ON RIEMANNIAN MANIFOLDS ADMITTING CERTAIN STRICTLY CONVEX FUNCTIONS

Atsushi KASUE

1. Introduction. Let M be an m-dimensional connected complete Riemannian manifold with metric g. For a smooth function f on M, the *Hessian* D^2f of f is defined by $D^2f(X,Y)=X(Yf)-D_XY\cdot f(X,Y\in TM)$. By a theorem of H.W. Wissner ([5; Satz. II. 1.3]), if there is a smooth function f on M such that $D^2f=g$ on M, then M is isometric to Euclidean space. In this note, we shall prove that if the Hessian of a smooth function f on M is close enough to g, then M is quasi-isometric to Euclidean space in the following sense: There exists a diffeomorphism $F: M \rightarrow R^m$ and some positive constant μ such that for each tangent vector X on M, $\mu^{-1}||X||_M \leq ||F_*X||_{R^m} \leq ||X||_M$. Our result contains the above theorem by Wissner as a special case ($\mu=1$), and generalises Yagi's theorem ([7]). Our theorem is stated as follows.

Theorem. *Let M be an m-dimensional connected complete Riemannian manifold with metric g. Suppose there exists a smooth function f on M which satisfies the following conditions*:

(i) $(1-H_1(f(x)))g(X, X) \leq \dfrac{1}{2} D^2f(X, X) \leq (1+H_2(f(x)))g(X, X)$,

where $X \in T_xM (x \in M)$ and each H_i $(i=1,2)$ is a nonnegative continuous function on R,

(ii) $1-H_1(t)>0$ *for* $t \in R$ *and* $\lim\limits_{t \to \infty} H_i(t) = 0$ $(i = 1, 2)$,

(iii) $\begin{cases} \displaystyle\int^{\infty} H_i(s)/s\, ds < +\infty\,, \\ \displaystyle\int^{\infty}(\int_0^s H_i(u)du/s^2)ds < +\infty\ (i = 1, 2)\,. \end{cases}$

Then M is quasi-isometric to Euclidean space.

2. Proof of theorem and corollaries. Let M be an m-dimensional connected complete Riemannian manifold with metric g.

Lemma 1. *Let M and g be as above. Let f be a smooth function on M such*

that the eigenvalues of $D^2 f$ are bounded from below by some positive constant 2ν outside a compact subset C. Then f is an exhaustion function, that is, $\{x \in M : f(x) \leqq t\}$ is compact for each $t \in R$. In particular, f takes the minimum on M.

Proof. Suppose $f \gneqq \lambda = \inf \{f(x) : x \in M\}$ $(-\infty \leqq \lambda < \infty)$. Then there is a divergent sequence $\{p_n\}_{n \in N}$ in M with $\lim_{n \to \infty} f(p_n) = \lambda$. Fix any point $o \in M$. Let $\gamma_n : [0, 1_n] \to M$ be a minimizing geodesic joining o to p_n for each $n \in N$, where $1_n = \mathrm{dis}(o, p_n)$. Then by the assumption of Lemma 1, we can choose sufficiently large N and T so that $\gamma_n(t) \in M - C$ and $f(\gamma_n)''(t) = D^2 f(\dot{\gamma}_n, \dot{\gamma}_n)(t) \geqq \nu$ for any $n \geqq N$ and $t \in [T, 1_n]$. This implies $f(\gamma_n)(t) \geqq f(\gamma_n)(T) + f(\gamma_n)'(T)(t - T)$ $+ \dfrac{\nu}{2}(t - T)^2$ for $t \in [T, 1_n]$. Taking $t = 1_n$, we have $f(p_n) \geqq f(\gamma_n)(T) + f(\gamma_n)'(T)$ $(1_n - T) + \dfrac{\nu}{2}(1_n - T)^2$. Since the distance between o and $\gamma_n(T)$ equals T for each $n \geqq N$, $\{f(\gamma_n)'(T)\}_{n \in N}$ is bounded. In the preceding inequality, the left side tends to λ and the right side goes to infinity as $n \to \infty$. This is a contradiction. Therefore f takes the minimum at some points. By the same way, we see that f is an exhaustion function on M. This completes the proof of Lemma 1.

Proof of Theorem. By (ii) in Theorem and Lemma 1, we can see that f is a strictly convex exhaustion function on M. Let λ be the minimum of f on M and $o \in M$ be the only one point such that $f(o) = \lambda$. Set $\tilde{f}(x) = f(x) - \lambda$, $k(x) = \tilde{f}(x)^{1/2}$, and $h_i(t) = H_i(t^2 + \lambda)$ $(i = 1, 2)$. Then the conditions (i)~(iii) in Theorem can be rewritten as follows:

(i)′ $(1 - h_1(k(x)))g(X, X) \leqq \dfrac{1}{2} D^2 \tilde{f}(X, X) \leqq (1 + h_2(k(x)))g(X, X)$,

where $X \in T_x M$ $(x \in M)$ and each $h_i(t)$ $(i = 1, 2)$ is a nonnegative continuous function on $[0, \infty)$,

(ii)′ $1 - h_1(t) > 0$ for $t \in 0$ and $\lim_{t \to \infty} h_i(t) = 0$ $(i = 1, 2)$

(iii)′ $\begin{cases} \displaystyle\int^{\infty} h_i(s)/s \, ds < +\infty, \\ \displaystyle\int^{\infty} \left(\int_0^s u \cdot h_i(u) \, du/s^3\right) ds < +\infty \ (i = 1, 2). \end{cases}$

Since $o \in M$ is a nondegenerate critical point of $\tilde{f}$, there exists a coordinate system $x : U \to R^m$, where U is a neighborhood of o, with $x(o) = (0, \cdots, 0)$ and $\tilde{f}(p) = \sum_{i=1}^m x_i(p)^2$ for all $p \in U$ where $x(p) = (x_1(p), \cdots, x_m(p))$ (cf. [3; p. 6]). Let δ be a positive number such that $\{(x_1, \cdots, x_m) \in R^m : \sum_{i=1}^m x_i^2 < \delta\} \subset x(U)$. We construct a metric $\tilde{g}$ on M with $\tilde{g}\left(\dfrac{\partial}{\partial x_i}, \dfrac{\partial}{\partial x_j}\right) = \delta_{ij}$ on $U_{\delta/2} = \left\{p \in U : \sum_{i=1}^m x_i(p)^2 < \dfrac{\delta}{2}\right\}$ and

$\tilde{g}=g$ on $M-U$; such a metric can be constructed by the standard partition-of-unity extention process. By the construction of $\tilde{g}$, it suffices to prove that M with the metric $\tilde{g}$ is quasi-isometric to Euclidean space. Let $a>0$ be such that $k^{-1}(a)=\{x\in M: k(x)=a\}\subset U_\delta$. For each $p\in k^{-1}(a)$, let $\lambda_p(t)$ be the maximal integral curve of grad $k/\|\text{grad } k\|^2$ with $\lambda_p(a)=p$. Then we have $\dfrac{d}{dt}k(\lambda_p(t))=1$ and hence $k(\lambda_p(t))=t$ $(t>0)$. Define $F_1: k^{-1}(a)\times(0,\infty)\to M-\{o\}$ by $F_1(p,t)=\lambda_p(t)$, and $F_2: k^{-1}(a)\times(0,\infty)\to R^m-\{(0,\cdots,0)\}$ by $F_2(p,t)=\dfrac{t}{a}(x_1(p),$ $\cdots,x_m(p))$. It follows that F_1 and F_2 are diffeomorphisms and $F_2\circ F_1^{-1}$ can be extended to the diffeomorphism $F: M\to R^m$ (cf. [3; p. 221]). We shall now show that F is a required quasi-isometry. Let $\lambda: [0,\varepsilon]\to k^{-1}(a)$ be any smooth regular curve. Define a smooth map $G: [0,\varepsilon]\times[0,\infty)\to M-\{o\}$ by $G(t,s)=F_1$ $(\lambda(t),s)$, and vector fields X and Y along G by $X=F^*\left(\dfrac{\partial}{\partial s}\right)=\text{grad } k/\|\text{grad } k\|^2$ and $Y=F_*\left(\dfrac{\partial}{\partial t}\right)$. Fix any $b>0$ such that $k^{-1}(b)\subset M-U_\delta$. Then we have for $s\geq b$,

$$(1)\qquad \frac{\partial}{\partial s}\|Y\|(t,s) = (D_X Y, Y)/\|Y\|(t,s)$$
$$= (D_Y X, Y)/\|Y\|(t,s)$$
$$= (D_Y\text{grad } k, Y)/\|Y\| \|\text{grad } k\|^2 (t,s)$$
$$= D^2 k(Y, Y)/\|Y\| \|\text{grad } k\|^2 (t,s)$$

By the assumption (i)$'$, we have on $\{x\in M: k(x)\geq b\}$

$$\{(1-h_1(k))\cdot g-dk\cdot dk\}/k\leq D^2 k\leq \{(1+h_2(k))\cdot g-dk\cdot dk\}/k .$$

Therefore we get

$$(2)\quad \{(1-h_1(k))/k\cdot\|Y\|^2\}(t,s)\leq D^2 k(Y,Y)(t,s)\leq \{(1+h_2(k))/k\cdot\|Y\|^2\}(t,s)$$

for $s\geq b$. Now we need the following

Lemma 2. *On* $\{x\in M: k(x)\geq b\}$, *we have the following estimate*:

$$(3)\qquad 1-2\int_b^k u\cdot h_1(u)\, du/k^2+(B-4b^2)/4k^2\leq$$
$$\|\text{grad } k\|^2\leq 1+2\int_b^k u\cdot h_2(u)\, du/k^2+(C-4b^2)/4k^2 ,$$

where $B=\min\{\|\text{grad }\tilde{f}\|^2(x): x\in k^{-1}(b)\}$ and $C=\max\{\|\text{grad }\tilde{f}\|^2(x): x\in k^{-1}(b)\}$.

We leave a proof of this lemma later. By (1), (2) and (3), we have

$$(4)\quad (1-\chi_1(s))/s\cdot\|Y\|\leq(t,s)\leq\frac{\partial}{\partial s}\|Y\|(t,s)\leq(1+\chi_2(s))/s\cdot\|Y\|(t,s) ,$$

where $\quad \mathcal{X}_1(s) = (8\int_0^s u \cdot h_2(u)\, du + 4s^2 h_1(s) + C - 4b^2)/$

$$(8\int_0^s u \cdot h_2(u)\, du + 4s^2 + C - 4b^2)$$

and $\quad \mathcal{X}_2(s) = (8\int_0^s u \cdot h_1(u)\, du + 4s^2 h_2(s) - B + 4b^2)/$

$$(-8\int_0^s u \cdot h_1(u)\, du + 4s^2 + B - 4b^2).$$

It follows that

(5) $\quad \dfrac{s}{b}\exp\left(\int_b^s -\mathcal{X}_1(u)/u\, du\right) \leq \|Y\|(t, s)/\|Y\|(t, b) \leq \dfrac{s}{b}\exp\left(\int_b^s \mathcal{X}_2(u)/u\, du\right).$

By the assumption (iii)′, there exists some positive constant ξ such that

(6) $\quad \exp\left(\int_b^s \mathcal{X}_2(u)/u\, du\right) \leq \xi \quad \text{and} \quad \xi^{-1} \leq \exp\left(\int_b^s -\mathcal{X}_1(u)/u\, du\right).$

By (5) and (6), we get

(7) $\quad \xi^{-1}\|Y\|(t, 0)/b \leq \|Y\|(t, s)/s \leq \xi\|Y\|(t, 0)/b.$

The assumption (iii)′ implies that for some positive constant ζ

(8) $\quad \zeta^{-1} \leq \|\text{grad } k\| \leq \zeta.$

Inequalities (7) and (8) show that $F: M \to R^m$ is quasi-isometric. This completes the proof of Theorem.

Proof of Lemma 2. For each $p \in k^{-1}(b)$, let $\gamma_p(t)$ be the maximal integral curve of $\text{grad } \tilde{f}/\|\text{grad } \tilde{f}\|^2$ with $\gamma_p(0) = p$. Then $\dfrac{d}{dt}\tilde{f}(\gamma_p(t)) = 1$ and hence $\tilde{f}(\gamma_p(t)) = t + b^2 \ (t \geq 0)$. From the assumption (i)′, we obtain the inequality:

$$(1 - h_1(k))\|\text{grad } \tilde{f}\|^2 \leq \frac{1}{2}D^2\tilde{f}(\text{grad } \tilde{f}, \text{grad } f) \leq (1 + h_2(k))\|\text{grad } \tilde{f}\|^2$$

on $\{x \in M: k(x) \geq b\}$. Noting $D^2\tilde{f}(\text{grad } \tilde{f}, \text{grad } \tilde{f}) = \dfrac{1}{2}\text{grad } \tilde{f}(\|\text{grad } \tilde{f}\|^2)$ we see

$$4(1 - h_1(\sqrt{t+b^2})) \leq \frac{d}{dt}\|\text{grad } \tilde{f}\|^2(\gamma_p(t)) \leq 4(1 + h_2(\sqrt{t+b^2}))$$

for $t \geq 0$. Therefore we get the inequalities:

$$8\int_b^{\sqrt{t+b^2}} u(1 - h_1(u))\, du + \|\text{grad } \tilde{f}\|^2(p) \leq \|\text{grad } \tilde{f}\|^2(\gamma_p(t))$$

$$\leq 8\int_b^{\sqrt{t+b^2}} u(1 + h_2(u))\, du + \|\text{grad } \tilde{f}\|^2(p)$$

for $t \geq 0$. Since $k(\gamma_p(t)) = \sqrt{t+b^2}$ and $\text{grad } f = 2\, k \cdot \text{grad } k$ we get the required

estimate.   This completes the proof of Lemma 2.

By Moser's theorem ([4]), we have the following

Corollary 1.   *Let M be as in Theorem.   Then on M there are no positive harmonic functions other than constants.   If M is in addition a Kaehler manifold, then it has no nonconstant bounded holomorphic functions.*

We shall derive the theorem of Wissner ([5; Satz. II. 1.3]) as follows.

Corollary 2.   *If M is a connected complete Riemannian manifold and f is a smooth function on M whose Hessian is equal to the metric tensor on M, then M is isometric to Euclidean space.*

Proof.   By Lemma 1 and the strictly convexity of f, f attains its minimum λ at the one and only one point $o \in M$.   Replacing f for $\frac{1}{2}(f-\lambda)$, we may assume that f is a smooth function such that $D^2 f = \frac{1}{2} g$ on M and $f(x) \gneqq f(o) = 0$ for any $x \in M - \{o\}$.   Let $\gamma: [0, \infty) \to M$ be any arc-length parametrized geodesic issuing from o.   Then $D^2 f(\dot{\gamma}, \dot{\gamma}) = f(\gamma(t))'' = \frac{1}{2}$ for $t \geq 0$ and hence $f(\gamma(t)) = t^2$. That is, $f(x)$ equals $\mathrm{dis}(x, o)^2$ near $o \in M$.   Therefore the same arguments as in the proof of Theorem can be applied without any change of metric and we see that the exponential mapping at $o \in M$ is an isometry.   This completes the proof of Corollary 2.

EXAMPLE.   Let M be C^m with the Kaehler metric g defined by $g_{ij} = \dfrac{\partial^2}{\partial z_i \partial z_j}$ $(|z|^2 + \log(1 + |z|^2))$, where $(z_1, \cdots, z_m)$ is the canonical holomorphic coordinates on C^m and $|z|^2 = |z_1|^2 + \cdots + |z_m|^2$.   Then M is a Kaehler manifold with a pole $o = (0, \cdots 0)$, that is, the exponential mapping at o induces a global diffeomorphism between $T_o M$ and M.   Let $r(x)$ be the distance between o and $x \in M$.   By computing the (radial) curvatures, we can see r^2 satisfies all the conditions required in Theorem (cf. [1; Theorem C]).

Corollary 3 ([7]).   *Let M be a Riemannian manifold with a pole $o \in M$ and $r(x)$ be the distance between o and $x \in M$.   Suppose there exists a continuous function $h: [0, \infty) \to [0, \infty)$ satisfying the following conditions:*

(i) $(1 - h(r(x)))g(X, X) \leq \dfrac{1}{2} D^2 r^2(X, X) \leq (1 + h(r(x)))g(X, X)$

where $X \in T_x M (x \in M)$, and

(ii) $\displaystyle\int_1^\infty h(t)/t \, dt < +\infty$.

Then the exponential mapping at $o \in M$ is quasi-isometric.

Proof. Noting $\|\operatorname{grad} r\| = 1$ on $M - \{o\}$, we see the result easily from the proof of Theorem.

References

[1] R.E. Greene and H. Wu: Function theory on manifolds which posses a pole, Mathematics Lecture Notes No. 699, Springer-Verlag, 1979.

[2] R.E. Greene and H. Wu: C^∞ *convex function and manifolds of positive curvature,* Acta Math. **137** (1976), 209–245.

[3] J. Milnor: Morse theory, Ann. of Math. Studies, No. 51, Princeton University Press, Princeton, 1963.

[4] J. Moser: *On Harnack's theorem for elliptic differential equations,* Comm. Pure Appl. Math. **14** (1961), 577–591.

[5] H.W. Wissner: *Geodätische Konvexität in Riemannsche Mannigfältigkeiten,* Doctoral Disseration, Berlin, 1979.

[6] H. Wu: *On a problem concerning the intrinsic characterization of C^n,* (to appear).

[7] K. Yagi: *On the Hessian of the square of the distance on a manifold with a pole,* Proc. Japan Acad. Ser. A **56** (1980), 332–337.

Department of Mathematics
University of Tokyo
Hongo, Tokyo 113
Japan