}_i)_1{}^+ = \mathfrak{r}_1{}^+\cap\mathfrak{a}_i,\ (\mathfrak{r}_i)_2{}^+ = \mathfrak{r}_2{}^+\cap\mathfrak{a}_i\,.$$

Then

$$\begin{aligned}\mathfrak{r} &= \mathfrak{r}_1\cup\cdots\cup\mathfrak{r}_r\,,\\ \mathfrak{r}_1{}^+ &= (\mathfrak{r}_1)_1{}^+\cup\cdots\cup(\mathfrak{r}_r)_1{}^+\,,\\ \mathfrak{r}_2{}^+ &= (\mathfrak{r}_1)_2{}^+\cup\cdots\cup(\mathfrak{r}_r)_2{}^+\,,\end{aligned}$$

We say that the R-space M has the *property* $(*)$, if the following condition is satisfied:

$$(*) \qquad (\mathfrak{r}_i)_2{}^+\neq\phi, \qquad i=1,2,\cdots,r\,.$$

Since $ad(\mathfrak{g}_i)$ acts on $\mathfrak{m}_i$ irreducibly, the R-space M in S is full, if and only if M has the property $(*)$.

Proposition 8.3.1. *Let M be a minimal symmetric R-space with the property* $(*)$. *Then the operator S_1 in subsection* 3.3 *of* [*I*] *vanishes.*

Proof. The assumptions of Theorem 1 of [I] are satisfied by the arguments of subsections 8.1 and 8.2. Since $[\mathfrak{g}_\lambda, \mathfrak{a}]\subset\mathfrak{m}_\lambda$ and $[\mathfrak{g}_\lambda, \mathfrak{m}_\mu]\subset\mathfrak{m}_{\lambda+\mu}+\mathfrak{m}_{\lambda-\mu}$ for λ, $\mu\in\mathfrak{r}^+$, we have for $X\in \sum_{\lambda\in\mathfrak{r}_2{}^+}\mathfrak{g}_\lambda$, $v\in\mathfrak{a}_H+\sum_{\lambda\in\mathfrak{r}_1{}^+}\mathfrak{m}_\lambda$

$$[X, v] \in \sum_{\lambda \in r_2^+} \mathfrak{m}_\lambda .$$

Therefore the operator S_1 vanishes by Lemma 3.3.1 of [I], (8.1.1) and (8.1.2). Q.E.D.

REMARK 8.3.1. Let $\mathfrak{h}$ be the Lie algebra $\mathfrak{su}(n+1)$ of the special unitary group $SU(n+1)$, σ the complex conjugation of $\mathfrak{su}(n+1)$, *i. e.* $(\mathfrak{h}, \sigma)$ is an irreducible orthogonal symmetric Lie algebra of type $A\ I$. Then

$$G = SO(n+1),$$

$$\mathfrak{m} = \{\sqrt{-1}\, X;\ X \text{ is a real symmetric matrix and } Tr\, X = 0\}.$$

The representation $\rho: SO(n+1) \to GL(\mathfrak{m})$, $\rho(x)X = xXx^{-1}$ $X \in \mathfrak{m}$, is the spherical representation ρ_2 in (1) of Remark 3.3.2 of [I]. Therefore by the above proposition we have $S_1 = 0$ in the case (1) of Remark 3.3.2 of [I].

9. Representations of the special orthogonal group $SO(n+1)$

9.1. In this section we assume the followings. Let $G = SO(n+1)$ and $K = \left\{\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \in G;\ A \in SO(n)\right\}$. Let $\langle\ ,\ \rangle$ be the $Ad(G)$-invariant inner product on the Lie algebra $\mathfrak{g} = \mathfrak{so}(n+1)$ of G, which is defined by

$$\langle X, Y\rangle = -\frac{1}{2}\, Tr(XY) \qquad \text{for } X, Y \in \mathfrak{g}. \tag{9.1.1}$$

The Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is given by

$$\mathfrak{k} = \left\{\begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix};\ X \in \mathfrak{so}(n)\right\} \cong \mathfrak{so}(n),$$

$$\mathfrak{p} = \left\{\begin{pmatrix} 0 & -{}^t v \\ v & 0 \end{pmatrix};\ v \in \boldsymbol{R}^n\right\}.$$

Put

$$\mathfrak{t} = \left\{ \begin{pmatrix} (0) & & & 0 \\ & \boxed{\begin{matrix} 0 & -\lambda_h \\ \lambda_h & 0 \end{matrix}} & & \\ & & \ddots & \\ 0 & & & \boxed{\begin{matrix} 0 & -\lambda_1 \\ \lambda_1 & 0 \end{matrix}} \end{pmatrix};\ \lambda_1, \lambda_2, \cdots, \lambda_h \in \boldsymbol{R}, \right\}$$

where $h = \left[\frac{n+1}{2}\right]$ and $[*]$ denotes the Gauss symbol. Then $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$. Put

$$\phi_i = \begin{pmatrix} (0) & \overset{1}{\vee} & & \overset{h+1-i}{\vee} & & \overset{h}{\vee} \\ & \boxed{\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}} & & & 0 & \\ & & \ddots & & & \\ & & & \boxed{\begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix}} & & \\ & & & & \ddots & \\ & 0 & & & & \boxed{\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}} \end{pmatrix}.$$

Then $\{\phi_1, \phi_2, \cdots, \phi_h\}$ is an orthonormal basis of $\mathfrak{t}$. We introduce a linear order $>$ on $\mathfrak{t}$ such that $0<\phi_1<\phi_2<\cdots<\phi_h$.

The root system $\mathfrak{r}$ of $\mathfrak{g}^C$ with respect to $\mathfrak{t}$ is given by

$$\mathfrak{r} = \begin{cases} \{\pm\phi_i\pm\phi_j;\, i, j = 1, 2, \cdots, h \text{ with } i\neq j\}, & \text{if } n = 2h-1\,, \\ \{\pm\phi_i,\, \pm\phi_i\pm\phi_j;\, i, j = 1, 2, \cdots, h \text{ with } i\neq j\}, & \text{if } n = 2h\,. \end{cases}$$

Put for $i, j=1,2,\cdots,h$ with $i<j$

$$X_{\phi_j-\phi_i} = \frac{1}{2}\begin{pmatrix} (0) & \overset{h+1-j}{\vee} & \overset{h+1-i}{\vee} \\ 0 & & \boxed{\begin{matrix} 1 & \sqrt{-1} \\ -\sqrt{-1} & 1 \end{matrix}} \\ & \boxed{\begin{matrix} -1 & \sqrt{-1} \\ -\sqrt{-1} & -1 \end{matrix}} & \\ & & 0 \end{pmatrix} \begin{matrix} \\ <h+1-j \\ <h+1-i \\ \\ \end{matrix},$$

$$X_{\phi_i-\phi_j} = \frac{1}{2}\begin{pmatrix} (0) & \overset{h+1-j}{\vee} & \overset{h+1-i}{\vee} \\ 0 & & \boxed{\begin{matrix} -1 & \sqrt{-1} \\ -\sqrt{-1} & -1 \end{matrix}} \\ & \boxed{\begin{matrix} 1 & \sqrt{-1} \\ -\sqrt{-1} & 1 \end{matrix}} & \\ & & 0 \end{pmatrix} \begin{matrix} \\ <h+1-j \\ <h+1-i \\ \\ \end{matrix},$$

$$X_{\phi_i+\phi_j} = \frac{1}{2}\begin{pmatrix} (0) & \overset{h+1-j}{\vee} & \overset{h+1-i}{\vee} \\ 0 & & \boxed{\begin{matrix} 1 & -\sqrt{-1} \\ -\sqrt{-1} & -1 \end{matrix}} \\ & \boxed{\begin{matrix} -1 & \sqrt{-1} \\ \sqrt{-1} & 1 \end{matrix}} & \\ & & 0 \end{pmatrix} \begin{matrix} \\ <h+1-j \\ <h+1-i \\ \\ \end{matrix},$$

$$X_{-\phi_i-\phi_j}=\frac{1}{2}\begin{pmatrix} (0) & \overset{h+1-j}{\vee} & & \overset{h+1-i}{\vee} \\ & 0 & & \boxed{\begin{matrix} 1 & \sqrt{-1} \\ \sqrt{-1} & -1 \end{matrix}} \\ & \boxed{\begin{matrix} -1 & -\sqrt{-1} \\ -\sqrt{-1} & 1 \end{matrix}} & & \\ & & & 0 \end{pmatrix}\begin{matrix} \\ <h+1-j \\ <h+1-i \\ \end{matrix}.$$

Put for $i=1,2,\cdots,h-1$ in the case of $n=2h-1$(resp. for $i=1,2,\cdots,h$ in the case of $n=2h$)

$$X_{\phi_i}=\sqrt{\frac{1}{2}}\begin{pmatrix} (0) & \overset{h+1-i}{\vee} \\ & \boxed{\begin{matrix} -\sqrt{-1} & -1 \end{matrix}} \\ \boxed{\begin{matrix} \sqrt{-1} \\ 1 \end{matrix}} & 0 \end{pmatrix}\begin{matrix} \\ \\ <h+1-i \end{matrix},$$

$$X_{-\phi_i}=\sqrt{\frac{1}{2}}\begin{pmatrix} (0) & \overset{h+1-i}{\vee} \\ & \boxed{\begin{matrix} \sqrt{-1} & -1 \end{matrix}} \\ \boxed{\begin{matrix} -\sqrt{-1} \\ 1 \end{matrix}} & 0 \end{pmatrix}\begin{matrix} \\ \\ <h+1-i \end{matrix}.$$

Then for $\lambda\in\mathfrak{r}$, X_λ is a λ-root vector with $|X_\lambda|=\sqrt{\langle X_\lambda, X_\lambda\rangle}=1$. Let τ be the conjugation of $\mathfrak{g}^C$ with respect to $\mathfrak{g}$. Then we have

$$\text{(9.1.2)} \qquad \tau X_{\pm\phi_i}=X_{\mp\phi_i}\,.$$

Put

$$\mathfrak{h}=\begin{cases}\{\phi_1,\phi_2,\cdots,\phi_{h-1}\}_R & \text{if } n=2h-1\,,\\ \mathfrak{t} & \text{if } n=2h\,.\end{cases}$$

Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{k}$. Let $\mathfrak{S}$ be the one in subsection 7.1. Then we have

$$\mathfrak{S}=\begin{cases}\{0,\phi_i;\, i=1,2,\cdots,h-1\} & \text{if } n=2h-1\,,\\ \{\phi_i;\, i=1,2,\cdots,h\} & \text{if } n=2h\,.\end{cases}$$

Every dominant integral form Λ of G with respect to $\mathfrak{t}$ is uniquely expressed as follows:

$$\Lambda=k_1\phi_1+k_2\phi_2+\cdots+k_h\phi_h\,,$$

where $k_1, k_2, \cdots, k_h$ are integers satisfying

$$(9.1.3)\qquad \begin{cases} |k_1| \leqq k_2 \leqq \cdots \leqq k_h & \text{if } n = 2h-1\,, \\ 0 \leqq k_1 \leqq k_2 \leqq \cdots \leqq k_h & \text{if } n = 2h\,. \end{cases}$$

REMARK 9.1.1. Suppose that $n \geqq 2$. Then the Riemannian symmetric pair (G,K) is of rank 1, and the dominant integral form ϕ_h is the fundamental weight of the pair (G,K) (cf. Takeuchi [10] p. 118). It follows from Remark 3.2.2 of [I] that when we consider a full equivariant minimal isometric immersion of S^n into a unit sphere, it is sufficient to consider the following real representations ρ_k of G, $k=2,3,\cdots$: The representation ρ_k is the real spherical representation of (G,K) whose complexification has the highest weight $k\phi_h$.

We denote by δ_G the half sum of all positive roots of $\mathfrak{g}^C$. Then we have

$$(9.1.4)\qquad \delta_G = \begin{cases} \phi_2 + 2\phi_3 + \cdots + (h-1)\phi_h & \text{if } n = 2h-1\,, \\ \dfrac{1}{2}(\phi_1 + 3\phi_2 + \cdots + (2h-1)\phi_h) & \text{if } n = 2h\,. \end{cases}$$

Let W_G be the Weyl group of G. For an element $\lambda \in \mathfrak{t}$ we denote by ξ_λ the principal alternating sum associated to λ defined by

$$\xi_\lambda = \sum_{\tau \in W_G} \det(\tau)\, e(\tau\lambda)\,,$$

where $e(\lambda)(H) = \exp\sqrt{-1}\,(\lambda, H)$ for $H \in \mathfrak{t}$. For a complex irreducible representation σ of G, we denote by Λ_σ(resp. by χ_σ) the highest weight of σ(resp. the pull back to $\mathfrak{t}$ via $\exp\colon \mathfrak{t} \to G$ of the character of σ). Then we have by the character formula of Weyl (cf. Takeuchi [10] p. 153)

$$(9.1.5)\qquad \xi_{\Lambda_\sigma + \delta_G} = \chi_\sigma \xi_{\delta_G}\,.$$

For complex irreducible representations σ and σ' of G, the character $\chi_{\sigma\otimes\sigma'}$ of the tensor product $\sigma \otimes \sigma'$ is given by

$$(9.1.6)\qquad \chi_{\sigma\otimes\sigma'} = \chi_\sigma \chi_{\sigma'}\,.$$

Let V_μ and V_ν be complex irreducible G-modules with the highest weights μ and ν respectively. For an integral form λ of G we define a non-negative integer $m(\lambda; \mu)$ to be the multiplicity of λ in the G-module V_μ. Let Λ be a dominant integral form of G and U_Λ a complex irreducible G-module with the highest weight Λ. We denote by m_Λ the number of times that U_Λ is contained in the G-module $V_\mu \otimes V_\nu$. Then we have

Lemma 9.1.1 (Bourbaki [3] pp. 153–154).

$$m_\Lambda = \sum_{\tau \in W_G} \det(\tau) m(\Lambda + \delta_G - \tau(\nu + \delta_G); \mu)\,.$$

9.2. The following proposition gives the decomposition of an irreducible G-module into a direct sum of irreducible K-submodules.

Proposition 9.2.1 (Boerner [2] pp. 267–269). *Let σ: $G\to GL(W)$ be a complex irreducible representation with the highest weight $\Lambda_\sigma=k_1\phi_1+k_2\phi_2+\cdots+k_h\phi_h\in\mathfrak{t}$. Then W is decomposed into a direct sum of irreducible K-submodules of W as follows:*

(1) *The case $n=2h-1$:*

$$W=\sum W_{k'_1\phi_1+k'_2\phi_2+\cdots+k'_{h-1}\phi_{h-1}},$$

where the summation runs over all integers $k'_1,k'_2,\cdots,k'_{h-1}$ such that

$$|k_1|\leqq k'_1\leqq k_2\leqq k'_2\leqq\cdots\leqq k_{h-1}\leqq k'_{h-1}\leqq k_h,$$

and $W_{k'_1\phi_1+k'_2\phi_2+\cdots+k'_{h-1}\phi_{h-1}}$ denotes the irreducible K-submodule of W with the highest weight $k'_1\phi_1+k'_2\phi_2+\cdots+k'_{h-1}\phi_{h-1}$.

(2) *The case $n=2h$:*

$$W=\sum W_{k'_1\phi_1+k'_2\phi_2+\cdots+k'_h\phi_h},$$

where the summation runs over all integers $k'_1,k'_2,\cdots,k'_h$ such that

$$|k'_1|\leqq k_1\leqq k'_2\leqq k_2<\cdots\leqq k_{h-1}\leqq k'_h\leqq k_h.$$

The following corollary is an immediate consequence of the above proposition.

Corollary. *Every complex irreducible representation of G has the property (P_1).*

Let $\boldsymbol{C}^{n+1}$ be the vector space of ($(n+1)$-tuples) of complex numbers, and $\langle\ ,\ \rangle$ the Hermitian inner product on $\boldsymbol{C}^{n+1}$ defined by $\langle u,v\rangle={}^t u\bar{v}$. Let ι: $G\to GL(\boldsymbol{C}^{n+1})$ be the canonical representation of G. Put

$$v_i=\sqrt{\frac{1}{2}}\begin{pmatrix} (0) \\ \boxed{\begin{matrix}0\\0\end{matrix}} \\ \vdots \\ \boxed{\begin{matrix}\sqrt{-1}\\1\end{matrix}} \\ \vdots \\ \boxed{\begin{matrix}0\\0\end{matrix}}\end{pmatrix}\begin{matrix} \\ <1 \\ \\ <h+1-i \\ \\ <h\end{matrix},\quad v_{-i}=\sqrt{\frac{1}{2}}\begin{pmatrix} (0) \\ \boxed{\begin{matrix}0\\0\end{matrix}} \\ \vdots \\ \boxed{\begin{matrix}-\sqrt{-1}\\1\end{matrix}} \\ \vdots \\ \boxed{\begin{matrix}0\\0\end{matrix}}\end{pmatrix}\begin{matrix} \\ <1 \\ \\ <h+1-i \\ \\ <h\end{matrix},\quad v_0=\begin{pmatrix} 1 \\ \boxed{\begin{matrix}0\\0\end{matrix}} \\ \vdots \\ \vdots \\ \boxed{\begin{matrix}0\\0\end{matrix}}\end{pmatrix}.$$

Then we have

Lemma 9.2.2. (1) *The case $n=2h-1$: The vector v_i(resp. v_{-i}) is a ϕ_i-weight vector(resp. $-\phi_i$-weight vector), $i=1,2,\cdots,h$, and $\{v_i, v_{-i}; i=1,2,\cdots,h\}$ is* *an orthonormal basis of $\boldsymbol{C}^{n+1}$. Therefore the character χ_ι of ι is given by*

(9.2.1) $$\chi_\iota = \sum_{i=1}^{h}(e(\phi_i)+e(-\phi_i)).$$

(2) *The case $n=2h$: The vector v_i(resp. v_{-i} and v_0) is a ϕ_i-weight vector (resp. $-\phi_i$-weight vector and 0-weight vector), $i=1,2,\cdots,h$, and $\{v_0, v_i, v_{-i};\ i=1,2,\cdots,h\}$ is an orthonormal basis of $\boldsymbol{C}^{n+1}$. Therefore we have*

(9.2.2) $$\chi_\iota = 1+\sum_{i=1}^{h}(e(\phi_i)+e(-\phi_i)).$$

We have the following lemma by straightforward calculation.

Lemma 9.2.3. (a) *The both cases $n=2h-1$ and $n=2h$:*

$$\begin{cases} X_{\phi_i-\phi_j}v_k = \delta_{jk}v_i\,, \\ X_{\phi_j-\phi_i}v_k = \delta_{ik}v_j\,, \\ X_{\phi_i+\phi_j}v_k = 0\,, \\ X_{-\phi_i-\phi_j}v_k = \delta_{jk}v_{-i}-\delta_{ik}v_{-j}\,, \end{cases} \qquad \begin{cases} X_{\phi_i-\phi_j}v_{-k} = -\delta_{ik}v_{-j}\,, \\ X_{\phi_j-\phi_i}v_{-k} = -\delta_{jk}v_{-i}\,, \\ X_{\phi_i+\phi_j}v_{-k} = \delta_{jk}v_i-\delta_{ik}v_j\,, \\ X_{-\phi_i-\phi_j}v_{-k} = 0\,, \end{cases}$$
$$\text{for } i,j,k=1,2,\cdots,h \text{ with } i<j\,.$$

(b) The case $n=2h$:

$$\begin{cases} X_{\phi_i}v_k = 0\,, \\ X_{-\phi_i}v_k = -\delta_{ik}v_0\,, \end{cases} \qquad \begin{cases} X_{\phi_i}v_{-k} = -\delta_{ik}v_0\,, \\ X_{-\phi_i}v_{-k} = 0\,, \end{cases}$$
$$\text{for } i,j=1,2,\cdots,h\,.$$

$$\begin{cases} X_{\phi_i}v_0 = v_i\,, \qquad X_{-\phi_i}v_0 = v_{-i}\,, \\ X_{\phi_i-\phi_j}v_0 = X_{\phi_j-\phi_i}v_0 = X_{\phi_i+\phi_j}v_0 = X_{\phi_i-\phi_j}v_0 = 0\,, \end{cases}$$
$$\text{for } i,j,k=1,2,\cdots,h \text{ with } i<j\,.$$

Lemma 9.2.4. *Let $\sigma\colon G\to GL(W)$ be a complex irreducible representation with the highest weight $\Lambda_\sigma=k_1\phi_1+k_2\phi_2+\cdots+k_h\phi_h$. Then the G-module $\boldsymbol{C}^{n+1}\otimes W$ is decomposed into a direct sum of irreducible G-submodules as follows:*

(1) *The case $n=2h-1$:*

$$\boldsymbol{C}^{n+1}\otimes W = \sum W_{k'_1\phi_1+k'_2\phi_2+\cdots+k'_h\phi_h}\,,$$

where $W_{k'_1\phi_1+k'_2\phi_2+\cdots+k'_h\phi_h}$ is the irreducible G-submodule of $\boldsymbol{C}^{n+1}\otimes W$ with the highest weight $k'_1\phi_1+k'_2\phi_2+\cdots+k'_h\phi_h$ and the summation runs over all integers k'_1, k'_2, $\cdots$, k'_h satisfying (9.1.3) *and the following additional condition* (*): *There exists i, $1\leqq i\leqq h$, such that*

$$\begin{cases} k'_i = k_i+1 \text{ or } k_i-1\,, \\ k'_j = k_j \qquad \text{for } j=1,2,\cdots,h \text{ with } j\neq i\,. \end{cases}$$

(2) *The case* $n=2h$:

$$C^{n+1}\otimes W=\begin{cases}\sum W_{k'_1\phi_1+k'_2\phi_2+\cdots+k'_h\phi_h}, & \text{if } k_1=0,\\ W_{k_1\phi_1+k_2\phi_2+\cdots+k_h\phi_h}+\sum W_{k'_1\phi_1+k_2\phi_2+\cdots+k'_h\phi_h}, & \\ & \text{if } k_1>0,\end{cases}$$

where the summation runs over all integers $k'_1, k'_2, \cdots, k'_h$ *satisfying the same condition as above* (1).

Proof. It follows from (9.1.5) and (9.1.6) that

$$\chi_{\iota\otimes\sigma}\xi_{\delta_G}=\chi_\iota\xi_{\Lambda_\sigma+\delta_G}.$$

We denote by $\mathfrak{S}_h$ the symmetric group of degree h. Let $\lambda=m_1\phi_1+m_2\phi_2+\cdots+m_h\phi_h\in\mathfrak{t}$ be an integral form of G.

(1) The principal alternating sum ξ_λ associated to λ is given by

$$\begin{aligned}\xi_\lambda&=\sum_{0\leq 2s\leq h}\ \sum_{1\leq i_1<\cdots<i_{2s}\leq h}\ \sum_{\tau\in\mathfrak{S}_h}\operatorname{sgn}(\tau)e(m_{\tau(1)}\phi_1+\cdots\\ &\qquad -m_{\tau(i_1)}\phi_{i_1}+\cdots-m_{\tau(i_{2s})}\phi_{i_{2s}}+\cdots+m_{\tau(h)}\phi_h)\\ &=\sum_{0\leq 2s\leq h}\ \sum_{1\leq i_1<\cdots<i_{2s}\leq h}\ \sum_{\tau\in\mathfrak{S}_h}\operatorname{sgn}(\tau)e(m_{\tau(j_1)}\phi_{j_1}+\cdots\\ &\qquad +m_{\tau(j_{h-2s})}\phi_{j_{h-2s}}-m_{\tau(i_1)}\phi_{i_1}-\cdots-m_{\tau(i_{2s})}\phi_{i_{2s}}),\end{aligned}$$

where $\{i_1, i_2, \cdots, i_{2s}, j_1, j_2, \cdots, j_{h-2s}\}=\{1, 2, \cdots, h\}$ and $1\leqq j_1<j_2<\cdots<j_{h-2s}\leqq h$. Therefore we have by (9.2.1)

$$\begin{aligned}\chi_\iota\xi_\lambda&=\sum_{0\leq 2s\leq h}\ \sum_{1\leq i_1<\cdots<i_{2s}\leq h}\ \sum_{\tau\in\mathfrak{S}_h}\operatorname{sgn}(\tau)\times\\ &\Big\{\sum_{q=1}^{h-2s}e(m_{\tau(j_1)}\phi_{j_1}+\cdots+(m_{\tau(j_q)}+1)\phi_{j_q}+\cdots+m_{\tau(j_{h-2s})}\phi_{j_{h-2s}}\\ &\qquad\qquad -m_{\tau(i_1)}\phi_{i_1}-\cdots-m_{\tau(i_{2s})}\phi_{i_{2s}})\\ &+\sum_{r=1}^{2s}e(m_{\tau(j_1)}\phi_{j_1}+\cdots+m_{\tau(j_{h-2s})}\phi_{j_{h-2s}}\\ &\qquad -m_{\tau(i_1)}\phi_{i_1}-\cdots-(m_{\tau(i_r)}+1)\phi_{i_r}-\cdots-m_{\tau(i_{2s})}\phi_{i_{2s}})\\ &+\sum_{q=1}^{h-2s}e(m_{\tau(j_1)}\phi_{j_1}+\cdots+(m_{\tau(j_q)}-1)\phi_{j_q}+\cdots+m_{\tau(j_{h-2s})}\phi_{j_{h-2s}}\\ &\qquad\qquad -m_{\tau(i_1)}\phi_{i_1}-\cdots-m_{\tau(i_{2s})}\phi_{i_{2s}})\\ &+\sum_{r=1}^{2s}e(m_{\tau(j_1)}\phi_{j_1}+\cdots+m_{\tau(j_{h-2s})}\phi_{j_{h-2s}}\\ &\qquad -m_{\tau(i_1)}\phi_{i_1}-\cdots-(m_{\tau(i_r)}-1)\phi_{i_r}-\cdots-m_{\tau(i_{2s})}\phi_{i_{2s}})\Big\}.\end{aligned}$$

Put

$$\begin{cases}m^{(\pm j)}{}_i=\begin{cases}m_i & \text{if } i\neq j,\\ m_j\pm 1 & \text{if } i=j,\end{cases}\\ \lambda^{(\pm j)}=\sum_{j=1}^{h}m^{(\pm j)}{}_i\phi_i.\end{cases}$$

Then we have

$$\chi_\iota \xi_\lambda = \sum_{0\leqq 2s\leqq h} \sum_{1\leqq i_1<\cdots<i_{2s}\leqq h} \sum_{j=1}^{h} \sum_{\tau\in\mathfrak{S}_h} \mathrm{sgn}(\tau)\times$$
$$\{e(m^{(+j)}{}_{\tau(1)}\phi_1+\cdots-m^{(+j)}{}_{\tau(i_1)}\phi_{i_1}+\cdots-m^{(+j)}{}_{\tau(i_{2s})}\phi_{i_{2s}}+\cdots$$
$$+m^{(+j)}{}_{\tau(h)}\phi_h)$$
$$+e(m^{(-j)}{}_{\tau(1)}\phi_1+\cdots-m^{(-j)}{}_{\tau(i_1)}\phi_{i_1}+\cdots-m^{(-j)}{}_{\tau(i_{2s})}\phi_{i_{2s}}+\cdots$$
$$+m^{(-j)}{}_{\tau(h)}\phi_h)\}$$
$$=\sum_{j=1}^{h}(\xi_{\lambda^{(+j)}}+\xi_{\lambda^{(-j)}}).$$

Therefore we have

$$\chi_{\iota\otimes\sigma}\xi_{\delta_G}=\sum_{j=1}^{h}(\xi_{\Lambda_\sigma^{(+j)}+\delta_G}+\xi_{\Lambda_\sigma^{(-j)}+\delta_G}).$$

The integral forms $\Lambda_\sigma{}^{(\pm j)}+\delta_G (j=1, 2, \cdots, h)$ of G are all dominant. Since a dominant integral form $m_1\phi_1+m_2\phi_2+\cdots+m_h\phi_h+\delta_G$ of G is regular if and only if the integers $m_1, m_2, \cdots, m_h$ satisfy (9.1.3), we obtain the assertion.

(2) We have in the similar way to the proof of (1)

$$\chi_{\iota\otimes\sigma}\xi_{\delta_G}=\xi_{\Lambda_\sigma+\delta_G}+\sum_{j=1}^{h}(\xi_{\Lambda_\sigma^{(+j)}+\delta_G}+\xi_{\Lambda_\sigma^{(-j)}+\delta_G}).$$

Suppose that $k_1=0$. Then the integral forms $\Lambda_\sigma{}^{(\pm j)}+\delta_G (j=1,2,\cdots,h)$ of $\mathfrak{g}$ except for $\Lambda_\sigma{}^{(-1)}+\delta_G$ are dominant. Let $\{1\}$ be the element of W_G such that $\{1\}(\phi_1)=-\phi_1$ and $\{1\}(\phi_i)=\phi_i$, $i=2,3,\cdots,h$. Then we have

$$\{1\}(\Lambda_\sigma+\delta_G)=\Lambda_\sigma{}^{(-1)}+\delta_G.$$

Therefore we have

$$\chi_{\iota\otimes\sigma}\xi_{\delta_G}=\xi_{\Lambda_\sigma^{(+1)}+\delta_G}+\sum_{j=2}^{h}(\xi_{\Lambda_\sigma^{(+j)}+\delta_G}+\xi_{\Lambda_\sigma^{(-j)}+\delta_G}).$$

Considering the regularity of the integral forms, we obtain the assertion. Suppose that $k_1>0$. Then the integral forms $\Lambda_\sigma{}^{(\pm j)}+\delta_G (j=1, 2, \cdots, h)$ of $\mathfrak{g}$ are dominant. Considering the regularity of the integral forms, we obtain the assertion. Q.E.D.

Corollary. *The pair (G,K) has the property (P_2).*

Proof. Since there exists a canonical unitary K-isomorphism of $\mathfrak{p}^{\boldsymbol{C}}$ onto $\boldsymbol{C}^n$ which sends $X_{\pm\phi_i}$ to $v_{\pm i}$, $i=1,2,\cdots,h$, we have the corollary by the above lemma. Q.E.D.

9.3. In this subsection we assume that $n\geqq 2$. Then the canonical representation ι of G is irreducible. Let $S^k(\boldsymbol{C}^{n+1})$ be the space of symmetric k-tensors over $\boldsymbol{C}^{n+1}$. The space $S^k(\boldsymbol{C}^{n+1})$ has the Hermitian inner product, also denoted by $\langle\ ,\ \rangle$, induced from the Hermitian inner product $\langle\ ,\ \rangle$ on C^{n+1}. Suppose that $n=2h-1$. Let $(i_1,i_2,\cdots,i_k)$ be a k-tuple of integers such that $1\leqq|i_s|\leqq h$, $s=1,2,\cdots,k$. If i(resp. $-i$) is contained a_i-times(resp. b_i-times) in $(i_1,i_2,\cdots,i_k)$, $i=1,2,\cdots,h$, we denote it by

$$\{i_1, i_2, \cdots, i_k\} = \{1^{a_1}, (-1)^{b_1}, 2^{a_2}, (-2)^{b_2}, \cdots, h^{a_h}, (-h)^{b_h}\}\ .$$

We define a vector $v_1{}^{a_1}\cdot v_{-1}{}^{b_1}\cdot v_2{}^{a_2}\cdot v_{-2}{}^{b_2}\cdots v_h{}^{a_h}\cdot v_{-h}{}^{b_h}$ in $S^k(C^{n+1})$ by

$$v_1{}^{a_1}\cdot v_{-1}{}^{b_1}\cdots v_h{}^{a_h}\cdot v_{-h}{}^{b_h} = \sqrt{\frac{a_1!\,b_1!\cdots a_h!\,b_h!}{k!}}\sum v_{i_1}\otimes v_{i_2}\otimes\cdots\otimes v_{i_k}\,,$$

where the summation runs over k-tuples $(i_1,i_2,\cdots,i_k)$ such that $\{i_1,i_2,\cdots,i_k\}$ $=\{1^{a_1}, (-1)^{b_1}, 2^{a_2}, (-2)^{b_2}, \cdots, h^{a_h}, (-h)^{b_h}\}$. Then

$$A_k = \{v_1{}^{a_1}\cdot v_{-1}{}^{b_1}\cdots v_h{}^{a_h}\cdot v_{-h}{}^{b_h};\ a_1+a_2+\cdots+a_h+b_1+b_2+\cdots+b_h = k\}$$

is an orthonormal basis of $S^k(C^{n+1})$. Suppose that $n=2h$. We define a vector $v_0{}^c\cdot v_1{}^{a_1}\cdot v_{-1}{}^{b_1}\cdots v_h{}^{a_h}\cdot v_{-h}{}^{b_h}$ in $S^k(C^{n+1})$ in the same way as above. Then

$$A_k = \{v_0{}^c\cdot v_1{}^{a_1}\cdot v_{-1}{}^{b_1}\cdots v_h{}^{a_h}\cdot v_{-h}{}^{b_h};\ a_1+\cdots+a_h+b_1+\cdots+b_h+c = k\}$$

is an orthonormal basis of $S^k(C^{n+1})$.

Let $\otimes^k C^{n+1}$ be the space of k-tensors over C^{n+1} and $\otimes^k\iota\colon G\to GL(\otimes^k C^{n+1})$ the k-*th* tensor product of ι. Then the space $S^k(C^{n+1})$ is a G-submodule of $\otimes^k C^{n+1}$. We denote by σ_k this representation $\otimes^k\iota\colon G\to GL(S^k(C^{n+1}))$. Then we have

Theorem 9.3.1 (cf. Takeuchi [10] p. 255). *Suppose that $k\geqq 2$. Let σ: $G\to GL(W)$ be a complex irreducible representation with the highest weight $k\phi_h$. Then the G-module $S^k(C^{n+1})$ is G-isomorphic to the direct sum $W+S^{k-2}(C^{n+1})$.*

We also denote by ρ_k the complexification of the k-*th* real spherical representation ρ_k. We have

Proposition 9.3.2. *Suppose that $k\geqq 2$.*

(1) *The case $n=2h-1$: The set of weights of the representation ρ_k is*

$$\left\{m_1\phi_1+m_2\phi_2+\cdots+m_h\phi_h;\ m_1, m_2, \cdots, m_h \text{ are integers such that } k-\sum_{i=1}^{h}|m_i| \text{ is a non-negative even integer}\right\}.$$

The multiplicity $m(m_1\phi_1+m_2\phi_2+\cdots m_h\phi_h;\ k\phi_h)$ is given by

$$m(m_1\phi_1+m_2\phi_2+\cdots+m_h\phi_h;\ k\phi_h) = {}_hH_q-{}_hH_{q-1}\,,$$

where $2q=k-\sum_{i=1}^{h}|m_i|$, ${}_hH_{-1}=0$ and ${}_hH_i$, $i\geqq 0$, denotes the number of ways of choos-

ing i elements, allowing repetition, from a set of h elements.

(2) *The case $n=2h$: The set of weights of ρ_k is*

$$\left\{m_1\phi_1+m_2\phi_2+\cdots+m_h\phi_h;\ m_1, m_2, \cdots, m_h \text{ are integers such that } k-\sum_{i=1}^{h}|m_i| \text{ is non-negative}\right\}.$$

The multiplicity $m(m_1\phi_1+m_2\phi_2+\cdots+m_h\phi_h; k\phi_h)$ is given by

$$m(m_1\phi_1+m_2\phi_2+\cdots+m_h\phi_h; k\phi_h) = {}_hH_{[q'/2]},$$

where $q'=k-\sum_{i=1}^{h}|m_i|$ and $[*]$ is the Gauss symbol.

Proof. (1) The vector $v_1^{a_1}\cdot v_{-1}^{b_1}\cdot v_2^{a_2}\cdot v_{-2}^{b_2}\cdots v_h^{a_h}\cdot v_{-h}^{b_h}$ of $S^k(C^{n+1})$ is a weight vector belonging to the weight $(a_1-b_1)\phi_1+(a_2-b_2)\phi_2+\cdots+(a_h-b_h)\phi_h$ of the representation σ_k. Therefore the set of weights of σ_k is

$$\left\{m_1\phi_1+m_2\phi_2+\cdots+m_h\phi_h;\ m_1, m_2, \cdots, m_h \text{ are integers such that } k-\sum_{k=1}^{h}|m_i| \text{ is a non-negative even integer}\right\}.$$

We denote by $m_k(m_1\phi_1+m_2\phi_2+\cdots+m_h\phi_h)$ the multiplicity of an integral form $m_1\phi_1+m_2\phi_2+\cdots+m_h\phi_h$ in the G-module $S^k(C^{n+1})$. Let $m_1\phi_1+m_2\phi_2+\cdots+m_h\phi_h$ be a weight of σ_k. Put $2q=k-\sum_{i=1}^{h}|m_i|$ and

$$p_i = \begin{cases} m_i & \text{if } m_i>0, \\ 0 & \text{if } m_i\leqq 0, \end{cases}$$

$$p_{-i} = \begin{cases} 0 & \text{if } m_i>0, \\ -m_i & \text{if } m_i\leqq 0, \end{cases}$$

Then the weight vectors in A_k belonging to the weight $m_1\phi_1+m_2\phi_2+\cdots+m_h\phi_h$ are

$$\left\{v_1^{(p_1+a_1)}\cdot v_{-1}^{(p_{-1}+a_1)}\cdots v_h^{(p_h+a_h)}\cdot v_{-h}^{(p_{-h}+a_h)};\ a_1, a_2, \cdots, a_h \text{ are non-negative integers such that } a_1+a_2+\cdots+a_h = q\right\}.$$

Therefore we have

$$m_k(m_1\phi_1+m_2\phi_2+\cdots+m_h\phi_h) = {}_hH_q,$$

and hence

$$m_{k-2}(m_1\phi_1+m_2\phi_2+\cdots+m_h\phi_h) = \begin{cases} 0 & \text{if } q=0, \\ {}_hH_{q-1} & \text{if } q\geqq 1. \end{cases}$$

We have the assetion by the above equalities and Theorem 9.3.1.

(2) We have the followings in the similar way to the proof of (1). The set of weights of σ_k is

$$\left\{m_1\phi_2+m_2\phi_2+\cdots+m_h\phi_h;\ m_1, m_2, \cdots, m_h \text{ are integers such that } k-\sum_{i=1}^{h}|m_i| \text{ is non-negative}\right\},$$

and the weight vectors in A_k belonging to the weight $m_1\phi_1+m_2\phi_2+\cdots+m_h\phi_h$ are

$$\left\{v_0{}^c\cdot v_1{}^{(p_1+a_1)}\cdot v_{-1}{}^{(p_{-1}+a_1)}\cdots v_h{}^{(p_h+a_h)}\cdot v_{-h}{}^{(p_{-h}+a_h)};\ a_1, a_2, \cdots, a_h, c \text{ are non-negative integers such that } 2a_1+2a_2+\cdots+2a_h+c=q'\right\},$$

where $q'=k-\sum_{i=1}^{h}|m_i|$. Therefore

$$c=q', q'-2, \cdots, q'-2\left[\frac{q'}{2}\right].$$

Since the number of the weight vectors in A_k belonging to the weight $m_1\phi_1+m_2\phi_2+\cdots+m_h\phi_h$ with $c=q'-2i$ is equal to ${}_hH_i$, the multiplicity $m_k(m_1\phi_1+m_2\phi_2+\cdots+m_h\phi_h)$ is given

$$m_k(m_1\phi_1+m_2\phi_2+\cdots+m_h\phi_h)=\sum_{i=1}^{[q'/2]}{}_hH_i\,.$$

Hence

$$m_{k-2}(m_1\phi_1+m_2\phi_2+\cdots+m_h\phi_h)=\begin{cases}0 & \text{if } q'=0, 1\,,\\ \sum_{i=1}^{[q'/2]-1}{}_hH_i & \text{if } q'\geqq 2\,.\end{cases}$$

Therefore we have the assertion in the same way as in (1). Q.E.D.

Let V_k be the irreducible G-submodule of $S^k(\boldsymbol{C}^{n+1})$ with the highest weight $k\phi_h$, and $V_k(\lambda)$ the λ-weight space of V_k, i. e. $V_k(\lambda)=\{\lambda\text{-weight vectors of } V_k\}\cup\{0\}$. Then we have

Lemma 9.3.3. *Suppose that* $k\geqq 2$.

(1) *The case* $n=2h-1$:

(a) $V_k(\pm\phi_i+(k-1)\phi_h)=\{v_{\pm i}\cdot v_h{}^{k-1}\}_C \qquad i=1, 2, \cdots, h-1\,.$

(b) $$V_k((k-2)\phi_h)=\left\{\sqrt{\frac{k-1}{k}}\,v_i\cdot v_{-i}\cdot v_h{}^{k-2}-\sqrt{\frac{1}{k}}\,v_h{}^{k-1}\cdot v_{-h};\ i=1, 2, \cdots, h-1\right\}_C.$$

(2) *The case* $n=2h$:

(a) $V_k(\pm\phi_i+(k-1)\phi_h)=\{v_{\pm i}\cdot v_h{}^{k-1}\}_C \qquad i=1, 2, \cdots, h-1\,.$

(b) $V_k((k-1)\phi_h)=\{v_0\cdot v_h{}^{k-1}\}_C\,.$

(c) $$V_k((k-2)\phi_h) = \left\{\sqrt{\frac{2(k-1)}{2k-1}}\, v_0^2\cdot v_h^{k-2} - \sqrt{\frac{1}{2k-1}}\, v_h^{k-1}\cdot v_{-h},\ \sqrt{\frac{k-1}{k}}\, v_i\cdot v_{-i}\cdot v_h^{k-2} - \sqrt{\frac{1}{k}}\, v_h^{k-1}\cdot v_{-h};\ i=1,2,\cdots,h-1\right\}_C .$$

Proof. (1) We have by the above proposition

$$m(\pm\phi_i+(k-1)\phi_h;\, k\phi_h) = 1,$$
$$m((k-2)\phi_h;\, k\phi_h) = h-1.$$

Applying Lemma 9,2.3, we have

$$\begin{cases} d\sigma_k(X_{\pm\phi_i-\phi_h})(v_h^k) = \sqrt{k}\, v_{\pm i}\cdot v_h^{k-1}, \\ d\sigma_k(X_{\phi_i-\phi_h})(v_{-i}\cdot v_h^{k-1}) = \sqrt{k-1}\, v_i\cdot v_{-i}\cdot v_h^{k-2} - v_h^{k-1}\cdot v_{-h}. \end{cases}$$

Therefore we have the assertion.

(2) We have the followings in the similar way to above:

$$\begin{cases} m(\pm\phi_i+(k-1)\phi_h;\, k\phi_h) = m((k-1)\phi_h;\, k\phi_h) = 1, \\ m((k-2)\phi_h;\, k\phi_h) = h, \end{cases}$$

$$\begin{cases} d\sigma_k(X_{-\phi_h})(v_h^k) = -\sqrt{k}\, v_0\cdot v_h^{k-1}, \\ d\sigma_k(X_{-\phi_h})(v_0\cdot v_h^{k-1}) = v_h^{k-1}\cdot v_{-h} - \sqrt{2(k-1)}\, v_0^2\cdot v_h^{k-2}. \end{cases}$$

Therefore we have the assertion. Q.E.D.

9.4. In this subsection we shall compute the components of the vector $v_{-h}\otimes v_h^k$ of $C^{n+1}\otimes V_k$ with respect to the decomposition of the G-module $C^{n+1}\otimes V_k$. This is important for the later computation. The G-module $C^{n+1}\otimes V_k$ has the Hermitian inner product, also denoted by $\langle\ ,\ \rangle$, induced from the Hermitian inner products on C^{n+1} and $S^k(C^{n+1})$. We denote by ψ_k the tensor product $\iota\otimes\sigma_k\colon G\to GL(C^{n+1}\otimes V_k)$. We have by Lemma 9.2.4

$$(9.4.1)\quad C^{n+1}\otimes V_k = \begin{cases} W_{(k+1)\phi_1}+W_{(k-1)\phi_1} & \text{if } n=1, \\ W_{(k+1)\phi_1}+W_{k\phi_1}+W_{(k-1)\phi_1} & \text{if } n=2 \text{ and } k>0, \\ W_{(k+1)\phi_2}+W_{\phi_1+k\phi_2}+W_{-\phi_1+k\phi_2}+W_{(k-1)\phi_2} & \text{if } n=3 \text{ and } k>0, \\ W_{(k+1)\phi_h}+W_{\phi_{h-1}+k\phi_h}+W_{(k-1)\phi_h} & \text{if } n\geqq 4 \text{ and } k>0. \end{cases}$$

Since the Hermitian inner product $\langle\ ,\ \rangle$ on $C^{n+1}\otimes V_k$ is G-invariant, any pair of the irreducible components of $C^{n+1}\otimes V_k$ is orthogonal. For a G-module U and an integral form λ of G, we denote by $U(\lambda)$ the λ-weight space of U.

Suppose that $n\geqq 4$. We define some vectors of $C^{n+1}\otimes V_k$ as follows:

(a) The case $n=2h-1$ or $n=2h$: For $i=1,2,\cdots,h-1$

$$\alpha_i=\begin{cases} v_i\otimes v_{-i}+v_{-i}\otimes v_i-v_h\otimes v_{-h}-v_{-h}\otimes v_h & \text{if } k=1\,,\\ \sqrt{k}\,(v_i\otimes v_{-i}\cdot v_h^{\,k-1}+v_{-i}\otimes v_i\cdot v_h^{\,k-1}) & \\ \quad +kv_h\otimes\left(\sqrt{\frac{k-1}{k}}\,v_i\cdot v_{-i}\cdot v_h^{\,k-2}-\sqrt{\frac{1}{k}}\,v_h^{\,k-1}\cdot v_{-h}\right)-v_{-h}\otimes v_h^{\,k} & \\ & \text{if } k\geqq 2\,, \end{cases}$$

$$\beta_i=\begin{cases} v_i\otimes v_{-i}-v_{-i}\otimes v_i+v_h\otimes v_{-h}-v_{-h}\otimes v_h & \text{if } k=1\,,\\ kv_i\otimes v_{-i}\cdot v_h^{\,k-1}-v_{-i}\otimes v_i\cdot v_h^{\,k-1} & \\ \quad -\sqrt{k}\,v_h\otimes\left(\sqrt{\frac{k-1}{k}}\,v_i\cdot v_{-i}\cdot v_h^{\,k-2}-\sqrt{\frac{1}{k}}\,v_h^{\,k-1}\cdot v_{-h}\right)-\sqrt{k}\,v_{-h}\otimes v_h^{\,k} & \\ & \text{if } k\geqq 2\,, \end{cases}$$

$$\gamma_i=\begin{cases} -v_i\otimes v_{-i}+v_{-i}\otimes v_i+v_h\otimes v_{-h}-v_{-h}\otimes v_h & \text{if } k=1\,,\\ -v_i\otimes v_{-i}\cdot v_h^{\,k-1}+kv_{-i}\otimes v_i\cdot v_h^{\,k-1} & \\ \quad -\sqrt{k}\,v_h\otimes\left(\sqrt{\frac{k-1}{k}}\,v_i\cdot v_{-i}\cdot v_h^{\,k-2}-\sqrt{\frac{1}{k}}\,v_h^{\,k-1}\cdot v_{-h}\right)-\sqrt{k}\,v_{-h}\otimes v_h^{\,k} & \\ & \text{if } k\geqq 2\,. \end{cases}$$

(b) The case $n=2h-1$:

$$\delta=\begin{cases} \sum_{i=1}^{h}(v_i\otimes v_{-i}+v_{-i}\otimes v_i) & \text{if } k=1\,,\\ \sqrt{k}\sum_{i=1}^{h-1}(v_i\otimes v_{-i}\cdot v_h^{\,k-1}+v_{-i}\otimes v_i\cdot v_h^{\,k-1}) & \\ \quad -\frac{k}{k+h-2}\sum_{i=1}^{h-1}v_h\otimes\left(\sqrt{\frac{k-1}{k}}\,v_i\cdot v_{-i}\cdot v_h^{\,k-2}-\sqrt{\frac{1}{k}}\,v_{-h}\cdot v_h^{\,k-1}\right) & \\ \quad +kv_{-h}\otimes v_h^{\,k} & \text{if } k\geqq 2\,. \end{cases}$$

(c) The case $n=2h$:

$$\alpha_0=\begin{cases} 2v_0\otimes v_0-v_h\otimes v_{-h}-v_{-h}\otimes v_h & \text{if } k=1\,,\\ 2\sqrt{k}\,v_0\otimes v_0\cdot v_h^{\,k-1} & \\ \quad +\sqrt{k(2k-1)}\,v_h\otimes\left(\sqrt{\frac{2(k-1)}{2k-1}}\,v_0^{\,2}\cdot v_h^{\,k-2}-\sqrt{\frac{1}{2k-1}}\,v_h^{\,k-1}\cdot v_{-h}\right) & \\ \quad -v_{-h}\otimes v_h^{\,k} & \text{if } k\geqq 2\,. \end{cases}$$

$$\beta_0=\begin{cases} -v_{h-1}\otimes v_{-(h-1)}+v_{-(h-1)}\otimes v_{h-1} & \text{if } k=1\,,\\ (k-1)v_0\otimes v_0\cdot v_h^{\,k-1}-kv_{h-1}\otimes v_{-(h-1)}\cdot v_h^{\,k-1}+v_{-(h-1)}\otimes v_{h-1}\cdot v_h^{\,k-1} & \\ \quad -\sqrt{2k-1}\,v_h\otimes\left(\sqrt{\frac{2(k-1)}{2k-1}}\,v_0^{\,2}\cdot v_h^{\,k-2}-\sqrt{\frac{1}{2k-1}}\,v_h^{\,k-1}\cdot v_{-h}\right) & \\ \quad +\sqrt{k}\,v_h\otimes\left(\sqrt{\frac{k-1}{k}}\,v_{h-1}\cdot v_{-(h-1)}\cdot v_h^{\,k-2}-\sqrt{\frac{1}{k}}\,v_h^{\,k-1}\cdot v_{-h}\right) & \\ & \text{if } k\geqq 2\,, \end{cases}$$

$$\delta = \begin{cases} v_0\otimes v_0+\sum_{i=1}^{h}(v_i\otimes v_{-i}+v_{-i}\otimes v_i) & \text{if } k=1\,, \\ \sqrt{k}\,v_0\otimes v_0\cdot v_h^{k-1}+\sqrt{k}\sum_{i=1}^{h-1}(v_i\otimes v_{-i}\cdot v_h^{k-1}+v_{-i}\otimes v_i\cdot v_h^{k-1}) & \\ \quad -\frac{\sqrt{k(2k-1)}}{2k+2h-3}v_h\otimes\left(\sqrt{\frac{2(k-1)}{2k-1}}\,v_0^2\cdot v_h^{k-2}-\sqrt{\frac{1}{2k-1}}\,v_h^{k-1}\cdot v_{-h}\right) & \\ \quad -\frac{2k}{2k+2h-3}\sum_{i=1}^{h-1}v_h\otimes\left(\sqrt{\frac{k-1}{k}}\,v_i\cdot v_{-i}\cdot v_h^{k-2}-\sqrt{\frac{1}{k}}\,v_h^{k-1}\cdot v_{-h}\right) & \\ \quad +kv_{-h}\otimes v_h^k & \text{if } k\geqq 2\,. \end{cases}$$

Then we have

Lemma 9.4.1. *Snppose that* $n\geqq 4$.

(1) *The case* $n=2h-1(h\geqq 3)$:

$$\begin{cases} W_{(k+1)\phi_h}((k-1)\phi_h)=\{\alpha_1,\alpha_2,\cdots,\alpha_{h-1}\}_C\,, \\ W_{\phi_{h-1}+k\phi_h}((k-1)\phi_h)=\{\beta_1,\beta_2,\cdots,\beta_{h-1},\gamma_1,\gamma_2,\cdots,\gamma_{h-1}\}_C\,, \\ W_{(k-1)\phi_h}((k-1)\phi_h)=\{\delta\}_C\,. \end{cases}$$

(2) *The case* $n=2h(h\geqq 2)$;

$$\begin{cases} W_{(k+1)\phi_h}((k-1)\phi_h)=\{\alpha_0,\alpha_1,\cdots,\alpha_{h-1}\}_C\,, \\ W_{\phi_{h-1}+k\phi_h}((k-1)\phi_h)=\{\beta_0,\beta_1,\beta_2,\cdots,\beta_{h-1},\gamma_1,\gamma_2,\cdots,\gamma_{h-1}\}_C\,, \\ W_{(k-1)\phi_h}((k-1)\phi_h)=\{\delta\}_C\,. \end{cases}$$

Proof. (1) It follows from Lemma 9.3.3 that the space $(\boldsymbol{C}^{n+1}\otimes V_k)((k-1)\phi_h)$ is spanned by

$$\begin{cases} \{v_i\otimes v_{-i},\,v_{-i}\otimes v_i;\,i=1,2,\cdots,h\} & \text{if } k=1\,, \\ \left\{\begin{array}{l} v_i\otimes v_{-i}\cdot v_h^{k-1},\,v_{-i}\otimes v_i\cdot v_h^{k-1}\,, \\ v_h\otimes\left(\sqrt{\frac{k-1}{k}}\,v_i\cdot v_{-i}\cdot v_h^{k-2}-\sqrt{\frac{1}{k}}\,v_h^{k-1}\cdot v_{-h}\right),\,v_{-h}\otimes v_h^k; \\ \qquad\qquad i=1,2,\cdots,h-1 \end{array}\right\} & \text{if } k\geqq 2\,. \end{cases}$$

Therefore we have by Lemma 9.3.3 and (9.4.1)

$$\dim W_{\phi_{h-1}+k\phi_h}((k-1)\phi_h)=\begin{cases} h & \text{if } k=1\,, \\ 2h-2 & \text{if } k\geqq 2\,. \end{cases} \tag{9.4.2}$$

Applying Lemma 9.2.3, we have

$$d\psi_k(X_{\phi_i-\phi_h})d\psi_k(X_{-\phi_i-\phi_h})(v_h\otimes v_h^k)=\alpha_i \qquad i=1,2,\cdots,h-1\,.$$

Therefore by the proof of Lemma 9.3.3 we obtain the first assertion. Since

$\dim(\boldsymbol{C}^{n+1}\otimes V_k)(\phi_{h-1}+k\phi_h)=2$ and

$$d\psi_k(X_{\phi_{h-1}-\phi_h})(v_h\otimes v_h^k) = v_{h-1}\otimes v_h^k+\sqrt{k}\, v_h\otimes v_{h-1}\cdot v_h^{k-1},$$

the vector $\omega=\sqrt{k}\, v_{h-1}\otimes v_h^k - v_h\otimes v_{h-1}\cdot v_h^{k-1}$ is a highest weight vector of $W_{\phi_{h-1}+k\phi_h}$. Applying Lemma 9.2.3, we have

$$\begin{cases} d\psi_k(X_{-\phi_i-\phi_h})d\psi_k(X_{\phi_i-\phi_{h-1}})(\omega) = \beta_i & i=1,2,\cdots,h-2, \\ d\psi_k(X_{\phi_i-\phi_h})d\psi_k(X_{-\phi_i-\phi_{h-1}})(\omega) = \gamma_i & i=1,2,\cdots,h-2, \\ d\psi_k(X_{-\phi_{h-1}-\phi_h})(\omega) = \beta_{h-1}, \\ d\psi_k(X_{\phi_{h-1}-\phi_h})d\psi_k(X_{-\phi_i-\phi_{h-1}})d\psi_k(X_{\phi_i-\phi_{h-1}})(\omega) = -\gamma_{h-1}. \end{cases}$$

If $k=1$(resp. $k\geqq 2$), $\{\beta_1,\beta_2,\cdots,\beta_{h-1},\gamma_1\}$ (resp. $\{\beta_1,\beta_2,\cdots,\beta_{h-1},\gamma_1,\gamma_2,\cdots,\gamma_{h-1}\}$) is linear independent. Therefore by (9.4.2) we obtain the second assertion. Since the vector δ is orthogonal to α_i, β_i, γ_i, $i=1,2,\cdots,h-1$, we have the last assertion.

(2) We have the followings in the same way as above: The space $(\boldsymbol{C}^{n+1}\otimes V_k)((k-1)\phi_h)$ is spanned by

$$\begin{cases} \{v_0\otimes v_0,\ v_i\otimes v_{-i},\ v_{-i}\otimes v_i;\ i=1,2,\cdots,h\} & \text{if } k=1, \\ \left\{\begin{array}{l} v_0\otimes v_0\cdot v_h^{k-1},\ v_i\otimes v_{-i}\cdot v_h^{k-1},\ v_{-i}\otimes v_i\cdot v_h^{k-1}, \\ v_h\otimes\left(\sqrt{\dfrac{2(k-1)}{2k-1}}\, v_0^2\cdot v_h^{k-2}-\sqrt{\dfrac{1}{2k-1}}\, v_h^{k-1}\cdot v_{-h}\right), \\ v_h\otimes\left(\sqrt{\dfrac{k-1}{k}}\, v_i\cdot v_{-i}\cdot v_h^{k-2}-\sqrt{\dfrac{1}{k}}\, v_h^{k-1}\cdot v_{-h}\right),\ v_{-h}\otimes v_h^k; \\ \qquad i=1,2,\cdots,h-1 \end{array}\right\} & \text{if } k\geqq 2. \end{cases}$$

We have

$$\dim W_{\phi_{h-1}+k\phi_h}((k-1)\phi_h) = \begin{cases} h & \text{if } k=1, \\ 2h-1 & \text{if } k\geqq 2, \end{cases}$$

and the vector $\omega=\sqrt{k}\, v_{h-1}\otimes v_h^k - v_h\otimes v_{h-1}\cdot v_h^{k-1}$ is a highest weight vector of $W_{\phi_{h-1}+k\phi_h}$. Applying Lemma 9.2.3, we have

$$\begin{cases} d\psi_k(X_{-\phi_h})^2(v_h\otimes v_h^k) = \alpha_0, \\ d\psi_k(X_{-\phi_{h-1}})d\psi_k(X_{-\phi_h})(\omega) = \beta_0. \end{cases}$$

Therefore we obtain the first and the second assertions. Since the vector δ is orthogonal to α_0, β_0, α_i, β_i, γ_i, $i=1,2,\cdots,h-1$, we have the last assertion. Q.E.D.

Suppose that $n=2$. Put

$$\alpha_{(2)} = \begin{cases} 2v_0\otimes v_0 - v_1\otimes v_{-1} - v_{-1}\otimes v_1 & \text{if } k=1, \\ 2\sqrt{k}\, v_0\otimes v_0\cdot v_1^{k-1} & \\ \quad +\sqrt{k(2k-1)}\, v_1\otimes\left(\sqrt{\frac{2(k-1)}{2k-1}}\, v_0^2\cdot v_1^{k-2} - \sqrt{\frac{1}{2k-1}}\, v_1^{k-1}\cdot v_{-1}\right) & \\ \quad - v_{-1}\otimes v_1^k & \text{if } k\geqq 2, \end{cases}$$

$$\beta_{(2)} = \begin{cases} v_1\otimes v_{-1} - v_{-1}\otimes v_1 & \text{if } k=1, \\ (k-1)v_0\otimes v_0\cdot v_1^{k-1} & \\ \quad -\sqrt{2k-1}\, v_1\otimes\left(\sqrt{\frac{2(k-1)}{2k-1}}\, v_0^2\cdot v_1^{k-2} - \sqrt{\frac{1}{2k-1}}\, v_1^{k-1}\cdot v_{-1}\right) & \\ \quad -\sqrt{k}\, v_{-1}\otimes v_1^k & \text{if } k\geqq 2, \end{cases}$$

$$\gamma_{(2)} = \begin{cases} v_0\otimes v_0 + v_1\otimes v_{-1} + v_{-1}\otimes v_1 & \text{if } k=1 \\ (2k-1)v_0\otimes v_0\cdot v_1^{k-1} & \\ \quad -\sqrt{2k-1}\, v_1\otimes\left(\sqrt{\frac{2(k-1)}{2k-1}}\, v_0^2\cdot v_1^{k-2} - \sqrt{\frac{1}{2k-1}}\, v_1^{k-1}\cdot v_{-1}\right) & \\ \quad +(2k-1)\sqrt{k}\, v_{-1}\otimes v_1^k & \text{if } k\geqq 2. \end{cases}$$

We have the following lemma in the similar way to Lemma 9.4.1.

Lemma 9.4.2. *If $n=2$, we have*

$$\begin{cases} W_{(k+1)\phi_1}((k-1)\phi_1) = \{\alpha_{(2)}\}_C, \quad W_{k\phi_1}((k-1)\phi_1) = \{\beta_{(2)}\}_C, \\ W_{(k-1)\phi_1}((k-1)\phi_1) = \{\gamma_{(2)}\}_C. \end{cases}$$

Suppose that $n=3$. Put

$$\alpha_{(3)} = \begin{cases} v_1\otimes v_{-1} + v_{-1}\otimes v_1 - v_2\otimes v_{-2} - v_{-2}\otimes v_2 & \text{if } k=1, \\ \sqrt{k}\,(v_1\otimes v_{-1}\cdot v_2^{k-1} + v_{-1}\otimes v_1\cdot v_2^{k-1}) & \\ \quad +kv_2\otimes\left(\sqrt{\frac{k-1}{k}}\, v_1\cdot v_{-1}\cdot v_2^{k-2} - \sqrt{\frac{1}{k}}\, v_2^{k-1}\cdot v_{-2}\right) - v_{-2}\otimes v_2^k & \\ & \text{if } k\geqq 2, \end{cases}$$

$$\beta_{(3)} = \begin{cases} v_1\otimes v_{-1} - v_{-1}\otimes v_1 + v_2\otimes v_{-2} - v_{-2}\otimes v_2 & \text{if } k=1, \\ kv_1\otimes v_{-1}\cdot v_2^{k-1} - v_{-1}\otimes v_1\cdot v_2^{k-1} & \\ \quad -\sqrt{k}\, v_2\otimes\left(\sqrt{\frac{k-1}{k}}\, v_1\cdot v_{-1}\cdot v_2^{k-2} - \sqrt{\frac{1}{k}}\, v_2^{k-1}\cdot v_{-2}\right) & \\ \quad -\sqrt{k}\, v_{-2}\otimes v_2^k & \text{if } k\geqq 2, \end{cases}$$

$$\gamma_{(3)} = \begin{cases} -v_1\otimes v_{-1} + v_{-1}\otimes v_1 + v_2\otimes v_{-2} - v_{-2}\otimes v_2 & \text{if } k=1, \\ -v_1\otimes v_{-1}\cdot v_2^{k-1} + kv_{-1}\otimes v_1\cdot v_2^{k-1} & \\ \quad -\sqrt{k}\, v_2\otimes\left(\sqrt{\frac{k-1}{k}}\, v_1\cdot v_{-1}\cdot v_2^{k-2} - \sqrt{\frac{1}{k}}\, v_2^{k-2}\cdot v_{-2}\right) & \\ \quad -\sqrt{k}\, v_{-2}\otimes v_2^k & \text{if } k\geqq 2, \end{cases}$$

$$\delta_{(3)}=\begin{cases} v_1\otimes v_{-1}+v_{-1}\otimes v_1+v_2\otimes v_{-2}+v_{-2}\otimes v_2 & \text{if } k=1\,, \\ \sqrt{k}\,(v_1\otimes v_{-1}\cdot v_2^{k-1}+v_{-1}\otimes v_1\cdot v_2^{k-1}) & \\ \quad -v_2\otimes\left(\sqrt{\frac{k-1}{k}}\,v_1\cdot v_{-1}\cdot v_2^{k-2}-\sqrt{\frac{1}{k}}\,v_2^{k-1}\cdot v_{-2}\right) & \\ \quad +kv_{-2}\otimes v_2^{k} & \text{if } k\geqq 2\,. \end{cases}$$

We have the following lemma in the similar way to Lemma 9.4.1.

Lemma 9.4.3. *If $n=3$, we have*

$$\begin{cases} W_{(k+1)\phi_2}((k-1)\phi_2)=\{\alpha_{(3)}\}_C, & W_{\phi_1+k\phi_2}((k-1)\phi_2)=\{\beta_{(3)}\}_C\,, \\ W_{-\phi_1+k\phi_2}((k-1)\phi_2)=\{\gamma_{(3)}\}_C, & W_{(k-1)\phi_2}((k-1)\phi_2)=\{\delta_{(3)}\}_C\,. \end{cases}$$

Suppose that $n\geqq 4$. Define as follows:

(a) The case $n=2h-1$ or $n=2h$: For $i=1,2,\cdots,h-1$

$$\alpha'_i=\begin{cases} \sqrt{\frac{1}{2i(i+1)}}\,\{-\sum_{p=1}^{i-1}(v_p\otimes v_{-p}+v_{-p}\otimes v_p)+i(v_i\otimes v_{-i}+v_{-i}\otimes v_i) & \\ \quad -v_h\otimes v_{-h}-v_{-h}\otimes v_h\} & \text{if } k=1\,, \\ \sqrt{\frac{1}{(k+1)(k+i)(k+i-1)}}\left\{-\sum_{p=1}^{i-1}(v_p\otimes v_{-p}\cdot v_h^{k-1}+v_{-p}\otimes v_p\cdot v_h^{k-1})\right. & \\ \quad +(k+i-1)(v_i\otimes v_{-i}\cdot v_h^{k-1}+v_{-i}\otimes v_i\cdot v_h^{k-1}) & \\ \quad -\sqrt{k}\sum_{p=1}^{i-1}v_h\otimes\left(\sqrt{\frac{k-1}{k}}\,v_p\cdot v_{-p}\cdot v_h^{k-2}-\sqrt{\frac{1}{k}}\,v_h^{k-1}\cdot v_{-h}\right) & \\ \quad +(k+i-1)\sqrt{k}\,v_h\otimes\left(\sqrt{\frac{k-1}{k}}\,v_i\cdot v_{-i}\cdot v_h^{k-2}-\sqrt{\frac{1}{k}}\,v_h^{k-1}\cdot v_{-h}\right) & \\ \quad \left.-\sqrt{k}\,v_{-h}\otimes v_h^{k}\right\} & \text{if } k\geqq 2\,, \end{cases}$$

$$\beta'_i=\begin{cases} \sqrt{\frac{1}{2}}\,(v_h\otimes v_{-h}-v_{-h}\otimes v_h) & \text{if } k=1\,, \\ \sqrt{\frac{1}{2(k-1)(k+1)(k+2i-1)(k+2i-3)}}\times & \\ \quad \left\{-2(k-1)\sum_{p=1}^{i-1}(v_p\otimes v_{-p}\cdot v_h^{k-1}+v_{-p}\otimes v_p\cdot v_h^{k-1})\right. & \\ \quad +(k-1)(k+2i-3)(v_i\otimes v_{-i}\cdot v_h^{k-1}+v_{-i}\otimes v_i\cdot v_h^{k-1}) & \\ \quad +4\sqrt{k}\sum_{p=1}^{i-1}v_h\otimes\left(\sqrt{\frac{k-1}{k}}\,v_p\cdot v_{-p}\cdot v_h^{k-2}-\sqrt{\frac{1}{k}}\,v_h^{k-1}\cdot v_{-h}\right) & \\ \quad -2(k+2i-3)\sqrt{k}\,v_h\otimes\left(\sqrt{\frac{k-1}{k}}\,v_i\cdot v_{-i}\cdot v_h^{k-2}-\sqrt{\frac{1}{k}}\,v_h^{k-1}\cdot v_{-h}\right) & \\ \quad \left.-2(k-1)\sqrt{k}\,v_{-h}\otimes v_h^{k}\right\} & \text{if } k\geqq 2\,, \end{cases}$$

$$\gamma'_i=\sqrt{\frac{1}{2}}\,(v_i\otimes v_{-i}\cdot v_h^{k-1}-v_{-i}\otimes v_i\cdot v^{k-1})\qquad \text{if } k\geqq 1\,.$$

(b) The case $n=2h-1$:

$$\delta'=\sqrt{\frac{k+h-2}{k(k+h-1)\,(k+2h-3)}}\,\delta\qquad \text{if } k\geqq 1\,.$$

(c) The case $n=2h$:

$$\alpha'_0=\begin{cases}\sqrt{\dfrac{1}{2h(2h+1)}}\,\{2hv_0\otimes v_0-\sum\limits_{i=1}^{h}(v_i\otimes v_{-i}+v_{-i}\otimes v_i)\} & \text{if } k=1\,,\\ \sqrt{\dfrac{1}{(k+1)\,(k+h-1)\,(2k+2h-1)}}\Big\{2(k+h-1)v_0\otimes v_0\cdot v_h^{k-1} & \\ \quad -\sum\limits_{i=1}^{h-1}(v_i\otimes v_{-i}\cdot v_h^{k-1}+v_{-i}\otimes v_i\cdot v_h^{k-1}) & \\ \quad +(k+h-1)\sqrt{2k-1}\,v_h\otimes\Big(\sqrt{\dfrac{2(k-1)}{2k-1}}\,v_0^2\cdot v_h^{k-2}-\sqrt{\dfrac{1}{2k-1}}\,v_h^{k-1}\cdot v_{-h}\Big) & \\ \quad -\sqrt{k}\sum\limits_{k=1}^{h-1}v_h\otimes\Big(\sqrt{\dfrac{k-1}{k}}\,v_i\cdot v_{-i}\cdot v_h^{k-2}-\sqrt{\dfrac{1}{k}}\,v_h^{k-1}\cdot v_{-h}\Big) & \\ \quad -\sqrt{k}\,v_{-h}\otimes v_h^{k}\Big\} & \text{if } k\geqq 2\,,\end{cases}$$

$$\begin{aligned}\beta'_0=\sqrt{\frac{1}{(k-1)\,(k+1)\,(k+2h-2)\,(k+2h-3)}}\Big\{&(k-1)\,(k+2h-3)v_0\otimes v_0\cdot v_h^{k-1}\\ &-(k-1)\sum_{i=1}^{h-1}(v_i\otimes v_{-i}\cdot v_h^{k-1}+v_{-i}\otimes v_i\cdot v_h^{k-1})\\ &-(k+2h-3)\sqrt{2k-1}\,v_h\otimes\Big(\sqrt{\frac{2(k-1)}{2k-1}}\,v_0^2\cdot v_h^{k-2}-\sqrt{\frac{1}{2k-1}}\,v_h^{k-1}\cdot v_{-h}\Big)\\ &+2\sqrt{k}\sum_{i=1}^{h-1}v_h\otimes\Big(\sqrt{\frac{k-1}{k}}\,v_i\cdot v_{-i}\cdot v_h^{k-2}-\sqrt{\frac{1}{k}}\,v_h^{k-1}\cdot v_{-h}\Big)\\ &-(k-1)\sqrt{k}\,v_{-h}\otimes v_h^{k}\Big\}\qquad \text{if } k\geqq 2\,,\end{aligned}$$

$$\delta'=\sqrt{\frac{2k+2h-3}{k(k+2h-2)\,(2k+2h-1)}}\,\delta\qquad \text{if } k\geqq 1\,.$$

The following lemma gives orthonormal bases of $W_{(k+1)\phi_h}((k-1)\phi_h)$, $W_{\phi_{h-1}+k\phi_h}((k-1)\phi_h)$ and $W_{(k-1)\phi_h}((k-1)\phi_h)$.

Lemma 9.4.4. (1) *The case* $n=2h-1(h\geqq 3)$:

orthonormal basis	*space*	
$\{\alpha'_1, \alpha'_2, \cdots, \alpha'_{h-1}\}$	$W_{(k+1)\phi_h}((k-1)\phi_h)$	*if* $k\geqq 1$,
$\{\beta'_1, \gamma'_1, \gamma'_2, \cdots, \gamma'_{h-1}\}$	$W_{\phi_{h-1}+k\phi_h}((k-1)\phi_h)$	*if* $k=1$,
$\{\beta'_1, \beta'_2, \cdots, \beta'_{h-1}, \gamma'_1, \gamma'_2, \cdots, \gamma'_{h-1}\}$	$W_{\phi_{h-1}+k\phi_h}((k-1)\phi_h)$	*if* $k\geqq 2$,

$$
\begin{array}{lll}
\{\delta'\} & W_{(k-1)\phi_h}((k-1)\phi_h) & \textit{if } k \geqq 1 .
\end{array}
$$

(2) *The case* $n=2h(h\geqq 2)$:

$$
\begin{array}{lll}
\textit{orthonormal basis} & \textit{space} & \\
\{\alpha'_0, \alpha'_1, \cdots, \alpha'_{h-1}\} & W_{(k+1)\phi_h}((k-1)\phi_h) & \textit{if } k \geqq 1 , \\
\{\beta'_0, \gamma'_1, \gamma'_2, \cdots, \gamma'_{h-1}\} & \left.\right\} & \textit{if } k = 1 , \\
\{\beta'_0, \beta'_1, \beta'_2, \cdots, \beta'_{h-1}, & W_{\phi_{h-1}+k\phi_h}((k-1)\phi_h) & \\
\quad \gamma'_1, \gamma'_2, \cdots, \gamma'_{h-1}\} & & \textit{if } k \geqq 2 , \\
\{\delta'\} & W_{(k-1)\phi_h}((k-1)\phi_h) & \textit{if } k \geqq 1 .
\end{array}
$$

For the proof of the lemma, we need the following lemma.

Lemma 9.4.5. *We define* (p,p)*-matrices* $A_p(m)$ *and* $B_p(m)$ *by*

$$
A_p(m) = \begin{pmatrix}
m+1 & 1 & \cdots & & 1 \\
1 & m+1 & \ddots & & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \ddots & 1 \\
1 & \cdots & \cdots & 1 & m+1
\end{pmatrix} ,
$$

$$
B_p(m) = \begin{pmatrix}
1 & \cdots & \cdots & \cdots & \cdots & 1 \\
m+1 & \ddots & & & & \vdots \\
1 & \ddots & \ddots & & & \vdots \\
\vdots & \ddots & \ddots & \ddots & & \vdots \\
1 & \cdots & \cdots & 1 & m+1 & 1
\end{pmatrix} .
$$

Then we have

$$
\begin{aligned}
\det A_p(m) &= m^{p-1}(m+p) , \\
\det B_p(m) &= (-1)^{p-1} m^{p-1} .
\end{aligned}
$$

Proof. We shall prove the above equalities by the induction on p. If $p=1$, the equalities are evident. Suppose that the equalities hold for $p-1$. Then we have

$$
\det A_p(m) = \sum_{j=1}^{p} (-1)^{p+i} \det
\begin{pmatrix}
m+1 & 1 \cdot \cdot & \overset{i}{\hat{1}} & \cdot \cdot 1 \\
1 & m+1 \ddots & \hat{1} & 1 \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & 1 \\
\hat{1} & \hat{1} \cdot \cdot & \hat{1} & m\hat{+}1
\end{pmatrix}
$$

$$= \sum_{i=1}^{p-1} (-1)^{p+i}(-1)^{i-1} \, det \, B_{p-1}(m) + (m+1) \, det \, A_{p-1}(m)$$
$$= m^{p-1}(m+p) ,$$

$$det \, B_p(m) = \sum_{i=1}^{p} (-1)^{1+i} \, det \begin{pmatrix} \hat{1} & \hat{1} \cdot & \cdot \overset{i}{\hat{1}} & \cdot & \cdot & \cdot & \cdot & \hat{1} \\ m+1 & 1 \; . & \hat{1} & & & & & 1 \\ 1 & . \, m+1 \, . & \cdot & . & & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot & & \cdot \\ 1 & \cdot \; \cdot \; \cdot & \hat{1} & \cdot \, \cdot \, 1 & & m+1 & & 1 \end{pmatrix}$$

$$= \sum_{i=1}^{p-1} (-1)^{1+i}(-1)^{i-1} \, det \, B_{p-1}(m) + (-1)^{p+1} \, det \, A_{p-1}(m)$$
$$= (-1)^{p-1} m^{p-1} .$$

In the above matrices the symbol $\hat{}$ means that the components are omitted. Thus the lemma is proved. Q.E.D.

Proof of Lemma 9.4.4. Put $a_i = \sqrt{\frac{1}{k+1}} \, \alpha_i$ for $i=1, 2, \cdots, h-1$. Then we have for $k \geqq 1$

$$\langle a_i, a_j \rangle = \begin{cases} 1 & \text{if } i \neq j , \\ k+1 & \text{if } i = j . \end{cases}$$

Put for $i=1, 2, \cdots, h-1$

$$a'_i = \det \begin{pmatrix} k+1 & 1 & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & k+1 & \cdot & & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & & \cdot \\ 1 & \cdot & \cdot & 1 & k+1 & & 1 \\ a_1 & \cdot & \cdot & a_{i-2} & a_{i-1} & & a_i \end{pmatrix} .$$

Then $\{a'_1, a'_2, \cdots, a'_{h-1}\}$ is an orthogonal system. We have by Lemma 9.4.5

$$\begin{cases} a'_i = k^{i-2}\{(k+i-1)a_i - \sum_{p=1}^{i-1} a_p\} , \\ \langle a'_i, a'_i \rangle = \det A_{i-1}(k) \det A_i(k) = k^{2i-3}(k+i)(k+i-1) . \end{cases}$$

It follows that $\alpha'_i = \frac{1}{|a'_i|} a'_i$, and therefore $\{\alpha'_1, \alpha'_2, \cdots, \alpha'_{h-1}\}$ is an orthonormal system in $W_{(k+1)\phi_h}((k-1)\phi_h)$. We have for $k \geqq 1$

$$\gamma'_i = \frac{1}{\sqrt{2(k+1)}} (\beta_i - \gamma_i) .$$

Put

$$b_i=\sqrt{\frac{1}{k+1}}\,\{\beta_i-\sum_{p=1}^{k-1}\langle\beta_i,\gamma'_p\rangle\gamma'_p\}\qquad i=1,2,\cdots,h-1\,.$$

Then

$$b_i=\begin{cases}\sqrt{\frac{1}{2}}\,(v_h\otimes v_{-h}-v_{-h}\otimes v_h)=\beta'_i & \text{if } k=1\,,\\ \sqrt{\frac{1}{k+1}}\Big\{\frac{k-1}{2}\,(v_i\otimes v_{-i}\cdot v_h^{k-1}+v_{-i}\otimes v_i\cdot v_h^{k-1}) & \\ \quad -\sqrt{k}\,v_h\otimes\Big(\sqrt{\frac{k-1}{k}}\,v_i\cdot v_{-i}\cdot v_h^{k-2}-\sqrt{\frac{1}{k}}\,v_h^{k-1}\cdot v_{-h}\Big) & \\ \quad -\sqrt{k}\,v_{-h}\otimes v_h^{k}\Big\} & \text{if } k\geqq 2\,.\end{cases}$$

If $k=1$, $\{\beta'_1,\gamma'_1,\gamma'_2,\cdots,\gamma'_{h-1}\}$ is an orthonormal system in $W_{\phi_{h-1}+k\phi_h}((k-1)\phi_h)$. Suppose that $k\geqq 2$. We have for $i,j=1,2,\cdots,h-1$

$$\begin{cases}\langle b_i,\gamma'_j\rangle=0\,,\\ \langle b_i,b_j\rangle=\begin{cases}1 & \text{if } i\neq j\,,\\ \frac{k+1}{2} & \text{if } i=j\,.\end{cases}\end{cases}$$

Put for $i=1,2,\cdots,h-1$

$$b'_i=\det\begin{pmatrix}\frac{k+1}{2} & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1\\ 1 & \ddots & \ddots & & & & & \vdots\\ \vdots & \ddots & \ddots & \ddots & & & & \vdots\\ \vdots & & \ddots & \ddots & \ddots & & & \vdots\\ 1 & \cdot & \cdot & \cdot & \cdot & 1 & \frac{k+1}{2} & 1\\ b_1 & \cdot & \cdot & \cdot & \cdot & \cdot & b_{i-1} & b_i\end{pmatrix}.$$

Applying Lemma 9.4.5, we have $\frac{1}{|b'_i|}\,b'_i=\beta_i$. Therefore if $k\geqq 2$, $\{\beta'_1,\beta'_2,\cdots,\beta'_{h-1},\gamma'_1,\gamma'_2,\cdots,\gamma'_{h-1}\}$ is an orthonormal system in $W_{\phi_{h-1}+k\phi_h}((k-1)\phi_h)$.

(1) It follows from Lemma 9.4.1 and the above arguments that the first and the second assertions are valid. Since $\langle\delta,\delta\rangle=\frac{k(k+h-1)(k+2h-3)}{k+h-2}$ for $k\geqq 1$, we have the third assertion.

(2) Put $a_0=\sqrt{\frac{1}{k+1}}\,\alpha_0$ for $k\geqq 1$. Then

$$\begin{cases}\langle a_0,a_0\rangle=2k+1\,,\\ \langle a_0,a_i\rangle=1 \qquad\qquad \text{for } i=1,2,\cdots,h-1\,.\end{cases}$$

Put

$$a'_0 = \det \begin{pmatrix} k+1 & 1 & \cdots & \cdots & 1 \\ 1 & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ 1 & \cdots & 1 & k+1 & 1 \\ a_1 & \cdots & \cdots & a_{h-1} & a_0 \end{pmatrix} .$$

Then we have by Lemma 9.4.5

$$\langle a'_0, a'_0\rangle = \det A_{h-1}(k) \det \begin{pmatrix} k+1 & 1 & \cdots & \cdots & 1 \\ 1 & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & k+1 & 1 \\ 1 & \cdots & \cdots & 1 & 2k+1 \end{pmatrix}$$

$$= \det A_{h-1}(k) \left\{ \det \begin{pmatrix} k+1 & 1 & \cdots & \cdots & 1 \\ 1 & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 1 \\ 1 & \cdots & \cdots & 1 & k+1 \end{pmatrix} + \det \begin{pmatrix} k+1 & 1 & \cdots & \cdots & \cdots & 1 \\ 1 & \ddots & & & & \vdots \\ \vdots & & \ddots & & & \vdots \\ 1 & \cdots & 1 & k+1 & & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & k \end{pmatrix} \right\}$$

$$= k^{2h-3}(k+h+1)(2k+2h-1) .$$

It follows that $\frac{1}{|a'_0|} a'_0 = \alpha'_0$. Therefore by Lemma 9.4.1 $\{\alpha'_0, \alpha'_1, \cdots, \alpha'_{h-1}\}$ is an orthonormal basis of $W_{(k+1)\phi_h}((k-1)\phi_h)$. If $k=1$, $\{\beta'_1, \gamma'_1, \gamma'_2, \cdots, \gamma'_{h-1}\}$ is an orthonormal basis of $W_{\phi_{h-1}+k\phi_h}((k-1)\phi_h)$. Suppose that $k \geqq 2$. Put

$$b_0 = \sqrt{\frac{1}{k+1}} \{\beta_0 - \sum_{i=1}^{h-1} \langle \beta_0, \gamma'_i\rangle \gamma'_i\} .$$

Then

$$\begin{cases} \langle b_0, \gamma'_i\rangle = 0 & i = 1, 2, \cdots, h-1 , \\ \langle b_0, b_0\rangle = \frac{3}{2}(k-1) , & \\ \langle b_0, b_i\rangle = \begin{cases} 0 & \text{if } i = 1, 2, \cdots, h-2 , \\ -\frac{1}{2}(k-1) & \text{if } i = h-1 . \end{cases} \end{cases}$$

Put

$$b'_0=\det\begin{pmatrix} \frac{k+1}{2} & 1 & \cdots & \cdots & 1 & 0 \\ 1 & \ddots & \ddots & & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 & 0 \\ 1 & \cdots & \cdots & 1 & \frac{k+1}{2} & -\frac{k-1}{2} \\ b_1 & \cdots & \cdots & \cdots & b_{h-1} & b_0 \end{pmatrix}.$$

Then we have

$$b'_0=\sum_{i=1}^{h-2}(-1)^{h+i+i-1}\det\begin{pmatrix} \overbrace{1 \quad \cdots \quad \cdots \quad \cdots \quad 1}^{h-1} & 0 \\ \frac{k+1}{2} \quad \ddots \qquad\qquad \vdots & \vdots \\ 1 \qquad \ddots \quad \ddots \qquad \vdots & \vdots \\ \vdots \quad \ddots \qquad \ddots \quad 1 & 0 \\ 1 \quad \cdots \quad \cdots \quad 1 \quad \frac{k+1}{2} & -\frac{k-1}{2} \end{pmatrix} b_i$$

$$+(-1)^{2h-1}\det\begin{pmatrix} \frac{k+1}{2} & 1 & \cdots & \cdots & 1 & 0 \\ 1 & \ddots & \ddots & \ddots & 1 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \frac{k+1}{2} & 0 \\ 1 & \cdots & \cdots & \cdots & 1 & -\frac{k-1}{2} \end{pmatrix} b_{h-1}$$

$$+A_{h-1}\left(\frac{k-1}{2}\right)b_0$$

$$=\left(\frac{k-1}{2}\right)^{h-2}\left(\frac{k+2h-3}{2}b_0+\frac{k+2h-5}{2}b_{h-1}-\sum_{i=2}^{h-2}b_i\right),$$

and

$$\langle b'_0, b'_0\rangle=\det A_{h-1}\left(\frac{k-1}{2}\right)\det\begin{pmatrix} \frac{k+1}{2} & 1 & \cdots & \cdots & 1 & 0 \\ 1 & \ddots & & & \vdots & \vdots \\ \vdots & & \ddots & \ddots & 1 & 0 \\ 1 & \cdots & \cdots & 1 & \frac{k+1}{2} & -\frac{k-1}{2} \\ 0 & \cdots & \cdots & 0 & -\frac{k-1}{2} & \frac{3(k-1)}{2} \end{pmatrix}$$

$$=\left(\frac{k-1}{2}\right)^{2h-3}\frac{(k+2h-2)(k+2h-3)}{2}.$$

It follows that $\frac{1}{|b'_0|}b'_0=\beta'_0$. Therefore by Lemma 9.4.1 $\{\beta'_0,\beta'_1,\beta'_2,\cdots,\beta'_{h-1},\gamma'_1,\gamma'_2,\cdots,\gamma'_{h-1}\}$ is an orthonormal basis of $W_{\phi_{h-1}+k\phi_h}((k-1)\phi_h)$. Since $\langle\delta,\delta\rangle=\frac{k(k+2h-2)(2k+2h-1)}{2k+2h-3}$ for $k\geqq 1$, we have the third assertion. Q.E.D.

Put $w=v_{-h}\otimes v_h{}^k$. We denote by w_λ the W_λ-component of w with respect to the decomposition (9.4.1). Then we have

Lemma 9.4.6. (1) *The case* $n=2$:

$$|w_{(k+1)\phi_1}|=\sqrt{\frac{1}{(k+1)(2k+1)}},\ |w_{k\phi_1}|=\sqrt{\frac{1}{k+1}},$$
$$|w_{(k-1)\phi_1}|=\sqrt{\frac{2k-1}{2k+1}}.$$

(2) *The case* $n=3$:

$$|w_{(k+1)\phi_2}|=\frac{1}{k+1},\ |w_{\phi_1+k\phi_2}|=\frac{\sqrt{k}}{k+1},$$
$$|w_{-\phi_1+k\phi_2}|=\frac{\sqrt{k}}{k+1},\ |w_{(k-1)\phi_2}|=\frac{k}{k+1}.$$

(3) *The case* $n=2h-1(h\geqq 3)$:

$$|w_{(k+1)\phi_h}|=\sqrt{\frac{h-1}{(k+1)(k+h-1)}},\ |w_{\phi_{h-1}+k\phi_h}|=\sqrt{\frac{2k(h-1)}{(k+1)(k+2h-3)}},$$
$$|w_{(k-1)\phi_h}|=\sqrt{\frac{k(k+h-2)}{(k+h-1)(k+2h-3)}}.$$

(4) *The case* $n=2h(h\geqq 2)$:

$$|w_{(k+1)\phi_h}|=\sqrt{\frac{2h-1}{(k+1)(2k+2h-1)}},\ |w_{\phi_{h-1}+k\phi_h}|=\sqrt{\frac{k(2h-1)}{(k+1)(k+2h-2)}},$$
$$|w_{(k-1)\phi_h}|=\sqrt{\frac{k(2k+2h-3)}{(k+2h-2)(2k+2h-1)}}.$$

Proof. It follows from Lemma 9.4.2 (resp. from Lemma 9.4.3) that $\left\{\sqrt{\frac{1}{(k+1)(2k+1)}}\alpha_{(2)},\sqrt{\frac{1}{k(k+1)}}\beta_{(2)},\sqrt{\frac{1}{k(2k-1)(2k+1)}}\gamma_{(2)}\right\}$ $\left(\text{resp. }\left\{\frac{1}{k+1}\alpha_{(3)},\frac{1}{k+1}\beta_{(3)},\frac{1}{k+1}\gamma_{(3)},\frac{1}{k+1}\delta_{(3)}\right\}\right)$ is an orthonormal basis of $(\boldsymbol{C}^3\otimes V_k)((k-1)\phi_1)$ (resp. of $(\boldsymbol{C}^4\otimes V_k)((k-1)\phi_2)$) for $k\geqq 1$. Therefore we have (1) and (2) easily.

(3) It follows from Lemma 9.4.4 that $\{\alpha'_1,\alpha'_2,\cdots,\alpha'_{h-1},\beta'_1,\gamma'_1,\gamma'_2,\cdots,\gamma'_{h-1},\delta'\}$ (resp. $\{\alpha'_1,\alpha'_2,\cdots,\alpha'_{h-1},\beta'_1,\beta'_2,\cdots,\beta'_{h-1},\gamma'_1,\gamma'_2,\cdots,\gamma'_{h-1},\delta'\}$) is an

orthonormal basis of $(C^{n+1}\otimes V_k)((k-1)\phi_h)$, if $k=1$(resp. if $k\geqq 2$). Therefore we have by Lemma 9.4.4

$$\langle w_{(k+1)\phi_h}, w_{(k+1)\phi_h}\rangle = \sum_{i=1}^{h-1}\frac{k}{(k+1)(k+i)(k+i-1)} = \frac{h-1}{(k+1)(k+h-1)} \qquad \text{for } k\geqq 1,$$

$$\langle w_{\phi_{h-1}+k\phi_h}, w_{\phi_{h-1}+k\phi_h}\rangle = \begin{cases} \dfrac{1}{2} & \text{if } k=1, \\ \displaystyle\sum_{i=1}^{h-1}\frac{2k(k-1)}{(k+1)(k+2i-1)(k+2i-3)} = \frac{2k(h-1)}{(k+1)(k+2h-3)} & \text{if } k\geqq 2, \end{cases}$$

$$|w_{(k-1)\phi_h}| = \sqrt{\frac{k(k+h-2)}{(k+h-1)(k+2h-3)}} \qquad \text{for } k\geqq 1.$$

Therefore we have the assertion.

(4) We obtain the equalities in the same way as above. Q.E.D.

Bibliography

[1] S. Araki: *On root systems and infinitesimal classifications of irreducible symmetric spaces,* Math. J. Osaka City Univ. **13** (1962), 1–34.

[2] H. Boerner: Representations of groups, North-Holland, Amsterdam, 1963.

[3] N. Bourbaki: Groupes et algèbres de Lie, Chapitres 7 et 8, Hermann, Paris, 1975.

[4] S. Helgason: Differential geometry and symmetric spaces, Academic Press, New York, 1962.

[5] T. Nagura: *On the lengths of the second fundamental forms of R-spaces,* Osaka J. Math. **14** (1977), 207–223.

[6] ————: *On the Jacobi differential operators associated to minimal isometric immersions of symmetric spaces into spheres,* I. Osaka J. Math. **18** (1981), 115–145.

[7] I. Satake: *On representations and compactifications of symmetric Riemannian spaces,* Ann. of Math. **71** (1960), 77–110.

[8] J.P. Serre: Algèbres de Lie semi-simples complexes, W.A. Benjamin, New York, 1966.

[9] M. Takeuchi: *Cell decompositions and Morse equalities on certain symmetric spaces,* J. Fac. Sci. Univ. Tokyo, **12** (1965), 81–192.

[10] ————: Theory of spherical functions (in Japanese), Iwanami, Tokyo, 1974.

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