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ANALYTICITY OF SOLUTIONS OF QUASILINEAR EVOLUTION EQUATIONS

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0. Introduction

In this paper we establish analyticity in t of solutions to quasilinear evolution equations

$$\frac{du}{dt}+A(t,u)u=f(t,u),\quad 0\leqq t\leqq T, \tag{0.1}$$

$$u(0)=u_0\,. \tag{0.2}$$

The unknown, u, is a function of t with values in a Banach space X. For fixed t and $v\in X$, the linear operator $-A(t,v)$ is the generator of an analytic semigroup in X and $f(t,v)\in X$. Several authors Ōuchi [6], Hayden and Massey [2], have considered analyticity for semilinear equations $du/dt+A(t)u=f(t,u)$. And Massey [5] discussed analyticity for quasilinear equations (0.1) when the domain $D(A(t,u))$ of $A(t,u)$ does not depend on t, u.

In the present paper, we consider analyticity for (0.1), (0.2) under the assumption that $D(A(t,u)^h)$ is independent of t, u for some $h=1/m$ where m is a positive integer. In order to prove it we shall make use of the linear theory of Kato [3].

In the following $L(X,Y)$ is the space of linear operators from a normed space X to another normed space Y, and $B(X,Y)$ is the space of bounded linear operators belonging to $L(X,Y)$. $L(X)=L(X,X)$ and $B(X)=B(X,X)$. $\|\ \|$ will be used for the norm both in X and $B(X)$; it should be clear from the context which is intended.

We shall make the following assumptions:

A-1°) $u_0\in D(A_0)$ and $A_0^{-\alpha}$ is a well-defined operator $\in B(X)$ where $A_0\equiv A(0,u_0)$.
A-2°) There exist $h=1/m$, where m is an integer, $m\geqq 2$, $R>0$, $T_0>0$, $\phi_0>0$ and $0\leqq\alpha<h$, such that $A(t,A_0^{-\alpha}w)$ is a well-defined operator $\in L(X)$ for each $t\in\Sigma_0\equiv\{t\in \boldsymbol{C};\ |\arg t|<\phi_0,\ 0\leqq|t|<T_0\}$ and $w\in N\equiv\{w\in X;\ \|w-A_0^\alpha u_0\|<R\}$.
A-3°) For any $t\in\Sigma_0$ and $w\in N$

$$(0.3)\quad \begin{cases} \text{the resolvent set of } A(t, A_0^{-\alpha}w) \text{ contains the left half-plane} \\ \text{and there exists } C_1 \text{ such that } \|(\lambda - A(t, A_0^{-\alpha}w))^{-1}\| \leqq C_1(1+|\lambda|)^{-1},\ \mathrm{Re}\,\lambda \leqq 0. \end{cases}$$

A-4°) The domain $D(A(t, A_0^{-\alpha}w)^h) = D$ of $A(t, A_0^{-\alpha}w)^h$ is independent of $t \in \Sigma_0$ and $w \in N$.

A-5°) The map $\Phi: (t, w) \mapsto A(t, A_0^{-\alpha}w)^h A_0^{-h}$ is analytic from $(\Sigma_0 \backslash \{0\}) \times N$ to $B(X)$.

A-6°) There exist C_2, C_3, σ, $1-h < \sigma \leqq 1$ such that

$$(0.4)\qquad \|A(t, A_0^{-\alpha}w)^h A(s, A_0^{-\alpha}v)^{-h}\| \leqq C_2 \qquad t, s \in \Sigma_0,\ w, v \in N,$$

$$(0.5)\qquad \|A(t, A_0^{-\alpha}w)^h A(s, A_0^{-\alpha}v)^{-h} - I\| \leqq C_3\{|t-s|^\sigma + \|w-v\|\} \qquad t, s \in \Sigma_0,\ w, v \in N.$$

A-7°) $f(t, A_0^{-\alpha}w)$ is defined and belongs to X for each $t \in \Sigma_0$ and $w \in N$, and there exists C_4 such that

$$(0.6)\qquad \|f(t, A_0^{-\alpha}w) - f(s, A_0^{-\alpha}v)\| \leqq C_4\{|t-s|^\sigma + \|w-v\|\} \qquad t, s \in \Sigma_0,\ w, v \in N.$$

A-8°) The map $\Psi: (t, w) \mapsto f(t, A_0^{-\alpha}w)$ is analytic from $(\Sigma_0 \backslash \{0\}) \times N$ into X.
These constants $C_i (i \in \boldsymbol{N}_+)$ do not depend on t, s, w, v.

The main result of this paper is the following theorem.

Theorem 1. *Let the assumptions* A-1°)—A-8°) *hold. Then there exist* T, $0 < T \leqq T_0$, ϕ, $0 < \phi \leqq \phi_0$, $K > 0$, k, $1-h < k < 1$ *and a unique continuous function* u *mapping* $\Sigma \equiv \{t \in \boldsymbol{C};\ |\arg t| < \phi,\ 0 \leqq |t| < T\}$ *into* X *such that* $u(0) = u_0$, $u(t) \in D(A(t, u(t)))$ *and* $\|A_0^\alpha u(t) - A_0^\alpha u_0\| < R$ *for* $t \in \Sigma \backslash \{0\}$; $u: \Sigma \backslash \{0\} \to X$ *is analytic,* $\dfrac{du(t)}{dt} + A(t, u(t))u(t) = f(t, u(t))$ *for* $t \in \Sigma \backslash \{0\}$, *and* $\|A_0^\alpha u(t) - A_0^\alpha u_0\| \leqq K|t|^k$ *for* $t \in \Sigma$.

REMARK. Under the assumption that $D(A(t, u)^h)$ is constant, Sobolevskii [8] gave the existence of solutions to (0.1) with *differentiable* coefficients. But, as far as the author knows, the proof of [8] (or similar results) is not published yet. In this paper we give the existence of local solutions to (0.1) for $A(t, u)$ differentiable in t, u (Theorem 2). But in this case, the condition (3.5) seems to be too restrictive to apply Theorem 2 to the Neumann problems. The condition may be reasonable when $A(t, u)$ is *analytic* in u and *differentiable* in t.

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1. Fractional powers of operators which generate analytic semigroups

Assume that A is a closed operator in Banach space X with domain, $D(A)$, dense in X and that the resolvent set of A contains the left half-plane and $(1+$

$|\lambda|)(A-\lambda)^{-1}$ is uniformly bounded in $Re\,\lambda \leqq 0$. Then there exist M, θ, $0<\theta<\frac{\pi}{2}$ such that the resolvent set of A contains closed sectorial domain $\Sigma \equiv \{\lambda \in \boldsymbol{C};\ |arg\,\lambda| \geqq \theta\} \cup \{0\}$ and

$$(1.1)\qquad \|(A-\lambda)^{-1}\| \leqq M(1+|\lambda|)^{-1} \qquad \lambda \in \Sigma\,,$$

$$(1.1)'\qquad \|A(A-\lambda)^{-1}\| = \|I-\lambda(A-\lambda)^{-1}\| \leqq 1+M = \tilde{M} \quad \lambda \in \Sigma\,.$$

$-A$ is a generator of an analytic semigroup in X, and the fractional powers $A^{\alpha}(\alpha \in \boldsymbol{R})$ are defined as follows;

$$(1.2)\qquad A^{\alpha} = \begin{cases} (A^{-\alpha})^{-1} & \alpha>0 \\ I & \alpha=0 \\ \dfrac{1}{2\pi i}\displaystyle\int_{\Gamma} \lambda^{\alpha}(A-\lambda)^{-1} d\lambda & \alpha<0 \end{cases}$$

where the integration path Γ consists of the two rays $a+re^{\pm i\phi}[\theta<\phi<\pi,\ a>0,\ 0\leqq r<\infty]$ and run in the resolvent of A from $\infty e^{-i\phi}$ to $\infty e^{i\phi}$. We define that λ^{α} attain positive values when $\lambda>0$.

A^{α} have the following properties;

1) For $\alpha<0$, $A^{\alpha} \in B(X)$.
2) For $\alpha>0$, A^{α} is a closed operator in X with domain, $D(A^{\alpha})$, dense in X.
3) $D(A^{\alpha}) \supset D(A^{\beta})$ for $\beta>\alpha>0$.
4) For any $\alpha>0$, $\beta>0$, $A^{\alpha+\beta}=A^{\alpha}A^{\beta}=A^{\beta}A^{\alpha}$ holds.

It follows from (1.1) that there exist $\delta>0$, $C>0$ such that

$$(1.3)\qquad \|\exp(-\tau A)\| \leqq Ce^{-\delta\tau}\,,$$

$$(1.4)\qquad \|A\exp(-\tau A)\| \leqq Ce^{-\delta\tau}\tau^{-1}\,.$$

For an operator A satisfying (1.3) we can give an equivalent definition of the fractional powers A^{α} as follows;

$$(1.5)\qquad A^{\alpha} = \begin{cases} (A^{-\alpha})^{-1} & \alpha>0 \\ I & \alpha=0 \\ \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^{\infty} \exp(-sA(t))s^{-\alpha-1} ds & \alpha<0\,. \end{cases}$$

For any $\alpha<\beta<\gamma$ an inequality of moments

$$(1.6)\qquad \|A^{\beta}u\| \leqq C(\alpha,\beta,\gamma)\|A^{r}u\|^{(\beta-\alpha)/(\gamma-\alpha)}\|A^{\alpha}u\|^{(\gamma-\beta)/(\gamma-\alpha)} \qquad [u \in D(A^{\gamma})]$$

holds. (Krein[4] Chapter 1. Theorem 5.2)

For $0\leqq\alpha<1$, $-A^{\alpha}$ is also the generator of an analytic semigroup in X and has similar properties as A with θ replaced by $\alpha\theta$.

Assume that A and B are closed operators in X with domain, $D(A)$ and $D(B)$, dense in X and with property (1.1), and that $D(A)\subset D(B)$. Then $D(A^{\beta})\subset D(B^{\alpha})$ for $0<\alpha<\beta\leqq 1$. (Krein [4] Chapter 1. Lemma 7.3)

For these and other properties of analytic semigroups, see Tanabe [9] Sobolevskii [7] Krein [4] Friedman [1] etc..

2. Kato's results

We shall make the following assumptions:

1°) For each $t\in[0,T]$, $A(t)$ is a densely defined, closed linear operator in X with its spectrum contained in a fixed sector $S_{\theta}=\{z\in \boldsymbol{C};\ |arg\ z|<\theta\leqq\frac{\pi}{2}\}$. The resolvent of $A(t)$ satisfies the inequality

$$||[z-A(t)]^{-1}||\leqq M_0/|z| \qquad \text{for } z\notin S_{\theta} \tag{2.1}$$

where M_0 is a constant independent of t. Furthermore, $z=0$ also belongs to the resolvent set of $A(t)$ and

$$||A(t)^{-1}||\leqq M_1 \tag{2.2}$$

M_1 being independent of t.

2°) For some $h=1/m$, where m is a positive integer, $\geqq 2$, $D(A(t)^{h})=D$ is independent of t, and there are constants k, M_2 and M_3 such that

$$||A(t)^{h}A(s)^{-h}||\leqq M_2, \qquad 0\leqq t\leqq T,\ 0\leqq s\leqq T. \tag{2.3}$$

$$||A(t)^{h}A(s)^{-h}-I||\leqq M_3|t-s|^{k}, \quad 0\leqq t\leqq T,\ 0\leqq s\leqq T,\ 1-h<k\leqq 1. \tag{2.4}$$

REMARK. From (2.2) there exists $C_h>0$ such that

$$||A(t)^{-h}||\leqq C_h \qquad \text{for } t\in[0, T] \tag{2.2$'$}$$

C_h being independent of t.

Under these assumptions, we get the following theorems. They are due to Kato.

Theorem A. *Let the conditions* 1°) *and* 2°) *be satisfied. Then there exists a unique evolution operator* $U(t,s)\in B(X)$ *defined for* $0\leqq s\leqq t\leqq T$, *with following properties.* $U(t,s)$ *is strongly continuous for* $0\leqq s\leqq t\leqq T$ *and*

$$U(t,r)=U(t,s)U(s,r), \qquad r\leqq s\leqq t, \tag{2.5}$$

$$U(t,t)=I. \tag{2.6}$$

For $s<t$, *the range of* $U(t,s)$ *is a subset of* $D(A(t))$ *and*

$$A(t)U(t,s)\in B(X),\ ||A(t)U(t,s)||\leqq M|t-s|^{-1}, \tag{2.7}$$

where M is a constant depending only on θ, h, k, T, M_0, M_1, M_2 and M_3. Furthermore, $U(t,s)$ is strongly continuously differentiable in t for $t>s$ and

$$\frac{\partial}{\partial t}U(t,s)+A(t)U(t,s)=0\,. \tag{2.8}$$

If $u\in D$, $U(t,s)u$ is strongly continuously differentiable in s for $s<t$. If in particular $u\in D(A(s_0))$, then

$$\frac{\partial}{\partial s}U(t,s)u\,|_{s=s_0}=U(t,s_0)A(s_0)u\,. \tag{2.9}$$

If $f(t)$ is continuous in t, any strict solution of

$$\frac{du}{dt}+A(t)u=f(t) \tag{2.10}$$

must be expressible in the form

$$u(t)=U(t,0)u(0)+\int_0^t U(t,s)f(s)ds\,. \tag{2.11}$$

Conversely, the $u(t)$ given by (2.11) *is a strict solution of* (2.10) *if $f(t)$ is Hölder continuous on $[0,T]$; here $u(0)$ may be an arbitrary element of X.*

Proof. See, [3].

Theorem B. *Assume that $A(t)$ can be continued to a complex neighborhood Δ of the interval $[0,T]$ in such a way that the conditions* 1°), 2°) *are satisfied for $t,s\in\Delta$. Furthermore, let $A(t)^{-h}$ be holomorphic for $t\in\Delta$. Then the evolution operator $U(t,s)$ exists for $s\leqq t$, satisfies the assertions of Theorem A and is holomorphic in s and t for $s<t$. (Here "$s<t$" should be interpreted as meaning "$t-s\in\Sigma$", where Σ is the sector $|\arg t|<\pi/2-\theta$ of the t-plane, and "$s\leqq t$" as "$s<t$ or $s=t$".) If $f(t)$ is holomorphic for $t\in\Delta$, $t>0$, and Hölder continuous at $t=0$, every solution of* (2.10) *has a continuation holomorphic for $t\in\Delta$, $t>0$.*

Proof. See, [3].

It follows from 1°) and 2°) that

$$\|A(t)^{\alpha}\exp(\tau A(t))\|\leqq N_6|\tau|^{-\alpha}:\ 0\leqq\alpha\leqq 2,\ |\arg\tau|\leqq\frac{\pi}{2}-\theta \tag{2.12}$$

$$\|A(t)^{\alpha}U(t,s)\|\leqq(h+k-\alpha)^{-1}N_{18}(t-s)^{-\alpha}:\ 0\leqq\alpha<k+h \tag{2.13}$$

$$\|A(t)^{\alpha+h}U(t,s)A(s)^{-h}\|\leqq(k-\alpha)^{-1}N_{19}(t-s)^{-\alpha}:\ 0\leqq\alpha<k,\ 0\leqq s\leqq t\leqq T \tag{2.14}$$

Here the constants $N_i(i\geqq 4, i\in\boldsymbol{N})$ are determined by $M_0, M_1, M_2, M_3, \theta, h, k, T$. For a proof of the above estimates, (2.12)–(2.14), see the argument in [3]. In addition to these, we shall prove some estimates which will be used in the following.

Proposition 1. *If* $1-h<k<1$, $0<\alpha<\alpha'<1-k$, *then for any* $0\leqq r\leqq s\leqq t\leqq T$, *the following inequalities hold*:

$$\|A(0)^{\alpha}[U(t,0)-U(s,0)]A(0)^{-1}\|\leqq C(t-s)^{1-\alpha'} \tag{2.15}$$

$$\|A(0)^{\alpha}[U(t,r)-U(s,r)]\|\leqq C(t-s)^{1-\alpha'}(s-r)^{-1}, \tag{2.16}$$

where the constant C *is determined by* $M_0, M_1, M_2, M_3, \theta, h, k, \alpha, T$.

Proof of (2.15). Actually, by (2.5), the identity

$$\begin{aligned}
&A(0)^{\alpha}[U(t,0)-U(s,0)]A(0)^{-1}\\
&=\{A(0)^{\alpha}A(t)^{-\alpha'}A(t)^{\alpha'}[U(t,s)-e^{-(t-s)A(t)}]A(s)^{-1}\\
&\quad -A(0)^{\alpha}A(t)^{-\alpha'}\int_0^{t-s}A(t)^{\alpha'}e^{-rA(t)}dr\\
&\quad +A(0)^{\alpha}A(t)^{-\alpha'}\int_0^{t-s}A(t)^{1+\alpha'-h}e^{-rA(t)}A(t)^{h}[A(t)^{-1}-A(s)^{-1}]dr\}\\
&\quad \times A(s)U(s,0)A(0)^{-1}
\end{aligned} \tag{2.17}$$

holds.

For any $0\leqq t\leqq T$ the following inequality holds:

$$\|A(0)^{\alpha}A(t)^{-\alpha'}\|\leqq M_{\alpha\alpha'} \tag{2.18}$$

where the constant $M_{\alpha\alpha'}$ depends on α and α', but is independent of t.

In fact, from formula (1.5) and from the inequalities (1.6), (1.3), (1.4) and (2.3) it follows that for any $v\in X$, we have $A(t)^{-\alpha'}v\in D(A(0)^{\alpha})$ and there exist $C>0$, $\tilde{\delta}>0$ such that

$$\begin{aligned}
&\|A(0)^{\alpha}A(t)^{-\alpha'}v\|\\
&=\|A(0)^{h(\alpha/h)}A(t)^{h(-\alpha'/h)}v\|\\
&=\|A(0)^{h(\alpha/h)}\frac{1}{\Gamma(\alpha'/h)}\int_0^{\infty}e^{-sA(t)^h}s^{(\alpha'/h)-1}v\,ds\|\\
&\leqq \frac{C\left(0,\frac{\alpha}{h},1\right)}{\Gamma\left(\frac{\alpha'}{h}\right)}\int_0^{\infty}\|A(0)^hA(t)^{-h}A(t)^he^{-sA(t)^h}v\|^{\alpha/h}\|e^{-sA(t)^h}v\|^{1-(\alpha/h)}s^{(\alpha'/h)-1}ds\\
&\leqq \frac{C'\|v\|}{\Gamma\left(\frac{\alpha'}{h}\right)}\|A(0)^hA(t)^{-h}\|^{\alpha/h}\int_0^{\infty}\|A(t)^he^{-sA(t)^h}\|^{\alpha/h}\|e^{-sA(t)^h}\|^{1-(\alpha/h)}s^{(\alpha'/h)-1}ds\\
&\leqq \frac{\|v\|}{\Gamma\left(\frac{\alpha'}{h}\right)}M^{\alpha/h}C\int_0^{\infty}e^{-\tilde{\delta}s(\alpha/h)}s^{-(\alpha/h)}e^{-\tilde{\delta}s(1-\alpha/h)}s^{(\alpha'/h)-1}ds\\
&=\frac{\|v\|}{\Gamma\left(\frac{\alpha'}{h}\right)}M^{\alpha/h}C\tilde{\delta}^{(\alpha-\alpha')/h}\int_0^{\infty}e^{-t}t^{(\alpha'-\alpha)/h-1}dt
\end{aligned}$$

$$\leqq \Gamma\left(\frac{\alpha'}{h}\right)^{-1} M^{\alpha/h} C \delta^{(\alpha-\alpha')/h} \Gamma\left(\frac{\alpha'-\alpha}{h}\right) \|v\|$$
$$\leqq M_{\alpha\alpha'} \|v\| .$$

Thus we obtain (2.18).

In the following, the constants $C_1, C_2, \cdots$ do not depend on s, t.

We verify the following inequality:

$$\|A(t)^h [A(t)^{-1} - A(s)^{-1}]\| \leqq C_1 |t-s|^k \qquad 0 \leqq s \leqq t \leqq T . \tag{2.19}$$

From formula (1.2) and from the inequalities (2.1) and (2.4) it follows that $A(t)^{-1}v \in D(A(t)^h)$ and $A(s)^{-1}v \in D(A(t)^h)$ for any $v \in X$ and

$$\begin{aligned}
&\|A(t)^h [A(t)^{-1} - A(s)^{-1}] v\| \\
&= \|A(t)^h [A(t)^{h(-m)} - A(s)^{h(-m)}] v\| \\
&= \|A(t)^h \frac{1}{2\pi i} \int_\Gamma \lambda^{-m} [(A(t)^h - \lambda)^{-1} - (A(s)^h - \lambda)^{-1}] v \, d\lambda\| \\
&\leqq \frac{\|v\|}{2\pi} \int_\Gamma \|A(t)^h \lambda^{-m} (\lambda - A(t)^h)^{-1} (A(t)^h - A(s)^h) (\lambda - A(s)^h)^{-1}\| d|\lambda| \\
&\leqq \frac{\|v\|}{2\pi} \int_\Gamma |\lambda^{-m}| \|A(t)^h (\lambda - A(t)^h)^{-1}\| \cdot \|A(t)^h A(s)^{-h} - I\| \cdot \|A(s)^h (\lambda - A(s)^h)^{-1}\| d|\lambda| \\
&\leqq C_1 |t-s|^k \|v\| .
\end{aligned}$$

Thus we have (2.19).

For any $0 \leqq s \leqq t \leqq T$, the inequality

$$\|A(t) e^{-tA(t)} A(s)^{-1}\| \leqq C_2 e^{-\delta t} \tag{2.20}$$

holds. In fact, from (1.3), (1.4), (2.19) and $k > 1-h$ it follows that

$$\begin{aligned}
&\|A(t) e^{-tA(t)} A(s)^{-1}\| \\
&= \|e^{-tA(t)} - A(t)^{1-h} e^{-tA(t)} A(t)^h [A(t)^{-1} - A(s)^{-1}]\| \\
&\leqq \|e^{-tA(t)}\| + \|A(t)^{1-h} e^{-tA(t)}\| \cdot \|A(t)^h [A(t)^{-1} - A(s)^{-1}]\| \\
&\leqq C e^{-\delta t} + C e^{-\delta t} t^{h-1} C_1 (t-s)^k \\
&\leqq C(1 + C_1 T^{k-(1-h)}) e^{-\delta t} \\
&\leqq C_2 e^{-\delta t} .
\end{aligned}$$

Thus we get (2.20).

For any $0 \leqq s \leqq t \leqq T$ we get the bound

$$\|A(r) U(r, s) A(s)^{-1}\| \leqq C_3 \tag{2.21}$$

Actually, for any $v \in X$ we have

$$
\begin{aligned}
& A(r)U(r,s)A(s)^{-1}v \\
&= A(r)\,[e^{-(s-r)A(r)}A(s)^{-1}v+\int_s^r e^{-(r-\zeta)A(r)}[A(r)-A(\zeta)]U(\zeta,s)A(s)^{-1}v\,d\zeta] \\
&= A(r)e^{-(s-r)A(r)}A(s)^{-1}v \\
&\quad +\int_s^r A(r)e^{-(r-\zeta)A(r)}\sum_{p=1}^m A(r)^{1-ph}[A(r)^h A(\zeta)^{-h}-I]A(\zeta)^{ph}U(\zeta,s)A(s)^{-1}v\,d\zeta \\
&= A(r)e^{-(s-r)A(r)}A(s)^{-1}v \\
&\quad +\sum_{p=1}^m\int_s^r A(r)^{2-ph}e^{-(r-\zeta)A(r)}[A(r)^h A(\zeta)^{-h}-I]A(\zeta)^{ph}U(\zeta,s)A(s)^{-1}v\,d\zeta\,.
\end{aligned}
$$

Applying (2.20), (1.4) and (2.4), we get

$$
\begin{aligned}
& ||A(r)U(r,s)A(s)^{-1}v|| \\
&\leqq C_2e^{-\delta(r-s)}||v|| \\
&\quad +\sum_{p=1}^m\int_s^r C(r-\zeta)^{ph-2}e^{-\delta(r-\zeta)}M_3|r-\zeta|^k||A(\zeta)^{(p-m)h}||\cdot||A(\zeta)U(\zeta,s)A(s)^{-1}v||d\zeta \\
&\leqq C_2e^{-\delta(r-s)}||v|| \\
&\quad +\int_s^r C_4\sum_{p=1}^m (r-\zeta)^{ph-2+k}e^{-\delta(r-\zeta)}C_h^{m-p}||A(\zeta)U(\zeta,s)A(s)^{-1}v||d\zeta \\
&\leqq C_2e^{-\delta(r-s)}||v|| \\
&\quad +\int_s^r C_5(r-\zeta)^{h-2+k}\sum_{p=1}^m T^{(p-1)h}e^{-\delta(r-\zeta)}\max_{1\leqq p\leqq m}C_h^{m-p}||A(\zeta)U(\zeta,s)A(s)^{-1}v||d\zeta \\
&\leqq C_2e^{-\delta(r-s)}||v|| \\
&\quad +\int_s^r C_5(r-\zeta)^{h-2+k}e^{-\delta(r-\zeta)}||A(\zeta)U(\zeta,s)A(s)^{-1}v||d\zeta\,.
\end{aligned}
$$

Therefore, applying Gronwall's Lemma, we have

$$
\begin{aligned}
& ||A(r)U(r,s)A(s)^{-1}|| \\
&\leqq C_2e^{-\delta(r-s)}\exp\Big|\int_s^r C_5(r-\zeta)^{h-2+k}e^{-\delta(r-\zeta)}d\zeta\Big| \\
&= C_2e^{-\delta(r-\zeta)}\exp\Big|\delta^{-k+(1-h)}C_5\int_0^{(r-s)/\delta}e^{-t}t^{k-(1-h)-1}dt\Big| \\
&\leqq C_3\,.
\end{aligned}
$$

Thus (2.21) is proved.

Next, for any $0\leqq s\leqq t\leqq T$, the inequality

$$
||A(t)^{\alpha'}[U(t,s)-e^{-(t-s)A(t)}]A(s)^{-1}||\leqq C_6(t-s)^{k+h-\alpha'} \tag{2.22}
$$

holds. In fact we can write

$$
\begin{aligned}
& A(t)^{\alpha'}[U(t,s)-e^{-(t-s)A(t)}]A(s)^{-1} \\
&= A(t)^{\alpha'}[\int_s^t \exp(-(t-r)A(t))\,[A(t)-A(r)]U(\imath,s)dr]A(s)^{-1}
\end{aligned}
$$

$$= \int_s^t A(t)^{\alpha'} e^{-(t-r)A(t)} \sum_{p=1}^m A(t)^{1-ph}[A(t)^h A(r)^{-h} - I]A(r)^{ph} U(r, s)A(s)^{-1} dr$$

$$= \sum_{p=1}^m \int_s^t A(t)^{\alpha'+1-ph} e^{-(t-r)A(t)}[A(t)^h A(r)^{-h} - I]A(r)^{(p-m)h} A(r) U(r,s)A(s)^{-1} dr\,.$$

Therefore, from (2.12), (2.4) and (2.21), it follows that

$$\begin{aligned}
&||A(t)^{\alpha'}[U(t, s) - e^{-(t-s)A(t)}]A(s)^{-1}|| \\
&\leqq \sum_{p=1}^m \int_s^t N_6(t-r)^{ph-1-\alpha'} M_3(t-r)^k ||A(r)^{(p-m)h}|| C_3 dr \\
&\leqq N_6 M_3 C_3 \int_s^t (t-r)^{k+h-\alpha'-1} \sum_{p=1}^m T^{(p-1)h} \max_{1\leqq p\leqq m} ||A(r)^{(p-m)h}|| dr \\
&\leqq C_6(t-s)^{k+h-\alpha'}\,.
\end{aligned}$$

Thus (2.22) is obtained.

$$||\int_0^{t-s} A(t)^{\alpha'} e^{-rA(t)} dr|| \leqq \int_0^{t-s} N_6 r^{-\alpha'} dr \leqq C_7(t-s)^{1-\alpha'}\,. \tag{2.23}$$

This follows from (2.12).

Finally, from (2.12) and (2.19) it follows that

$$\begin{aligned}
&||\int_0^{t-s} A(t)^{1+\alpha'-h} e^{-rA(t)} A(t)^h [A(t)^{-1} - A(s)^{-1}] dr|| \\
&\leqq \int_0^{t-s} N_6 r^{h-1-\alpha'} C_1(t-s)^k dr \\
&\leqq C_8(t-s)^{k+h-\alpha'}\,.
\end{aligned} \tag{2.24}$$

Then from (2.17), (2.18), (2.22), (2.23), (2.24) and (2.21), we get

$$\begin{aligned}
&||A(0)^{\alpha}[U(t, 0) - U(s, 0)]A(0)^{-1}|| \\
&\leqq \{M_{\alpha\alpha'} C_6(t-s)^{k+h-\alpha'} + M_{\alpha\alpha'} C_7(t-s)^{1-\alpha'} + M_{\alpha\alpha'} C_8(t-s)^{k+h-\alpha'}\} C_2 \\
&\leqq C_3 M_{\alpha\alpha'} \{C_6 T^{k-(1-h)} + C_7 + C_8 T^{k-(1-h)}\}(t-s)^{1-\alpha'} \\
&\leqq C(t-s)^{1-\alpha'}
\end{aligned}$$

Thus, (2.15) is proved.

Proof of (2.16). Actually, by (2.5), the identity

$$\begin{aligned}
&A(0)^{\alpha}[U(t, r) - U(s, r)] \\
&= \{A(0)^{\alpha} A(t)^{-\alpha'} A(t)^{\alpha'}[U(t, s) - e^{-(t-s)A(t)}]A(s)^{-1} \\
&\quad - A(0)^{\alpha} A(t)^{-\alpha'} \int_0^{t-s} A(t)^{\alpha'} e^{-\zeta A(t)} d\zeta \\
&\quad + A(0)^{\alpha} A(t)^{-\alpha'} \int_0^{t-s} A(t)^{1+\alpha'-h} e^{-\zeta A(t)} A(t)^h [A(t)^{-1} - A(s)^{-1}] d\zeta\} A(s) U(s,r)
\end{aligned} \tag{2.25}$$

holds. By (2.13)

$$\|A(s)U(s,r)\| \leqq (h+k-1)^{-1}N_{18}(s-r)^{-1}. \tag{2.26}$$

Then, from (2.25), (2.18), (2.22), (2.23), (2.24) and (2.26), we have

$$\begin{aligned}&\|A(0)^{\alpha}[U(t,r)-U(s,r)]\| \\ &\leqq \{M_{\alpha\alpha'}C_6(t-s)^{k+h-\alpha'}+M_{\alpha\alpha'}C_7(t-r)^{1-\alpha'}+M_{\alpha\alpha}C_8(t-s)^{k+h-\alpha'}\} \\ &\qquad\qquad \times(h+k-1)^{-1}N_{18}(s-r)^{-1} \\ &\leqq C(t-s)^{1-\alpha'}(s-r)^{-1}\end{aligned}$$

Thus, (2.16) is proved.

REMARK. Even if $0<\alpha<\alpha'\leqq h$, (2.18) holds good.

Proposition 2. *Let the function $f(t)$ be continuous on $[0,T]$. Then for any $0\leqq s\leqq t\leqq T$, $0<\alpha<\alpha'<\alpha''<h$, the following inequality holds*:

$$\|A_0^{\alpha}[\int_0^t U(t,r)f(r)dr-\int_0^s U(s,r)f(r)dr]\| \leqq C_{\alpha\alpha'}|t-s|^{1-\alpha''}(|\log(t-s)|+1). \tag{2.27}$$

Proof. In fact, first let $s\leqq t-s$. Then from (2.18) and (2.13) it follows that

$$\begin{aligned}&\|A_0^{\alpha}[\int_0^t U(t,r)f(r)dr-\int_0^s U(s,r)f(r)dr]\| \\ &\leqq \int_0^t\|A_0^{\alpha}U(t,r)\|\cdot\|f(r)\|dr+\int_0^s\|A_0^{\alpha}U(s,r)\|\cdot\|f(r)\|dr \\ &\leqq \|A_0^{\alpha}A(t)^{-\alpha'}\|\int_0^t\|A(t)^{\alpha'}U(t,r)\|\cdot\|f(r)\|dr \\ &\qquad\qquad +\|A_0^{\alpha}A(s)^{-\alpha'}\|\int_0^s\|A(s)^{\alpha'}U(s,r)\|\cdot\|f(r)\|dr \\ &\leqq M_{\alpha\alpha'}(h+k-\alpha')^{-1}N_{18}[\int_0^t(t-r)^{-\alpha'}\|f(r)\|dr+\int_0^s(s-r)^{-\alpha'}\|f(r)\|dr] \\ &\leqq M_{\alpha\alpha'}(h+k-\alpha')^{-1}N_{18}(1-\alpha')^{-1}[t^{1-\alpha'}+s^{1-\alpha'}]\max_{0\leqq r\leqq t}\|f(r)\|.\end{aligned}$$

And $t\leqq 2(t-s)$ since $s\leqq t-s$. Therefore

$$t^{1-\alpha'}+s^{1-\alpha'}\leqq[2(t-s)]^{1-\alpha'}+(t-s)^{1-\alpha'}\leqq(2^{1-\alpha'}+1)(t-s)^{1-\alpha'}$$

hence, put

$$C_{\alpha\alpha'}=M_{\alpha\alpha'}(h+k-\alpha')^{-1}(1-\alpha')^{-1}(2^{1-\alpha'}+1)\max_{0\leqq r\leqq T}\|f(r)\|N_{18}$$

and we obtain (2.27) for $s\leqq t-s$.

If $s\geqq t-s$, then $s-(2s-t)\leqq t-s$ and from (2.18) and (2.16)

$$\|A_0^{\alpha}[\int_0^t U(t,r)f(r)dr-\int_0^s U(s,r)f(r)dr]\|$$

$$\leqq \|A_0^\alpha[\int_{2s-t}^t U(t,r)f(r)dr - \int_{2s-t}^s U(s,r)f(r)dr]\|$$
$$+\|A_0^\alpha A(t)^{-\alpha'}\int_0^{2s-t} A(t)^{\alpha'}[U(t,r)-U(s,r)]f(r)dr\|$$
$$\leqq C'_{\alpha\alpha'}|t-s|^{1-\alpha'}+M_{\alpha\alpha'}C\int_0^{2s-t}(t-s)^{1-\alpha''}(s-r)^{-1}\|f(r)\|dr$$
$$\leqq C'_{\alpha\alpha'}|t-s|^{1-\alpha'}+C''_{\alpha\alpha'}(t-s)^{1-\alpha''}[|\log(t-s)|+1]\max_{0\leqq r\leqq t}\|f(r)\|$$

Hence, put

$$C_{\alpha\alpha'} = C'_{\alpha\alpha'}+C''_{\alpha\alpha'}\max_{0\leqq r\leqq T}\|f(r)\|$$

and we obtain (2.27).

Proposition 3. *If* $0<\alpha'<\alpha''<h$, *then for any* $0\leqq r\leqq t\leqq T$, *the following inequality holds*:

$$\|A(t)^{\alpha'}U(t,r)A(r)^{1-ph}\| \leqq E(t-r)^{ph-\alpha''-1} \qquad p=1,2,\cdots,m. \tag{2.28}$$

Proof. First we note the following identity:

$$\begin{aligned}(2.29)\quad & A(t)^{\alpha'}U(t,r)A(r)^{1-ph}\\ &= A(t)^{\alpha'}\{\exp(-(t-r)A(r))\\ &\quad -\sum_{l=1}^m\int_r^t U(t,s)A(s)^{1-lh}[A(s)^hA(r)^{-h}-I]A(r)^{lh}\\ &\qquad\qquad \times\exp(-(s-r)A(r))ds\}A(r)^{1-ph}\\ &= A(t)^{\alpha'}\exp(-(t-r)A(r))A(r)^{1-ph}\\ &\quad -\sum_{l=1}^m\int_r^t A(t)^{\alpha'}U(t,s)A(s)^{1-lh}[A(s)^hA(r)^{-h}-I]A(r)^{1-ph+lh}\\ &\qquad\qquad \times\exp(-(s-r)A(r))ds.\end{aligned}$$

Set

$$\begin{cases}X_p(t,s) = A(t)^{\alpha'}U(t,s)A(s)^{1-ph}\\ X_{p,0}(t,s) = A(t)^{\alpha'}A(s)^{1-ph}\exp(-(t-s)A(s))\\ K_{l,p}(s,r) = -[A(s)^hA(r)^{-h}-I]A(r)^{1-ph+lh}\exp(-(s-r)A(r)).\end{cases} \tag{2.30}$$

We obtain a system of integral equations satisfied by $X_p, p=1,\cdots,m$.

In writing down these integral equations, we find it convenient to introduce the following notation. For any two operator-valued functions $K'(t,s)$, $K''(t,s)$ defined for $0<s<t<T$, we define their convolution by

$$K = K'*K'',\ K(t,r) = \int_r^t K'(t,s)K''(s,r)ds.$$

Then the system of integral equations for X_p has the form

$$(2.31) \qquad X_p = X_{p,0} + \sum_{l=1}^{m} X_l * K_{l,p} \qquad P = 1,2,\cdots,m.$$

From (2.30), (2.18) and (2.12) it follows that

$$(2.32) \qquad \begin{aligned} &\|X_{p,0}(t,r)\| \\ &\leqq \|A(t)^{\alpha'} A(r)^{-\alpha''}\| \cdot \|A(r)^{1-ph+\alpha''} \exp(-(t-r)A(r))\| \\ &\leqq M_{\alpha'\alpha''} N_6 (t-r)^{ph-1-\alpha''} \leqq E_1 (t-r)^{ph-\alpha''-1}. \end{aligned}$$

From (2.30), (2.4) and (2.12) it follows that

$$(2.33) \qquad \begin{aligned} \|K_{l,p}(s,r)\| &\leqq \|A(s)^h A(r)^{-h} - I\| \cdot \|A(r)^{1-ph+lh} \exp(-(s-r)A(r))\| \\ &\leqq M_3 (s-r)^k N_6 (s-r)^{ph-lh-1} \leqq E_2 (s-r)^{k+ph-lh-1}. \end{aligned}$$

Suppose that the system (2.31) has been solved for X_p by successive approximation in the form

$$(2.34) \qquad X_p(t,r) = \sum_{i=0}^{\infty} X_{p,i}(t,r),$$

$$(2.35) \qquad X_{p,i+1} = \sum_{l=1}^{m} X_{l,i} * K_{l,p}.$$

Applying (2.32) and (2.33), we shall show that the series (2.34) are in fact convergent, with the rate of convergence determined by the constants, T, θ, h, k, α', $\alpha'', M_0, M_1, M_2, M_3$ alone. For convenience in this estimation, we further introduce the following notation. We denote by $P(a, M)$ the set of all operator-valued function $K(t,s)$, defined and strongly continous for $0 \leqq s \leqq t \leqq T$ such that

$$\|K(t,s)\| \leqq M(t-s)^{a-1}.$$

In particular, $K \in P(a, M)$ with $a > 1$ implies that $K(t,s)$ is continous even for $s=t$ and $K(t,t)=0$. The following Lemma is a direct consequence of the definition.

Lemma 1. *If $K' \in P(a', M')$ and $K'' \in P(a'', M'')$ with a' and a'' positive, then $K' * K'' \in P(a'+a'', B(a', a'')M'M'')$. Here B denotes the beta function.*

Now we have from (2.32) and (2.33)

$$(2.36) \qquad X_{p,0} \in P(ph-\alpha'', E_1),$$

$$(2.37) \qquad K_{l,p} \in P(k+ph-lh, E_2);$$

(2.36) and (2.37) lead to the following estimate on $X_{p,i}$:

$$(2.38) \qquad X_{p,i} \in P(ph-\alpha''+ik, L_i E_1 (mE_2)^i) \qquad i \in \boldsymbol{N}$$

where $\{L_i\}$ is a sequence defined successively by

$$L_0 = 1,\ L_{i+1}/L_i = B(h-\alpha''+ik,\ h+k-1)\,. \tag{2.39}$$

(2.38) can be proved by mathematical induction. For $i=0$, it coincides with (2.36). Assuming that it was proved for i, we have from (2.35) and (2.37), using Lemma 1,

$$\begin{aligned}X_{l,i}*K_{l,p}&\in P(ph-\alpha''+(i+1)k,\ C_{l,p,i})\,,\\ C_{l,p,i}&= B(lh-\alpha''+ik,\ k+ph-lh)L_iE_1m^iE_2^{i+1}\\ &\leqq B(h-\alpha''+ik,\ h+k-1)L_iE_1m^iE_2^{i+1}\end{aligned}$$

from which (2.38) follows for i replaced by $i+1$ in virtue of (2.39). Here it should be noted that $lh-\alpha''+ik\geqq h-\alpha''+ik>0$, $k+ph-lh\geqq k-mh+h=k-1+h>0$.

It follows from (2.39) that

$$L_{i+1}/L_i = 0(i^{-(h+k-1)}) \qquad [i\to+\infty]\,.$$

Since $h+k-1>0$, we see from (2.38) that the series in (2.34) are absolutely convergent for $s<t$, the convergence being uniform for $t-s\geqq a>0$. Noting that the first term in each of these series is estimated by (2.36), we thus obtain the estimates

$$X_p\in P(ph-\alpha'',\ E) \qquad p=1,2,\cdots,m\,.$$

where E may depend on $\alpha,\theta,h,k,M_0,\cdots$ alone. Thus (2.28) is proved.

Proposition 4. *Let the function $f(t)$ be Hölder continuous on $[0,T]$. Then for any $0\leqq r\leqq T$, the following inequality holds:*

$$\|A(r)^{ph}\int_0^r U(r,s)f(s)ds\| \leqq E'r^{1-ph} \quad :p=1,2,\cdots,m\,. \tag{2.40}$$

Proof. Actually, the identity

$$\begin{aligned}&\int_0^r U(r,s)f(s)ds \qquad\qquad (2.41)\\ &=\int_0^r[\exp(-(r-s)A(r))+\int_s^r\exp(-(r-\zeta)A(r))[A(r)-A(\zeta))]\\ &\qquad\qquad\qquad\qquad \times U(\zeta,s)d\zeta]f(s)ds\\ &=\int_0^r\exp(-(r-s)A(r))f(s)ds\\ &\quad+\sum_{p=1}^m\int_0^r\int_s^r A(r)^{1-ph}\exp(-(r-\zeta)A(r))[A(r)^hA(\zeta)^{-h}-I]\\ &\qquad\qquad\qquad\qquad \times A(\zeta)^{ph}U(\zeta,s)d\zeta f(s)ds\\ &=\int_0^r\exp(-(r-s)A(r))f(s)ds\end{aligned}$$

$$+\sum_{p=1}^{m}\int_0^r A(r)^{1-ph}\exp(-(r-\zeta)A(r))[A(r)^h A(\zeta)^{-h}-I]$$
$$\times\int_0^\zeta A(\zeta)^{ph}U(\zeta,s)f(s)dsd\zeta\,.$$

Multiplying (2.41) from left by $A(r)^{qh}$, we obtain a system of integral equations

$$Y_q = Y_{q,0}+\sum_{p=1}^{m} H_{q,p}*Y_p \qquad q=1,2,\cdots,m\,, \tag{2.42}$$

where

$$\begin{cases} Y_q(r,0) = A(r)^{qh}\int_0^r U(r,s)f(s)ds\,, \\ Y_{q,0}(r,0) = A(r)^{qh}\int_0^r \exp(-(r-s)A(r))f(s)ds\,, \\ H_{q,p}(r,s) = A(r)^{1+qh-ph}\exp(-(r-s)A(r))[A(r)^h A(s)^{-h}-I]\,. \end{cases} \tag{2.43}$$

In the following the constants $E_3, E_4, \cdots$ do not depend on r,s.
We get

$$||Y_{q,0}(r,0)|| \leqq E_3 r^{1-ph} \qquad q=1,2,\cdots,m\,. \tag{2.44}$$

In fact, for $q=1,2,\cdots m-1$, from (2.12) it follows that

$$\begin{aligned} ||Y_{q,0}(r,0)|| &\leqq \int_0^r ||A(r)^{qh}\exp(-(r-s)A(r))||\cdot||f(s)||ds \\ &\leqq \int_0^r N_6(r-s)^{-qh}ds \max_{0\leqq t\leqq T}||f(t)|| \leqq E_4 r^{1-qh}\,. \end{aligned}$$

The case $q=m$. Noting that there exists $E_5>0$, $0<k\leqq 1$ such that $||f(t)-f(s)|| \leqq E_5|t-s|^k$ for every s,t in $[0,T]$, from (2.12), we have

$$\begin{aligned} &||Y_{m,0}(r,0)|| \\ &= ||\int_0^r A(r)\exp(-(r-s)A(r))[f(s)-f(r)]ds+\int_0^r A(r)\exp(-(r-s)A(r))f(r)ds|| \\ &\leqq \int_0^r ||A(r)\exp(-(r-s)A(r))||\cdot||f(s)-f(r)||ds+||\int_0^r \frac{\partial}{\partial s}\exp(-(r-s)A(r))dsf(r)|| \\ &\leqq \int_0^r N_6(r-s)^{-1}E_5|s-r|^k ds+2N_6\max_{0\leqq t\leqq T}||f(t)|| \\ &\leqq E_6\,. \end{aligned}$$

Hence for a constant $E_3 \geqq \max\{E_4, E_6\}$ we obtain (2.44).

From (2.11) and (2.4) it follows that

$$\begin{aligned} &||H_{q,p}(r,s)|| \\ &\leqq ||A(r)^{1+qh-ph}\exp(-(r-s)A(r))||\cdot||A(r)^h A(s)^{-h}-I|| \\ &\leqq N_6(r-s)^{-1-qh+ph}M_3(r-s)^k \\ &\leqq E_7(r-s)^{k+ph-qh-1}\,. \end{aligned} \tag{2.45}$$

Suppose that the system (2.42) has been solved for Y_q by successive approximation in the form

$$Y_q(r,0)=\sum_{i=0}^{\infty} Y_{q,i}(r,0)\,, \tag{2.46}$$

$$Y_{q,i+1}=\sum_{p=1}^{m} H_{q,p}*Y_{p,i}\,. \tag{2.47}$$

Applying (2.44) and (2.45), we shall show that the series (2.46) are in fact convergent.

We have from (2.44) and (2.45)

$$Y_{q,0}\in P(2-qh,\, E_3) \tag{2.48}$$

$$H_{q,p}\in P(k+ph-qh,\, E_7) \tag{2.49}$$

(2.48) and (2.49) lead to the following estimates on $Y_{q,i}$:

$$Y_{q,i}\in P(2-qh+ik,\, L_iE_3(mE_7)^i) \qquad i\in N \tag{2.50}$$

where L_i is a sequence defined successively by

$$L_0=1,\ L_{i+1}/L_i=B(1+ik,\, k+h-1) \tag{2.51}$$

It follows from (2.51) that

$$L_{i+1}/L_i=0(i^{-(h+k-1)}) \qquad [i\to+\infty]$$

Since $h+k-1>0$ we see from (2.50) that the series in (2.46) are absolutely convergent for $s<t$, the convergence being uniform for $t-s\geqq a>0$. Noting that the first term in each of these series is estimated by (2.48), we thus obtain the estimates

$$Y_q\in P(2-qh,\, E_8) \qquad q=1,2,\cdots,m \tag{2.52}$$

Hence put $E'=E_8$, and (2.40) is proved.

3. Existence of the solutions on the real axis

We consider the Cauchy problem

$$\frac{du}{dt}+A(t,u)u=f(t,u) \qquad 0\leqq t\leqq T \tag{3.1}$$

$$u(0)=u_0\,. \tag{3.2}$$

We shall make the following assumptions:

3°) For some $0<\alpha<h=1/m$, where m is an integer, $\geqq 2$, and $R>0$ and for any $v\in N(R)\equiv\{v\in X;\,||v||<R\}$ the operator $A(t,A_0^{-\alpha}v)=A(t,A(0,u_0)^{-\alpha}v)$ is well defined on $D(A(t,A_0^{-\alpha}v))$, for all $0\leqq t\leqq T$.

4°) For any $t\in[0,T]$ and $v\in N(R)$, the operator $A(t,A_0^{-\alpha}v)$ is a closed operator from X to X with a domain $D(A(t,A_0^{-\alpha}v))$ dense in X and

$$(3.3)\qquad \|(\lambda I-A(t,A_0^{-\alpha}v))^{-1}\|\leqq C_1/(1+|\lambda|)\quad \text{for all } \lambda \text{ with } Re\,\lambda\leqq 0$$

where C_1 is a constant independent of t,v.

5°) For every $t\in[0,T]$ and $v\in N(R)$, the domain $D(A(t,A_0^{-\alpha}v)^h)\equiv D$ of $A(t,A_0^{-\alpha}v)^h$ does not depend on t,v. Furthermore, for any $t,s\in[0,T]$ and $v,w\in N(R)$

$$(3.4)\qquad \|A(t,A_0^{-\alpha}v)^h A(s,A_0^{-\alpha}w)^{-h}\|\leqq C_2$$

$$(3.5)\qquad \|A(t,A_0^{-\alpha}v)^h A(s,A_0^{-\alpha}w)^{-h}-I\|\leqq C_3\{|t-s|^\sigma+\|v-w\|\}$$

where $1-h<\sigma\leqq 1$.

(6°) For every $t,s\in[0,T]$ and $v,w\in N(R)$

$$(3.6)\qquad \|f(t,A_0^{-\alpha}v)-f(s,A_0^{-\alpha}w)\|\leqq C_4\{|t-s|^\sigma+\|v-w\|\}$$

7°) $u_0\in D(A_0)$ and

$$(3.7)\qquad A_0^\alpha u_0\in N(R).$$

Theorem 2. *Let the assumptions* 3°)-7°) *hold. Then there exists a unique solution of* (3.1) *which is continuously differentiable for* $0<t\leqq t^*$, *continuous for* $0\leqq t\leqq t^*$ *and satisfies* (3.2).

Proof. We first introduce sets $Q(s,L,k)$. Here k is any number satisfying $1-h<k<\min\{1-\alpha,\sigma\}$ and L is any positive number. A function $v(t)$, defined for $0\leqq t\leqq s$, is said to belong to $Q(s,L,k)$ if

$$v(0)=A_0^\alpha u_0$$

and if for any t_1,t_2 in $[0,s]$

$$(3.8)\qquad \|v(t_1)-v(t_2)\|\leqq L|t_1-t_2|^k.$$

Suppose $s_1\in(0,T]$. Then for any $v\in Q(s_1,L,k)$

$$(3.9)\qquad \|v(t)\|\leqq L|t-0|^k+\|v(0)\|\leqq Lt^k+\|A_0^\alpha u_0\|.$$

From (3.7) and (3.8) it follows that if $0<s_2<\min\{s_1,[L^{-1}(R-\|A_0^\alpha u_0\|)]^{1/k}\}$, then

$$\|v(t)\|<L[L^{-1}(R-\|A_0^\alpha u_0\|)]+\|A_0^\alpha u_0\|=R\quad \text{for } t\in[0,s_2].$$

Hence the operator

$$(3.10)\qquad A_v(t)=A(t,A_0^{-\alpha}v(t))$$

is well defined for $t\in[0,s_2]$ and, by (3.3)

$$||(\lambda I-A_v(t))^{-1}||\leqq C_1/(1+|\lambda|) \quad \text{if } Re\,\lambda\leqq 0,\ t\in[0,s_2]\,.$$

From (3.4) we obtain

$$||A_v(t)^h A_v(s)^{-h}||\leqq C_2 \quad \text{if } t,s\in[0,s_2]\,.$$

From (3.5) and (3.8) we also get

$$\begin{aligned}||A_v(t)^h A_v(s)^{-h}-I|| &\leqq C_3\{|t-s|^{\sigma}+||v(t)-v(s)||\}\\ &\leqq C_3\{T^{\sigma-k}+L\}\,|t-s|^{k}\,.\end{aligned}$$

By Theorem A, there exists a fundamental solution $U_v(t,s)$ corresponding to $A_v(t)$ and all the estimates for fundamental solutions derived in previous section hold uniformly with respect to v in $Q(s_2,L,k)$. In paritcular, from (2.15) and (2.16) we get for $0<\alpha<\alpha'<1-k$, $0\leqq r\leqq s\leqq t\leqq s_2$

$$||A_0^{\alpha}[U_v(t,0)-U_v(s,0)]A_0^{-1}||\leqq \tilde{C}\,|t-s|^{1-\alpha'} \tag{3.11}$$

$$||A_0^{\alpha}[U_v(t,r)-U_v(s,r)]||\leqq \tilde{C}\,|t-s|^{1-\alpha'}\,|s-r|^{-1} \tag{3.12}$$

where $\tilde{C}$ is the constant depending on $\theta,h,k,\alpha,C_1,C_2,C_3,s_2$.

Setting $f_v(t)=f(t,A_0^{-\alpha}v(t))$, it follows from (3.6) and (3.8) that

$$\begin{aligned}||f_v(t)-f_v(s)|| &\leqq C_4\{|t-s|^{\sigma}+||v(t)-v(s)||\}\\ &\leqq C_4\{T^{\sigma-k}+L\}\,|t-s|^{k}\,.\end{aligned} \tag{3.13}$$

Since $f_v(0)=f(0,A_0^{-\alpha}v(0))=f(0,u_0)$ is independent of v, (3.13) implies that

$$\max_{0\leqq t\leqq s_2}||f_v(t)||\leqq ||f(0,u_0)||+C_4\{s_2^{\sigma-k}+L\}\,s_2^{k}\leqq C_5 \tag{3.14}$$

Set $w_v(t)=A_0^{\alpha}w(t)$, where w is the unique solution of

$$\frac{dw}{dt}+A_v(t)w=f_v(t) \qquad t\in[0,s_2] \tag{3.15}$$

$$w(0)=u_0\,. \tag{3.16}$$

Then from (3.13) and Theorem A, w_v is given by

$$w_v(t)=A_0^{\alpha}U_v(t,0)u_0+A_0^{\alpha}\int_0^t U_v(t,s)f_v(s)ds\,. \tag{3.17}$$

In view of (3.17), for any t_1,t_2 in$[0,s_2]$ we obtain

$$\begin{aligned}&||w_v(t_1)-w_v(t_2)||\\ &\leqq ||A_0^{\alpha}[U_v(t_1,0)-U_v(t_2,0)]A_0^{-1}||\cdot||A_0u_0||\\ &\quad+||A_0^{\alpha}[\int_0^{t_1}U_v(t_1,s)f_v(s)ds-\int_0^{t_2}U_v(t_2,s)f_v(s)ds]||\end{aligned} \tag{3.18}$$

Making use of (3.13), (3.14) and (2.27), we find that

$$\begin{aligned}(3.19)\qquad & \|A_0^{\alpha}[\int_0^{t_1} U_v(t_1,s)f_v(s)ds-\int_0^{t_2} U_v(t_2,s)f_v(s)ds]\| \\ & \leqq \tilde{C}|t_1-t_2|^{1-\alpha'}(|\log(t_1-t_2)|+1)\end{aligned}$$

Therefore from (3.18), (3.11) and (3.19) it follows that

$$\begin{aligned}& \|w_v(t_1)-w_v(t_2)\| \\ & \leqq \tilde{C}|t_1-t_2|^{1-\alpha'}\|A_0u_0\|+C|t_1-t_2|^{1-\alpha'}(|\log(t_1-t_2)|+1)\end{aligned}$$

Hence if $s_3>0$ satisfies $\tilde{C}s_3^{1-k-\alpha'}\|A_0u_0\|+Cs_3^{1-k-\alpha'-\varepsilon}|t_1-t_2|^{\varepsilon}(|\log(t_1-t_2)|+1)\leqq L$ where $0<\varepsilon<1-k-\alpha'$ and if $s_3\leqq s_2$, the inequality

$$(3.20)\qquad \|w_v(t_1)-w_v(t_2)\|\leqq L|t_1-t_2|^k \quad \text{for } t_1,t_2\in[0,s_3]$$

holds.

Since (3.16) implies

$$(3.21)\qquad w_v(0)=A_0^{\alpha}w(0)=A_0^{\alpha}u_0\,,$$

we get $w_v\in Q(s_3,L,k)$.

We set $F_3=Q(s_3,L,k)$ and define a transformation $w_v=Tv$ for $v\in F_3$. Then from (3.21) and (3.20) we have

$$\begin{aligned}& (Tv)(0)=w_v(0)=A_0^{\alpha}u_0\,, \\ & \|(Tv)(t_1)-(Tv)(t_2)\|\leqq L|t_1-t_2|^k \quad \text{for } t_1,t_2\in[0,s_3]\end{aligned}$$

that is, T maps F_3 into itself.

We now consider F_3 as a subset of the Banach space $Y\equiv C([0,s_3];X]$ consisting of all the continuous functions $v(t)$ from $[0,s_3]$ into X with norm

$$|||v|||=\sup_{0\leqq t\leqq s_3}\|v(t)\|\,.$$

We shall prove that T is a continuous mapping in F_3 (with the topology induced by Y) and that furthermore, if s_3 is sufficiently small, then T is a contraction mapping.

i) The case of bounded $A(t,A_0^{-\alpha}v)$.

If $A(t,A_0^{-\alpha}v)$ is assumed to be bounded for some $t\in[0,s_3]$ and some $v\in N(R)$, in addition to the assumptions 4°) and 5°), it follows that $A(t,A_0^{-\alpha}v)\in B(X)$ for all $t\in[0,s_3]$ and $v\in N(R)$. In fact the boundedness of $A(t,A_0^{-\alpha}v)$ implies that of $A(t,A_0^{-\alpha}v)^h$ so that the constant domain $D=D(A(t,A_0^{-\alpha}v)^h)$ must coincide with X. From (3.4) it follows that for any s in $[0,s_3]$ and $w\in N(R)$

$$\begin{aligned}\|A(s,A_0^{-\alpha}w)^h\| & \leqq \|A(s,A_0^{-\alpha}w)^hA(t,A_0^{-\alpha}v)^{-h}\|\cdot\|A(t,A_0^{-\alpha}v)^h\| \\ & \leqq C_2\|A(t,A_0^{-\alpha}v)^h\|\end{aligned}$$

Thus $A(s,A_0^{-\alpha}w)^h\in B(X)$ and hence $A(s,A_0^{-\alpha}w)\in B(X)$ for all s and w.

Let v_1, v_2 belong to F_3 and set

$$
(3.22)\qquad \begin{cases} A_i(t) = A(t, A_0^{-\alpha} v_i(t)) \\ U_i(t,s) = U_{v_i}(t,s) \\ f_i(t) = f(t, A_0^{-\alpha} v_i(t)) \\ z_i(t) = A_0^{-\alpha} w_{v_i}(t) \qquad\qquad i = 1, 2\,. \end{cases}
$$

Thus, for $i=1,2$.

$$
(3.23)\qquad \begin{cases} \dfrac{dz_i}{dt} + A_i(t) z_i = f_i(t) \\ z_i(0) = u_0\,. \end{cases}
$$

Note that $z_1(t) \in D(A_2(t))$, $z_2(t) \in D(A_1(t))$ since $A_i(t) \in B(X)$ $[i=1,2]$, and we get

$$
(3.24)\qquad \frac{d}{dt}(z_1 - z_2) + A_1(t)(z_1 - z_2) = [A_2(t) - A_1(t)] z_2 + [f_1(t) - f_2(t)]\,.
$$

Now, we shall show the following,

Lemma 2. *$[A_2(t) - A_1(t)] z_2(t)$ is Hölder continuous in t for $0 \leqq t \leqq s_3$.*

Proof of Lemma. Write

$$
\begin{aligned}
(3.25)\qquad & [A_2(t) - A_1(t)] z_2(t) - [A_2(s) - A_1(s)] z_2(s) \\
&= [A_2(t) - A_2(s)] z_2(t) + A_2(s) [z_2(t) - z_2(s)] \\
&\quad - [A_1(t) - A_1(s)] z_2(t) - A_1(s) [z_2(t) - z_2(s)]\,.
\end{aligned}
$$

First we verify the following two inequalities:

$$
(3.26)\qquad ||[A_i(t) - A_i(s)] z_2(t)|| \leqq D_1 (t-s)^{\sigma} \qquad 0 \leqq s \leqq t \leqq s_3,\ i = 1, 2\,,
$$

$$
(3.27)\qquad ||A_i(s) [z_2(t) - z_2(s)]|| \leqq D_2 (t-s)^{1-h} \qquad 0 \leqq s \leqq t \leqq s_3,\ i = 1, 2\,,
$$

where the constants D_1, D_2 do not depend on v_i, s, t but depend on $||A_0^h||$.

In fact from (3.4), (3.5), (2.13) and (3.14) we have

$$
\begin{aligned}
& ||[A_i(t) - A_i(s)] z_2(t)|| \\
&= ||\sum_{p=1}^{m} A_i(t)^{1-ph} [A_i(t)^h A_i(s)^{-h} - I] A_i(s)^{ph} \{U_2(t,0) u_0 + \int_0^t U_2(t,r) f_2(r) dr\}|| \\
&\leqq \sum_{p=1}^{m} ||A_i(t)^h||^{m-p} ||A_i(t)^h A_i(s)^{-h} - I|| \cdot ||A_i(s)^h||^p \\
&\qquad\qquad \times [||U_2(t,0) u_0|| + \int_0^t ||U_2(t,r) f_2(r)|| dr] \\
&\leqq m C_2^m (t-s)^{\sigma} [(h+k)^{-1} N_{18} ||u_0|| + t(h+k)^{-1} N_{18} C_5] ||A_0^h||^m C_3 \\
&\leqq E_1 (t-s)^{\sigma}\,.
\end{aligned}
$$

In fact from (3.4), (3.11) and (3.19) we have

$$
\begin{aligned}
&\|A_i(s)[z_2(t)-z_2(s)]\| \\
&\leqq \|A_i(s)A_0^{-\alpha}\|\cdot\|A_0^{\alpha}\{U_2(t,0)u_0+\int_0^t U_2(t,r)f_2(r)dr \\
&\qquad\qquad -U_2(s,0)u_0-\int_0^s U_2(s,r)f_2(r)dr\}\| \\
&\leqq \|A_i(s)A_0^{-\alpha}\|\{\|A_0^{\alpha}[U_2(t,0)-U_2(s,0)]A_0^{-1}\|\cdot\|A_0u_0\| \\
&\qquad\qquad +\|A_0^{\alpha}[\int_0^t U_2(t,r)f_2(r)dr-\int_0^s U_2(s,r)f_2(r)dr]\|\} \\
&\leqq C_2^m\|A_0^h\|^m\|A_0^{-\alpha}\|\{\tilde{C}(t-s)^{1-\alpha'}\|A_0u_0\|+C(t-s)^{1-\alpha'}(|\log(t-s)|+1)\} \\
&\leqq D_2(t-s)^{1-h}.
\end{aligned}
$$

Thus using (3.25), (3.26) and (3.27) we obtain

$$
\begin{aligned}
&\|[A_2(t)-A_1(t)]z_2(t)-[A_2(s)-A_1(s)]z_2(s)]\| \\
&\leqq 2D_1|t-s|^{\sigma}+2D_2|t-s|^{1-h} \\
&\leqq D_3|t-s|^{1-h}
\end{aligned}
\tag{3.28}
$$

so that $[A_2(t)-A_1(t)]z_2(t)$ is Hölder continuous. q.e.d.

From (3.6) for any $0\leqq s\leqq t\leqq s_3$ it follows that

$$
\|[f_1(t)-f_2(t)]-[f_1(s)-f_2(s)]\|\leqq 2C_4|t-s|^{\sigma}. \tag{3.29}
$$

Hence from (3.28) and (3.29) the right-hand side of (3.24) is Hölder continuous. Then applying Theorem A to (3.23) and $z_1(0)-z_2(0)=0$ we can write

$$
z_1(t)-z_2(t)=\int_0^t U_1(t,r)\{[A_2(r)-A_1(r)]z_2(r)+[f_1(r)-f_2(r)]\}dr \tag{3.30}
$$

Therefore from the definition of w_v we get the identity

$$
\begin{aligned}
&w_{v_1}(t)-w_{v_2}(t) \\
&=A_0^{\alpha}z_1(t)-A_0^{\alpha}z_2(t) \\
&=-A_0^{\alpha}\int_0^t U_1(t,r)\{[A_1(r)-A_2(r)]z_2(r)+[f_2(r)-f_1(r)]\}dr \\
&=-A_0^{\alpha}\int_0^t U_1(t,r)\sum_{p=1}^m A_1(r)^{1-ph}[A_1(r)^hA_2(r)^{-h}-I]A_2(r)^{ph}z_2(r)dr \\
&\quad+A_0^{\alpha}\int_0^t U_1(t,r)[f_1(r)-f_2(r)]dr \\
&=-\sum_{p=1}^m\int_0^t A_0^{\alpha}U_1(t,r)A_1(r)^{1-ph}[A_1(r)^hA_2(r)^{-h}-I]A_2(r)^{ph}z_2(r)dr \\
&\quad+\int_0^t A_0^{\alpha}U_1(t,r)[f_1(r)-f_2(r)]dr.
\end{aligned}
\tag{3.31}
$$

In the following the constants $E_1, E_2, \cdots$ do not depend on s, t, v_i, $\|A_0^h\|$.

For any $0 \leqq t \leqq s_3$, the following inequality holds:

$$\left\| \int_0^t A_0^\alpha U_1(t,r)[f_1(r)-f_2(r)]dr \right\| \leqq E_1 t^{1-h} |||v_1-v_2||| . \tag{3.32}$$

We see this, using (2.18), (2.13) and (3.6) for $0<\alpha<\alpha'<h$, as follows;

$$\begin{aligned}
&\left\| \int_0^t A_0^\alpha U_1(t,r)[f_1(r)-f_2(r)]dr \right\| \\
&\leqq \int_0^t \|A_0^\alpha A_1(t)^{-\alpha'}\| \cdot \|A_1(t)^{\alpha'} U_1(t,r)\| \cdot \|f_1(r)-f_2(r)\| dr \\
&\leqq \int_0^t M_{\alpha\alpha'}(h+k-\alpha')^{-1} N_{18}(t-r)^{-\alpha'} C_4 \|v_1(r)-v_2(r)\| dr \\
&\leqq E_1 t^{1-h} |||v_1-v_2||| .
\end{aligned}$$

Here we cite (2.28) for $A=A_1$, $U=U_1$;

$$\|A_1(t)^{\alpha'} U_1(t,r) A_1(r)^{1-ph}\| \leqq E_2(t-r)^{ph-\alpha'-1} . \tag{3.33}$$

Note that

$$A_2(r)^{ph} z_2(r) = A_2(r)^{ph} U_2(r,0)u_0 + A_2(r)^{ph} \int_0^r U_2(r,s) f_2(s) ds \tag{3.34}$$

$$\begin{aligned}
\|A_2(r)^{ph} U_2(r,0)u_0\| &\leqq \|A_2(r)^{ph} U_2(r,0) A_0^{-h}\| \cdot \|A_0^h u_0\| \\
&\leqq (k-ph+h)^{-1} N_{19} r^{h-ph} \|A_0^h u_0\| \\
&\leqq E_3 r^{h-ph}
\end{aligned} \tag{3.35}$$

since by (2.14).

From (2.40) we find that

$$\left\| A_2(r)^{ph} \int_0^r U_2(r,s) f_2(s) ds \right\| \leqq E_4 r^{1-ph} . \tag{3.36}$$

Hence using (3.34), (3.35) and (3.36) we have

$$\begin{aligned}
\|A_2(r)^{ph} z_2(r)\| &\leqq E_3 r^{h-ph} + E_4 r^{1-ph} \\
&\leqq E_5 r^{h-ph}
\end{aligned} \tag{3.37}$$

Therefore from (3.31), (3.33), (3.5), (3.37) and (3.32) it follows that

$$\begin{aligned}
&\|w_{v_1}(t) - w_{v_2}(t)\| \\
&\leqq \sum_{p=1}^m \int_0^t \|A_0^\alpha U_1(t,r) A_1(r)^{1-ph}\| \cdot \|A_1(r)^h A_2(r)^{-h} - I\| \cdot \|A_2(r)^{ph} z_2(r)\| dr \\
&\quad + \left\| \int_0^t A_0^\alpha U_1(t,r)[f_1(r)-f_2(r)]dr \right\| \\
&\leqq \sum_{p=1}^m \int_0^t E_2(t-r)^{ph-\alpha''-1} \|v_1(r)-v_2(r)\| E_5 r^{h-ph} dr + E_1 t^{1-h} |||v_1-v_2||| \\
&\leqq E_6[t^{h-\alpha''} + t^{1-h}] |||v_1-v_2|||
\end{aligned} \tag{3.38}$$

$$\leqq E_7 t^{h-\alpha''}|||v_1-v_2|||\,.$$

Hence

$$|||Tv_1-Tv_2||| = \sup_{0\leqq t\leqq s_3}||w_{v_1}(t)-w_{v_2}(t)|| \leqq E_7 s_3^{h-\alpha''}|||v_1-v_2|||\,. \tag{3.39}$$

This means that T is a Lipschitz continuous operator.

Furthermore, if $0<s_3<E^{1/(\alpha''-h)}$ for $\theta=E_7 s_3^{h-\alpha''}<1$, we get

$$\begin{aligned}|||Tv_1-Tv_2||| &= \sup_{0\leqq t\leqq s_3}||w_{v_1}(t)-w_{v_2}(t)|| \\ &\leqq E_7 s_3^{h-\alpha''}|||v_1-v_2||| \leqq \theta|||v_1-v_2|||. \qquad v_1,v_2\in F_3\end{aligned} \tag{3.40}$$

So T is a contraction mapping, and by applying fixed point theorem we can prove that there exists unique point v in F_3 such that $Tv=v$.

ii) The general case.

We now turn to general case in which $A(t,A_0^{-\alpha}v)$ is not necessarily bounded. We first construct a sequence of bounded operators $A_n(t,A_0^{-\alpha}v)$ that approximate $A(t,A_0^{-\alpha}v)$ in a certain sense. We set

$$\begin{cases}A_n(t,A_0^{-\alpha}v) = A(t,A_0^{-\alpha}v)J_n(t,A_0^{-\alpha}v)\\ J_n(t,A_0^{-\alpha}v) = [1+n^{-1}A(t,A_0^{-\alpha}v)^h]^{-m} \qquad n=1,2,\cdots\end{cases} \tag{3.41}$$

Obviously $A_n(t,A_0^{-\alpha}v)$ belong to $B(X)$ and satisfy the assumptions 1°), 2°). Therefore, all the estimates deduced in the preceding section are valid with constants independent of n. Hence from i) there exist a fundermental solution $U_{i,n}(t,s)$ corresponding to $A_n(t,A_0^{-\alpha}v_i(t))$ and a solution $z_{i,n}$ of

$$\begin{cases}\dfrac{dz_{i,n}}{dt}+A_n(t,A_0^{-\alpha}v_i(t))z_{i,n}=f_i(t)\\ z_{i,n}(0)=u_0\end{cases} \qquad \begin{matrix}v_i\in F_3\\ i=1,2\,.\end{matrix}$$

Then, we get

$$||A_n(0,u_0)^\alpha[z_{1,n}(t)-z_{2,n}(t)]|| \leqq E_8 s_3^{h-\alpha''}|||v_1-v_2||| \qquad n\in \boldsymbol{N}_+ \tag{3.42}$$

Due to Kato [3], we obtain that $A_n(0,u_0)^\alpha U_{i,n}(t,0)\to A_0^\alpha U_i(t,0)$ as $n\to\infty$. Thus T is a Lipschitz continuous operator.

Furthermore, if $0<s_4<\min\{s_3,E_8^{1/(h-\alpha'')}\}$ and set $F_4=Q(s_4,L,k)$, same as i), there exists $0<\theta<1$ such that for any $v_1,v_2\in F_4$ the inequality $|||Tv_1-Tv_2||| <\theta|||v_1-v_2|||$ holds. Then there exists unique point v in F_4 such that $Tv=v$.

Thus in i) and ii) we have shown the existence of the fixed point v for T. Noting $Tv=w_v$ and $w_v(t)=A_0^\alpha w(t)$, we have $A_0^\alpha w(t)=v(t)$ or $w(t)=A_0^{-\alpha}v(t)$. Applying (3.15) we find that

$$\frac{d}{dt}A_0^{-\alpha}v(t)+A(t,A_0^{-\alpha}v(t))A_0^{-\alpha}v(t)=f(t,A_0^{-\alpha}v(t))\,.$$

This finishes the proof of Theorem 2 for $t^*=s_4$ and $u=A_0^{-\alpha}v$.

4 Further results on linear equations

In the proof of Theorem 1 we shall use some results on analyticity of solutions of linear evolution equations of the form

$$\begin{cases} \dfrac{du}{dt}+A(t)u=f(t) \\ u(0)=u_0\,. \end{cases} \tag{4.1}$$

We shall make the following assumtions:

8°) For each $t\in\Sigma\equiv\{t\in\boldsymbol{C};\ |\arg t|<\phi,\ 0\leqq|t|\leqq T\}$, $A(t)\in L(X)$ which has resolvent set containing the sector $Q\equiv\{\lambda\in\boldsymbol{C};\ |(\arg\lambda)-\pi|\leqq\pi/2+\phi\}$ and

$$\|(\lambda+A(t))^{-1}\|\leqq C(1+|\lambda|)^{-1}, \qquad \lambda\in Q,\ t\in\Sigma\,, \tag{4.2}$$

where C is a constant independent of λ and t.

9°) There exists $h=1/m$, where m is an integer, $\geqq 2$ such that the domain, D, of $A(t)^h$ is independent of t and dense in X.

10°) There exist C_1, C_2, C_3, k, $1-h<k<1$ such that

$$\|A(t)^hA(s)^{-h}\|\leqq C_1 \qquad t,s\in\Sigma,\ |\arg(t-s)|<\phi\,, \tag{4.3}$$

$$\|A(t)^hA(s)^{-h}-I\|\leqq C_2|t-s|^k \qquad t,s\in\Sigma,\ |\arg(t-s)|<\phi\,. \tag{4.4}$$

11°) The map $t\mapsto A(t)^hA(0)^{-h}$ is analytic from $\Sigma\backslash\{0\}$ to $B(X)$.

12°) f maps Σ into X with

$$\|f(t)-f(s)\|\leqq C_3|t-s|^k \qquad t,s\in\Sigma,\ |\arg(t-s)|<\phi\,, \tag{4.5}$$

13°) $f\colon\Sigma\backslash\{0\}\to X$ is analytic.

14°) $u_0\in D(A(0))$.

Theorem 3. *Let the assumptions 8°)–14°) hold. Then there exists a unique continuous function $u\colon\Sigma\to X$ such that $u\colon\Sigma\backslash\{0\}\to X$ is analytic, $u(t)\in D(A(t))$ with $du(t)/dt+A(t)u(t)=f(t)$ for $t\in\Sigma\backslash\{0\}$ and $u(0)=u_0$. Furthermore, $A(0)^hu\colon\Sigma\backslash\{0\}\to X$ is analytic and, for $0<\alpha<1-k$, there exists constant $G>0$, such that*

$$\|A(0)^\alpha u(t)-A(0)^\alpha u(s)\|\leqq G|t-s|^k, \qquad t,s\in\Sigma,\ |\arg(t-s)|<\phi\,. \tag{4.6}$$

Proof. We first restrict t to be real in (4.1), $t\in[0,T)$. Then the family $\{A(t);0\leqq t\leqq T\}$ and the function $f\colon[0,T)\to X$ satisfy the hypotheses of Theorem A. Thus there is a continuous function $u\colon[0,T)\to X$ which is a solution to (4.1).

From (2.15) and (2.27) for any $0<\alpha<1-k$ and s,t in $[0,T)$ we obtain

$$\|A(0)^\alpha u(t)-A(0)^\alpha u(s)\| \tag{4.7}$$

$$
\begin{aligned}
&= \|A(0)^{\alpha}[U(t,0)u_0+\int_0^t U(t,r)f(r)dr] \\
&\qquad\qquad -A(0)^{\alpha}[U(s,0)u_0+\int_0^s U(s,r)f(r)dr]\| \\
&\leqq \|A(0)^{\alpha}[U(t,0)-U(s,0)]A(0)^{-1}\|\cdot\|A(0)u_0\| \\
&\qquad\qquad +\|A(0)^{\alpha}[\int_0^t U(t,r)f(r)dr-\int_0^s U(s,r)f(r)dr]\| \\
&\leqq CT^{1-k-\alpha'}|t-s|^k+C|t-s|^{1-\alpha'}(|\log(t-s)|+1) \qquad 0<\alpha<\alpha'<1-k \\
&\leqq G_1|t-s|^k .
\end{aligned}
$$

We fix α, $0<\alpha<1-k$, and we have $\|A(0)^{\alpha}u(t)\|$ bounded on $[0,T)$. In fact for any t in $[0,T)$ from (4.7) we have

$$\|A(0)^{\alpha}u(t)\|\leqq G_1|t|^k+\|A(0)^{\alpha}u_0\|\leqq G_1T^k+\|A(0)^{\alpha}u_0\| .$$

For $0<\varepsilon<T/2$ we consider the sector $\Sigma_{\varepsilon}=\{t\in \boldsymbol{C};|\arg(t-\varepsilon)|<\phi,\ |t|<T-\varepsilon\}$. Since the functions $t\mapsto A(t)^hA(0)^{-h}$ and $t\mapsto f(t)$ are analytic in a neighborhood of the closure of Σ_{ε}, and by (4.5) $f(t)$ is Hölder continuous, we can apply Theorem B: u has an extention to $\cup\{\Sigma_{\varepsilon};\varepsilon>0\}=\Sigma\backslash\{0\}$ such that u: $\Sigma\backslash\{0\}\to X$ is analytic, $u(t)\in D(A(t))$ and $du(t)/dt+A(t)u(t)=f(t)$ for $t\in\Sigma\backslash\{0\}$.

Next we shall show that $A(0)^hu$: $\Sigma\backslash\{0\}\to X$ is analytic. Actually seeing that $t\mapsto A(t)^hA(0)^{-h}$ is analytic, $t\mapsto A(0)^hA(t)^{-h}$ is analytic. By rewriting the equation as $A(t)u(t)=f(t)-u'(t)$ and using the fact that $t\mapsto u(t)$ and $t\mapsto f(t)$ are analytic, we have that $t\mapsto A(t)^hu(t)=A(t)^{-1+h}[f(t)-u'(t)]$ is analytic. Then $t\mapsto A(0)^hu(t)$ is analytic from $\Sigma\backslash\{0\}$ to X as we see the differentiability of $A(0)^hu(t)$ in the following identity

$$
\begin{aligned}
&A(0)^hu(t+\Delta t)-A(0)^hu(t) \\
&= A(0)^hA(t+\Delta t)^{-h}[A(t+\Delta t)^hu(t+\Delta t)-A(t)^hu(t)] \\
&\qquad\qquad +[A(0)^hA(t+\Delta t)^{-h}-A(0)^hA(t)^{-h}]A(t)^hu(t) .
\end{aligned}
$$

It remains to show inequality (4.6). We do this in several steps.

i) First, suppose $s\in(0,T)$ and

$$t=s+\lambda_0e^{i\theta}\in\Sigma,\ |\theta|<\phi,\ \lambda_0>0 . \tag{4.8}$$

Then $v(\lambda)=u(s+\lambda e^{i\theta})$ is a solution of the equation

$$
\begin{cases}
\dfrac{d}{d\lambda}v(\lambda)+e^{i\theta}A(s+\lambda e^{i\theta})v(\lambda)=e^{i\theta}f(s+\lambda e^{i\theta}) & 0\leqq\lambda\leqq\lambda_0 , \\
v(0)=u(s) .
\end{cases}
\tag{4.9}
$$

The family $B(\lambda)=e^{i\theta}A(s+\lambda e^{i\theta})$, $0\leqq\lambda\leqq\lambda_0$, and the function $g(\lambda)=e^{i\theta}f(s+\lambda e^{i\theta})$, $0\leqq\lambda\leqq\lambda_0$, satisfy the hypotheses 1°) and 2°), and the various constants are in-

dependent of s,t. In fact, set $t_\lambda=s+\lambda e^{i\theta}\in\Sigma$, $0\leqq\lambda\leqq\lambda_0$. Then for any $\lambda\in[0,\lambda_0]$, $D(B(\lambda))=D(A(t_\lambda))$ is dense in X and $B(\lambda)$ has resolvent containing the sector $\tilde{Q}\equiv\{\gamma\in\boldsymbol{C}: Re\,\gamma\leqq 0\}$.
And for any $\gamma\in\tilde{Q}$ from (4.2) it follows that

$$\begin{aligned}\|(\gamma-B(\lambda))^{-1}\| &= \|(e^{-i\theta}\gamma-A(t_\lambda))^{-1}\| \\ &\leqq C(1+|e^{-i\theta}\gamma|)^{-1}=C(1+|\gamma|)^{-1}.\end{aligned}$$

Furthermore for any λ,μ in $[0,\lambda_0]$ from (4.3), (4.4) and (4.5) we get the followings,

$$\begin{aligned}\|B(\lambda)^hB(\mu)^{-h}\| &= \|e^{ih\theta}A(t_\lambda)^he^{-ih\theta}A(t_\mu)^{-h}\| \\ &= \|A(t_\lambda)^hA(t_\mu)^{-h}\| \qquad \leqq C_1, \\ \|B(\lambda)^hB(\mu)^{-h}-I\| &= \|A(t_\lambda)^hA(t_\mu)^{-h}-I\| \\ &\leqq C_2|s+\lambda e^{i\theta}-(s+\mu e^{i\theta})|^k=C|\lambda-\mu|^k, \\ \|g(\lambda)-g(\mu)\| &= \|f(s+\lambda e^{i\theta})-f(s+\mu e^{i\theta})\| \\ &\leqq C_2|s+\lambda e^{i\theta}-(s+\mu e^{i\theta})|^k=C_3|\lambda-\mu|^k.\end{aligned}$$

Thus $B(\lambda)$ satisfy 1°) and 2°) and g is a Hölder continuous mapping. Hence in the same way as (4.7), we find that

$$\|B(0)^{\alpha'}v(\lambda)-B(0)^{\alpha'}v(\mu)\|\leqq G_2|\lambda-\mu|^k \qquad \alpha<\alpha'<1-k \tag{4.20}$$
$$\text{where } B(0)=e^{i\theta}A(s).$$

Therefore from (2.18) and (4.10) we get

$$\begin{aligned}&\|A(0)^\alpha u(t)-A(0)^\alpha u(s)\| \\ &=\|A(0)^\alpha A(s)^{-\alpha'}A(s)^{\alpha'}u(s+\lambda e^{i\theta})-A(0)^\alpha A(s)^{-\alpha'}A(s)^{\alpha'}u(s+0e^{i\theta})\| \\ &\leqq \|A(0)^\alpha A(s)^{-\alpha'}\|\,|e^{-i\theta}|\,\|B(0)^{\alpha'}v(\lambda)-B(0)^{\alpha'}v(0)\| \\ &\leqq M_{\alpha\alpha'}G_2|\lambda|^k\leqq G_3|t-s|^k.\end{aligned} \tag{4.11}$$

ii) in the case of $s=0$.
From (4.7) for any $\varepsilon>0$, there exists $s\in(0,T)$, $|\arg(t-s)|<\phi$, such that $\|A(0)^\alpha u(s)-A(0)^\alpha u(0)\|<\varepsilon$.
Hence according to i) we get

$$\begin{aligned}&\|A(0)^\alpha u(t)-A(0)^\alpha u(0)\| \\ &\leqq \|A(0)^\alpha u(t)-A(0)^\alpha u(s)\|+\|A(0)^\alpha u(s)-A(0)^\alpha u(0)\| \\ &\leqq G_3|t-s|^k+\varepsilon \\ &\leqq G_3|t|^k+\varepsilon.\end{aligned}$$

Then as $\varepsilon\to 0$, we get

$$\|A(0)^\alpha u(t)-A(0)^\alpha u(0)\|\leqq G_3|t|^k \qquad |\arg t|<\phi,\ t\in\Sigma. \tag{4.12}$$

iii) The general case.
In the same way as in i) for with general $s, t\in\Sigma\backslash\{0\}$, $|\arg(t-s)|<\phi$, we obtain

$$\|A(0)^{\alpha}u(t)-A(0)^{\alpha}u(s)\| \leqq G_4|t-s|^{k}. \tag{4.13}$$

Thus for $G=\max\{G_1, G_3, G_4\}$ Theorem 3 is proved.

5. Proof of Theorem 1

From (0.3) there are constants C_4, $\phi_1>0$, $T_1>0$ such that for $t\in\Sigma_1$, $w\in N$ and $|\theta|<\phi_1$ the resolvent set of $e^{i\theta}A(t, A_0^{-\alpha}w)$ contains the left plane and

$$\|(\lambda-e^{i\theta}A(t, A_0^{-\alpha}w))^{-1}\| \leqq C_4(1+|\lambda|)^{-1} \qquad Re\,\lambda\leqq 0. \tag{5.1}$$

where $\Sigma_1\equiv\{t\in C;\ |\arg t|<\phi_1,\ 0\leqq|t|<T_1\}$.

We let $\phi=\min\{\phi_0, \phi_1\}$, and in (0.1) and (0.2) we make the change of variable $t=\tau e^{i\theta}$, $\tau\in[0, T_1]$, $|\theta|<\phi$, so equations (0.1) and (0.2) become

$$\begin{cases}\dfrac{\partial v}{\partial\tau}+e^{i\theta}A(\tau e^{i\theta}, v)v = e^{i\theta}f(\tau e^{i\theta}, v)\,, \\ v(0, e^{i\theta}) = u_0\,. \end{cases} \tag{5.2}$$

where $v(\tau, e^{i\theta})=u(\tau e^{i\theta})$, $u(t)=v(|t|, t/|t|)$.

We hold $|\theta|<\phi$ fixed and apply Theorem 2 to equation (5. 2). In order to make precise, let

$$B(\tau, w, \theta) = e^{i\theta}A(\tau e^{i\theta}, w),\ g(\tau, w, \theta) = e^{i\theta}f(\tau e^{i\theta}, w)$$

for $\tau\in[0, T_1]$, $\|A_0^{\alpha}w-A_0^{\alpha}u_0\|<R$, $|\theta|<\phi$. We shall show that for fixed θ, $B(\tau, w, \theta)$ and $g(\tau, w, \theta)$ satisfy the hypotheses 3°)–7°) of section 3 with constants independent of θ.

Since $A(t, A_0^{-\alpha}w)$ is well defined for any $w\in N$ and $t\in\Sigma$ and

$$B(\tau, B_0^{-\alpha}w, \theta) \equiv B(\tau, B(0, u_0, \theta)^{-\alpha}w, \theta) = e^{i\theta}A(\tau e^{i\theta}, A_0^{-\alpha}(e^{-i\alpha\theta}w))$$

$B(\tau, B_0^{-\alpha}w, \theta)$ is well defined for $w\in N$ and $\tau\in[0, T_1]$, which verifies 3°).

4°) is verified since by (5.1) and $D(B(\tau, B_0^{-\alpha}w, \theta))=D(A(\tau e^{i\theta}, A_0^{-\alpha}(e^{-i\alpha\theta}w)))$.

For any $w\in N$ and $\tau\in[0, T_1]$ we have

$$D(B(\tau, B_0^{-\alpha}w, \theta)^{h}) = D(e^{i\theta}A(\tau, A_0^{-\alpha}(e^{-i\alpha\theta}w))^{h}) \equiv D\,,$$

and from (0.4) and (0.5) it follows that

$$\begin{aligned} &\|B(\tau_1, B_0^{-\alpha}w, \theta)^{h}B(\tau_2, B_0^{-\alpha}v, \theta)^{-h}\| \\ &\leqq \|e^{ih\theta}A(\tau_1e^{i\theta}, A_0^{-\alpha}e^{-i\alpha\theta}w)^{h}e^{-ih\theta}A(\tau_2e^{i\theta}, A_0^{-\alpha}e^{-i\alpha\theta}v)^{-h}\| \\ &\leqq C_2 \end{aligned}$$

and

$$
\begin{aligned}
&||B(\tau_1, B_0^{-\alpha}w, \theta)^h B(\tau_2, B_0^{-\alpha}v, \theta)^{-h}-I||\\
&= ||A(\tau_1 e^{i\theta}, A_0^{-\alpha}e^{-i\alpha\theta}w)^h A(\tau_2 e^{i\theta}, A_0^{-\alpha}e^{-i\alpha\theta}v)^{-h}-I||\\
&\leqq C_3\{|\tau_1 e^{i\theta}-\tau_2 e^{i\theta}|^{\sigma}+||e^{-i\alpha\theta}w-e^{-i\alpha\theta}v||\}\\
&= C\{|\tau_1-\tau_2|^{\sigma}+||w-v||\} \qquad w, v\in N,\ \tau_1, \tau_2\in[0, T_1]\,.
\end{aligned}
$$

Therefore 5°) is verified.

Next from (0.6) we get

$$
\begin{aligned}
&||g(\tau_1, B_0^{-\alpha}w, \theta)-g(\tau_2, B_0^{-\alpha}v, \theta)||\\
&= ||e^{i\theta}f(\tau_1 e^{i\theta}, A_0^{-\alpha}e^{-i\alpha\theta}w)-e^{i\theta}f(\tau_2 e^{i\theta}, A_0^{-\alpha}e^{-i\alpha\theta}v)||\\
&\leqq C_4\{|\tau_1 e^{i\theta}-\tau_2 e^{i\theta}|^{\sigma}+||e^{-i\alpha\theta}w-e^{-i\alpha\theta}v||\}\\
&= C_4\{|\tau_1-\tau_2|^{\sigma}+||w-v||\} \qquad \tau_1, \tau_2\in[0, T_1],\ v, w\in N\,,
\end{aligned}
$$

which verifies 6°).

Finally, note that

$$
\begin{aligned}
&e^{i\theta}u_0\in D(e^{i\theta}A_0)=D(B_0)\,,\\
&||B_0^{\alpha}e^{i\theta}u_0-e^{i\theta}A_0^{\alpha}u_0||<R\,,
\end{aligned}
$$

and 7°) is verified.

Hence it follows from Theorem 2, that there exist $T, 0<T\leqq\min\{T_0, T_1\}$ and a unique solution $v(\tau, e^{i\theta})$ of (5.2) defined for $\tau\in[0, T]$, $|\theta|<\phi$, which also satisfies

$$
\begin{aligned}
&(5.3)\quad ||A_0^{\alpha}v(\tau_1, e^{i\theta})-A_0^{\alpha}v(\tau_2, e^{i\theta})||\leqq K|\tau_1-\tau_2|^{k} \qquad \tau_1, \tau_2\in[0, T]\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 1-h<k<\min\{1-\alpha, \sigma\}
\end{aligned}
$$

$$
(5.4)\quad ||A_0^{\alpha}v(\tau, e^{i\theta})-A_0^{\alpha}u_0||<R \qquad \tau\in[0, T]
$$

where the constant K does not depend on θ.

Let $\Sigma\equiv\{t\in \boldsymbol{C};\ |\arg t|<\phi,\ 0\leqq|t|\leqq T\}$ and

$$
(5.5)\qquad \begin{cases} u(t)=v(|t|, t/|t|) & t\in\Sigma\setminus\{0\}\\ u(0)=u_0\,. \end{cases}
$$

We shall show that u satisfies the conclusions of Theorem 1.

The fact that $u(t)\in D(A(t, u(t)))$ and

$$
\begin{aligned}
&||A_0^{\alpha}u(t)-A_0^{\alpha}u_0||<R \quad \text{for } t\in\Sigma\setminus\{0\}\,,\\
&||A_0^{\alpha}u(t)-A_0^{\alpha}u_0||\leqq K|t|^{k} \qquad \text{for } t\in\Sigma
\end{aligned}
$$

follow from the corresponding properties of v.

We now show that $A_0^{\alpha}u;\ \Sigma\setminus\{0\}\to X$ is analytic. Actually the proof of

Theorem 2 shows that $v(\tau, e^{i\theta})$ is the limit of a sequence $\{v_n(\tau, e^{i\theta})\}$ where $v_0(\tau, e^{i\theta})\equiv u_0$, $\tau\mapsto v_n(\tau, e^{i\theta})$ is the unique solution of the linear equation $\partial v_n/\partial\tau + e^{i\theta}A(\tau e^{i\theta}, v_{n-1})v_n = e^{i\theta}f(\tau e^{i\theta}, v_{n-1})$ $\tau\in[0, T]$, (set $A_0^\alpha v_{n+1} = T(A_0^\alpha v_n)$ for $n\in\boldsymbol{N}$,) and also $A_0^\alpha v_n(\tau, e^{i\theta})$ converges to $A_0^\alpha v(\tau, e^{i\theta})$ uniformly in $\tau\in[0, T]$, v_n also satisfies

$$(5.6)\qquad \|A_0^\alpha v_n(\tau, e^{i\theta}) - A_0^\alpha u_0\| < R \qquad \tau\in[0, T],$$

$$(5.7)\qquad \|A_0^\alpha v_n(\tau_1, e^{i\theta}) - A_0^\alpha v_n(\tau_2, e^{i\theta})\| \leqq K|\tau_1-\tau_2|^k \qquad \tau_1, \tau_2\in[0, T].$$

Since $A_0^\alpha v_n$ converges to $A_0^\alpha v$, we have

$$A_0^\alpha u(t) = \lim_{n\to\infty} A_0^\alpha u_n(t) \quad \text{where } u_n(t) = v_n(|t|, t/|t|).$$

Therefore, we get the following Lemma.

Lemma 3. *$A_0^\alpha u$ is analytic.*

Proof of Lemma. From (5.6) it follows that $\{\|A_0^\alpha u_n(t)\|\}$ is uniformly bounded in $n\in\boldsymbol{N}$ and $t\in\Sigma$. Therefore, in order to show $A_0^\alpha u$ is analytic, it suffices to show

$$(5.8)\qquad A_0^\alpha u_n\colon \Sigma\backslash\{0\} \to X \text{ is analytic for each } n.$$

We shall show (5.8) by induction, combining with the following inequality

$$(5.9)\qquad \|A_0^\alpha u_n(t) - A_0^\alpha u_n(s)\| \leqq K_n|t-s|^k \quad \text{for } t, s\in\Sigma,\ |\arg(t-s)|<\phi.$$

This is true for $n=0$ since $u_0(t) = v_0(|t|, t/|t|)\equiv u_0$. Suppose they are true for u_{n-1}. We shall apply Theorem 3 to the equation

$$(5.10)\qquad \begin{cases} \dfrac{dw}{dt} + A(t, u_{n-1})w = f(t, u_{n-1}) & t\in\Sigma \\ w(0) = u_0\,. \end{cases}$$

We must show

$$H(t) \equiv A(t, u_{n-1}(t)) \text{ and } h(t) \equiv f(t, u_{n-1}(t))$$

satisfy the hypotheses of Theorem 3.

The fact that each $H(t)$ has resolvent containing the sector $|(\arg\lambda)-\pi| \leqq \pi/2+\phi$ with the estimate

$$\|(\lambda-H(t))^{-1}\| = \|(\lambda-A(t, A_0^{-\alpha}A_0^\alpha u_{n-1}(t)))^{-1}\| \leqq C_1(1+|\lambda|)^{-1},\ t\in\Sigma$$

follows from (0,3) and the fact that

$$\|A_0^\alpha u_{n-1}(t) - A_0^\alpha u_0\| < R \qquad t\in\Sigma.$$

From (0.4), we have

$$||H(t)^h H(s)^{-h}|| = ||A(t, A_0^{-\alpha} A_0^\alpha u_{n-1}(t))^h A(s, A_0^{-\alpha} A_0^\alpha u_{n-1}(s))^{-h}|| \leqq C_2 \qquad t,s\in\Sigma,\ |\arg(t-s)|<\phi\,.$$

Using (0.5), $k\leqq\sigma$ and the induction hypothesis on u_{n-1} it follows that

$$\begin{aligned}||H(t)^h H(s)^{-h}-I|| &= ||A(t, A_0^{-\alpha} A_0^\alpha u_{n-1}(t))^h A(s, A_0^{-\alpha} A_0^\alpha u_{n-1}(s))^{-h}-I||\\ &\leqq C_3\{|t-s|^\sigma+||A_0^\alpha u_{n-1}(t)-A_0^\alpha u_{n-1}(s)||\\ &\leqq C_3\{T^{\sigma-k}+K_{n-1}\}\,|t-s|^k\\ &\leqq C_3'|t-s|^k \qquad t,s\in\Sigma,\ |arg(t-s)|<\phi\,.\end{aligned}$$

The analyticity of the map

$$H(t)^h H(0)^{-h} = A(t, A_0^{-\alpha} A_0^\alpha u_{n-1}(t))^h A_0^{-h}$$

follows from the analyticity of the maps $\Phi\colon (t,w)\mapsto A(t, A_0^{-\alpha}w)^h A_0^{-h}$

Applying (0.6), $h\leqq\sigma$ and the induction hypothesis on u_{n-1} and $t\mapsto A_0^\alpha u_{n-1}(t)$. we obtain

$$\begin{aligned}||h(t)-h(s)|| &= ||f(t, A_0^{-\alpha} A_0^\alpha u_{n-1}(t))-f(s, A_0^{-\alpha} A_0^\alpha u_{n-1}(s))||\\ &\leqq C_4\{|t-s|^\sigma+||A_0^\alpha u_{n-1}(t)-A_0^\alpha u_{n-1}(s)||\}\\ &\leqq C_4\{T^{\sigma-k}+K_{n-1}\}\,|t-s|^k\\ &\leqq C_4'|t-s|^k \qquad t,s\in\Sigma,\ |\arg(t-s)|<\phi\,.\end{aligned}$$

The analyticity of the map

$$h(t) = f(t, A_0^{-\alpha} A_0^\alpha u_{n-1}(t))$$

follows from the analyticity of the maps $\Psi\colon (t,w)\mapsto f(t, A_0^{-\alpha}w)$ and $t\mapsto A_0^\alpha u_{n-1}(t)$.

Therefore $H(t)$ and $h(t)$ satisfy the hypotheses of Theorem 3. So (5.10) has a unique solution w satisfying the conclutions of Theorem 3, i.e. w satisfies (5.10) and $w\colon \Sigma\setminus\{0\}\to X$ is analytic. Furthermore $A(0)^h w\colon \Sigma\setminus\{0\}\to X$ is analytic, and there exists $K_n>0$ such that

$$||A_0^\alpha w(t)-A_0^\alpha w(s)||\leqq K_n|t-s|^k \qquad t,s\in\Sigma,\ |\arg(t-s)|<\phi\,.$$

Next, we claim $u_n\equiv w$. We must show $v_n(\tau, e^{i\theta})=w(\tau e^{i\theta})$. This is true because the function $\tau\mapsto w(\tau e^{i\theta})$ is also a solution to $\dfrac{\partial v_n}{\partial\tau}+e^{i\theta}A(\tau e^{i\theta}, v_{n-1})v_n= e^{i\theta}f(\tau e^{i\theta}, v_{n-1})$ and hence $v_n(\tau, e^{i\theta})=w(\tau e^{i\theta})$ by uniqueness. Hence (5.8) and (5.9) are obtained.

This completes the proof that $A_0^\alpha u\colon \Sigma\setminus\{0\}\to X$ is analytic. q.e.d.

The continuity of $A_0^\alpha u\colon\Sigma\to X$ follows from the analyticity of $A_0^\alpha u\colon \Sigma\setminus\{0\}\to X$ and the estimate $||A_0^\alpha u(t)-A_0^\alpha u_0||\leqq K|t|^k$ for t in Σ. Finally the fact that u satisfies the differential equation (0.1) and (0.2) follows from the corresponding property of v.

To show that u is unique, it suffices to restrict to real t since u is analytic. However, for real t, uniqueness is included in Theorem 2.

This finishes the proof of Theorem 1.

References

[1] A. Friedman: Partial differential equations, Holt, Rinehart and Winston, New York, 1969.

[2] T.L. Hayden and F.J. Massey III: *Nonlinear holomorphic semigroups*, Pacific J. Math. **57** (1975), 423–439.

[3] T. Kato: *Abstract evolution equations of parabolic type in Banach and Hilbert spaces*, Nagoya Math. J. **5** (1961), 93–125.

[4] S.G. Krein: Linear differential equations in a Banach space, Izdatel'stov Nauka, Moscow. (in Russian); Japanese transl., Yoshioka-shoten, Kyoto, 1972.

[5] F.J. Massey III: *Analyticity of solutions of nonlinear evolution equations*, J. Differential Eqations **22** (1976), 416–427.

[6] S. Ōuchi: *On the analyticity in time of solutions of initial boundary value problems for semi-linear parabolic differential equations with monotone nonlinearity*, J. Fac. Sci. Univ. Tokyo, Sect. 1A **20** (1974), 19–41.

[7] P.E. Sobolevskii: *Equations of parabolic type in a Banach space*, Trudy Moscow Mat. Obsc. **10** (1961), 297–350. (in Russian); English transl., Amer. Math. Soc. Transl. Ser. II, **49** (1965), 1–62.

[8] P.E. Sobolevskii: *Parabolic equations in Banach space with an unbounded variable operator, a fractional power of which has a constant domain of definition*, Dokl. Akad. Nauk SSSR, **138** (1961), 59–62. (in Russian); English transl., Soviet. Math. Dokl. **2** (1961), 545–548.

[9] H. Tanabe: Equations of evolution, Iwanami-shoten, Tokyo. 1975. (in Japanese); English transl., Monogr. & Studies in Math. vol, 6. Pitman, 1979.

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