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ON THE GROUPS $J_G(*)$ FOR $G=SL(2, p)$

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0. Introduction

The homotopy theory of group representations has been studied by various authors (see [3], [4], [7], [8], [9], [10] and [11] for example). We are concerned with it in this paper.

Let G be a compact topological group. If V is a real G-representation, we denote by $S(V)$ its unit sphere with respect to some G-invariant inner product. Two real G-representations V and W are called *J_G-related* if and only if there exists a real G-representation U such that $S(V\oplus U)$ and $S(W\oplus U)$ are G-homotopy equivalent. Then the group $J_G(*)$ is defined as the quotient group of the real representation ring $RO(G)$ by the above relation. K. Kawakubo studied $J_G(*)$ for abelian groups G in [8] and [9]; S. Kakutani for some kind of metacyclic group in [7]. T. tom Dieck and T. Petrie made it clear in [3] and [4] that J_G-relation is deeply connected with field automorphisms.

The purpose of this paper is to determine $J_G(*)$ for $G=SL(2, p)$, where p is a prime greater than four. For this, Petrie's theorem introduced in section 2 is applicable. Our main results are Theorems 1.1 and 1.2.

The arrangement of this paper is as follows. In section 1 we determine the irreducible real $SL(2, p)$-representations and state Theorem 1.1 and 1.2. In section 2 we introduce our main tools (developed by Petrie, tom Dieck and Kawakubo). Section 3 is devoted to algebraic lemmas. In section 4 we study subgroups of $SL(2, p)$ up to conjugation. In section 5 we consider J_G-relation for generalized quaternion groups G, and get Corollary 5.5 as a by-product related to [3; Theorems 1 and 3]. In section 6 we list the restriction of $SL(2, p)$-representations to some subgroups. Putting all this together we prove Theorem 1.2 in sections 7 and 8.

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1. The main results and the real $SL(2, p)$-representations

Let $\boldsymbol{N}$ be the set of positive integers. Denote

$$D(n) = \{k\in \boldsymbol{N}\colon k \text{ divides } n \text{ and } 2k<n\}\ ,$$

$$D_o(n) = \{k \in D(n) \colon k \text{ is odd}\}\ ,$$
$$D_e(n) = \{k \in D(n) \colon k \text{ is even}\}\ ,$$

for each positive integer n. We define groups $J_{1,n}$ and $J_{2,n}$ for each integer $n>1$ and for each even integer $n>2$ respectively as follows. Let $n=2^k p_1^{r(1)}\cdots p_t^{r(t)}$ be the prime decomposition of n.

Case 1. $k\geqq 2$. We set

$$J_{1,n} = \boldsymbol{Z}\oplus\boldsymbol{Z}_{2^{k-2}}\oplus\bigoplus_{i=1}^{t}\boldsymbol{Z}_{p_i^{r(i)}-p_i^{r(i)-1}} \quad \text{and}$$

$$J_{2,n} = \boldsymbol{Z}\oplus\boldsymbol{Z}_{2^{k-2}}\oplus\bigoplus_{i=1}^{t}\boldsymbol{Z}_{(p_i^{r(i)}-p_i^{r(i)-1})/2}\ .$$

Case 2. $k=0$ or 1. We set

$$J_{1,n} = \boldsymbol{Z}\oplus(\bigoplus_{i=1}^{t}\boldsymbol{Z}_{p_i^{r(i)}-p_i^{r(i)-1}})/\boldsymbol{Z}_2 \quad \text{and}$$

$$J_{2,n} = \boldsymbol{Z}\oplus\bigoplus_{i=1}^{t}\boldsymbol{Z}_{(p_i^{r(i)}-p_i^{r(i)-1})/2}\ ,$$

where the inclusion of $\boldsymbol{Z}_2$ into $\bigoplus_{i=1}^{t}\boldsymbol{Z}_{p_i^{r(i)}-p_i^{r(i)-1}}$ is given by $1\mapsto\oplus(p_i^{r(i)}-p_i^{r(i)-1})/2$, $\boldsymbol{Z}$ is the group of integers, and $\boldsymbol{Z}_m$ are the quotient groups $\boldsymbol{Z}/(m)$ for positive integers m.

Then we have

Theorem 1.1. *Let $p>4$ be a prime. Then the group $J_{SL(2,p)}(*)$ is isomorphic to*

$$\boldsymbol{Z}\oplus\boldsymbol{Z}\oplus\boldsymbol{Z}\oplus\boldsymbol{Z}\oplus T$$
$$\oplus\bigoplus_{m\in D_e(p-1)}J_{1,(p-1)/m}\oplus\bigoplus_{m\in D_o(p-1)}J_{2,(p-1)/m}$$
$$\oplus\bigoplus_{m\in D_e(p+1)}J_{1,(p+1)/m}\oplus\bigoplus_{m\in D_o(p+1)}J_{2,(p+1)/m}\ ,$$

where T is $\boldsymbol{Z}_2\oplus\boldsymbol{Z}_2$ if $p\equiv 1 \bmod 4$, $\{0\}$ if $p\equiv 3 \bmod 4$ respectively.

This theorem is a consequence of Theorem 1.2. We, however, need some preparation to state Theorem 1.2.

Let G be a topological group. If V is a real G-representation, define $cV=\mathrm{C}\otimes_{\boldsymbol{R}}V$, regarded as a complex G-representation, where $\boldsymbol{R}$ and $\boldsymbol{C}$ are the classical fields of the real numbers and of the complex numbers respectively. Similarly, if V is a complex G-representation, let rV have the same underlying set as V and the same operations from G, but regard it as a vector space over $\boldsymbol{R}$. Let V be a real (resp. complex) G-representation. If H is a subgroup of G, then the restriction of the group action to H defines the real (resp. complex) H-representation $\mathrm{res}_H V$. If K is another topological group and f is a homomorphism from K to G, then the canonically induced real (resp. complex) K-representation is denoted by f^*V.

We will determine the irreducible real $SL(2, p)$-representations according to [5; §38].

Let p be a prime greater than four, G be $SL(2, p)$, F be a finite field of p elements and ν be a generator of the cyclic group $F^*=F-\{0\}$. Denote

$$1=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad z=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix},\quad c=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix},$$

$$d=\begin{pmatrix}1 & 0\\ \nu & 1\end{pmatrix},\quad a=\begin{pmatrix}\nu & 0\\ 0 & \nu^{-1}\end{pmatrix}.$$

Moreover G contains an element b of order $p+1$, and we identify F with $\boldsymbol{Z}_p=\boldsymbol{Z}/(p)$. For each element x of G, let (x) denote the conjugacy class of G containing x. Then G has exactly $p+4$ conjugacy classes (1), (z), (c), (d), (zc), (zd), (a), (a^2), $\cdots$, $(a^{(p-3)/2})$, (b), (b^2), $\cdots$, $(b^{(p-1)/2})$ satisfing

Table 1

x	1	z	c	d	zc	dz	a^m	b^n
$\lvert(x)\rvert$	1	1	$(p^2-1)/2$	$(p^2-1)/2$	$(p^2-1)/2$	$(p^2-1)/2$	$p(p+1)$	$p(p-1)$

for $1\leqq m\leqq(p-3)/2$, $1\leqq n\leqq(p-1)/2$.

Put $\varepsilon=(-1)^{(p-1)/2}$, $\rho=\exp(2\pi\sqrt{-1}/(p-1))$ and $\sigma=\exp(2\pi\sqrt{-1}/(p+1))$. We can choose ν and b so that the complex character table of G is

Table 2

	1	z	c	d	a^m	b^n
1_G	1	1	1	1	1	1
ψ	p	p	0	0	1	-1
χ_i	$p+1$	$(-1)^i(p+1)$	1	1	$\rho^{im}+\rho^{-im}$	0
θ_j	$p-1$	$(-1)^j(p-1)$	-1	-1	0	$-(\sigma^{jn}+\sigma^{-jn})$
ξ_1	$(p+1)/2$	$\varepsilon(p+1)/2$	$(1+\sqrt{\varepsilon p})/2$	$(1-\sqrt{\varepsilon p})/2$	$(-1)^m$	0
ξ_2	$(p+1)/2$	$\varepsilon(p+1)/2$	$(1-\sqrt{\varepsilon p})/2$	$(1+\sqrt{\varepsilon p})/2$	$(-1)^m$	0
η_1	$(p-1)/2$	$-\varepsilon(p-1)/2$	$(-1+\sqrt{\varepsilon p})/2$	$(-1-\sqrt{\varepsilon p})/2$	0	$(-1)^{n+1}$
η_2	$(p-1)/2$	$-\varepsilon(p-1)/2$	$(-1-\sqrt{\varepsilon p})/2$	$(-1+\sqrt{\varepsilon p})/2$	0	$(-1)^{n+1}$

for $1 \leqq i \leqq (p-3)/2$, $1 \leqq j \leqq (p-1)/2$, $1 \leqq m \leqq (p-3)/2$, $1 \leqq n \leqq (p-1)/2$. The columns for the classes (zc) and (zd) are missing in this table. These values are obtained from the relations

$$\chi(zc) = \chi(z)\chi(c)/(1), \quad \chi(zd) = \chi(z)\chi(d)/\chi(1)$$

for all irreducible characters χ of G. We usually identify the above characters with the corresponding complex representations.

According to [1; 3.62], an irreducible complex G-representation χ is real, not self-conjugate or quaternionic if and only if

$$\{\textstyle\sum_{x \in G} \chi(x^2)\}/|G|$$

is equal to 1, 0 or -1, respectively. By calculation of the above values, we determine the irreducible real G-representations as follows.

If $p \equiv 1 \bmod 4$, we have

Table 3

	R	Ψ	X_i (i: even)	Θ_j (j: even)	Ξ_1	Ξ_2	X_i (i: odd)	Θ_j (j: odd)	$\mathfrak{H}_1$	$\mathfrak{H}_2$
$\dim_R$	1	p	$p+1$	$p-1$	$(p+1)/2$	$(p+1)/2$	$2(p+1)$	$2(p-1)$	$p-1$	$p-1$
n.b.	$cR=1_G$	$c\Psi=\psi$	$cX_i=\chi_i$	$c\Theta_j=\theta_j$	$c\Xi_1=\xi_1$	$c\Xi_2=\xi_2$	$X_i=r\chi_i$	$\Theta_j=r\theta_j$	$\mathfrak{H}_1=r\eta_1$	$\mathfrak{H}_2=r\eta_2$

for $1 \leqq i \leqq (p-3)/2$, $1 \leqq j \leqq (p-1)/2$.

If $p \equiv 3 \bmod 4$, we have

Table 4

	R	Ψ	X_i (i: even)	Θ_j (j: even)	X_i (i: odd)	Θ_j (j: odd)	Ξ	$\mathfrak{H}$
$\dim_R$	1	p	$p+1$	$p-1$	$2(p+1)$	$2(p-1)$	$p+1$	$p-1$
n.b.	$cR=1_G$	$c\Psi=\psi$	$cX_i=\chi_i$	$c\Theta_j=\theta_j$	$X_i=r\chi_i$	$\Theta_j=r\theta_j$	$\Xi=r\xi_1$	$\mathfrak{H}=r\eta_1$

for $1\leqq i\leqq(p-3)/2$, $1\leqq j\leqq(p-1)/2$.

Put

$$[n: m] = \{k\in \boldsymbol{N}: 2k<n, (k, n) = m\}$$

for $n\in \boldsymbol{N}$ and $m\in D(n)$, where (k, n) is the greatest common divisor of k and n.

Now we state Theorem 1.2.

Theorem 1.2. *Let $V=\oplus_Y c(Y)Y$ and $W=\oplus_Y c'(Y)Y$, where Y runs through the irreducible real representations and the coefficients $c(Y)$ and $c'(Y)$ are non-negative integers. Then V and W are J_G-related if and only if all the following conditions* (I), (II), ⋯, (VII) *are satisfied.*

(I) $c(\boldsymbol{R}) = c'(\boldsymbol{R})$.

(II) $c(\Psi) = c'(\Psi)$.

(III) *For each element m of $D_e(p-1)$,*
(III, 0) $\sum_k c(X_k) = \sum_k c'(X_k)$ *and*
(III, 1) $\prod_k (k/m)^{c(X_k)} \equiv \pm \prod_k (k/m)^{c'(X_k)} \bmod (p-1)/m$,
where k runs through $[p-1: m]$.

(IV) *For each element m of $D_o(p+1)$,*
(IV, 0) $\sum_k c(X_k) = \sum_k c'(X_k)$ *and*
(IV, 1) $\prod_k (k/m)^{2c(X_k)} \equiv \prod_k (k/m)^{2c'(X_k)} \bmod 2(p-1)/m$,
where k runs throdugh $[p-1: m]$.

(V) *For each element m of $D_e(p+1)$,*
(V, 0) $\sum_k c(\Theta_k) = \sum_k c'(\Theta_k)$ *and*
(V, 1) $\prod_k (k/m)^{c(\Theta_k)} \equiv \pm \prod_k (k/m)^{c'(\Theta_k)} \bmod (p+1)/m$,
where k runs through $[p+1: m]$.

(VI) *For each element m of $D_o(p+1)$,*
(V, 0) $\sum_k c(\Theta_k) = \sum_k c'(\Theta_k)$
(V, 1) $\prod_k (k/m)^{2c(\Theta_k)} \equiv \prod_k (w/k)^{2c'(\Theta_k)} \bmod 2(p+1)/m$,
whre k runs through $[p+1: m]$.

(VII) *If $p\equiv 1 \bmod 4$,*
(VII, 0) $c(\Xi_1)+c(\Xi_2) = c'(\Xi_1)+c'(\Xi_2)$ *and*
(VII, 1) $c(\Xi_1)\equiv c'(\Xi_1) \bmod 2$.
If $p\equiv 3 \bmod 4$, $c(\Xi) = c'(\Xi)$.

$$
\text{(VIII)} \quad \begin{cases} \textit{If } p \equiv 1 \bmod 4\,, \\ \text{(VIII, 0) } c(\mathfrak{H}_1)+c(\mathfrak{H}_2) = c'(\mathfrak{H}_1)+c'(\mathfrak{H}_2) \quad \textit{and} \\ \text{(VIII, 1) } c(\mathfrak{H}_1) \equiv c'(\mathfrak{H}_1) \bmod 2\,. \\ \textit{If } p \equiv 3 \bmod 4,\ c(\mathfrak{H}) = c'(\mathfrak{H})\,. \end{cases}
$$

This theorem will be proved in sections 7 and 8. Theorem 1.1 can be obtained from the same argument in [9; 3]. The details are omitted.

2. Introduction of fundamental theorems

Let G be a finite group, V and W G-representations, and f a G-map from $S(V)$ to $S(W)$. For each subgroup H of G, let $S(V)^H$ and $S(W)^H$ be the H-fixed point sets of $S(V)$ and of $S(W)$ respectively, and f^H the induced map from $S(V)^H$ to $S(W)^H$. By the equivariant case of the theorem of J.H.C. Whitehead (see [6] and [12]), the G-map f is a G-homotopy equivalence if and only if $\deg f^H$ is equal to 1 or -1 for each subgroup H of G. When does a G-homotopy equivalence from $S(V)$ to $S(W)$ exist? If V and W are complex G-representations, Petrie's theorem below is applicable to this question. Now let V and W be complex G-representations. Then the H-fixed point sets $S(V)^H$ and $S(W)^H$ inherit the canonical orientations from the complex structures on V^H and W^H. To introduce Petrie's theorem, we assume that

$$\dim V^H = \dim W^H \ \text{ for each subgroup } H \text{ of } G\,.$$

Let K be a cyclic subgroup of G with a generator g of order n. As complex K-representation, V (resp. W) splits as $V=V^K \oplus V_K$ (resp. $W=W^K \oplus W_K$) which defines V_K (resp. W_K) as the complement of the K-fixed point set V^K (resp. W^K) of V (resp. W). Put $\lambda_{-1}(V_K)(g)=\sum_i (-1)^i \operatorname{trace}(g, \Lambda^i V_K)$, where i runs through the non-negative integers. Define $\chi(V-W; K)$ by

$$\chi(V-W; K) = \sum_{i \in \boldsymbol{Z}_n^*} \frac{\lambda_{-1}(V_K)(g^i)}{\lambda_{-1}(W_K)(g^i)}\,,$$

where $\boldsymbol{Z}_n^*$ is the set of units of $\boldsymbol{Z}_n$.

Theorem 2.1 (Petrie). *Let an integer $d(K)$ be given for each subgroup K of G. There exists a G-map f from $S(W)$ to $S(V)$ such that* $\deg f^K=d(K)$ *for every K, if and only if the following conditions are satisfied.*

(i) *If H and K are conjugate in G, then $d(H)=d(K)$.*

(ii) *If* $\dim V^K=0$, *then $d(K)=1$.*

(iii) *If $H<K$ and $V^H=V^K$, then $d(H)=d(K)$.*

(iv) *For each subgroup H,*

$$\textstyle\sum_K \chi(V^H-W^H; K/H)d(K) \equiv 0 \bmod |NH/H|\,,$$

where the summation is taken over the subgroups K of NH such that K include H and K/H are cyclic. Here NH is the normalizer of H in G.

Let's call the above relation (iv) *the* Petrie *equation.* In order to calculate the Petrie equation, we express $\chi(V-W;\ K)$ in another form. Let $\alpha_1(g)$, $\alpha_2(g), \cdots, \alpha_s(g)$ be all the eigenvalues of g on V, $\beta_1(g), \beta_2(g), \cdots, \beta_s(g)$ all the eigenvalues on W, where s is the complex dimension of V. Assume that $\alpha_m(g)=\beta_m(g)=1$ for each $m<t$, $\alpha_m(g)\neq 1$ and $\beta_m(g)\neq 1$ for each $m\geqq t$. Then we have

$$\chi(V-W;\ K)=\sum_{i\in Z_n^*}\prod_{m=1}^{s}\frac{1-\alpha_m(g)^i}{1-\beta_m(g)^i},$$

where g is the generator as before and for $m<t$ we put $\{1-\alpha_m(g)^i\}/\{1-\beta_m(g)^i\}=1$ for convenience' sake. This yields

$$\chi(V-W;\ K)=\sum_{g'}\prod_{m=1}^{s}\frac{1-\alpha_m(g')}{1-\beta_m(g')},$$

where g' runs through the generators of K.

If we deal with complex G-representations, we call $S(V)$ and $S(W)$ *oriented G-homotopy equivalent* if there exists a G-homotopy equvalence f from $S(V)$ to $S(W)$ such that for each subgroup H of G, the restricted map f^H from $S(V)^H$ to $S(W)^H$ has degree one with respect to the canonical orientations. Let $R_h(G)$ be the subgroup of the complex G-representation ring $R(G)$, consisting of $\alpha=V-W$ such that $S(V)$ and $S(W)$ are oriented G-homotopy equivalent. Let $RO(G)$ be the real G-representation ring and $RO_h(G)$ be the subgroup of $RO(G)$ consisting of $\alpha=V-W$ such that V and W are J_G-related. The elements $\alpha=V-W$ of $R(G)$ such that $\dim V^H=\dim W^H$ for all subgroups H of G form a subgroup of $R(G)$ which is denoted by $R_0(G)$. The analogous subgroup of $RO(G)$ is denoted by $RO_0(G)$. If G has order $n=|G|$, then G-representations are realizable over the field $\boldsymbol{Q}(\zeta)$ where ζ is a primitive n-th root of unity. The Galois group Γ of $\boldsymbol{Q}(\zeta)$ over $\boldsymbol{Q}$ acts on $R(G)$ and $RO(G)$ via its action on character values. Let $\boldsymbol{Z}[\Gamma]$ be the integral group ring of Γ and $I(\Gamma)$ its augumentation ideal. It is well known that $R_0(G)=I(\Gamma)R(G)$ and $RO_0(G)=I(\Gamma)RO(G)$. Put $R_1(G)=I(\Gamma)R_0(G)$ and $RO_1(G)=I(\Gamma)RO_0(G)$. (Our notations follow tom Dieck [3].)

Then the following theorem is proved.

Theorem 2.2 (tom Dieck and Petrie). *For all finite groups G we have* $R_1(G)\subset R_h(G)$ *and* $RO_1(G)\subset RO_h(G)$.

We conclude this section by introducing a result for the cyclic group $\boldsymbol{Z}_n$ (see [8] and [9]). For an integer k, we define a complex $\boldsymbol{Z}_n$-character $v(n, k)$

which will be identified with the corresponding complex $\boldsymbol{Z}_n$-representation, by

$$v(n, k)(j) = \exp(2\pi jk\sqrt{-1}/n) \qquad \text{for } j \in \boldsymbol{Z}_n .$$

Put $V(n, k)=rv(n, k)$, and simply write $V(k)$ for $V(n, k)$ if there is no confusion.

Theorem 2.3 (Kawakubo). *Let $V=\oplus_k c(k)V(k)$ and $W=\oplus_k c'(k)V(k)$, where k runs through $[n: 1]$ and all the coefficients $c(k)$ and $c'(k)$ are non-negative integers. Then the following three conditions* (i), (ii) *and* (iii) *are equivalent.*

(i) *$S(V)$ and $S(W)$ are $\boldsymbol{Z}_n$-homotopy equivalent.*

(ii) *V and W are $J_{\boldsymbol{Z}_n}$-related.*

(iii) $\begin{cases} \sum_k c(k) = \sum_k c'(k) \quad \textit{and} \\ \prod_k k^{c(k)} \equiv \pm \prod_k k^{c'(k)} \bmod n . \end{cases}$

3. Algebraic lemmas

For an even integer $q \geqq 4$, put

$$O_q = \{k \in \boldsymbol{N}: 1 \leqq k \leqq q,\ k \text{ is odd}\} ,$$
$$E_q = \{k \in \boldsymbol{N}: 1 \leqq k \leqq q,\ k \text{ is even}\} ,$$

and $\mu=\exp(2\pi\sqrt{-1}/q)$.

Lemma 3.1. *Let r be a positive divisor of q with $2r<q$. Then we have*

$$\sum_{k\in O_q} \mu^{rk} = 0 .$$

Proof. (i) We firstly prove the relation for $r=1$. Observe

$$\sum_{k\in O_q \cup E_q} \mu^k = 0 .$$

On the other hand we have

$$\sum_{k\in E_q} \mu^k = \sum_{k=1}^{q/2} (\mu^2)^k = 0 .$$

Therefore we obtain $\sum_{k\in O_q} \mu^k=0$.

(ii) If q/r is even, then $q/r \geqq 4$. From (i) we have

$$\sum_{k\in O_q} \mu^{rk} = r \sum_{k\in O_{q/r}} (\mu^r)^k = 0 .$$

(iii) If q/r is odd, then we have

$$\sum_{k\in O_q} \mu^{rk} = (r/2) \sum_{r=1}^{q/r} (\mu^r)^k = 0 .$$

This completes the proof of Lemma 3.1.

Now let p be an odd prime with $p \equiv 1 \bmod 4$. Denote

$$K_s=\{n\in N\colon 1\leqq n\leqq p-1,\ n\equiv i^2 \bmod p \text{ for some integer } i\}\ ,$$
$$K_s'=\{n\in K_s\colon 1\leqq n\leqq (p-1)/2\}\ ,$$
$$K_r=\{n\in N\colon 1\leqq n\leqq p-1,\ n\notin K_s\}\ , \quad \text{and}$$
$$K_r'=\{n\in K_r\colon 1\leqq n\leqq (p-1)/2\}\ .$$

Then we have the following lemma.

Lemma 3.2. *It never holds that*

$$\prod_{n\in K_s'} n\equiv\pm\prod_{n\in K_r'} n \bmod p\ .$$

Proof. Let α be an integer which generates Z_p^*. For $i\in N$ there exists an integer m such that $\alpha^i-mp\in K_s$, if and only if $i\equiv 0 \bmod 2$. Thus for $\beta\in K_s$, there exists an integer m such that $\alpha\beta-mp\in K_r$. Since $p\equiv 1 \bmod 4$ imples $p-1\in K_s$, either $\alpha\beta-mp\in K_r'$ or $p-(\alpha\beta-mp)\in K_r'$ happens. We define a map α^* from K_s' to K_r' which maps β to $\alpha\beta-mp$ if $\alpha\beta-mp\in K_r'$, to $p-(\alpha\beta-mp)$ if $p-(\alpha\beta-mp)\in K_r'$ respectively. It is easy to check that α^* is bijective. Suppose that

$$\prod_{n\in K_s'} n\equiv\pm\prod_{n\in K_r'} n \bmod p\ .$$

Then we have

$$\prod_{n\in K_s'} n\equiv\pm\prod_{n\in K_r'} n\equiv\pm\prod_{n\in K_s'}\alpha^*(n)\equiv\pm\prod_{n\in K_s'}\alpha n\equiv\pm\alpha^{(p-1)/4}\prod_{n\in K_s'} n \bmod p\ .$$

This implies $\alpha^{(p-1)/4}\equiv\pm 1 \bmod p$. Therefore we have $\alpha^{(p-1)/2}\equiv 1 \bmod p$. This contradicts the fact that the order of α is $p-1$. This completes the proof.

Lastly we quote a lemma which contributes to the computation to determine the restriction of $SL(2, p)$-representations to subgroups in section 6. Put $\zeta=\exp(2\pi\sqrt{-1}/p)$, where p is an odd prime with $p\equiv 1 \bmod 4$.

Lemma 3.3 (Gauss, see [13]). *It holds that*

$$\sum_{n\in K_s}\zeta^n=\frac{-1+\sqrt{p}}{2} \quad \textit{and} \quad \sum_{n\in K_r}\zeta^n=\frac{-1-\sqrt{p}}{2}\ .$$

4. On subgroups of $SL(2, p)$

In this section we determine some subgroups of $SL(2, p)$ up to conjugation and their normalizers in order to use the Petrie equation. Here p is an odd prime greater than four.

Put

$$B=\left\{\begin{pmatrix}\alpha & 0\\ \beta & \alpha^{-1}\end{pmatrix}\colon \alpha\in F^*,\ \beta\in F\right\}.$$

Then B is the subgroup generated by a and zc defined in section 1. If $p\equiv 1 \bmod 4$, B has the subgroup

$$B_0=\left\langle x, y\colon x=zc, y=\begin{pmatrix} \nu^{(p-1)/4} & 0 \\ 0 & \nu^{-(p-1)/4}\end{pmatrix}\right\rangle.$$

We firstly consider subgroups H of $SL(2, p)$ such that p divides $|H|$. Since H contains a cyclic subgroup of order p, we may assume that H contains $\langle c\rangle$ (see Table 1). Let c act canonically on

$$F\oplus F=\left\{\begin{pmatrix} x \\ y\end{pmatrix}\colon x\in F\right\}.$$

Then $\langle c\rangle$ preserves the line

$$L=\left\{\begin{pmatrix} 0 \\ x\end{pmatrix}\colon x\in F\right\}.$$

We easily see that $N\langle c\rangle$ is B, where $N\langle c\rangle$ is the normalizer of $\langle c\rangle$ in $SL(2, p)$. If all the elements of H of order p are represented in the form

$$\begin{pmatrix} 1 & 0 \\ * & 1\end{pmatrix},$$

then for each element g of H it holds that $g^{-1}\langle c\rangle g=\langle c\rangle$. Therefore H is included in B. Otherwise H has an element of order p which never preserves the line L. According to the proof of [14, 2.4 Proposition 15] H must be $SL(2, p)$. Hence H is included in B, if H is different from $SL(2, p)$. Moreover $\langle c\rangle$ is the unique cyclic subgroup of B of order p, consequently of H. Thus $B=N\langle c\rangle$ includes the normalizer NH of H in $SL(2, p)$.

We have proved

Proposition 4.1. *For a subgroup H with $\langle c\rangle\subset H\subset B$, NH is included in B.*

Secondly we will determine the subgroups H of $SL(2, p)$ with $(|H|, 2p)=1$. In this case $|H|$ is divisible neither by 2 nor by p. Let f be the projection of $SL(2, p)$ to $PSL(2, p)=SL(2, p)/\langle z\rangle$. Observe

$$f(H)\subset PSL(2, p)\subset PGL(2, p).$$

Let K be a subgroup of $PGL(2, p)$ whose order is prime to p. If K is neither cyclic nor dihedral, then K is isomorphic to one of the groups $\mathfrak{A}_4$, $\mathfrak{S}_4$ and $\mathfrak{A}_5$ (see [14; Proposition 16]). Now $|f(H)|$ is prime to p, moreover to 2. It follows that $f(H)$ is cyclic. As $|H|$ is prime to 2, H is a cyclic group which does not contain z. If H is non-trivial, then we assume that H is one of $\langle a^s\rangle$ and $\langle b^t\rangle$, where $1\leqq s<(p-1)/2$, $1\leqq t<(p+1)/2$.

Proposition 4.2. *Provided* $1\leqq s<(p-1)/2$, *we have*

$$N\langle a^s\rangle=\left\{\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}, \begin{pmatrix}0 & -\alpha\\ \alpha^{-1} & 0\end{pmatrix}: \alpha\in F^*\right\}.$$

Since the proof is straightforward, we omit it.

From now on we will determine $N\langle b^t\rangle$ for $1\leqq t<(p+1)/2$. Following to Dornfhoff [5; p. 229], put $k=GF(p^2)$, then $k\supset F$. Choose a generator τ of k^* with $\tau^{p+1}=\nu$. Let b and e be the F-linear maps of the F-vector space k defined by $b(\gamma)=\tau^{p-1}\gamma$ and $e(\gamma)=\tau\gamma$, where γ are elements of k. If an element g of $SL(2, p)$ satisfies $gb^tg^{-1}=b^{rt}$, $1\leqq t<(p+1)/2$, then we have $gb^t=b^{rt}g$. Let ϕ: $k\rightarrow k$ be the map given by

$$\phi(\alpha+\tau^{(p-1)t}\beta)=\alpha+\tau^{(p-1)rt}\beta,$$

where α and β are elements of F. So we have $\phi(\alpha+b^t\beta)=\alpha+b^{rt}\beta$. If $\gamma=\alpha+\tau^{(p-1)t}\beta$ and $\mu\in k$, then it follows that $g(\gamma\mu)=g((\alpha+b^t\beta)\mu)=(\alpha+b^{rt}\beta)g(\mu)=\phi(\gamma)g(\mu)$. Therefore we have $g(\gamma\gamma')=\phi(\gamma\gamma')g(1)$ and $g(\gamma\gamma')=\phi(\gamma)g(\gamma')=\phi(\gamma)\phi(\gamma')g(1)$, and these relations imply $\phi(\gamma\gamma')=\phi(\gamma)\phi(\gamma')$. Hence ϕ is an automorphism of k over F.

Proposition 4.3. *For each element g of $N\langle b^t\rangle$, $1\leqq t<(p+1)/2$, g is included in the normalizer $N_{GL(2,p)}\langle e\rangle$ of $\langle e\rangle$ in $GL(2, p)$.*

Proof. Let ϕ be the map defined as above, ϕ' the map of k given by $\phi'(\gamma)=g(1)\gamma$ for elements γ of k. Then g is considered as the composition $\phi'\circ\phi$. It is sufflcient to show that both ϕ and ϕ' belong to the normalizer $N_{GL(2,p)}\langle e\rangle$. Since ϕ is an automorphsm, we have $(\phi e\phi^{-1})(\gamma)=\phi(\tau)\gamma$. $\phi(\tau)$ is an element of k^*. Therefore $\phi e\phi^{-1}=e^m$ follows for some integer m. As for ϕ', since $g(1)$ is an element of k^*, there exists an integer m such that $\phi'=e^m$. It is clear that ϕ' belongs to $N_{GL(2,p)}\langle e\rangle$. This completes the proof.

Proposition 4.4. *$N\langle b^t\rangle$, $1\leqq t<(p+1)/2$, is equal to $N\langle b\rangle$. Moreover $|N\langle b\rangle/\langle b\rangle|=2$ holds.*

Proof. By Proposition 4.3, $N\langle b^t\rangle$ is included in $N_{GL(2,p)}\langle e\rangle\cap SL(2, p)$. On the other hand, it is obvious $N\langle b^t\rangle$ includes $N_{GL(2,p)}\langle e\rangle\cap SL(2, p)$. Consequently we have $N\langle b^t\rangle=N_{GL(2,p)}\cap SL(2,p)$. The first part of Proposition 4.4 follows from this relation.

From the above fact we obtain

$$\begin{aligned}|N\langle b\rangle/\langle b\rangle| &= |N_{GL(2,p)}\langle e\rangle\cap SL(2, p)/\langle e\rangle\cap SL(2, p)|\\ &= |N_{GL(2,p)}\langle e\rangle/\langle e\rangle|\\ &=2.\end{aligned}$$

Proposition 4.5. *$N\langle a\rangle$ and $N\langle b\rangle$ are the generalized quaternion groups such that*

$$N\langle a\rangle = \langle x, y\colon x^{p-1} = 1,\ x^{(p-1)/2} = y^2,\ y^{-1}xy = x^{-1}\rangle\,,$$
$$N\langle b\rangle = \langle x, y\colon x^{p+1} = 1,\ x^{(p+1)/2} = y^2,\ y^{-1}xy = x^{-1}\rangle\,.$$

Proof. For $N\langle a\rangle$, put $x=a$ and $y=$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then the three relations $x^{p-1}=1$, $x^{(p-1)/2}=y^2$ and $y^{-1}xy=x^{-1}$ follow immediately.

For $N\langle b\rangle$, put $x=b$ and choose an arbitrary element as y in $N\langle b\rangle\backslash b$. Since $|Gal(k/F)|=2$, we get $y(\gamma)=\gamma^p y(1)$ for any $\gamma\in k$. Let ϕ and ϕ' be the maps of k defined by $\phi(\gamma)=\gamma^p$ and $\phi'(\gamma)=y(1)\gamma$ for $\gamma\in k$. Then we can consider y as the composition $\phi'\circ\phi$. As $\det y=1$ and $\det\phi=-1$, we get $\det\phi'=-1$. $\det e=v$ implies that the order of $y(1)$ as an element of k^* divides $2(p+1)$. Since we have $y^2(\gamma)=y(\gamma^p y(1))=\gamma y(1)^{p+1}$, the order of y^2 is at most 2. Since $y\notin\langle b\rangle$, we conclude that the order of y is 4 and $y^2=z$.

We complete the proof if we show the relation $y^{-1}xy=x^{-1}$. Notice that $y^{-1}=zy$ and $y(1)^{p+1}=-1$. For $\gamma\in k$, we have

$$\begin{aligned}(y^{-1}xy)(\gamma) &= (zyx)(y(\gamma)) = (zyx)(\gamma^p y(1))\\ &= (zy)(\tau^{p-1}\gamma^p y(1)) = z(\tau^{1-p}\gamma\, y(1)^{p+1})\\ &= \tau^{1-p}\gamma = x^{-1}(\gamma)\,.\end{aligned}$$

Hence we obtain $y^{-1}xy=x^{-1}$. This completes the proof.

5. J_G-relation for generalized quaternion groups G

We showed in the previous section that $SL(2, p)$ contains some generalized quaternion groups. In order to consider $J_{SL(2,p)}$-relation, we consider J_G-relation for the generalized quaternion groups. Let G be one of them with presentation

$$\langle x, y\colon x^{2s} = 1,\ x^s = y^2,\ y^{-1}xy = x^{-1}\rangle,$$

where s is an integer greater than 1. For an integer k with $1\leqq k<s$, define a complex matrix representation $T(G, k)$ by

$$T(G, k)(x) = \begin{pmatrix} \zeta^k & 0 \\ 0 & \zeta^{-k} \end{pmatrix} \text{ and}$$
$$T(G, k)(y) = \begin{pmatrix} 0 & (-1)^k \\ 1 & 0 \end{pmatrix},$$

where $\zeta=\exp(\pi\sqrt{-1}/s)$. The corresponding complex G-representation to $T(G, k)$ will be also denoted by $T(G, k)$. From [2], $T(G, k)$ is irreducible. If G is fixed and there is no confusion, we simply write $T(k)$ for $T(G, k)$.

Proposition 5.1. *If $rT(1)$ and $rT(k)$ are J_G-related, then it holds that* $k^2\equiv 1 \bmod 4s$.

We will prove this proposition later on, and we admit this proposition to be true for a moment.

Proposition 5.2. *Provided* $k^2\equiv 1 \bmod 4s$, *$S(T(1))$ and $S(T(k))$ are oriented G-homotopy equivalent.*

Proof. We show that there exists a G-map f from $S(T(1))$ to $S(T(k))$ such that $\deg f^K=1$ with respect to the canonical orientations for each subgroup K of G, using Theorem 2.1. Put $V=T(k)$, $W=T(1)$ and $d(K)=1$ for all $K<G$. Since G acts freely on $S(T(k))$ as well as $S(T(1))$, it is sufficient to check the Pertrie equation for $H=\{1\}$. That is we check the equation:

$$(*) \quad d_{\{1\}}+\sum_{\langle x^i\rangle\neq 1}\chi(T(k)-T(1);\langle x^i\rangle)d(\langle x^i\rangle)+sd(\langle xy\rangle)+sd(\langle y\rangle)\equiv 0 \bmod 4s\,,$$

where all $d(*)$ are equal to one. Since

$$\chi(T(k)-T(1);\langle x^i\rangle)=\sum_{j\in Z^*_{2s/i}}\frac{(1-\zeta^{ijk})(1-\zeta^{-ijk})}{(1-\zeta^{ij})(1-\zeta^{-ij})}$$

$$=\sum_{j\in Z^*_{2s/i}}\{1+\zeta^{ij}+\cdots+\zeta^{ij(k-1)}\}\{1+\zeta^{-ij}+\cdots+\zeta^{-ij(k-1)}\}\,,$$

the left hand side of (*)

$$=1+\sum_{\langle x^i\rangle\neq\{1\}}\chi(T(k)-T(1);\langle x^i\rangle)+2s$$

$$=1+\sum_{j=1}^{2s}\{1+\zeta^{j}+\cdots+\zeta^{j(k-1)}\}\{1+\zeta^{-j}+\cdots+\zeta^{-j(k-1)}\}-k^2+2s\,.$$

By the fact that for integers α and β, $\zeta^\alpha\zeta^{-\beta}=1$ implies $\zeta^\beta\zeta^{-\alpha}=1$, the above value is congruent to $1+2ks-k^2+2s \bmod 4s$, consequently to $0 \bmod 4s$. This completes the proof.

Corollary 5.3. *$T(k)-T(1)$ (resp. $rT(k)-rT(1)$) belongs to $R_h(G)$ (resp. $RO_h(G)$), if and only if* $k^2\equiv 1 \bmod 4s$.

Proposition 5.4. *If $\sum_{k\in[2s\,:\,1]}c(k)T(k)$ (resp. $\sum_{k\in[2s\,:\,1]}c(k)\{rT(k)\}$) belongs to $R_1(G)$ (resp. $RO_1(G)$), then we have*

$$\prod_k k^{c(k)}\equiv\pm 1 \bmod 2s\,,$$

where all the coefficients $c(k)$ are integers, and for a positive integer i, k^{-i} is an integer such that $k^ik^{-i}\equiv 1 \bmod 2s$.

Proof. Let ϕ be a field-automorphism and t an integer such that $\phi(\zeta)=\zeta^t$, then for $h\in[2s\colon 1]$ $T(h)$ is equal to $T(h')$, where $h'\in[2s\colon 1]$ and $h'\equiv\pm th \bmod 2s$. If we have $\sum_k c(k)T(k)=(1-\phi)(1-\phi')T(h)$ for some $h\in[2s\colon 1]$ and field-automorphisms ϕ and ϕ', then we easily get $\prod k^{c(k)}\equiv\pm 1 \bmod 2s$. Proposition 5.4 comes from this fact.

Corollary 5.5. *Assume s is odd. If we have $R_1(G)=R_h(G)$ or $RO_1(G)=RO_h(G)$, then s is a power of a prime.*

Proof. If s is odd and divisible by distinct more than two primes, then there exist more than two integers k such that $k^2\equiv 1 \bmod 4s$ and $1\leqq k<s$. If we take such a non-trivial k, $T(k)-T(1)$ (resp. $rT(k)-rT(1)$) belongs not to $R_1(G)$ (resp. $RO_1(G)$) but to $R_h(G)$ (resp. $RO_h(G)$).

Theorem 5.6. *Let $V=\oplus_{k\in[2s:1]}c(k)T(k)$ and $W=\oplus_{k\in[2s:1]}c'(k)T(k)$, where $c(k)$ and $c'(k)$ are non-negative integers. Then the following three statements* (i), (ii) *and* (iii) *are equivalent.*

(i) $\begin{cases} \sum_k c(k)=\sum_k c'(k) & \text{and} \\ \prod_k k^{2c(k)}\equiv\prod_k k^{2c'(k)} \bmod 4s\,. \end{cases}$

(ii) *rV and rW are J_G-related.*

(iii) *$S(V)$ and $S(W)$ are oriented G-homotopy equivalent.*

Proof. Firstly we prove that (i) implies (iii). Since $\sum_k c(k)=\sum_k c'(k)$, $\sum_k (c(k)-c'(k))T(k)$ belongs to $R_0(G)$. There exists $i\in[2s\colon 1]$ such that

$$\sum_k (c(k)-c'(k))T(k)\equiv T(i)-T(1) \bmod R_1(G)$$

(see [3; section 1]). By Proposition 5.4 and (i), we have $i^2\equiv 1 \bmod 4s$. Combining Theorems 2.1 and 2.2 and Proposition 5.2 we see that $S(V)$ and $S(W)$ are oriented G-homotopy equivalent.

It is clear from the definition that (iii) implies (ii).

We complete the proof by showing that (ii) implies (i). It is easy to get $\sum_k c(k)=\sum_k c'(k)$ from (ii). Therefore we have $\sum_k (c(k)-c'(k))T(k)\equiv T(i)-T(1) \bmod R_1(G)$ for some $i\in[2s\colon 1]$. Since $rT(1)$ and $rT(i)$ are J_G-related, $i^2\equiv 1 \bmod 4s$. By Proposition 5.4 it holds that $\prod k^{2c(k)}\equiv\prod k^{2c'(k)} \bmod 4s$.

We have proved Theorem 5.6 assuming Proposition 5.1.

Proof of Proposition 5.1. For s a power of two, Proposition 5.1 is given in tom Dieck [3; section 4].

Lemma 5.7. *If s is a power of an odd prime, $rT(1)$ and $rT(k)$ are J_G-related, then $k^2\equiv 1 \bmod 4s$ holds.*

If we admit this lemma to be true, we can prove Proposition 5.1 in general as follows. Let $s=p_1^{r(1)}p_2^{r(2)}\cdots p_t^{r(t)}$ be the prime decomposition of s. For each integer i, $1\leqq i\leqq t$, restrict $T(1)$ and $T(k)$ to the subgroup $H(i)$ of G generated by x^j and y, where $j=s/p_i^{r(i)}$. If $rT(1)$ and $rT(k)$ are J_G-related, then $r(\mathrm{res}_{H(i)}T(1))$ and $r(\mathrm{res}_{H(i)}T(k))$ are $J_{H(i)}$-related. From the case of a power of a prime, we obtain that $k^2\equiv 1 \bmod 4p_i^{s(i)}$. This yields that $k^2\equiv 1 \bmod 4s$.

Proof of Lemma 5.7. Let $s=q^n$ be the prime decomposition of s. From the assumption such that $rT(1)$ and $rT(k)$ are J_G-related, there exists a complex G-representation U such that $S(T(1)\oplus U)$ and $S(T(k)\oplus U)$ are G-homotopy equivalent. Therefore there exists $(d(K))_K$ consisting of 1 and -1, and satisfying the Petrie equation (iv) of Theorem 2.1 for $V=T(k)\oplus U$ and $W=T(1)\oplus U$, where K runs through the subgroups of G. The following assertion is a key to complete the proof of Lemma 5.7.

Assertion 5.8. *In the above situation, it holds that*
(i) $d(\langle x^{q^m}\rangle) = d(\langle x\rangle)$ *for* $0\leqq m\leqq n$,
(ii) $d(\langle x^{2q^m}\rangle)=d(\langle x\rangle)$ *for* $0\leqq m\leqq n-1$.

Proof. We prove (i) by induction on m. If $m=0$, then (i) holds trivialy. For fixed m, $0\leqq m\leqq n-1$, putting the inductive assumption such that

$$d(\langle x^{q^i}\rangle) = d(\langle x\rangle)$$

holds for each i, $0\leqq i\leqq m$, we prove $d(\langle x^{q^{m+1}}\rangle)=d(\langle x\rangle)$. The Petrie equation for $H=\langle x^{q^{m+1}}\rangle$ is

$$d(\langle x^{q^{m+1}}\rangle)+\sum_{i=0}^{m}\phi(q^{m+1-i})d(\langle x^{q^i}\rangle)+q^{m+1}d(\langle x^{q^{m+1}}, y\rangle)\equiv 0 \bmod 2q^{m+1}\,,$$

where ϕ is the Euler function. By the inductive assumption, we get

$$d(\langle x^{q^{m+1}}\rangle)+(q^{m+1}-1)d(\langle x\rangle)+q^{m+1}d(\langle x^{q^{m+1}}, y\rangle)\equiv 0 \bmod 2q^{m+1}\,.$$

Since $d(K)=1$ or -1, we obtain

$$d(\langle x^{q^{m+1}}\rangle) = d(\langle x\rangle)\,.$$

This completes the proof of (i).

Next we prove (ii) by induction on m. The Petrie equation for $H=\langle x^2\rangle$ is

$$d(\langle x^2\rangle)+d(\langle x\rangle)+2d(G)\equiv 0 \bmod 4\,.$$

Since $d(K)=1$ or -1, we obtain $d(\langle x^2\rangle)=d(\langle x\rangle)$. This shows (ii) for $m=0$. As the rest of the proof of (ii) is quite similar to that of (i), we omit it.

We return to the proof of Lemma 5.7. The Petrie equation for $H=\{1\}$ is

$$d(\{1\})+\sum_{\langle x^i\rangle\neq\{1\}}\chi(T(k)-T(1);\langle x^i\rangle)d(\langle x^i\rangle)+2sd(\langle y\rangle)\equiv 0 \bmod 4s\,.$$

From the definition we obtain

$$\chi(T(k)-T(1);\langle x^i\rangle=\sum_{j\in Z^*_{2s/i}}\frac{(1-\zeta^{ijk})(1-\zeta^{-ijk})}{(1-\zeta^{ij})(1-\zeta^{-ij})}$$
$$=\sum_{j\in Z^*_{2s/i}}\{1+\zeta^{ij}+\cdots+\zeta^{ij(k-1)}\}\{1+\zeta^{-ij}+\cdots+\zeta^{-ij(k-1)}\}\,.$$

By Assertion 5.8 $d(\langle x^i\rangle)=d(\langle x\rangle)$ holds for $\langle x^i\rangle\neq\{1\}$. Therefore it holds that

$$\sum_{\langle x^i\rangle\neq\{1\}}\chi(T(k)-T(1);\langle x^i\rangle)d(\langle x^i\rangle)$$
$$=\sum_{\langle x^i\rangle\neq\{1\}}\chi(T(k)-T(1);\langle x^i\rangle)d(\langle x\rangle)$$
$$=\ d(\langle x\rangle)[\sum_{j=1}^{2s}\{1+\zeta^{j}+\cdots+\zeta^{j(k-1)}\}\{1+\zeta^{-j}+\cdots+\zeta^{-j(k-1)}\}-k^2]\,.$$

Since for integers α and β, $\zeta^{\alpha}\zeta^{-\beta}=1$ implies $\zeta^{\beta}\zeta^{-\alpha}=1$, we get

$$\sum_{\langle x^i\rangle\neq\{1\}}\chi(T(k)-T(1);\langle x^i\rangle)d(\langle x^i\rangle)\equiv(2ks-k^2)d(\langle x\rangle)\bmod 4s\,.$$

Therefore the Petrie equation shows that

$$d(\{1\})+(2ks-k^2)d(\langle x\rangle)+2sd(\langle y\rangle)\equiv 0 \bmod 4s\,.$$

As $d(k)=1$ or -1, we obtain $d(\{1\})\equiv k^2d(\langle x\rangle)\bmod 4s$. Consequently we have not only $d(\{1\})=d(\langle x\rangle)$ but also $k^2\equiv 1\bmod 4s$. This completes the proof of Lemma 5.7.

6. Restriction of $SL(2, p)$-representations to subgroups

In section 4 we considered subgroups of $SL(2, p)$. Restriction of irreducible $SL(2, p)$-representations to those subgroups is listed below. Since this can be obtained by easy calculation using Lemmas 3.1 and 3.3, the proof is omited.

Suppose $p\equiv 1$ (resp. 3) mod 4.

(6.1) $$\mathrm{res}_{\langle a\rangle}X_{2i}=V(p-1,2i)\oplus 2\{\bigoplus_{k=1}^{(p-5)/4}V(p-1,2k)\oplus\boldsymbol{R}\oplus\boldsymbol{R}'\}\,,$$

where $\boldsymbol{R}'$ is the one dimensional real representation such that a acts as -1, and $V(*, *)$ are ones defined in section 2,

$$(\text{resp. }\mathrm{res}_{\langle a\rangle}X_{2i}=V(p-1,2i)\oplus 2\{\bigoplus_{k=1}^{(p-3)/4}V(p-1,2k)\oplus\boldsymbol{R}\})\,.$$

$\mathrm{res}_{\langle a\rangle}\Theta_{2j}$ and $\mathrm{res}_{\langle a\rangle}\Xi_k$ are independent of j and k respectively as real $\langle a\rangle$-representation.

(6.2) $$\mathrm{res}_{\langle b\rangle}\Theta_{2j}=V(p+1,2j)\oplus 2\{\bigoplus_{k=1,k\neq j}^{(p-1)/4}V(p+1,2k)\oplus\boldsymbol{R}\}\,,$$

$$(\text{resp. } \mathrm{res}_{\langle b\rangle}\Theta_{2j} = V(p+1,\, 2j)\oplus 2\{\bigoplus_{k=1,k\neq j}^{(p-3)/4} V(p+1,\, 2k)\oplus \boldsymbol{R}\oplus \boldsymbol{R}'\}\,,$$

where $\boldsymbol{R}'$ is the one dimensional real representation such that b acts as -1). $\mathrm{res}_{\langle b\rangle}X_{2i}$ and $\mathrm{res}_{\langle b\rangle}\Xi_k$ are independent of i and k respectively as real $\langle b\rangle$-representation.

(6.3)
$$\mathrm{res}_{\langle c\rangle}\Xi_1 = \boldsymbol{R}\oplus \oplus_{k\in K'_s} V(p,\, k)\,, \quad \text{and}$$
$$\mathrm{res}_{\langle c\rangle}\Xi_2 = \boldsymbol{R}\oplus \oplus_{k\in K'_r} V(p,\, k)\,,$$

where K'_s and K'_r are the sets defined in section 3. $\mathrm{res}_{\langle c\rangle}X_{2i}$ and $\mathrm{res}_{\langle c\rangle}\Theta_{2i}$ are independent of i and j respectively as real $\langle c\rangle$-representation.

Let B_0 be the subgroup in section 4, $T(B_0,\, k)$ the complex B_0-representations defined as in section 5 for $G=B_0$ and $1\leqq k<p-1$. Denote

$$K''_s = \{n\in \boldsymbol{Z}\colon 0<n<p,\, n \text{ is odd and } n+p\equiv 2i^2 \bmod p \text{ for some } i\in \boldsymbol{Z}\}\,, \quad \text{and}$$
$$K''_r = \{n\in \boldsymbol{Z}\colon 0<n<p,\, n \text{ is odd and } n\notin K''_s\}\,.$$

(6.4)
$$\mathrm{res}_{B_0}\, \eta_1 = \oplus_{k\in K''_s} T(B_0,\, k), \quad \text{and}$$
$$\mathrm{res}_{B_0}\, \eta_2 = \oplus_{k\in K''_r} T(B_0,\, k)\,.$$

$\mathrm{res}_{B_0}\, X_{2i-1}$ and $\mathrm{res}_{B_0}\, \Theta_{2j-1}$are independent of i and j as real B_0-representation.

(6.5)
$$\mathrm{res}_{N\langle a\rangle}\, X_{2i-1} = T(N\langle a\rangle,\, 2i-1)\oplus \bigoplus_{k=1}^{(p-1)/2} T(N\langle a\rangle,\, 2k-1)\,.$$

$\mathrm{res}_{N\langle a\rangle}\, \Theta_{2j-1}$ are independent of j as real $N\langle a\rangle$-representation.

(6.6)
$$\mathrm{res}_{N\langle b\rangle}\, \theta_{2j-1} = \bigoplus_{k=1,k\neq j}^{(p+1)/2} T(N\langle b\rangle,\, 2k-1)\,,$$

$\mathrm{res}_{N\langle b\rangle}X_{2i-1}$ are independent of i as real $N\langle b\rangle$-representation.

7. The Proof of the suffcient condition in Theorem 1.2

Let G be $SL(2,p)$ in this section. We prove that if (I), (II), $\cdots$, (VIII) in Theorem 1.2 all are satisfied, then V and W are J_G-related.
Suppose $p\equiv 1 \bmod 4$.

Proposition 7.1. *Let $V=\oplus_{k\in[p-1:m]}c(X_k)X_k$ and $W=\oplus_{k\in[p-1:m]}c'(X_k)X_k$ for an element m of $D_e(p-1)$. If for $c(X_k)$ and $c'(X_k)$ the condition* (III) *in Theorem* 1.2 *is satisfied, then V and W are J_G-related.*

Proof. We consder the submodule $M=\{\sum_{k\in[p-1:m]}a_kX_k\colon a_k\in \boldsymbol{Z}\}$ of $RO(G)$. A homomorphism $\mathrm{res}_{\langle a\rangle}$ from $RO(G)$ to $RO(\langle a\rangle)$ is canonically defined by restriction. The restricted map $\mathrm{res}_{\langle a\rangle}|M$ over M is injective from (6.1). The assumption (III) implies that $\mathrm{res}_{\langle a\rangle}(\sum_k(c(k)-c'(k))X_k)$ belongs to

$RO_1(\langle a\rangle)$, by Theorem 2.3. As $\mathrm{res}_{\langle a\rangle}|M$ is injective, we have $\sum_k (c(X_k)-c'(X_k))X_k$ in $RO_1(G)$. Then the conclusion follows from Theorem 2.2.

Proposition 7.2. *Let $V=\oplus_{k\in[p+1:m]}c(\Theta_k)\Theta_k$ and $W=\oplus_{k\in[p+1:m]}c'(\Theta_k)\Theta_k$ for an element m of $D_e(p+1)$. If for $c(\Theta_k)$ and $c'(\Theta_k)$ he condition* (V) *in Theorem* 1.2 *is satisfied, then V and W are J_G-related.*

Proof. Applying the previous argument to b and (6.2) instead of a and (6.1) respectively, we obtain Proposition 7.2.

Proposition 7.3. *Let $V=c(\Xi_1)\Xi_1\oplus c(\Xi_2)\Xi_2$ and $W=c'(\Xi_1)\Xi_1\oplus c'(\Xi_2)\Xi_2$. If for $c(\Xi_1)$, $c(\Xi_2)$, $c'(\Xi_1)$ and $c'(\Xi_2)$ the condition* (VII) *in Theorem* 1.2 *satisfied, then V and W are J_G-related.*

Proof. As $2\Xi_1-2\Xi_2$ belongs to $RO_1(G)$, $2\Xi_1$ and $2\Xi_2$ are J_G-related. Proposition 7.3 follows from this fact.

Proposition 7.4. *Let $V=c(\mathfrak{H}_1)\eta_1\oplus c(\mathfrak{H}_2)\eta_2$ and $W=c'(\mathfrak{H}_1)\eta_1\oplus c'(\mathfrak{H}_2)\eta_2$. If for $c(\mathfrak{H}_1)$, $c(\mathfrak{H}_2)$, $c'(\mathfrak{H}_1)$ and $c'(\mathfrak{H}_2)$ the condition* (VIII) *in Theorem* 1.2 *is satisfied, then $S(V)$ and $S(W)$ are oriented G-homotopy equivalent.*

REMARK. By the definition $\mathfrak{H}_1=r\eta_1$ and $\mathfrak{H}_2=r\eta_2$.

Proof. As $2\eta_1-2\eta_2$ belongs to $R_1(G)$, $S(2\eta_1)$ and $S(2\eta_2)$ are oriented G-homotopy equivalent. Proposition 7.4 follows from this fact.

Proposition 7.5. *Let $V=\oplus_{k\in[p-1:m]}c(X_k)\chi_k$ and $W=\oplus_{k\in[p-1:m]}c'(X_k)\chi_k$ for an element m of $D_o(p-1)$. If for $c(X_k)$ and $c'(X_k)$ the condition* (IV) *in Theorem* 1.2 *satisfied, then $S(V)$ and $S(W)$ are oreinted G-homotopy equivalent.*

Lemma 7.6. *If $\sum_{k\in[p-1:m]}c(k)\chi_k$ belongs to $R_1(G)$ for $m\in D_o(p-1)$, then we have*

$$\sum_{k\in[p-1:m]} (k/m)^{c(k)}\equiv \pm 1 \bmod (p-1)/m\,,$$

where all the coefficients $c(k)$ are integers, and for a positive integer i, $(k/m)^{-i}$ is an integer such that $(k/m)^i(k/m)^{-i}\equiv 1 \bmod (p-1)/m$.

Proof. By the same argument as the proof of Proposition 5.4, this lemma can be obtained.

Lemma 7.7. *Provided $(k/m)^2\equiv 1 \bmod 2(p-1)/m$ for $m\in D_o(p-1)$ and $k\in[p-1:m]$, $S(\chi_m)$ and $S(\chi_k)$ are oriented G-homotopy equivalent.*

Proof of Proposition 7.5. We admit Lemma 7.7 to be true for a moment. By the same argument as the proof of Theorem 5.6, we obtain Proposition 7.5 from Lemmas 7.6 and 7.7.

Proof of Lemma 7.7. We prove Lemma 7.7 using Theorem 2.1 for $V=\chi_k$, $W=\chi_m$ and $d(K)=1$ for all $K<G$. We have to check the Petrie equation (iv) of Theorem 2.1. For $H<G$ such that $S(\chi_m)^H=\phi$, there is no need to check it. Therefore we consider it for $H<G$ such that $S(\chi_m)^H \neq \phi$. Since $S(\chi_m)^{\langle z \rangle}=\phi$, $S(\chi_m)^H \neq \phi$ implies that H does not contain z. By the consideration in section 4, it is sufficient to check the Petrie equation for $H=\{1\}$ and to prove the following assertion.

Assertion 7.8. *If H is one of the subgroups B, $N\langle a\rangle$ and $N\langle b\rangle$, then $S(\mathrm{res}_H \chi_m)$ and $S(\mathrm{res}_H \chi_k)$ are oriented H-homotopy equivalent.*

We prove this assertion later on. Firstly we check the Petrie equation for $H=\{1\}$. That is, we show

$$\textstyle\sum_K \chi(\chi_k-\chi_m;\, K)\equiv 0 \bmod p(p^2-1)\,,$$

where K runs through the cyclic subgroups of G, and $p(p^2-1)=|G|$. From Tables 1 and 2, we obtain

$$\begin{aligned}&\textstyle\sum_K \chi(\chi_k-\chi_m;\, K)\\ &= 1+1+(p^2-1)/2+(p^2-1)/2+(p^2-1)/2+(p^2-1)/2\\ &\quad +p(p+1)\left\{(m-1)/2+(m-1)/2+m\sum_{i=1}^{(p-1)/2m-1}\frac{(1-\rho^{ik})(1-\rho^{-ik})}{(1-\rho^{im})(1-\rho^{-im})}\right\}\\ &\quad +p(p-1)(p-1)/2\,.\end{aligned}$$

Put $h=k/m$, then we have

$$\begin{aligned}&\textstyle\sum_K \chi(\chi_k-\chi_m;\, K)\\ &= 2p^2+p(p-1)^2/2+p(p+1)(m-1)\\ &\quad +mp(p+1)\sum_{i=1}^{(p-1)/2m-1}\{1+\rho^{im}+\cdots+\rho^{im(h-1)}\}\{1+\rho^{-im}+\cdots+\rho^{-im(h-1)}\}\end{aligned}$$

We can show that this value is congruent to 0 mod $p(p^2+1)$, by using the fact that for integers α and β, $\rho^\alpha\rho^{-\beta}=1$ implies $\rho^\beta\rho^{-\alpha}=1$.

By proving Assertion 7.8 we complete the proof of Lemma 7.7.

Proof of Assertion 7.8. First we consider the case that $H=B$. Let ν be an odd integer with $1\leqq\nu<2p$ which represents the generator of F^* defined in section 1. Put $\zeta=\exp(\pi\sqrt{-1}/p)$, $r=\nu^2$, $s=(p-1)/2$, $x=zc$ and $y=a$. For integers i define complex B-representations $T(B, i)$ by the corresponding matrix representations (also denoted by $T(B, i)$) such that

$$T(B, i)(x)=\begin{pmatrix}\zeta^i & & & \\ & \zeta^{ir} & & \\ & & \ddots & \\ & & & \zeta^{ir^{s-1}}\end{pmatrix},$$

$$T(B, i)(y) = \begin{pmatrix} 0 & & & & \zeta^{ip} \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix},$$

where all blancks are zero. Let λ be the homomorphism from B to $\boldsymbol{Z}_{p-1}$ given by

$$\lambda(\begin{pmatrix} \nu^h & 0 \\ * & \nu^{-h} \end{pmatrix}) = h \qquad \text{for } h \in \boldsymbol{Z}_{p-1}.$$

Put $\lambda(j)=\lambda^* v(p-1, j)$ (cf. section 2). Then we have

$$\mathrm{res}_B\, \chi_m = T(B, 1) \oplus T(B, \nu) \oplus \lambda(m) \oplus \lambda(-m),$$
$$\mathrm{res}_B\, \chi_k = T(B, 1) \oplus T(B, \nu) \oplus \lambda(k) \oplus \lambda(-k).$$

Since $v(p-1, m)+v(p-1, -m)-v(p-1, k)-v(p-1, -k)$ belongs to $R_1(\boldsymbol{Z}_{p-1})$ by $(k/m)^2 \equiv 1 \bmod 2(p-1)/m$, $\lambda(m)+\lambda(-m)-\lambda(k)-\lambda(-k)$ belongs to $R_1(G)$. We have proved the case that $H=B$.

If $H=N\langle a\rangle$ or $N\langle b\rangle$, then we obtain Assertion 7.8 from (6.5), (6.6) and Theorem 5.6 easily. Thus we complete the proof of Assertion 7.8.

In the same way as above we get the following result.

Proposition 7.9. *Let* $V=\oplus_{k\in[p+1:m]} c(\Theta_k)\theta_k$ *and* $W=\oplus_{k\in[p+1:m]} c'(\Theta_k)\theta_k$ *for an element* m *of* $D_0(p+1)$. *If for* $c(\Theta_k)$ *and* $c'(\Theta_k)$ *the condition* (VI) *in Theorem* 1.2 *is satisfied, then* $S(V)$ *and* $S(W)$ *are oriented G-homotopy equivalent.*

REMARK. Even if $p \equiv 3 \bmod 4$, Propositions 7.1, 7.2, 7.5 and 7.9 are valid.

Putting all propositions in this section together we see that V and W in Theorem 1.2 are J_G-related if all the conditions (I), (II), $\cdots$, (VIII) are satisfied.

8. The proof of the necessary condition in Theorem 1.2

Let G be $SL(2, p)$, V and W the real G-representations in Theorem 1.2. In this section we discuss that if V and W are J_G-related, the conditions (I), (II), $\cdots$, (VIII) in Theorem 1.2 hold Assume that V and W are J_G-related in this section

By the assumption $V-W$ belongs to $RO_0(G)$. Since $RO_0(G)=I(\Gamma)RO(G)$, we obtain

(I), (II), (III, 0) (IV, 0) (V, 0) (VI, 0), (VII, 0) and (VIII, 0) if $p \equiv 1 \bmod 4$,
(I), (II), (III, 0) (IV, 0), (V, 0), (VI, 0) (VII) and (VIII) if $p \equiv 3 \bmod 4$.

As $V-W$ belongs to $RO_h(G)$, $V^{\langle z\rangle}-W^{\langle z\rangle}$ and $V_{\langle z\rangle}-W_{\langle z\rangle}$ both belong to $RO_h(G)$. If $p \equiv 1$ (resp. 3) mod 4, we have

(8.1) $V^{\langle z\rangle}=c(\boldsymbol{R})\boldsymbol{R}\oplus c(\Psi)\Psi\oplus\bigoplus_{i=1}^{(p-5)/4}c(X_{2i})X_{2i}\oplus\bigoplus_{j=1}^{(p-1)/4}c(\Theta_{2j})\Theta_{2j}\oplus c(\Xi_1)\Xi_1\oplus c(\Xi_2)\Xi_2$
(resp. $V^{\langle z\rangle}=c(\boldsymbol{R})\boldsymbol{R}\oplus c(\Psi)\Psi\oplus\bigoplus_{i=1}^{(p-3)/4}c(X_{2i})X_{2i}\oplus\bigoplus_{j=1}^{(p-3)/4}c(\Theta_{2j})\Theta_{2j}\oplus c(\mathfrak{H})\mathfrak{H})$,

and

(8.2) $V_{\langle z\rangle}=\bigoplus_{i=1}^{(p-1)/4}c(X_{2i-1})X_{2i-1}\oplus\bigoplus_{j=1}^{(p-1)/4}c(\Theta_{2i-1})\Theta_{2i-1}\oplus c(\mathfrak{H}_1)\mathfrak{H}_1\oplus c(\mathfrak{H}_2)\mathfrak{H}_2$
(resp. $V_{\langle z\rangle}=\bigoplus_{i=1}^{(p-3)/4}c(X_{2i-1})X_{2i-1}\oplus\bigoplus_{i=1}^{(p+1)/4}c(\Theta_{2j-1})\Theta_{2j-1}\oplus c(\Xi)\Xi)$,

as real $SL(2, p)$-representation. Since $\mathrm{res}_{\langle a\rangle}(V^{\langle z\rangle}-W^{\langle z\rangle})\in RO_h(\langle a\rangle)$, we obtain (III, 1) from (6.1) and Theorem 2.3. Observing $\mathrm{res}_{\langle b\rangle}(V^{\langle z\rangle}-W^{\langle z\rangle})$, $\mathrm{res}_{N\langle a\rangle}(V_{\langle z\rangle}-W_{\langle z\rangle})$ and $\mathrm{res}_{N\langle b\rangle}(V_{\langle z\rangle}-W_{\langle z\rangle})$, we obtain (V, 1), (IV, 1) and (VI, 1) from Theorems 2.3 and 5.6.

To complete the proof of Theorem 1.2, we show (VII, 1) and (VIII, 1). By Propositions 7.3 and 7.4 we see that it is enough to show the following proposition.

Proposition 8.3. *Neither $\Xi_1-\Xi_2$ nor $\mathfrak{H}_1-\mathfrak{H}_2$ belongs to $RO_h(G)$.*

Proof. Assume that $\Xi_1-\Xi_2$ belongs to $RO_h(G)$. From (6.3) we obtain $\mathrm{res}_{\langle c\rangle}(\Xi_1-\Xi_2)=\sum_{k\in K'_s}V(p, k)-\sum_{k\in K'_r}V(p, k)$. Since $\mathrm{res}_{\langle c\rangle}(\Xi_1-\Xi_2)$ belongs to $RO_h(\langle c\rangle)$, we get

$$\prod_{k\in K'_s}k\equiv\pm\prod_{k\in K'_r}k \bmod p$$

from Theorem 2.3. This, however, contradicts Lemma 3.2.

Next assume that $\mathfrak{H}_1-\mathfrak{H}_2$ belongs to $RO_h(G)$. Since $\mathrm{res}_{B_0}(\mathfrak{H}_1-\mathfrak{H}_2)$ belongs to $RO_h(B_0)$, we get

$$\prod_{k\in K''_s}k^2\equiv\prod_{k\in K''_r}k^2 \bmod 4p$$

from Theorem 5.6. This yields

$$\prod_{k\in K''_s}k\equiv\pm\prod_{k\in K''_r}k \bmod p\,.$$

Consequently we have

$$\prod_{k\in K'_s}k\equiv\pm\prod_{k\in K'_r}k \bmod p\,.$$

This contradicts Lemma 3.2. Thus we complete the proof.

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