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ON GROUPS WITH A STANDARD COMPONENT OF KNOWN TYPE, II

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In [3] we considered those finite groups G having a standard subgroup A, such that $m_2(C_G(A))>1$ and $A/Z(A)$ is of known type. The goal of this paper is to settle certain ambiguities that were not dealt with in [3]. In the case $A\cong G_2(4)$ we showed that G was "of Conway type", although we did not actually prove that $G\cong Co_1$. For the case $A/Z(A)\cong L_3(4)$ we appealed to the results of Nah [7] to conclude that $\langle A^G\rangle\cong Suz$ or He. However, there were errors in [7] which put the results in question. Our main result is the following:

Theorem. *Let A be a standard subgroup of the finite group G. Suppose that $m_2(C_G(A))>1$ and $A/Z(A)\cong L_3(4)$ or $G_2(4)$. Then one of the following holds*:

i) $A\trianglelefteq G$;
ii) $A\cong G_2(4)$ *and* $\langle A^G\rangle\cong Co_1$;
iii) $A\cong L_3(4)$ *or* $SL_3(4)$ *and* $\langle A^G\rangle\cong Suz$ *or* Suz/Z_3; *or*
iv) $A/Z(A)\cong L_3(4)$, $Z(A)\cong Z_2\times Z_2$, *aand* $\langle A^G\rangle\cong He$.

The method of proof is to choose certain 2-groups in $AC_G(A)$ and push-up their normalizers. Eventually, we determine the structure of the centralizer of a central involution at which point we can quote an appropriate recognition theorem.

Throughout the paper we use the following notation. A is a standard subgroup of G, $R\in Syl_2(C_G(A))$ and $m(R)>1$. We assume $A\ntrianglelefteq G$ and that G is a minimal counterexample to this theorem.

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1. Pushing-up and cores

We have $A/Z(A) \cong L_3(4)$ or $G_2(4)$. In the first case let E, F be 2-subgroups of AR such that $R \leq E \cap F$ and such that $Z(A)E/RZ(A)$ and $Z(A)F/RZ(A)$ are the two E_{16} subgroups in a Sylow 2-subgroup of $RA/RZ(A)$. If $A/Z(A) \cong G_2(4)$, let $A_1/Z(A)$ be the subgroup generated by all long root subgroups in a fixed system of root subgroups of $G_2(4)$. Then $A_1/Z(A) \cong SL_3(4)$ and we may choose corresponding subgroups E and F of A_1R.

The first stage of the development of the 2-local structure of G is concerned with the groups $N_G(E)$ and $N_G(F)$. In this section we study these groups and make certain other observations that apply to each of the possible configurations. In later sections we look at individual cases.

(1.1) (i) R is elementary abelian.
(ii) There exists $g \in G - N(A)$ with $R^g \leq C(R)$. For any such g, $R^g \leq AR$.

Proof. The second assertion in (ii) follows from (20.1) of [2]. The rest of (ii) then follows from (3.3) of [3]. Also, (3.2) of [3] gives (i).

(1.2) Let X be a quasisimple group with $Z(X)$ an elementary abelian 2-group and $X/Z(X) \cong L_3(4)$. Let $H/Z(X)$ and $K/Z(X)$ be the E_{16} subgroups in a Sylow 2-subgroup of $X/Z(X)$. Then
(i) H and K are elementary abelian;
(ii) $H \cap K = Z(HK)$; and
(iii) $N_X(H)$ (resp. $N_X(K)$) is the split extension of H (resp. K) by $L_2(4)$.

Proof. $N_X(H)/Z(X)$ is the split extension of $H/Z(X)$ by $L_2(4)$, and $H/Z(X)$ is the natural module for $L_2(4)$. In particular, $N_X(H)$ is transitive on $(H/Z(X))^\#$. Thus, each coset of $Z(X)$ in H consists of involutions. This proves (i). (ii) follows from (i) and the fact that $(H \cap K)/Z(X) = Z(HK/Z(X))$. (iii) holds since a Sylow 2-subgroup of a complement to $H/Z(X)$ in $N_X(H)/Z(X)$ is conjugate to $(H \cap K)/Z(X)$.

(1.3) (i) $R \cong E_4$.
(ii) $R^g RZ(A)/RZ(A)$ is a root subgroup of $RA/RZ(A)$, and for suitable choice of g, it is a long root subgroup.
(iii) If $A/Z(A) \cong G_2(4)$, then $Z(A) = 1$.
(iv) If $|Z(A)|$ is odd, then $R^g \cap A = 1$ provided R^g projects to a long root subgroup of $A/Z(A)$.

Proof. (i) follows from (ii). Suppose $A/Z(A) \cong L_3(4)$. Choose $g \in G - N(A)$ with $R^g \leq AR$, and let $1 \neq x \in R^g$. By (1.2) we have x central in a Sylow 2-

subgroup, say D, of AR. Then $D \leq N(C(A^g))$. As D is generated by elementary subgroups of order $2^4|R|$, we conclude that $D \leq A^g R^g \leq C(R^g)$. (ii) follows. Suppose that $|Z(A)|$ is odd. Then $D \in \mathrm{Syl}_2(AR) \cap \mathrm{Syl}_2(A^g R^g)$ and $D' = Z(D) \cap A = Z(D) \cap A^g$. Consequently, (iv) holds.

Suppose $A/Z(A) \cong G_2(4)$. Then (iii), (ii), and (iv) follow from (8.3), (8.9), and (8.6) of [3], respectively.

(1.4) NOTATION. If $A/Z(A) \cong L_3(4)$, let $A_1 = A$. If $A/Z(A) \cong G_2(4)$, then $Z(A) = 1$ and we let A_1 be the group generated by all long root subgroups in a fixed system of root subgroups of A. In either case A_1 is quasisimple and $A_1/Z(A_1) \cong L_3(4)$. In the second case $A_1 \cong SL_3(4)$. Choose a fixed Sylow 2-subgroup of A_1R and let E/R and F/R be the corresponding E_{16} subgroups. By (1.2) and (1.3) $E \cong F \cong E_{64}$ and $E \cap F = Z(EF)$. Moreover, we may take $g \in G$ such that $E \cap F = R \times R^g$.

Let $\Omega = E^G \cup F^G$. We will refer to elements of Ω as *planes*, elements of R^G as *points*, and elements of $(E \cap F)^G$ as *lines*.

(1.5) Suppose that $|Z(A)|$ is odd. Then

(i) $E - A$ is partitioned by its 16 points.

(ii) $N(E) = P_0(N(E) \cap N(R))$, with $P_0 \trianglelefteq N(E)$ and $P_0/C_{P_0}(E) \cong E_{16}$, regular on the 16 points of E. $P_0 = O(C_G(E)) \times O_2(P_0)$.

Proof. By (1.3) (iv) and (3.6) of [3], $E \cap F$ contains 4 points and the non-identity elements of these points partition $(E \cap F) - A$. Now E contains 5 lines that contain R, these being conjugate under $N_{A_1}(E)$. This proves (i).

Since $R^g \cap A_1 = 1$, $N_{A_1}(E)$ is transitive on the 15 points of E, other than R. Since $E \cong E_{64}$, $E \leq A^g R^g$ and $N_{A^g}(E)$ is transitive on the 15 points of E other than R^g. Thus, $N(E)$ is 2-transitive on the 16 points in E. The 16 points and 20 lines in E form an affine plane, so all but the last sentence of (ii) follows from Theorem 1 of [8]. $P_0 = [N_A(E), P_0]C_{P_0}(E)$ and $C_{P_0}(E) = EO(C_G(E))$ with $[O(C_G(E)), N_A(E)] \leq [O(N_G(R)), N_A(E)] = 1$, so $P_0 = O(C(E)) \times O_2(P_0)$.

(1.6) Suppose $|Z(A)|$ is even. Then

(i) $R \leq A$.

(ii) E contains 6 points.

(iii) $N(E)/C(E)$ contains $\hat{A}_6$, the 3-fold cover of A_6, as a normal subgroup.

(iv) There is a 3-element acting as an outer diagonal automorphism of A and transitive on $R^\sharp$.

Proof. By (3.6) of [3] $E \cap F$ contains either 4 points or 2 points. In the first case we argue as in (1.5) to conclude that $N(E)$ is 2-transitive on the 16 points of E and there exists $D \trianglelefteq N(E)$ with D inducing a regular normal subgroup on $R^G \cap E$. Then $[D, E] \trianglelefteq N_G(E)$ and one checks that $E = [D, E] \times R$. But then EF

splits over R, contradicting $|Z(A)|$ even. Therefore, $E\cap F$ contains exactly 2 points, E contains exactly 6 points, and (ii) holds.

Let $L\in \mathrm{Syl}_3(N_A(EF))$. Then $L\leq N_G(E\cap F)$, so L stabilizes each of the two points in $E\cap F$. Therefore, $R^g=[L, E\cap F]$. By symmetry (iv) holds, and since $|Z(A)|$ is even, $R\leq A$, proving (i). Now $N(E)\cap N(R)$ contains a subgroup inducing $A_5\times Z_3$ on E, where the Z_3 factor stabilizes each point in E. Since $N_{A^g}(E)$ moves R, we conclude that $N(E)$ induces S_6 or A_6 on the points of E. Since $O^2(N(E))$ acts irreducibly on E as an $\boldsymbol{F}_2$-space, and since $N(E)/C(E)$ contains a normal subgroup of order 3, we see that E may be regarded as 3-dimensional $\boldsymbol{F}_4$-space for either $3\cdot A_6$ or $A_6\times Z_3$. But $SL_3(4)\not\geq A_6\times Z_3$, so the latter case is not possible. This proves (iii).

(1.7) Let $X\in N_G^*(E, 2')$ and $Y=\langle A^{N(X)}\rangle$. Then either

(i) $X=1$; or

(ii) $Y/Z(Y)\cong Suz$, He, or Co_1, and $X=O(C_G(A))$.

Proof. Suppose $X\neq 1$. Then $X=\Gamma_{1,R}(X)\leq N(A)$, and since $И^*_{N(A)}(E, 2')=\{O(C(A))\}$, $X=O(C(A))$. Similary, $X=O(C(A^g))$ for each $g\in N(E)$. As $N_G(E)\not\leq N(A)$, (ii) holds by minimality of $|G|$.

(1.8) Suppose G contains a 2-central involution, z, such that $(C_G(z)/OC_G(z)))^{(\infty)}$ is isomorphic to the centralizer of a 2-central involution in one of the groups Suz, He, or Co_1. Then $O(C_G(z))=1$.

Proof. We may assume that $z\in E$ is a 2-central involution in $N(A)$, and as $C_G(z)^{(\infty)}$ is 2-constrained, z is not conjugate to an involution in R. As $E\leq N(O_G(C(z))$, (1.7) imples that $O(C_G(z))\leq O(C_G(L))$. Suppose $O(C_G(z))\neq 1$, let $X=O(C_G(A))$ and $Y=\langle A^{N(X)}\rangle$. Then $[X, Y]=1$.

Suppose $R\leq N(Y^g)$. As $|\mathrm{Aut}(Y^g)\colon Y^g|\leq 2$, $R\cap Y^g$ contains an involution, r. Then $E(C_Y(r))\cong A$, so that $X^g\leq C(AR)$. Thus $X=X^g$ and $Y=Y^g$. That is, R fixes precisely one point in Y^G. Now suppose $z\in N(Y^g)$. Then z centralizes a Y^g-conjugate of R^g, and it follows from Gleason's lemma that $\langle R^{C_G(z)}\rangle$ is transitive on the elements of Y^G fixed by z. But $\langle R^{C_G(z)}\rangle\leq Y$. So z fixes a unique element of Y^G and the result follows from Holt's Theorem [6].

For the remainder of this section we operate under the following hypotheses:

(1.9) (i) z is a 2-central involution in G;

(ii) There is an extraspecial subgroup $X\leq C_G(z)$ such that $|X|=2^7$ or 2^9 and $\langle z\rangle\in \mathrm{Syl}_2(C(X))$;

(iii) X is weakly closed in a Sylow 2-subgroup of $C_G(z)$, with respect $C_G(z)$; and

(iv) If $g\in C_G(z)$ and $m(X\cap X^g)>1$, then $X=X^g$.

(1.10) Assume Hypothesis (1.9). Then X is strongly closed with respect to $C_G(z)$ in a Sylow 2-subgroup of $C_G(z)$.

The proof of (1.10) will be carried out in a series of steps. Assume the result to be false.

(1.11) There exists $g \in C_G(z)$ such that setting $Y=\langle X, X^g\rangle$, $B=N_X(X^g)$, $D=N_{X^g}(X)$, and $I=X\cap X^g$, the following hold:
 (i) $Y/BD\cong L_2(2^n)$, $Sz(2^n)$, or D_{2n} for n odd;
 (ii) BD/I is the sum of natural modules for Y/BD; and
 (iii) $I<D$.

Proof. Use (2.4) of [12].

(1.12) $I\cong Z_2$, Z_4, or Q_8.

Proof. This is (iv) of Hypotheses (1.9).

(1.13) $I\not\cong Z_2$.

Proof. Suppose otherwise and let bars denote images in $C(z)/\langle z\rangle$. We have $m(\bar{X})=m(\bar{B})+m(\bar{X}/\bar{B})=m(\bar{D})+m(\bar{X}/C_{\bar{X}}(\bar{D})$. Also, $m(\bar{D})\geq m(\bar{X}/\bar{B})=m(\bar{X}/C_{\bar{X}}(\bar{D}))$. For $\bar{d}\in\bar{D}^{\#}$, $[\bar{X},\bar{D}]=\bar{B}=C_{\bar{X}}(\bar{d})$, so by (7.6) of [2], B is abelian. We conclude from these facts that either $|X|=2^7$ with $m(\bar{D})=3$, or $|X|=2^9$ with $m(\bar{D})=4$. The first case is out since this would force each $1\neq\bar{d}\in\bar{D}$ to act on $\bar{X}$ as a b_3 involution of $O_6^{\pm}(2)$, whereas $\Omega_6^{\pm}(2)$ contains no such involutions. Hence $|X|=2^9$.

Now $Y/BD\cong L_2(2^4)$ and BD/I is the natural module, so there exists a subgroup $J\leq Y$ such that J induces Z_{15} on each of $\bar{B}$, $\bar{D}$, and $\bar{X}/\bar{B}$. Viewing $J\leq \mathrm{Aut}(\bar{X})$, we see that $\mathrm{Aut}(\bar{X})/\mathrm{Inn}(\bar{X})\cong O_8^+(2)$, $\bar{B}$ is a singular 4-space of $\bar{X}$, and $\bar{D}$ is contained in the unipotent radical of the stabilizer in $O_8^+(2)$ of $\bar{B}$. Let T be this unipotent radical. Then $T^{\#}$ consists of 28 a_4 involutions and 35 remaining involutions of type a_2. Also, $T=D\times D_1$, where $D_1\cong E_4$ and J induce Z_3 on D_1. Therefore, $D_1^{\#}$ consists of the 3 a_4 involutions fixed by $O_5(J)$ and J acts semiregularly on the a_2 involutions in T. This is numerically impossible.

(1.14) (i) $Y=C_Y(I)\circ I$ if and only if $I\cong Q_8$.
 (ii) $O^2(Y)\leq C(I)$.
 (iii) If $Y=O^2(Y)I$, then $I\cong Q_8$.

Proof. If $Y=C_Y(I)I$ and $I\cong Z_4$, then $X\leq Y\leq C(I)$, a contradiction. On the otherhand, if $I\cong Q_8$, then $Q=C_Q(I)I$, so $Y=C_Y(I)I$. Thus (i) holds. (iii) follows from (i) and (ii), and (ii) follows from the fact that Y centralizes both $\bar{I}$ and $\langle z\rangle$.

(1.15) $|X: B|=2$.

Proof. Suppose false. Then Y/BD is a Bender group and $Y=O^2(Y)I$. By (1.14) (iii) $I\cong Q_8$, and by (1.14) (i) $Y=C_Y(I)I$. Set $W=C_Y(I)$ and $V=W\cap X$. Then $m(\bar{V})=4$ or 6, and one of the following holds:

(a) $|\bar{X}|=2^6$, $W/O_2(W)\cong L_2(4)$, and $O_2(\bar{W})$ the natural module; or
(b) $|\bar{X}|=2^8$, $W/O_2(W)\cong L_2(8)$, and $O_2(\bar{W})$ is the natural module; or
(c) $|\bar{X}|=2^8$, $W/O_2(W)\cong L_2(4)$, and $O_2(\bar{W})$ is the sum of two copies of the natural module.

Set $E=D\cap W$ and consider the action of $\bar{E}$ on $\bar{X}$. Since $E\leq C(I)$, either $\bar{E}\leq O_4^{\pm}(2)$ or $\bar{E}\leq O_6^{\pm}(2)$, according to $|\bar{X}|=2^6$ or 2^8. If (b) holds, then $\bar{E}$ consists of b_3 involutions in $O_4^{\pm}(2)$, whereas $\Omega_4^{\pm}(2)$ contains no b_3 involutions. If (c) holds then $\bar{E}\cong E_{16}$ and $\bar{E}\leq C(\bar{B})$. Since $\bar{B}$ is a 4-space in the 6-space $\bar{V}$, $\bar{E}$ centralize a proper non-degenerate subspace of $\bar{V}$. However, $m(O^{\pm}(l,2))<4$ for $l<6$. Therefore, (c) does not hold. Suppose (a) holds. Then $O_2(W)\cong E_{32}$, $B\cap W\cong E_8$, and we may regard $\bar{E}\leq O_4^+(2)$. Then each $\bar{e}\in\bar{E}^\#$ is an a_2 involution in $O_4^+(2)$, and so $\bar{E}\leq\Omega_4^+(2)\cong S_3\times S_3$. But then $\bar{E}$ is a Sylow 2-subgroup of $\Omega_4^+(2)$, whereas $\Omega_4^+(2)$ contains c_2 involutions. This is a contradiotion.

(1.16) $I\cong Z_4$.

Proof. Otherwise $I\cong Q_8$ and by (1.15) $m(D\bar{X}/\bar{X})=3$ or 5, according to whether $|\bar{X}|=2^6$ or 2^8. By (1.14) (i), $\bar{D}$ centralize $\bar{I}$, so $\bar{D}\leq O_4^{\pm}(2)$, or $O_6^{\pm}(2)$, respectively. But $m(O_4^{\pm}(2))=2$ and $m(O_6^{\pm}(2))=4$. This is impossible.

(1.17) $I\not\cong Z_4$.

Proof. Suppose $I\cong Z_4$. Then by (1.15), $m(D/I)=m-2$, $m=m(\bar{X})$ while by (1.11), $B/I=C_{X/I}(D)$. This is impossible as $(\mathrm{Aut}(X)\cap N(I))/C(X/I)\cong Sp_{m-2}(2)$ is of 2-rank $m-3$.

In view of (1.16) and (1.17), the proof of (1.10) is now complete.

2. Suz

In this section we assume that $|Z(A)|$ is odd and $A/Z(A)\cong L_3(4)$. That is $A\cong L_3(4)$ or $SL_3(4)$. We maintain the notation of §1. In addition, we set $P=O_2(P_0)$, where P_0 is as in (1.5). Set $Z=A\cap Z(EF)$ and $S=FC_P(RZ/Z)$.

(2.1) (i) $E=C_{PF}(E)$;
(ii) $P/E=O_2(N_G(E)/E)\cong E_{16}$ and $P_0=P\times O(C_G(E))$, so $P=O_2(N_G(E))$.
(iii) $(S\cap P)/E\cong E_4$; and

(iv) $S/E \cong E_{16}$

Proof. These are all clear, given 1.5.

(2.2) (i) $S = N_{PS}(F)$;
(ii) $|F^P| = 4$.
(iii) S is a Sylow 2-subgroup of $C(Z) \cap C(RZ/Z) \cap N(E) \cap N(F)$.
(iv) $|\langle (F \cap A)^P \rangle| \geq 4^4$.

Proof. Since $S/E \cong E_{16}$, $EF \trianglelefteq S$. The groups E and F are the unique subgroups of EF isomorphic to E_{64}, and $S \leq N(E)$. Therefore, $S \leq N(F)$. (i) follows from this and the fact that $S/E = N_{PS/E}(FE/E)$. (ii) follows from (i). Let $S \leq T$, with T Sylow in $C(Z) \cap C(RZ/Z) \cap N(E) \cap N(F)$. As S is transitive on the points in RZ, $T \leq SN_T(R)$. But $N_T(R) = EF$, so (iii) holds.

To obtain (iv) let $T = \langle (F \cap A)^P \rangle$. Since $TE/E = S/E \cong E_{16}$, it will suffice to show that $E \cap A \leq T$. Suppose otherwise and let $W = [P, I]$, where $I \in \mathrm{Syl}_3(N_A(EF))$. $P/(E \cap A)$ is abelian since $N_A(E)$ is transitive on $(P/E)^\sharp$. Thus $|W| = 4^4$ and $W \cap R = 1$. As $Z \leq T$ and T is I-invariant, $T \cap (E \cap A) = Z$ and $T = (F \cap A)W_1$, where $W_1 = T \cap W$. As I acts irreducibly on W_1/Z and on Z, W_1 is abelian. Also $W_1 = T \cap W \trianglelefteq W$. Choosing an appropriate conjugate of F we obtain $W_2 \in W_1^{N(E)}$ with $W_2 \trianglelefteq W$ and $W_1 \cap W_2 = 1$. Therefore, W is abelian.

We show W is elementary abelian as follows. Let $f \in (F \cap A) - Z$. Let $g \in P$ such that $f^g = fw_1$, with $w_1 \in W_1 - Z$. As f^g is an involution, f inverts w_1. If W is not elementary, then $|w_1| = 4$ an**Z** letting g vary, f inverts W_1. Now let f vary and obtain a contradiction.

Consider $N = N(W)$ and let bars denote images in N/W. The involutions in WR are in $W \cup E$, so $R^G \cap WR = R^W$. We conclude that $\bar{N}$ has a standard subgroup $\bar{L} \cong L_2(4)$ with $\bar{R} \in \mathrm{Syl}_2(C_{\bar{N}}(\bar{L}))$. By [1], $E(\bar{N}) \cong L_2(4)$, A_9, HJ, or M_{12}. As $|W| = 2^8$ and 11 does not divide $|GL(8,2)|$, $E(\bar{N}) \not\cong M_{12}$. Suppose $E(\bar{N}) \cong A_9$. Then $\bar{R} \sim \bar{F}$ in $\overline{N(W)}$ and it follows that $R^G \cap A \neq \emptyset$, which is not the case. Next, suppose $E(\bar{N}) \cong HJ$. For $f \in (F \cap A) - Z$, we have $[f, W] = W_1 = C_W(f)$, and $\bar{f}$ is a 2-central involution of $E(\bar{N})$. Viewing $\bar{N} \leq \mathrm{Aut}(W)$ we then have $E(\bar{N}) = \langle C_{\bar{N}}(\bar{f}) \mid f \in (F \cap L) - Z \rangle \leq N(W_1)$. This is impossible.

We are left with the case $E(\bar{N}) \cong L_2(4)$. Clearly, W is weakly closed in a Sylow 2-subgroup of $N(W)$, and applying Theorem 4 of [5] we conclude that W is strongly closed in a Sylow 2-subgroup of C. The main theorem of [5] gives a contradiction.

Define $P(F) = O_2(N_G(F))$, so that (P, E) is symmetric to $(P(F), F)$. By 2.2 (i) and (iii), $S = FC_P(RZ/Z) = EC_{P(F)}(RZ/Z)$.

(2.3) Let $x \in P(F) - S$, $F_0 = (E \cap A)(E^x \cap P)$, and $H = \langle P, P(F) \rangle$. Then
(i) $E^x \cap E = Z$ and $S = EE^x$

(ii) $P\cap S=EF_0$ and E and F_0 are the maximal elementary abelian 2-subgroups of $P\cap S$. Also $E\cong F_0$.

(iii) $F^H=\{F_0, F^P\}$ and $E^H=\{E, (E^x)^P\}$.

(iv) $\Omega\cap S=F^H\cup E^H$ and $N_G(S)$ act on $\{F^H, E^H\}$.

(v) H induces A_5 on E^H.

Proof. Let $h\in P-S$. $F\cap F^h\cap E=Z$ and $F\cap F^h\leq E$, so $F\cap F^h=Z$. Then $|S|=|FF^h|$, so $S=FF^h$. So (i) follows from (2.2) (iii) which guarantees symmetry between E and F. (i) implies (ii).

If $U\cap P\neq 1$ for some point U in E^x, then $U\cap F_0\neq 1$, so as $m(F_0)=6$, $U\leq F_0$ and F_0 is a plane. On the otherhand if $U\cap P=1$ for each point U in E^x and each $x\in P(F)-S$, then $\langle(E\cap A)^{P(F)}\rangle=F_0$ is of order 64, contradicting 2.2 (iii) and (iv).

So F_0 is a plane. By (1.3) $E\cap A$ intersects each point of G trivially, and so $F_0-(E\cap A)$ is partitioned by its points and $E\cap A=F_0\cap A^y$ for each point $R^y\leq F_0$. $F_0E\trianglelefteq P$ so by (ii), $F_0\trianglelefteq P$. Then $P\leq O^{2'}(C(F_0\cap A^y)\cap N(F_0))=P(F_0)$, so $P=P(F_0)$.

Let V be a plane in S. If $V\leq P$, then $V=E$ or F_0 by (ii). Suppose $V\not\leq P$. $V=O^{2'}(C_G(V))$, so $Z\leq V$. As $V\not\leq P$ and $P=C_{SP}(e)$ for $e\in(E\cap A)-Z$, $V\cap(E\cap A)=Z$. If $V\cap E\neq Z$, then V contains some point R^j of E, for $j\in P$. Then $RZ\leq V^{j^{-1}}$, so $V\in F^P$. This leaves the case $V\cap E=Z$. The involutions in $S\cap P$ are $F_0^\#\cup E^\#$. Hence $|F_0: V\cap F_0|=4$, and as F_0-E is partitioned by its points, $V\cap F_0$ is a line. However, P is transitive on the lines in F_0, through Z, so $V\cap F_0\in(E^x\cup F_0)^P$. It follows that $V\in(E^x)^P$. It has now been shown that

$$\Omega\cap S=\{E, F_0\}\cup F^P\cup(E^x)^P\,.$$

Notice that $(E^x)^P$ is precisely the set of $V\in S\cap\Omega$ such that $V\cap E=Z$, while $F_0\cap F=Z$. By symmetry between E and F, $\{F_0\}\cup F^P=(F)\cup(F^h)^{P(F)}$, for $h\in P-S$. Therefore, $\{F_0\}\cup F^P=F^H$. By symmetry, $E^H=\{E\}\cup(E^x)^P$, and so (iii) and (iv) hold. (v) follows from (iii).

(2.4) S is special with $Z(S)=Z$.

Proof. $E/Z\leq Z(S/Z)$, so by (2.3) (i), $[S,S]\leq Z$. $[R,S]=Z$ so $[S,S]=\Phi(S)=Z$. $Z(S)\leq C_S(R)=EF$ with $C_E(S)=Z$, so the lemma holds.

(2.5) $Z(SP/Z)=(E\cap A)/Z$.

Proof. Set $SP/Z=\overline{SP}$. Then $Z(SP/E)=(S\cap P)/E$ so $Z(\overline{SP})\leq(S\cap P)/Z$. $C_{\bar{E}}(P)=C_{\bar{F}_0}(P)=(E\cap A)/Z$, since P is transitive on the lines through Z on E and F_0. On the otherhand if $x\in N_A(E)$ is of order 3 then $C_{S\cap P}(x)=R$ and $[S\cap P, x]=F_0$, so as $C_{\bar{R}}(\overline{SP})=1$, $Z(\overline{SP})=[Z(\overline{SP}), x]\leq\bar{F}_0$. Therefore $Z(\overline{SP})=$

$C_{\bar{F}_0}(\bar{P})=(E\cap A)/Z.$

(2.6) Choose notation as in (2.3) and set $\bar{S}=S/Z$ and $A(S)=\mathrm{Aut}(S)/C_{\mathrm{Aut}(S)}(\bar{S})$. Then

(i) S is the central product of two copies of the Sylow 2-group of $L_3(4)$.

(ii) $\bar{S}$ is an orthogonal space over $GF(4)$ with $(\bar{s},\bar{t})=0$ if and only if $[s,t]=1$ and $\bar{s}$ singular if and only if $s^2=1$. $\mathrm{Aut}(S)\cap C(Z)$ preserves this structure and $C_{A(S)}(Z)\cong O_4^+(4)$. $A(S)$ is $Z_3\times C_{A(S)}(Z)$ extended by a field automorphism of order 2, with $O_3(A(S))$ inducing scalar action on $\bar{S}$ corresponding to a generator of $GF(4)^\#$. $C_{\mathrm{Aut}(S)}(\bar{S})=V=\bar{S}\times U$, where $\bar{S}\cong U=C_V(O_3(A(S))\trianglelefteq\mathrm{Aut}(S)$ and for $z\in Z^\#$, the map $\bar{s}\to C_U(s\langle z\rangle)$ is a $C_{A(S)}(Z)$-isomorphism of $\bar{S}$ with the dual of U.

(iii) $H/S\cong A_5$ and $C_H(S)=Z\in\mathrm{Syl}_2(C_G(S))$ and $S\in\mathrm{Syl}_2(C_G(\bar{S}))$.

(iv) H is irreducible on $\bar{S}$ as a $GF(4)$-module.

(v) $\bar{S}$ is the sum of two natural modules for $S/H\cong A_5$, as a $GF(2)$-module.

(vi) $H\trianglelefteq N_G(S)$.

Proof. Let $S_0=\langle E\cap A, F\cap A\rangle$ and $S_1=\langle I,R\rangle$, where I is $F_0\cap C(F\cap A)$. Clearly S_0 is isomorphic to a Sylow 2-subgroup of $L_3(4)$ and this also holds for S_1 as $S_1=IR$ and $[i,R]=Z=Z(S_1)$ for $i\in I-Z$. Moreover, S is the central product of S_0 and S_1, proving (i). (i) implies (ii); the first two sentences of (ii) are reasonably clear; we supply a proof of the rest. Let $S=T_1*T_2$ with, $T_i\cong S_0$. Let $E_{16}\cong X_{ij}\le T_i$, $i,j\in\{1,2\}$. Each $v\in V^\#$ acts faithfully on some X_{ij}, say X. As $[v,S]\le Z$, $v\in C(Z)$. This determines $V/C_V(X)$ in $GL(X)\cong L_4(2)$, and we find $V/C_V(X)\le E_{16}$, and hence $|V|\le 2^{16}$. On the otherhand in the split extension of X_{ij} by $L_4(2)$ there is U_{ij} with $[U_{ij},X_{i3-j}]=1=U_{ij}\cap T_i=[U_{ij},y_{ij}]$, $[U_{ij},T_i]\le Z$, and $U_{ij}\cong E_4$, where y_{ij} is of order 3 with $C_{T_i}(y_{ij})=1$. Embed U_{ij} in $\mathrm{Aut}(S)$ by taking $[U_{ij},T_{3-i}]=1$; set $U=\langle U_{ij}: i,j\rangle$. $[U_{ij},U_{rs}]\le C(T_1)\cap C(T_2)=1$ for $(i,j)\neq(r,s)$, so U is elementary abelian. Similarly $U\cong E_{2^8}$ and $U\cap\bar{S}=1$. So $U\bar{S}\cong E_{2^{16}}$ and as $|V|\le 2^{16}$, $V=U\bar{S}$. Let y of order 3 with $\langle y\rangle V/V=O_3(A(S))$. Then $\langle y\rangle V/C_V(X_{ij})=\langle y_{ij}\rangle V/C_V(X_{ij})$, so $[y,U]=1$ and hence $U=C_V(y)\trianglelefteq\mathrm{Aut}(S)$. Finally let $z\in Z^\#$. If $s\in S$ with $[U,s]\le\langle z\rangle$, then as $C_{\mathrm{Aut}(S)}(z)$ is irreducible on $\bar{S}$, $[U,S]\le\langle z\rangle$, a contradiction. Thus $|U: C_U(s\langle z\rangle)|=2$, completing the proof of (ii).

Since $E\in\mathrm{Syl}_2(C_G(E))$, $Z\in\mathrm{Syl}_2(C_G(S))$. $C_G(\bar{S})\le N_G(R)S$ and $N_S(R)=EF\in\mathrm{Syl}_2(C_G(EF/Z)\cap N(R))$ so $S\in\mathrm{Syl}_2(C_G(\bar{S}))$. Thus $C_H(S)=XZ$, where $X=O(C_H(S))$. By (1.7) $X\le Z(H)$. We have $|PS/S|=4$ and $PS/S=[PS/S,u]$, when u is a 3-element in $N_A(S)$. So by (ii) together with (2.3) (v) and $H=O^{2'}(H)$, we have $H/S\cong A_5$. Therefore, (iii) holds.

By (ii) one of the following holds: H/S stabilizes a nonsingular 1-space of $\bar{S}$, H/S stabilizes a pair of complementary totally singular 2-spaces of $\bar{S}$, or H/S is irreducible on $\bar{S}$. The first two cases do not occur because of (2.5). Therefore, (iv) holds, and (iv) implies (v). Finally, (vi) follows from (2.3) (iv) and (1.7).

(2.7) Choose $u\in N_A(S)$ with $|u|=3$ and $[E, u]\neq 1$, and let $y\in N_H(R)\cap C(u)$ with $|y|=3$. Then $u=xy^{\pm 1}$, where $|x|=3$, x induces scalar action on S/Z as an $\boldsymbol{F}_4$-module, and $Z=[Z, x]$.

Proof. $Z=[Z, u]$ and $y\in C(Z)$, so $u\neq y$. Also, u acts on H and acts nontrivially on PS/S. Hence $u=xy^i$ w th x of order 3 in $C(H/S)$ and $i=\pm 1$. By (2.6) (v) $H\langle x\rangle$ acts irreducibly on S/Z as an $\boldsymbol{F}_2$-module, so Schur's lemma shows that x induces an $\boldsymbol{F}_4$ scalar on S/Z.

(2.8) Let $T_0\in \mathrm{Syl}_2(N_G(S))$ and $\bar{T}_0=T_0/Z$. Then
(i) $\bar{S}=J(\bar{T}_0)$;
(ii) $T_0\in \mathrm{Syl}_2(G)$; and
(iii) $Z\trianglelefteq N_G(T_0)$.

Proof. By 2.6. iii, $S=C_{T_0}(\bar{S})$. Thus if (i) fails there is a nontrivial elementary abelian 2-subgroup U of $\mathrm{Aut}_G(\bar{S})$ with $|U|\geq|\bar{S}: C_{\bar{S}}(U)|$, which is impossible from the structure of $\mathrm{Aut}(S)$ described in 2.6. ii.

Let $g\in N_G(T_0)$. We claim $Z^g=Z$. Either $Z=Z(T_0)$, in which case the claim is clear, or $|Z: Z(T_0)|=2$.

In the latter case, $Z(T_0)\leq Z^g$ and $Z^g/Z(T_0)\leq Z(T_0/Z(T_0))$. But using (i) and 2.6 (i), we see that $Z/Z(T_0)=Z(T_0/Z(T_0))$. This proves the claim, and so (ii) follows from (i).

(2.9) (i) $P\cap\Omega=\{E, F_0^{N(E)\cap N(P)}\}$ has order 6.
(ii) $P\in \mathrm{Syl}_2(C_G(E\cap A))$.
(iii) $N_G(P)$ is transitive on $P\cap\Omega$.

Proof. Let $V\in P\cap\Omega$ and B a point of V. Conjugating by $N(E)\cap N(P)$ we may take $B\cap S\neq 1$. Then $B\cap S\leq E$ or $B\cap S\leq F_0$ by (2.3) (ii). As each elementary subgroup of $N(R)$ of rank 6 is a plane through R, $B\leq E$ or $B\leq F_0$, so $V=(E\cap A)B=E$ or F_0. Hence (i) holds.

Clearly $P\in \mathrm{Syl}_2(C_G(E\cap A)\cap N(E))$. So if (ii) is false there is a 2-element $g\in N(P)\cap C(E\cap A)$ such that $E^g\neq E$. Therefore, $N(P)^{(P\cap\Omega)}=A_6$ or S_6. Let $I=N(P)\cap C(E\cap A)$. Then $I^{(P\cap\Omega)}\neq 1$ and is normal in $N(P)^{(P\cap\Omega)}$. So, $I^{(P\cap\Omega)}\geq A_6$ and this forces $S\leq I$, a contradiction. This proves (ii). (iii) now follows from (i), (ii), and the symmetry between E and F_0.

(2.10) $\Omega=E^G$.

Proof. See (2.9) and (2.3) (iii).

(2.11) Set $K=O^2(N_G(P))$. Then

(i) $K/PO(K)\cong 3A_6$.
(ii) $[y, K]\leq PO(K)$.
(iii) $P/(E\cap A)$ is the natural module for $K/PO(K)$.
(iv) $E\cap A$ is the natural module for $K/PO(K)\langle y\rangle\cong A_6$.

Proof. $N_K(E)^{(P\cap\Omega)}\geq A_5$, so by (2.9) $K^{(P\cap\Omega)}=A_6$. $N_K(E)\neq(N_K(E)\cap C(E\cap A))K_{P\cap\Omega}$ so $K\neq C_K(E\cap A)K_{P\cap\Omega}$. Hence $K/C_K(E\cap A)\cong A_6$ acts naturally on $E\cap A$.

$(KP)_{P\cap\Omega}=P(N_{KP}(R)_{P\cap\Omega})$ while $(N_{KP}(R)_{P\cap\Omega})/O(K)R$ acts faithfully on RZ, and hence is a subgroup of E_9. Thus $KP/PO(K)$ is a subgroup of $A_6\times E_9$ or of $3A_6\times Z_3$. Choose y as in 2.7. $y\in N_H(R)\leq N(E)\leq N(P)$, while by 2.6 parts (ii) and (v), $(E\cap A)/Z=[P, E/Z]\leq C_{P/Z}(y)$ and $C_{P/Z}(y)$ is a complement to R in $C_S(Z)$. Thus $[y, K]\leq PO(K)$, so $P/E\cap A$ is a faithful $GF(4)$-module for $K/PO(K)$, so $K/PO(K)\leq GL_3(4)$. Then as $K/PO(K)\leq A_6\times E_9$ or $3A_6\times Z_3$, the lemma holds.

(2.12) Let $PS\trianglerighteq T_0\in \mathrm{Syl}_2(N_G(S))$. Then
(i) $T_0\in\mathrm{Syl}_2(G)$;
(ii) $SP\leq T=T_0\cap O^2(N_G(P))$, $|T_0: T|\leq 2$, and $H\langle x\rangle T/S\cong S_3\times A_5$;
(iii) $Z_2(T)=Z\neq Z(T)$;
(iv) $E^T=\{E, F_0\}$; and
(v) $P\trianglelefteq T_0$.

Proof. (i) is just (2.8) (ii). $(E\cap A)/Z=Z(PS/Z)$, so $E\cap A\trianglelefteq T_0$. Thus (v) follows from (2.9) (ii). By (2.11) and (1.7), $O^2(N_G(P))=I\times O(C(R))$ where $y\in I$ is the split extension of P by A_6/Z_3. Let J be the setwise stabilizer in $O^2(N_G(P))$ of $\{E, F_0\}$. $PS\leq T=T_0\cap J\in\mathrm{Syl}_2(J)$, while with (2.11) (ii), $\langle y\rangle(N_A(S)\cap N(P))O(C(A))$ contains a Hall $2'$-group of J, so $J\leq N(S)$. $J/O_2(J)O(C(A))\cong Z_3\times S_3$ with $[y, J]\leq O_2(J)O(C(A))$, so $JH/SO(C(R))=S_3\times A_5$. Of course $JH=\langle X\rangle TH$. $T_0JH/S\leq S_3\times S_5$, so $|T_0: T|\leq 2$. Hence (ii) and (iv) hold. Finally J induces S_3 on Z, so $Z\neq Z(T)$. On the otherhand $Z_2(T)\leq C_{HT}(S/Z)=S$ while by (2.6) (i), $Z(S/Z(T))=Z/Z(T)$. Hence (iii) holds.

(2.13) Let $K=HT\langle x\rangle$. Then K is the semidirect product of $N_K(\langle x\rangle)$ with S and $N_K(\langle x\rangle)$ is determined up to conjugation in $\mathrm{Aut}(S)$, so that the isomorphism class of K is determined.

Proof. $C_S(x)=1$ and $C_K(S)=Z$, so K is the semidirect product of $N_K(\langle x\rangle)$ with S by a Frattini argument and we may regard K as a subgroup of $W=N_{\mathrm{Aut}(S)}(\langle x\rangle)$. Choose notation as in 2.6. ii and set $W^*=W/U$. By 2.6. ii and (v), and as the 1-cohomology of the natural module for A_5 is trivial, U is transitive on the complements to U in UH. Thus it remains to show K^* is determined up to conjugacy in W^*, since $C_U(H)=1$.

Let $t\in T$ invert x with $t^2\in S$ and $[H, t]\leq S$. As $C_{W^*}(t^*)^\infty\neq 1$, t interchanges the components of W^*, and then as t inverts x, t^* is determined up to conjugacy in W^*. Then $K^*=E(C_{W^*}(t^*))\langle t^*\rangle\langle x^*\rangle$ is determined up to conjugacy in W^*.

(2.14) (i) There exists a unique subgroup Q of T isomorphic to the central product of three quaternion groups and invariant under $\langle y\rangle$.

(ii) $Q\trianglelefteq HT$.

(iii) $|E\cap A\colon E\cap A\cap Q|=2$.

Proof. Let $D=Suz$. By (2.13) we may take $HT\langle x\rangle\leq D$. Set $\langle z\rangle=Z(T)$, $C=C_D(z)$ and $Q=O_2(C)$. Then $Q\cong(Q_8)^3$. Set $\tilde{C}=C/\langle z\rangle$ and $C^*=C/Q$. Suppose $B\leq T$ with $B\cong Q\neq B$. Then $\tilde{B}\cong E_{64}$, so $|B^*|\geq|\tilde{Q}\colon C_{\tilde{Q}}(B)|$. So as $C^*\cong\Omega_6^-(2)$ acts naturally on $\tilde{Q}$, $E_8\cong B^*\leq O_2(C_{C^*}(Z(T^*))$ with $B^*=C_{C^*}(B^*)$. Suppose $\langle y\rangle\leq N(B)$. Set $C_{C^*}(Z(T^*))=K^*$ and $\bar{K}=K^*/Z(T^*)$. Then $\bar{B}$ is a 4-subgroup of $\bar{K}$ invariant under $\langle\bar{y}\rangle$, so $\bar{B}=Z(\bar{T}^k)$ for some $k\in C_K(y)$, or $B^*\cong Q_8$. As $B^*\cong E_8$, the first case holds. But then $B^*\neq C_G(B^*)\cong E_{16}$. Thus Q is uniquely determined.

As $Q\trianglelefteq C\geq HT$, $Q\trianglelefteq HT$. $(Q\cap S)/Z$ is an irreducible $GF(2)$-module of S/Z of rank 4 for H/S, so $(E\cap A\cap Q)/Z=C_{Q\cap S/Z}(P)$ is of order 2, and (iii) holds.

(2.15) Set $K=O^2(N_G(P))$ and $\langle z\rangle=Z(T)$. Then

(i) $T\in\mathrm{Syl}_2(K)$.

(ii) $E\cap A\cap Q=Z_3(T)\cap E\cap A$.

(iii) $Q\trianglelefteq C_K(z)$.

Proof. $T_0\leq N(K)$ by (2.13) and $T\in\mathrm{Syl}_2(K)$ from the definition of T. By (2.11) (iv), $Z_3(T)\cap E\cap A$ is a hyperplane of $E\cap A$. By (2.14) (iii), $E\cap A\cap Q$ is a hyperplane of $E\cap A$ in $Z_3(T)$, so (ii) holds. Then $[Q, Z_3(T)\cap E\cap A]\leq\langle z\rangle$, so $Q\leq O_2(C_K(z))$ by (2.11) (iv). But $C_K(z)=O_2(C_K(z))C_K(\langle z,y\rangle)$, and for $g\in C_K(\langle z,y\rangle)$, $Q^g\leq T$ and $y\in N(Q^g)$, so $Q=Q^g$ by (2.14) (i). Thus $Q\trianglelefteq C_K(z)$.

(2.16) Set $M=\langle T^{N(Q)}\rangle$. Then $M/QO(Z(M))\cong\Omega_6^-(2)$ acts naturally on $Q/\langle z\rangle$.

Proof. Out $(Q)\cong O_6^-(2)$ with HT/Q a maximal parabolic of $E(\mathrm{Out}(Q))$. So by (2.15) (iii), $\mathrm{Out}_M(Q)\cong\Omega_6^-(2)$. $C_M(Q)=O(M)\langle z\rangle$ and by (1.7), $O(M)\leq Z(M)$.

(2.17) (i) M is transitive on $Z^{C(z)}\cap Q$

(ii) $N(Z)\cap C(z)$ is transitive on the $C(z)$-conjugates of Q containing Z.

Proof. (2.16) implies (i) and (i) implies (ii),

(2.18) (i) $N_G(Z)=HT_0\langle x\rangle O(N_G(Z))$ with $HT\trianglelefteq N_G(Z)$.

(ii) If $g\in C(z)$ and $m(Q\cap Q^g)>1$, then $Q=Q^g$.

Proof. Set $X=N(Z)$, $\bar{X}=X/Z$. Then by (2.8), $\bar{S}=J(\bar{T}_0)$, so $\bar{S}$ is weakly closed in $N_{\bar{X}}(\bar{S})$. We next show $\bar{S}$ to be strongly closed. If not by Corollary 4 of [5], there is $\bar{B}\leq\bar{S}$ and $g\in X$ such that $\bar{D}=\bar{B}^g\not\leq\bar{S}$ and for $d\in\bar{D}-\bar{S}$, $m([\bar{S},d])\leq m(\bar{D}/\bar{D}\cap\bar{S})$. But $m([\bar{S},t])\geq 2$ for each involution $t\in T_0-S$ by (2.6), so $m(\bar{D}/\bar{D}\cap\bar{S})>1$. Hence by (2.6) there is $d\in\bar{D}-\bar{S}$ with $m([\bar{S},d])=4$, so $m(\bar{D}/\bar{D}\cap\bar{S})\geq 4>m(T_0/S)$, a contradiction.

So $\bar{S}$ is strongly closed. Now by Goldschmidt's fusion Theorem [5], and the action of H on $\bar{S}$, $\bar{S}O(\bar{X})\trianglelefteq\bar{X}$. By (1.7), $S\trianglelefteq X$, so (i) follows from (1.7) and (2.6).

Choose g as in (ii). Then as $m(Q\cap Q^g)>1$, we may take $Z\leq Q^g$. So by (2.17) we may take $g\in X$. Now as $C_X(z)=C_H(z)T_0O(N(Z))$ with $[HT, O(C(Z))]=1$ and $Q\trianglelefteq C_H(z)T_0$, $Q=Q^g$.

Set $X=C(z)$, $\tilde{X}=X/\langle z\rangle$, $N_X(Q)^*=N_X(Q)/Q$.

(2.19) Q is weakly closed in X.

Proof. If $g\in X$ with $Q\neq Q^g\leq N(Q)$, then $\tilde{Q}^g\cong E_{64}$, so as $N_X(Q)^*\cong O_6^-(2)$ or $\Omega_6^-(2)$ acts naturally on $\tilde{Q}$, $m(Q\cap Q^g)>1$. This contradicts (2.18) (ii).

We can now obtain a contradiction. By (2.18) (ii), (2.19), and (1.10), Q is strongly closed in $C_G(z)$. So by Goldschmidt's fusion theorem [5], $QO(X)\trianglelefteq X$. Then (2.16) and (1.8) imply $O(X)=1$, and $M\trianglelefteq X$. By Theorem 2 in [11], and Theorem B of [10], we have $\langle A^G\rangle\cong Suz$, which we are assuming false.

3. Co_1

In this section we assume $A/Z(A)\cong G_2(4)$ and obtain a contradiction; we continue the notation in §1. In particular, let $A_1\cong SL_3(4)$ be as in (1.4) and $\langle x\rangle=Z(A_1)$. Inaddition we set $B=E(C_G(x))$ By (1.3) (iii) $Z(A)=1$.

(3.1) $B=3Suz$, the covering group of the Suzuki group.

Proof. This follows from (8.14) of [3] and the result established in §2.

Since A_1 is standard in B and $R\in\mathrm{Syl}_2(C_B(A_1))$ the entire analysis of §2 applies to the triple (R, A_1, B), replacing (R, A, G). We will make use of the subgroups Z, E, F, P, Q, and T as defined in §1 or constructed in §2. Then $E\cap A$ is the direct product of two long root subgroups of A(or A_1). Let $B_0=C_B(z)'$, for $z\in Z^\#$. Then $Q\trianglelefteq B_0$ and $B_0/Q\cong\Omega_6^-(2)$.

(3.2) Let $I=N_A(E\cap A)$.

(i) $I=D(J\times\langle x\rangle)$, where $D=O_2(I)$ and $J\cong SL_2(4)$.

(ii) $(E\cap A)=Z(D)$, $D/(E\cap A)$ is elementary of order 4^3, and $D/(E\cap A)$

is generated by the images of 3 short root subgroups.

(iii) $Z(DJ/(E\cap A))=U_\alpha(E\cap A))=U_\alpha(E\cap A)/(E\cap A)$ for U_α a short root subgroup.

(iv) $[D, U_\alpha]=E\cap A$.

(v) I/D acts indecomposably on $D/(E\cap A)$.

(vi) I/D acts on $D/U_\alpha(E\cap A)$ as on the natural module for $GL_2(4)$.

Proof. These facts are elementary consequences of the Chevalley commutator relations for $G_2(4)$.

(3.3) (i) $E-(E\cap A)$ is partitioned by the sixteen members of $R^G\cap E=\Delta$.

(ii) $N(E)=D(N(E)\cap N(\langle x\rangle))$. In particular $N(E)^\Delta=(N(E)\cap N(\langle x\rangle))^\Delta$, $N_B(E)^\Delta\trianglelefteq N(E)^\Delta$, and $N_B(E)^\Delta$ is $GL_2(4)$ acting on its natural module.

(iii) $\hat{P}=PD=O_2(N(E)\cap C(E\cap A))\in \mathrm{Syl}_2(N(E)\cap C(E\cap A))$ and $\hat{P}^\Delta=P^\Delta$ is regular.

(iv) $C_{\hat{P}}(E)=DC_P(E)=D\times R$.

Proof. (i) is just (1.5) (i). $X=N(E)\cap N(\langle x\rangle)$ is transitive on Δ, so $N(E)=X(N(E)\cap N(R))$. By a Frattini argument and (3.2) (i), $N(E)\cap N(R)=DN_X(R)$, so $N(E)=DX$. Now (ii) follows, and implies (iii) and (iv).

(3.4) (i) $\hat{P}=DP$ with $D\cap P=E\cap A$.

(ii) $D=[\hat{P}, x]$.

(iii) $Z(\hat{P}/(E\cap A))\geq U_\alpha R(E\cap A)/(E\cap A)$.

(iv) $[\hat{P}, \hat{P}]\leq U_\alpha(E\cap A)$.

Proof. By (3.3) (iii), $\hat{P}=DP$, while $D\cap P=C_{P\cap A}(E)=E\cap A$. By (i), $[\hat{P}, x]\leq D$, while by (3.2), $D=[D, x]$, so (ii) holds. J acts on $C_{D/(E\cap A)}(P)$, so by (3.2), $[U_\alpha, P]\leq E\cap A$. Of course $[P, R]\leq E\cap A$, so (iii) holds. Then (3.2) (vi) implies $[P, D]\leq U_\alpha(E\cap A)$, while by (3.2) (ii), $[D, D]\leq E\cap A$, and by (2.11) (iii), $[P, P]\leq E\cap A$. Hence (iv) holds.

(3.5) $D=O_2(C_G(P))\in \mathrm{Syl}_2(C_G(P))$.

Proof. We first show that $[D, P]=1$. Choose $Y\leq C_G(x)$ such that $|Y|=3$, Y is transitive on $R^\#$ and $[R, A]=1$ (for example $Y=\langle y\rangle$, with y as in (2.11)). Then $Y\times\langle x\rangle$ contains a subgroup Y_1 of order 3 such that $Y_1\leq C_G(A)$. Then Y_1 acts on $\hat{P}$, $[Y_1, D]=1$ and $[Y_1, P]=P$. Therefore, $[P, Y_1, D]=[P, D]$, $[Y_1, D, P]=[1, P]=1$, and $[D, P, Y_1]\leq[D, Y_1]=1$. By the 3-subgroups lemma, $[P, D]=1$.

Finally, $C_G(P)\leq C_G(R)$ so that $C_G(P)=C(P)\cap C(R)=C_D(P)O(C(A))=DO(C(A))$ by (1.7), so the lemma holds.

(3.6) Let $T_1=T\cap J\in \mathrm{Syl}_2(J)$, $V_0/(E\cap A)=C(T_1)\cap D/E\cap A)$, and $V=[V_0, \langle x\rangle]$.

Then V contains a unique $\langle x\rangle$-invariant subgroup Q_0 such that $Q_0 \cong Q_8$ and $Z(Q_0)=Z(Q)$.

Proof. The action of $J\times\langle x\rangle$ on $D/(E\cap A)$ is easily determined from the Chevalley commutator relations. The group V_0 is the product of $E\cap A$ together with the product of two short root subgroups, where the short roots add to a long root. Then V is the group generated by these two short root subgroups.

The group $V/Z(V)\cong E_{16}$ and $Z(V)=C_V(x)$ is a long root subgroup. Since $\langle x\rangle$ acts without fixed points on $V/Z(V)$, $\langle x\rangle$ stabilizes precisely five 4-subgroups of Y/Z. Aside from the images of the two short root subgroups, there are three subgroups each having preimage containing a unique $\langle x\rangle$-invariant Q_8 and having center of order 2 in $Z(V)=Z$. Since $Z(Q)\leq Z$, the result follows.

(3.7) (i) $T\leq C_G(Q_0)$.
(ii) Q_0Q is extraspecial of order 2^9.
(iii) $Q_0\in \mathrm{Syl}_2(C_G(Q))$.
(iv) $B_0\leq C_G(Q_0)$.

Proof. By (3.5) and the fact that $PT_1\trianglelefteq T$, we have $T\leq N(V_0)$. Since also $T\leq C(x)$, by (3.6), $T\leq N(Q_0)$. As $\langle x\rangle\times T$ acts on Q_0, we necessarily, have (i). In particular, $Q\leq C(Q_0)$, proving (ii).

Let $C=C_G(Q)$ and suppose $Q_0\notin \mathrm{Syl}_2(C)$. Consider $N_{C\langle r\rangle}(Q_0\langle r\rangle)=N$, where $r\in R^{\#}$. First we claim that $Q_0\langle r\rangle$ has index at most 2 in a Sylow 2-subgroup of N. So suppose otherwise and let $Y=C\cap N\cap C(r)$. Then $|C\cap N\colon Y|\leq 2$ so $Q_0O(Y)<Y$ and $Y\leq C(Q)<C(E\cap A\cap Q)$. By (2.14) (iii), $Y\leq DRO(C(A))\langle x\rangle$. By (1.7) $Y=(\langle x\rangle O_2(Y))\times O(Y)$. Now $AR\cap C(Q)\trianglelefteq AR\cap N(Q)$, so it follows from (3.2) and $Y^X=Y$, that $U_\alpha\leq Y$. However, the commutator relations show $U_\alpha\not\leq N(Q_0)$, a contradiction. Therefore, the claim holds. We conclude that $N/Q_0\langle r\rangle$ has a 2-complement of index 2.

Both N and $Q_0\langle r\rangle$ are invariant under $\langle x\rangle\times E$. By (1.7) and the above claim we conclude that $|N_C(\langle x\rangle)|$ is divisible by 4. As $N(\langle x\rangle)/O(N(\langle x\rangle))\leq \mathrm{Aut}(Suz)$, this is impossible. This establishes (iii).

To obtain (iv) consider the group C. If $O(C)\neq 1$, the assertion follows from (1.7) and the structure of Co_1. Suppose $O(C)=1$. If $E(C)=1$, then $C=Q_0\langle x\rangle$ and (iv) holds. If $E(C)\neq 1$, then $O^{2'}(C)\cong SL_2(q)$ for some $q\equiv 3,5 \pmod 8$ and $[Q_0, B_0]\leq [O^{2'}(C), B_0]=1$.

(3.8) Let $F=N_G(Q_0Q)^{(\infty)}$. Then $Q_0Q\trianglelefteq F$ and $F/Q_0Q\cong \Omega_8^+(2)$.

Proof. By (3.7) $\langle x\rangle\times B_0\leq N_G(Q_0Q)$. Let $M=O^2(N_G(Q_0Q)/C_G(Q_0Q/\langle z\rangle))$. Then $M\leq \Omega_8^+(2)$ and $\langle x\rangle\times B_0$ induces a subgroup M isomorphic to $Z_3\times\Omega_6^-(2)$. Easy arguments show that $(Z_3\times\Omega_6^-(2))\langle t\rangle=M_1$ is maximal in $\Omega_8^+(2)$, where

τ inverts the Z_3 factor and induces a transvection on the $\Omega_6^-(2)$ factor. It will suffice to show that M contains such an element τ and $M>M_1$.

To get τ, use the fact that $N_A(\langle x\rangle)$ contains an involution inverting x. Thus $M_1\leq M$. The argument in the first paragraph of the proof of (3.7) shows that $[V, T]\leq V$. Since $\langle x\rangle$ acts irreducibly on V/Q_0Z, $[V, T]\leq Q_0Z\leq Q_0Q$. Hence $V\leq N_G(Q_0Q)$ and V induces on $Q_0Q/\langle z\rangle$ a subgroup of M not contained in M_1. This proves (3.8).

(3.9) $N_G(Q_0Q)/Q_0QO(N_G(Q_0Q)\cong\Omega_8^+(2)$.

Proof. Otherwise $\langle x\rangle\times B_0\langle g\rangle\leq C_G(x)$, where g induces a transvection on $Q/\langle z\rangle$. On the otherhand $N_G(\langle x\rangle)/O(N_G(\langle x\rangle)=\mathrm{Aut}(Suz)$, so no such g exists.

(3.10) $C_F(Z)$ contains a normal subgroup $\hat{S}$ such that

(i) $\hat{S}$ is special with $Z(\hat{S})=Z$, and $\hat{S}$ is the central product of three copies of a Sylow 2-group of $L_3(4)$.

(ii) $C_F(Z)/\hat{S}\cong\Omega_6^+(2)$ has two noncentral chief factors on $\hat{S}/Z$, both of which are natural.

(iii) $\hat{S}$ is weakly closed in $N_G(\hat{S})$ with respect to $N_G(Z)$.

(iv) $N_G(\hat{S})/\hat{S}O(C_G(\hat{S}))\cong S_3\times\Omega_6^+(2)$ and $C_F(Z)O(C(Z))=N_G(\hat{S})\cap C(Z)$.

Proof. F acts on $Q_0Q/\langle z\rangle$ as the natural module for $\Omega_8^+(2)$ and the image of Z is a singular point. So $N_F(Z)/Q_0Q$ is a parabolic subgroup of $\Omega_8^+(2)$ isomorphic to $Q_6^+(2)$ on its natural module. Set $U=C_{Q_0Q}(Z)$ and $\hat{S}=O_2(C_F(Z))$. Then $1\trianglelefteq Z\trianglelefteq U\trianglelefteq\hat{S}\trianglelefteq C_F(Z)$ is a normal series with U/Z and $\hat{S}/U$ the natural module for $C_F(Z)/\hat{S}\cong\Omega_6^+(2)$. That is (ii) holds.

Next $S=C_{\hat{S}}(x)$ and $\hat{S}=S[\hat{S}, x]$. Moreover by 2.6, $B_1=C_B(Z)^\infty$ is a subgroup of F acting as $\Omega_4^-(2)$ on S/Z as the sum of two natural modules, and S is the central product of two copies of the Sylow 2-group of $L_3(4)$. Also there is $g\in C_F(Z)$ with $[\hat{S}, x]\leq C_{\hat{S}}(x^g)=S^g$, so $[\hat{S}, x]$ is isomorphic to a Sylow 2-group of $L_3(4)$. As $[\hat{S}, x, B_1]=1$, $S=[S, B]\leq C([\hat{S}, x])$. Therefore (i) holds.

$V=\hat{S}/Z$ is elementary abelian and if $g\in N(Z)$ with $\hat{S}^g\leq F$ and $\hat{S}\neq\hat{S}^g$, then $V\neq V^g$ and $m(V^g/V\cap V^g)=m(V/V\cap V^g)\geq m(V/C_V(V^g))$, which is impossible by (ii). Thus $\hat{S}$ is weakly closed in F with respect to $N(Z)$.

Let $j\in Q_0Q-C(Z)$ be an involution. Then $[Z,j]=z$ and $[j, C_F(Z)]\leq C_{Q_0Q}(Z)\leq\hat{S}$, so $j\in N(\hat{S})$ and $\langle C_F(Z), j\rangle/\hat{S}\cong Z_2\times\Omega_6^+(2)$. However from (i), $\mathrm{Out}(\hat{S})$ is the extension of $Z_3\times O_6^+(4)$ by a field automorphism, so as $[Z,j]\neq 1$, j induces a field or graph-field automorphism, and as j centralizes $C_F(Z)/\hat{S}$, it is the former. In particular $C_F(Z)/\hat{S}=E(\mathrm{Out}(\hat{S}))\cap C(j)$ is maximal in $E(\mathrm{Out}(\hat{S}))$, so if $N_G(\hat{S})^\infty\neq C_F(Z)$, then $N_G(\hat{S})^\infty/\hat{S}\cong\Omega_6^+(4)$. But then as RZ/Z and $(E\cap A)/Z$ are singular points in $\hat{S}/Z$, $RZ\in(E\cap A)^{N(\hat{S})}$, contradiction.

So $C_F(Z)=N_G(\hat{S})^\infty$, and hence by (1.7), $N_G(\hat{S})\cap C(Z)=C_F(Z)O(N_G(\hat{S}))\langle t\rangle$, where either $t=1$ or t induces a $GF(4)$-transvection on $\hat{S}/Z$. In the latter case t acts on $\langle j, U\rangle=Q_0Q$, and (3.9) supplies a contradiction. In particular the second part of (iv) holds. In addition as $\hat{S}$ is weakly closed in F with respect to $N(Z)$, (iii) holds. There is an element of order 3 in A acting nontrivially on Z, so by (iii) and a Frattini argument some 3-element in $N(\hat{S})$ is nontrivial on Z, so that the proof of (iv) is complete.

(3.11) (i) $\hat{S}=O_2(C_G(Z))$.
(ii) $N_G(\hat{S})$ contains a Sylow 2-group of G.

Proof. Claim $\hat{S}$ is strongly closed in $N(\hat{S})$ with respect to $C(Z)$. Assume not. By 3.10, $\hat{S}$ is weakly closed, while $V=\hat{S}/Z$ is an elementary subgroup of $C(Z)^*=C(Z)/Z$. So by Theorem 4 in [5] there is $U\leq V$ and $W=U^g\leq N(V)$ such that $m([V,w])\leq m(W/W\cap V)$ for each $w\in W$. But by 3.10, $m([V,w])\geq 4$ for each involution $w\in N(V)/V$, so $m(W/W\cap V)\geq 4$. As $(N(V)\cap C(Z)^*)/V\cong\Omega_6^+(2)$ has 2-rank 4, $m(W/W\cap V)=4$ and $WV/V=O_2(X/V)$ where X is the stabilizer of a singular point of V. Now if $w\in W-V$ then $m(C_V(w))=8$, so by symmetry between V and V^g, $m(W)\geq 8$. Thus $m(C_V(W))\geq m(V\cap W)\geq 4$, impossible as $m(C_V(W))=2$.

So the claim is established. Now by Goldschmidt's fusion theorem [5] and (1.7) and (3.10) (iv), (i) holds. Moreover if $I\in\mathrm{Syl}_2(N_G(Z))$, then $Z=Z_2(I)$, so (3.10) (iii) and (i) imply (ii).

(3.12) Let $Q_1=Q_0Q$.
(i) If $g\in C(z)$ and $m(Q_1\cap Q_1^g)>1$, then $Q_1=Q_1^g$.
(ii) Q_1 is weakly closed in a Sylow 2-subgroup of $C_G(z)$.

Proof. Suppose $g\in C(z)$ and $m(Q_1\cap Q_1^g)>1$. By (3.8) we may assume $Z\leq Q_1\cap Q_1^g$, and applying (3.8) to $N(Q_1^g)$ we may take $g\in N(Z)$. By (3.11) and (3.10) $C_G(z)\cap N(Z)=C_F(Z)\langle j\rangle O(N(Z))$ and by (1.7) $[O(N/Z)), C_F(Z)\langle j\rangle]=1$. Since $C_F(Z)\langle j\rangle\leq N(Q_1)$ we conclude that $g\in N(Q_1)$, proving (i).

To prove (ii), suppose $g\in C(z)$ and $Q_1^g\leq N(Q_1)$. By (3.9) $Q_1^gQ_1/Q_1\leq\Omega_8^+(2)$. If $Q_1^g\neq Q_1$, then by (i) $m(Q_1\cap Q_1^g)=1$, so $(Q_1^g\cap Q_1)/\langle z\rangle$ is an anisotropic 1-space or 2-space. In the first case $m(Q_1^gQ_1/Q_1)=7$ and $Q_1^gQ_1/Q_1$ is a subgroup of $Sp_6(2)$, while in the second case $m(Q_1^gQ_1/Q_1)=6$ and $Q_1^gQ_1/Q_1$ is a subgroup of $O_6^-(2)$. In either case we have a contradiction.

As in §2 we can now reach a contradiction. By (3.12) and (1.10), Q_1 is strongly closed in $C_G(z)$, so by Goldschmidt's fusion theorem [5] $Q_1O(C_G(z))\trianglelefteq C_G(z)$. By (1.7) and (1.8) $O(C_G/(z)=1$. Finally, (3.9) and Patterson's theorem [9] yield $G\cong Co_1$, which we have assumed to be false.

4. He

In this section we assume $|Z(A)|$ is even. By (1.6) $Z_2\times Z_2\cong R\in \mathrm{Syl}_2(Z(A))$.

(4.1) (i) $N(E)/C(E)$ contains $3A_6$ and induces S_6 on $R^G\cap E$. Similarly for F.
(ii) There is an element g of order 3 and an involution y such that $\langle g,y\rangle\cong S_3$ and $\langle g,y\rangle$ induces S_3 on R.

Proof. By (1.6) $N(E)/C(E)$ and $N(F)/C(F)$ contain $3A_6$. By (1.7) $N(E)^{(\infty)}=EL$, where $L\cong 3A_6$ and $\langle g\rangle=Z(L)$ acts as an outer diagonal automorphism of A. Now $C_A(g)\cong A_5$ and we may assume that $F_1=F\cap C_A(g)\in \mathrm{Syl}_2(C_A(g))$. Set $J=N_L(F_1)\cong S_4\times Z_3$. Then $EJ\leq N(C_E(F_1)F_1)=N(F)$.

Let bars denote images in $N(F)/C(F)$ and suppose $\overline{N(F)}=3A_6$. Then $\overline{EJ}\cong S_4\times Z_3$ and $Z(\overline{EJ})=Z(\overline{N(F)})$. This forces $\langle\bar g\rangle=Z(\overline{N(F)})$, whereas $[\bar E,\bar g]=\bar E$. Consequently $N(F)$ induces S_6 on $R^G\cap F$. By symmetry, (i) holds. Consequently, $N(E)\cap N(R)$ induces S_5 on $R^G\cap E-\{R\}$. and (ii) follows.

(4.2) Let $S=EF$ and $y\in S_1\in \mathrm{Syl}_2(N(S))$. Then either
(i) $S_1\in \mathrm{Syl}_2(G)$ and $S_1/S\cong E_4$, or
(ii) $S_1/S\cong D_8$ and $E\sim F$ in $N(A)$.

Proof. Let $S_2=N_{S_1}(E)$. By 4.1, $S_2/S\cong E_4$. As E and F are the unique elementary abelian subgroups of S of order 2^6 we conclude either $S_1/S\cong D_8$ or $S_1=S_2$. In the first case $E\in F^{N(S)}$ and as $N(E)$ is transitive on $R^G\cap E$, $N(R)$ is transitive on $E^G\cap N(R)$, so $E\in F^{N(A)}$ and (ii) holds. In the second case we show $S=J(S_1)$, to conclude $S_1\in \mathrm{Syl}_2(G)$, so that (i) holds. If not there exists $E_{2^6}\cong U\leq S_1$ with $U\neq E$ or F. Then

$$(*)\qquad |\mathrm{Aut}_U(E)|\geq |E\colon C_E(U)|.$$

But by 4.1.i, the representation of $\mathrm{Aut}_G(E)$ on E is determined and $(*)$ forces $\mathrm{Aut}_U(E)=\mathrm{Aut}_F(E)$, so that $U\leq UE=FE=S$.

(4.3) $S_1\in \mathrm{Syl}_2(G)$.

Proof. Suppose otherwise and let $g\in N(S_1)-S_1$ with $g^2\in S_1$. Then $S^g\neq S$. Let $Z=Z(S)=E\cap F$. If $Z^g=Z$, then g stabilizes the two element set $R^G\cap Z$. So, for some $s\in S_1$, $gs\in N(R)$ and it follows that $g\in S_1$. Suppose, then, that $Z^g\neq Z$.

We have $Z=S'$, so $Z^g=(S')^g$. By (4.2) $|E^g\cap S|\geq 2^4$ and so either $(E^g\cap S)Z$ or $(F^g\cap S)Z$ is elementary of order at least 2^5, say the former. Therefore, $(E^g\cap S)Z\leq E$ or F and $S^g\leq N(E)$ or $N(F)$. Apply (4.2) to conclude that $Z^g=(S^g)'\leq S$. Now $S\cap S^g\leq C(ZZ^g)$ and $ZZ^g\leq E$ or F. Since $|S^gS\colon S|\leq 4$ we necessarily have $|S\cap S^g|=2^6$ and $|S^gS\colon S|=4$. Then

$S\cap S^g=E$ or F, so $g\in N(E)$ or $N(F)$. But this is not the case.

(4.4) (i) $N_G(S)/SO(C(S))\cong S_3\times S_3$ or $S_3 \wr Z_2$

(ii) The structure of S_1 is uniquely determined by $|S_1|=2^{10}$ or 2^{11}.

Proof. Let $A(S)=\mathrm{Aut}(S)/C_{\mathrm{Aut}(S)}(S/Z(S))$. As $S\in \mathrm{Syl}_2(A)$ and $E_4\cong R\in \mathrm{Syl}_2(Z(A))$ with $A/Z(A)\cong L_3(4)$, we may calculate in A to determine $Z(S)=E\cap F$ is partitioned by

$$\{R, R_0\}\cup\{[E, s]: s\in S\}$$

where $R_0=[Z(S), x]$ and x is of order 3 in $N_A(S)-Z(A)$. $N_G(S)\le N_G(Z(S))$, so $N_G(S)$ acts transitively on the two member set $R^G\cap Z(S)=\{R, R_0\}$ and $|N_G(S): N(R)\cap N_G(S)|=2=|\mathrm{Aut}(S): N_{\mathrm{Aut}(S)}(R)|$. $\mathrm{Out}_{\mathrm{Aut}(A)}(S)\cong S_3\times S_3\cong A(S/R)$ and $N_{A(S)}(R)$ is isomorphic to a subgroup of $A(S/R)$, so $A(S)\cong S_3\wr Z_2$ and $N_{A(S)}(R)\cong S_3\times S_3$. $\mathrm{Out}_{N(E)}(S)\cong S_3\times S_3$, so (i) holds.

Let $T\in \mathrm{Syl}_3(A\langle g\rangle\cap N(S))$, and choose T so that $S_1=SN_{S_1}(T)$. $C_S(T)=1=C(T)\cap C_{\mathrm{Aut}(S)}(S/Z(S))$ as T is irreducible on $S/Z(S)$. Thus the product is semidirect and $N_{S_1}(T)\le A(S)\cong S_3\wr Z_2$. Next by 4.2, $N_{S_1}(T)\cong E_4$ or D_8, and in the former case $N_{S_1}(T)\le N(E)$. Thus $|S_1|=2^{10}$ or 2^{11}, $TN_{S_1}(T)=N_{A(S)}(E)$ or $A(S)$, and S_1T, and hence also S_1, is uniquely determined by $|S_1|$.

(4.5) (i) S_1 is isomorphic to a Sylow 2-group of He or Aut(He).

(ii) S_1 contains a unique extraspecial 2-subgroup Q of order 2^7 with $Z(Q)=Z(S_1)$.

(iii) $Q\le N(E)\cap N(F)$.

(iv) $S_1/Q\cong D_8$ or D_{16}.

(v) $Q\cong (D_8)^3$.

Proof. (i) follows from (4.4) and the fact that the results obtained so far apply to He and Aut(He). In particular we can embed S_1 as a Sylow 2-group of $G_1=He$ or Aut(He). Let $\langle z\rangle=Z(S_1)$, $C=C_{G_1}(z)$, and $Q=O_2(C)$. Then (iii), (iv), and (v) follow from the structure of G_1. Moreover $C/Q\cong L_3(2)$ or $PGL_2(7)$, with $E(C/Q)$ acting on $V=Q/\langle z\rangle$ as the sum of the natural module and its dual. In particular this forces $V=J(S/\langle z\rangle)$, so Q is unique, and (ii) holds.

(4.6) Let $\langle z\rangle=Z(Q)$, $X=E$ or F, and $I_X=O^{2'}(C(z)\cap N(X))$. Then

(i) $I_X\cong E_{64}(S_4\times Z_2)$.

(ii) $I_X\not\le N(R)$.

(iii) $|Q\cap X|=16$.

(iv) $Y=\langle I_E, I_F\rangle\le N(Q)$.

Proof. By (4.1) and (1.7) $O^{2'}(N_G(X))=L\cong S_6/Z_3/E_{64}$, and $E(L/X)$ acts naturally on X. In particular $I_X=C_L(z)\cong E_{64}(S_4\times Z_2)$. As $S_1\cap N(X)\not\le N(R)$,

(ii) holds. By (4.5) (v), $m(X\cap Q)\leq 4$ and by (4.5) (iv), $m(X/X\cap Q)\leq 2$, so (iii) holds. By (iii), $QX/X\cong E_8$, so as $L/X\cong S_6/Z_3$, $N_L(QX)/X\cong Z_2\times S_4$. As $\langle z\rangle=Z(QX)$, $\langle z\rangle\trianglelefteq N_L(QX)$, so $I_X=N_L(QX)$. Hence (iv) holds.

(4.7) (i) $Y/Q\cong L_3(2)$.
(ii) $Q/\langle z\rangle$ is the sum of the natural module for Y/Q and its dual.
(iii) $N(Q)/QO(N(Q))\cong L_3(2)$ or $PGL_2(7)$.

Proof. By (1.7) we may take $O(N(Q))=1$. Embed S_1 in G_1 as in 4.5, and adopt the notation of that lemma. Let V_1 and V_2 be the two $E(C/Q)$-chief factors in $V=Q/\langle z\rangle$. Then EQ/Q centralizes a hyperplane E_1 of V_1 and a point E_2 of V_2, with $E_1E_2=[V,E]$. As $[E,Q]\leq E\cap Q\cong E_{16}$, $E_1E_2=(E\cap Q)/\langle z\rangle$. In particular each member of $E-Q$ induces an involution of type a_2 on V, and EF induces automorphisms in $\Omega_6^-(2)$ on V. Therefore $Y=\langle E^Y, F^Y\rangle$ induces automorphisms in $\Omega_6^-(2)\cong A_8$ on V. $EFQ=S_1\cap YQ$ with $EF/Q\cong D_8$ and $Y=O^{2'}(Y)=O^2(Y)$, so $YQ/Q\cong A_6$, A_7, or $L_3(2)$. However there is one class each of A_6's and A_7's and two classes of $L_3(2)$'s in A_8. As the involutions in EFQ/Q are of type a_2, we conclude (i) and (ii) holds. Similarly as $S_1/Q\cong D_8$ or D_{16} and $Y/Q\cong L_3(2)$ is a transitive subgroup of $N_G(Q)^\infty/Q\leq A_8$, (iii) holds.

(4.8) Q is strongly closed in S_1 with respect to $C(z)$.

Proof. By (4.5) (ii), Q is weakly in S_1 with respect to $C(z)$. Set $\bar{N}(Q)=N(Q)/QO(N(Q))$ and $C(z)^*=C(z)/\langle z\rangle$, so that $V=Q^*\cong E_{64}$. Assume Q is not strongly closed. By (2.4) of [12], there exists $g\in C(z)$ such that, setting $L=\langle Q, Q^g\rangle$, $B=N_Q(Q^g)$, $D=Q^g\cap N(Q)$, and $I=Q\cap Q^g$, the following hold:
(1) $L/BD\cong L_2(2^n)$, $Sz(2^n)$, or D_{2m}, m odd;
(2) BD/I is the sum of natural modules for L/BD; and
(3) $I\neq D$.

$m(\bar{D})\leq m(\bar{S})=2$. But by Corollary 4 in [5], $m([V,d])\leq m(\bar{D})$ for each $d\in D-I$, while by (4.7), $m([V,s])\geq 2$ for each $s\in S_1-Q$. Hence $m(\bar{D})=2$ and $m([V,d])=2$ for each $d\in D-I$. By (4.7) it follows that $\bar{D}\leq E(\bar{N}(Q))$ and that $[D,V]=C_V(D)$ is of rank 3. But $B=[Q,V]I$, so $B^*=C_V(D)$ is of codimension at most 2 in V, a contradiction.

(4.9) (i) $Q=F^*(C_G(z))$.
(ii) $C_G(z)/Q\cong PGL_2(7)$.

Proof. By 4.8 and Goldschmidt's fusion theorem [5], $QO(C(z))\trianglelefteq C(z)$. By (4.7) and (1.8) $O(C(z))=1$. If $C(z)=Y$, then by [4], $G\cong He$, contrary to our assumption that G is a counter example to the Main Theorem. So (4.7) completes the proof.

(4.10) $G\neq O^2(G)$.

Proof. All involutions in EF are fused to z or $r \in R^{\sharp}$ in $N_G(E)$ and $N_G(F)$. All involutions in Y are fused into EF under Y. But by (4.9) (ii) $|S_1|=2^{11}$, so as $R^G \cap Z(S)$ is of order 2, $|S_1 \cap N(R)|=2^{10}$. In particular some involution $t \in S_1 \cap N(R)-Y$ induces a graph-field automorphism on A. Then $[R, t]=1$ and $C_A(t)/R \cong E_9Q_8$. Then $m_3(C_G(t))>1$, so by (4.9) $t \notin z^G$. Hence if (4.10) is false, $t \in r^G$ by Thompson transfer. As $[R, t]=1$, this contradicts (1.1).

As G is simple, (4.10) yields a contradiction. This completes the proof of the Main Theorem.

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