

ON THE EQUIVALENCE PROBLEM FOR A CERTAIN CLASS OF GENERALIZED SIEGEL DOMAINS, III

AKIO KODAMA*

(Received December 19, 1979)

Introduction. The notion of "generalized Siegel domains in $\mathbf{C}^n \times \mathbf{C}^m$ with exponent c " was introduced by Kaup, Matsushima and Ochiai [3]. In the previous paper [6], we studied exclusively the structure of generalized Siegel domains in $\mathbf{C} \times \mathbf{C}^m$ with exponent $1/2$. Since then, as an application of the results obtained in [6], we considered the equivalence problem and showed that two generalized Siegel domains in $\mathbf{C}^n \times \mathbf{C}^m$ with exponent $1/2$ are holomorphically equivalent only if they are linearly equivalent [7], [8].

In this paper we study the equivalence problem for generalized Siegel domains in $\mathbf{C} \times \mathbf{C}^m$ with arbitrary exponent. To state our results, we need a few preparations. Let \mathcal{D} be a generalized Siegel domain in $\mathbf{C}^n \times \mathbf{C}^m$ with exponent c and $\mathfrak{g}(\mathcal{D})$ the real Lie algebra consisting of all complete holomorphic vector fields on \mathcal{D} . Then, by the definition of \mathcal{D} , the Lie algebra $\mathfrak{g}(\mathcal{D})$ contains the following vector field E on \mathcal{D} (see section 1):

$$E = \sum_{k=1}^n z_k \frac{\partial}{\partial z_k} + c \sum_{\alpha=1}^m w_\alpha \frac{\partial}{\partial w_\alpha},$$

where $(z_1, \dots, z_n, w_1, \dots, w_m)$ is the natural coordinate system in $\mathbf{C}^n \times \mathbf{C}^m$. We put, for any $\lambda \in \mathbf{R}$

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g}(\mathcal{D}) \mid [E, X] = \lambda X\}.$$

Now we can state our results. First of all, we shall prove the following proposition in section 2 (see Proposition 2.6):

Proposition. *Let \mathcal{D} be a generalized Siegel domain in $\mathbf{C}^n \times \mathbf{C}^m$ with exponent $c=1$. Then $\mathfrak{g}(\mathcal{D})$ has the following graded structure:*

$$\mathfrak{g}(\mathcal{D}) = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1, \quad [\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}.$$

Combining the results obtained in [3], [4] and [13] with this fact, we obtain the following

* The author is partially supported by the Sakkokai Foundation. Work also supported in part by Grant-in-Aid for Scientific Research.

Theorem 1. *Let \mathcal{D} be a generalized Siegel domain in $\mathbf{C}^n \times \mathbf{C}^m$ with exponent c .*

(1) *If $c \neq 1/2$, then we have*

$$\mathfrak{g}(\mathcal{D}) = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1, \quad [\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}.$$

Moreover, in the case when $n=1$, we have the following direct sum decomposition of $\mathfrak{g}(\mathcal{D})$:

$$\mathfrak{g}(\mathcal{D}) = (\mathfrak{g}_{-1} + \mathfrak{g}'_0 + \mathfrak{g}_1) + \mathfrak{g}'_0,$$

where \mathfrak{g}'_0 and \mathfrak{g}_0 are vector subspaces of \mathfrak{g}_0 such that both $\mathfrak{g}_{-1} + \mathfrak{g}'_0 + \mathfrak{g}_1$ and \mathfrak{g}'_0 are ideals of $\mathfrak{g}(\mathcal{D})$.

(2) *If $c=1/2$, then we have*

$$\mathfrak{g}(\mathcal{D}) = \mathfrak{g}_{-1} + \mathfrak{g}_{-1/2} + \mathfrak{g}_0 + \mathfrak{g}_{1/2} + \mathfrak{g}_1, \quad [\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}.$$

Making use of this theorem, we can prove the following

Theorem 2. *Let \mathcal{D} and \mathcal{D}' be generalized Siegel domains in $\mathbf{C} \times \mathbf{C}^m$ with exponent c and c' , respectively. Then \mathcal{D} and \mathcal{D}' are holomorphically equivalent if and only if there exists a non-singular linear mapping $\mathcal{L}: \mathbf{C} \times \mathbf{C}^m \rightarrow \mathbf{C} \times \mathbf{C}^m$ such that $\mathcal{L}(\mathcal{D}) = \mathcal{D}'$. Moreover, if \mathcal{D} and \mathcal{D}' are holomorphically equivalent, we have $c=c'$.*

After some preliminaries in sections 1 and 2, these two theorems will be proved in sections 3 and 4 respectively.

Now, the following generalization of the classical result due to H. Cartan [2], which states that two bounded circular domains D and D' in \mathbf{C}^N containing the origin o are linearly equivalent if there exists a biholomorphic isomorphism f of D onto D' such that $f(o)=o$, will play an important role in the proof of Theorem 2.

Theorem 3. *Let D and D' be two circular domains in \mathbf{C}^N containing the origin o of \mathbf{C}^N . Suppose that D admits an $\text{Aut}(D)$ -invariant Kähler metric ds_D^2 , where $\text{Aut}(D)$ denotes the group of all biholomorphic transformations of D onto itself. Then D and D' are holomorphically equivalent if and only if they are linearly equivalent.*

In his letter of May 16, 1978, Dr. K. Nakajima kindly announced, but without proof, that this fact is true for bounded circular domains in \mathbf{C}^N containing the origin. Since we do not know his proof, we present our proof of this theorem in section 1.

1. Preliminaries

Let \mathbf{R} (resp. \mathbf{C}) denotes the field of real (resp. complex) numbers as usual.

Let $(z_1, \dots, z_n, w_1, \dots, w_m)$ be the natural coordinate system in $\mathbf{C}^n \times \mathbf{C}^m$.

DEFINITION 1. A domain \mathcal{D} in $\mathbf{C}^n \times \mathbf{C}^m$ is called a *generalized Siegel domain with exponent c* if the following conditions are satisfied:

(1) \mathcal{D} is holomorphically equivalent to a bounded domain in \mathbf{C}^{n+m} and $\mathcal{D} \cap (\mathbf{C}^n \times \{o\}) \neq \emptyset$, where $\{o\}$ denotes the origin of \mathbf{C}^m .

(2) \mathcal{D} is invariant by the transformations of \mathbf{C}^{n+m} of the following types:

- (a) $(z, w) \mapsto (z+a, w)$ for all $a \in \mathbf{R}^n$;
- (b) $(z, w) \mapsto (z, e^{\sqrt{-1}t}w)$ for all $t \in \mathbf{R}$;
- (c) $(z, w) \mapsto (e^t z, e^{ct}w)$ for all $t \in \mathbf{R}$,

where c is a fixed real number depending only on \mathcal{D} . We call c the *exponent of \mathcal{D}* .

It is obvious from the definition that the following vector fields on \mathcal{D} are contained in $\mathfrak{g}(\mathcal{D})$:

- (a)' $\partial/\partial z_k$ for $k = 1, 2, \dots, n$;
- (b)' $I = \sqrt{-1} \sum_{\alpha=1}^m w_\alpha \frac{\partial}{\partial w_\alpha}$;
- (c)' $E = \sum_{k=1}^n z_k \frac{\partial}{\partial z_k} + c \sum_{\alpha=1}^m w_\alpha \frac{\partial}{\partial w_\alpha}$.

For later use, we here study the structure of circular domains in \mathbf{C}^N .

DEFINITION 2. A domain D in \mathbf{C}^N is called a *circular domain* if D is invariant by the rotations

$$(1.1) \quad l_t: (z_1, \dots, z_N) \mapsto (e^{\sqrt{-1}t}z_1, \dots, e^{\sqrt{-1}t}z_N), \quad t \in \mathbf{R},$$

where (z_1, \dots, z_N) is a fixed coordinate system in \mathbf{C}^N .

Let D be a circular domain in \mathbf{C}^N which admits an $\text{Aut}(D)$ -invariant Kähler metric ds_D^2 . Then we have $\text{Aut}(D) \subset \text{Iso}(D)$, where $\text{Iso}(D)$ denotes the group of isometries of D with respect to ds_D^2 . Therefore, being a closed subgroup of the Lie group $\text{Iso}(D)$, $\text{Aut}(D)$ is also a real Lie group. Moreover, the isotropy subgroup of $\text{Aut}(D)$ at a point p of D is compact, since the isotropy subgroup of $\text{Iso}(D)$ at p is so. We may identify the Lie algebra of $\text{Aut}(D)$ with the real Lie algebra $\mathfrak{g}(D)$ consisting of all complete holomorphic vector fields on D . Using the coordinate system (z_1, \dots, z_N) , any vector field X in $\mathfrak{g}(D)$ can be written in the form

$$(1.2) \quad X = \sum_{k=1}^N f_k \frac{\partial}{\partial z_k},$$

where $f_k (k=1, 2, \dots, N)$ are holomorphic functions on D . Now, suppose fur-

ther that D contains the origin o of \mathbb{C}^N . Then it is easy to see (cf. [6], section 1) that any vector field $X \in \mathfrak{g}(D)$ is a polynomial vector field, that is, in the expression (1.2) of X every component f_k is a polynomial. A vector field X is called a homogeneous polynomial vector field of degree ν , if any component f_k of X in (1.2) is a homogeneous polynomial of degree ν . In this terminology, we put

$$(1.3) \quad \mathfrak{B}_\nu = \begin{cases} \text{the set of all homogeneous polynomial vector} \\ \text{fields of degree } \nu \end{cases}$$

and

$$(1.4) \quad \partial = \sqrt{-1} \sum_{k=1}^N z_k \frac{\partial}{\partial z_k},$$

which is the vector field in $\mathfrak{g}(D)$ induced by the global one-parameter subgroup $\{l_t\}_{t \in \mathbb{R}}$ defined in (1.1). Then we can show the following

Lemma 1.1 (cf. [6], [11]). *With the same assumptions on D and notation as above, we define an endomorphism J of $\mathfrak{g}(D)$ by $J(X) = [\partial, X]$ for $X \in \mathfrak{g}(D)$. Then, denoting by \mathfrak{k} the Lie subalgebra of $\mathfrak{g}(D)$ corresponding to the isotropy subgroup K of $\text{Aut}(D)$ at the origin $o \in D$, we have*

$$(1.5) \quad \mathfrak{k} = \text{Ker } J = \mathfrak{g}(D) \cap \mathfrak{B}_1,$$

where $\text{Ker } J$ denotes the kernel of J ; and

(1.6) if we put $\mathfrak{p} = \{X \in \mathfrak{g}(D) \mid J^2(X) = -X\}$, then

$$\begin{cases} \mathfrak{p} = \mathfrak{g}(D) \cap (\mathfrak{B}_0 + \mathfrak{B}_2); \\ \mathfrak{g}(D) = \mathfrak{k} + \mathfrak{p} \quad (\text{direct sum}). \end{cases}$$

Lemma 1.2. *Let D be a circular domain in \mathbb{C}^N containing the origin, which admits an $\text{Aut}(D)$ -invariant Kähler metric ds_D^2 . Let G be the identity component of $\text{Aut}(D)$ and D_0 the G -orbit passing through the origin o . Then D_0 is a complex submanifold of D . Moreover, it is a Hermitian symmetric space of non-compact type.*

Proof. First we notice that, being a G -orbit passing through the origin $o \in D$, D_0 is a Riemannian submanifold of D . Let g be the Riemannian metric on D_0 induced from ds_D^2 . For each element σ of G , we denote by $r(\sigma)$ the restriction of σ to D_0 , that is, $r(\sigma) \cdot x = \sigma \cdot x$ for all $x \in D_0$. It is then obvious that r is a Lie group homomorphism of G into the Lie group $\text{Iso}(D_0)$ consisting of all isometries of D_0 with respect to the metric g and $r(G)$ acts transitively on D_0 .

Now, assuming that $D_0 \cong \{o\}$, we shall show that the orbit D_0 is a non-compact

complex submanifold of D . Let K be the isotropy subgroup of G at the origin o . We may identify D_0 with the quotient space G/K . Let $\mathfrak{g}(D) = \mathfrak{k} + \mathfrak{p}$ be the direct sum decomposition of $\mathfrak{g}(D)$ as in Lemma 1.1 and $T_0(D_0)$ the tangent space to D_0 at the origin o . Then we have

$$(1.7) \quad T_0(D_0) = \{X(o) \mid X \in \mathfrak{p}\},$$

where $X(o)$ denotes the value at o of the vector field X . We now assert that

$$(1.8) \quad T_0(D_0) \text{ is a complex subspace of } T_0(D),$$

where $T_0(D)$ is the tangent space to D at the origin o . In view of (1.7), it is sufficient to verify the following

$$(1.9) \quad \sqrt{-1}X(o) \in T_0(D_0) \quad \text{for every } X \in \mathfrak{p}.$$

For this, take an arbitrary vector field X on D belonging to \mathfrak{p} . Then, by Lemma 1.1, X can be written in the form

$$(1.10) \quad X = X_0 + X_2 \quad \text{for some } X_0 \in \mathfrak{B}_0 \text{ and } X_2 \in \mathfrak{B}_2.$$

By a straightforward computation we have

$$(1.11) \quad J(X) = -\sqrt{-1}X_0 + \sqrt{-1}X_2 \in \mathfrak{p},$$

where J is the endomorphism of $\mathfrak{g}(D)$ defined in Lemma 1.1. It follows then that

$$(1.12) \quad \sqrt{-1}X(o) = \sqrt{-1}X_0(o) = (-J(X))(o) \in T_0(D_0),$$

as desired. Now, let I be the G -invariant complex structure on D . By virtue of (1.8) we can define an $r(G)$ -invariant tensor field \tilde{I} on $D_0 = G \cdot o$ by requiring that, for any point p of D_0 and any vector $X \in T_p(D_0)$,

$$(1.13) \quad \tilde{I}_p(X) = I_p(X).$$

Obviously \tilde{I} then defines a complex structure on D_0 , so that D_0 is a complex submanifold of D . Since D is an open subset of \mathbf{C}^N and D_0 is a complex submanifold of D of positive dimension, it is evident that D_0 is non-compact.

It remains to prove that D_0 is a Hermitian symmetric space. Let g be the Kähler metric on D_0 induced from ds_D^2 . Since ds_D^2 is G -invariant, the group $r(G)$ acts transitively on D_0 as a group of holomorphic isometries. Now, recalling that D is a circular domain in \mathbf{C}^N , we see that G contains the following element

$$(1.14) \quad l_\pi: (z_1, \dots, z_N) \mapsto (e^{\sqrt{-1}\pi}z_1, \dots, e^{\sqrt{-1}\pi}z_N).$$

It is an easy matter to see that $r(l_\pi)$ is an involutive holomorphic isometry of D_0

and the origin $o \in D_0$ is an isolated fixed point of $r(l_\theta)$. From this our last assertion is obvious, since the group of all holomorphic isometries acts transitively on D_0 . q.e.d.

Lemma 1.3. *Let D be a circular domain in \mathbb{C}^N as in Lemma 1.2. We denote by G the identity component of $\text{Aut}(D)$ and K the isotropy subgroup of G at the origin o as before. Let K_1 be any compact subgroup of G . Then there exists an element $g \in G$ such that $g^{-1} \cdot K_1 \cdot g \subset K$. In particular, K is a maximal compact subgroup of G and every maximal compact subgroup of G is conjugate to K under an inner automorphism of G .*

Proof. If $G=K$, our assertion is trivial. So we may assume that $G \not\supset K$. Then, by Lemma 1.2 the G -orbit $D_0=G/K$ passing through the origin o is a Hermitian symmetric space of non-compact type with $r(G)$ -invariant Kähler metric g , where $r: G \rightarrow \text{Iso}(D_0)$ is the Lie group homomorphism defined in the proof of Lemma 1.2. Consequently, $D_0=G/K$ is a complete simply connected Riemannian manifold of non-positive sectional curvature. On the other hand, being a subgroup of G , $r(K_1)$ acts on $D_0=G/K$ as a group of isometries. Hence, by a classical result due to E. Cartan [1] we conclude that $r(K_1)$ has a fixed point $p=g \cdot o \in D_0$ ($g \in G$), that is, $k \cdot g \cdot o = r(k) \cdot (g \cdot o) = g \cdot o$ for every $k \in K_1$. Clearly this implies our assertion. q.e.d.

Proof of Theorem 3. It is trivial that D and D' are holomorphically equivalent, if they are linearly equivalent. Thus we have only to prove the converse.

Suppose that there exists a biholomorphic isomorphism $\Phi: D \rightarrow D'$ of D onto D' . Let G (resp. G') be the identity component of $\text{Aut}(D)$ (resp. $\text{Aut}(D')$) and K (resp. K') the isotropy subgroup of G (resp. G') at the origin o . Now we have two cases to consider. Consider first the case where $G \cdot o = o$, that is, the origin o is invariant under G . In this case we have

$$(1.15) \quad \Phi(o) = \Phi(G \cdot o) = G' \cdot \Phi(o).$$

Since the group G' contains the global one-parameter subgroup $\{l_t\}_{t \in \mathbb{R}}$ as defined in (1.1), this means that $\Phi(o) = o$. Taking a real number θ arbitrarily, we now consider the following biholomorphic transformation f of D onto itself defined by the composition

$$(1.16) \quad f = \Phi^{-1} \cdot l_{-\theta} \cdot \Phi \cdot l_\theta$$

where $\Phi^{-1}: D' \rightarrow D$ denotes the inverse mapping of Φ and l_θ the rotation defined in (1.1). Then we have

$$(1.17) \quad f(o) = o \quad \text{and}$$

(1.18) the differential $(f_*)_o: T_o(D) \rightarrow T_o(D)$ of f at o is the identity mapping. Noting that the isotropy subgroup K is compact, we can see from the proof of Theorem 3.3, Chap. V of [5] that, under these two conditions, f must be the identity transformation of D . Hence, if we put $\Phi = (\Phi_1, \dots, \Phi_N)$, we have from (1.16) that

$$(1.19) \quad \Phi_j(e^{\nu^{-1}\theta z}) = e^{\nu^{-1}\theta} \Phi_j(z) \quad \text{for all } \theta \in \mathbf{R},$$

from which we conclude that each component Φ_j is linear (cf. [10], p. 67).

We next consider the case where $G \cdot o \cong \{o\}$. By virtue of Lemma 1.3, we can choose an element g of G' in such a way that $g^{-1} \cdot (\Phi \cdot K \cdot \Phi^{-1}) \cdot g = K'$, since $\Phi \cdot K \cdot \Phi^{-1}$ is a maximal compact subgroup of G' . Considering a biholomorphic isomorphism $\tilde{\Phi}: D \rightarrow D'$ defined by $\tilde{\Phi} = g^{-1} \cdot \Phi$, we have

$$(1.20) \quad \tilde{\Phi}(o) = \tilde{\Phi}(K \cdot o) = K' \cdot \tilde{\Phi}(o).$$

Since the isotropy subgroup K' contains the global one-parameter subgroup as defined in (1.1), it follows from (1.20) that $\tilde{\Phi}(o) = o$. Repeating exactly the same arguments as in the first case, we conclude that $\tilde{\Phi}$ is linear, completing the proof.

We finish this section by a recent result on the cancellation problem due to Urata. This will be used in the proof of Theorem 2 in section 4.

Theorem U (Urata [12]). *Let X, Y and V be complex analytic spaces such that $V \times X$ is biholomorphic to $V \times Y$. If V is hyperbolic in the sense of Kobayashi [6], then X is biholomorphic to Y .*

2. The structure of generalized Siegel domains in $\mathbf{C}^n \times \mathbf{C}^m$ with exponent $c=1$

Throughout this section we denote by \mathcal{D} a generalized Siegel domain in $\mathbf{C}^n \times \mathbf{C}^m$ with exponent $c=1$.

Let $Z_{\mu\nu}$ (resp. $W_{\mu\nu}$) be a polynomial vector field on \mathcal{D} having the following form

$$(2.1) \quad Z_{\mu\nu} = \sum_{k=1}^n P_{\mu\nu}^k \frac{\partial}{\partial z_k} \quad \left(\text{resp. } W_{\mu\nu} = \sum_{\alpha=1}^m Q_{\mu\nu}^\alpha \frac{\partial}{\partial w_\alpha} \right),$$

where $P_{\mu\nu}^k$ (resp. $Q_{\mu\nu}^\alpha$) are homogeneous polynomials of degree μ in z_l ($1 \leq l \leq n$) and of degree ν in w_β ($1 \leq \beta \leq m$). We denote by $\mathfrak{Z}_{\mu\nu}$ (resp. $\mathfrak{W}_{\mu\nu}$) the set of all vector fields of the form (2.1), that is,

$$(2.2) \quad \mathfrak{Z}_{\mu\nu} = \{Z_{\mu\nu}\} \quad \left(\text{resp. } \mathfrak{W}_{\mu\nu} = \{W_{\mu\nu}\} \right).$$

Then, as we have observed in section 1 of [9], we have the following bracket

relation in the case $c=1$:

$$(2.3) \quad \begin{cases} [E, Z_{\mu\nu}] = (\mu + \nu - 1)Z_{\mu\nu}; \\ [E, W_{\mu\nu}] = (\mu + \nu - 1)W_{\mu\nu}; \\ [I, Z_{\mu\nu}] = \sqrt{-1}\nu Z_{\mu\nu}; \\ [I, W_{\mu\nu}] = \sqrt{-1}(\nu - 1)W_{\mu\nu}, \end{cases}$$

where E and I are vector fields on \mathcal{D} defined in section 1.

Now, as in the case where \mathcal{D} is a generalized Siegel domain with exponent $1/2$ (cf. [3], Lemma 3.1), we can see that every holomorphic vector field X in $\mathfrak{g}(\mathcal{D})$ can be written in the form

$$(2.4) \quad X = \sum_{\mu \geq 0} \{Z_{\mu_0} + Z_{\mu_1} + W_{\mu_0} + W_{\mu_1} + W_{\mu_2}\}.$$

Using (2.3), we have then

$$(2.5) \quad adE \cdot X = \sum_{\mu \geq 0} \{(\mu - 1)Z_{\mu_0} + \mu Z_{\mu_1} + (\mu - 1)W_{\mu_0} + \mu W_{\mu_1} + (\mu + 1)W_{\mu_2}\}.$$

Hence, putting

$$(2.6) \quad X_\mu = Z_{(\mu+1)_0} + Z_{\mu_1} + W_{(\mu+1)_0} + W_{\mu_1} + W_{(\mu-1)_2}$$

for $\mu = -1, 0, 1, 2, \dots$, we can verify easily that

$$(2.7) \quad X = \sum_{\mu \geq -1} X_\mu$$

and

$$(2.8) \quad \Phi(adE) \cdot X = \sum_{\mu \geq -1} \Phi(\mu)X_\mu$$

for every polynomial $\Phi(x) \in \mathbf{R}[x]$. Thus, by the same reasoning as in section 3 of [3], we obtain the following proposition (cf. [3], Theorem 2):

Proposition 2.1. *Let \mathcal{D} be a generalized Siegel domain in $\mathbf{C}^n \times \mathbf{C}^m$ with exponent $c=1$. For each $\mu \geq -1$, let \mathfrak{g}_μ be the vector subspace of $\mathfrak{g}(\mathcal{D})$ consisting of all vector fields in $\mathfrak{g}(\mathcal{D})$ of the form (2.6). Then we have*

$$(2.9) \quad \mathfrak{g}_\mu \text{ is the eigen space of } adE \text{ for the eigen-value } \mu;$$

$$(2.10) \quad \mathfrak{g}(\mathcal{D}) = \sum_{\mu \geq -1} \mathfrak{g}_\mu;$$

$$(2.11) \quad [\mathfrak{g}_\mu, \mathfrak{g}_\nu] \subset \mathfrak{g}_{\mu+\nu}.$$

Lemma 2.2. *For $\mu = -1, 0, 1, 2, \dots$, we have*

$$\mathfrak{g}_\mu = \mathfrak{g}'_\mu + \mathfrak{g}''_\mu \text{ (direct sum), where}$$

$$\begin{cases} \mathfrak{g}'_\mu = \mathfrak{g}(\mathcal{D}) \cap (\mathfrak{Z}_{(\mu+1)_0} + \mathfrak{W}_{\mu_1}); \\ \mathfrak{g}''_\mu = \mathfrak{g}(\mathcal{D}) \cap (\mathfrak{Z}_{\mu_1} + \mathfrak{W}_{(\mu-1)_2}). \end{cases}$$

Moreover, we have $\mathfrak{g}_{-1} = \left\{ \sum_{k=1}^n a_k \frac{\partial}{\partial z_k} \mid (a_1, \dots, a_n) \in \mathbf{R}^n \right\}$.

Proof. Let X be an arbitrary vector field on \mathcal{D} belonging to \mathfrak{g}_μ . Then, assuming that X has the form as in (2.6), we have by a routine calculation that

$$(2.12) \quad \begin{cases} adI \cdot X = \sqrt{-1}Z_{\mu_1} - \sqrt{-1}W_{(\mu+1)_0} + \sqrt{-1}W_{(\mu-1)_2}; \\ (adI)^2 \cdot X = -\{Z_{\mu_1} + W_{(\mu+1)_0} + W_{(\mu-1)_2}\}, \end{cases}$$

from which we obtain $\mathfrak{g}_\mu = \mathfrak{g}'_\mu + \mathfrak{g}''_\mu$ (direct sum), where

$$(2.13) \quad \begin{cases} \mathfrak{g}'_\mu = \mathfrak{g}(\mathcal{D}) \cap (\mathfrak{Z}_{(\mu+1)_0} + \mathfrak{W}_{\mu_1}); \\ \mathfrak{g}''_\mu = \mathfrak{g}(\mathcal{D}) \cap (\mathfrak{Z}_{\mu_1} + \mathfrak{W}_{(\mu+1)_0} + \mathfrak{W}_{(\mu-1)_2}). \end{cases}$$

Now, we shall prove that $\mathfrak{g}''_\mu = \mathfrak{g}(\mathcal{D}) \cap (\mathfrak{Z}_{\mu_1} + \mathfrak{W}_{(\mu-1)_2})$ in (2.13). The proof is by induction on μ . Let W_{00} be an arbitrary element of \mathfrak{g}'_{-1} . Then we see from (2.12) that W_{00} and $\sqrt{-1}W_{00}$ are contained in $\mathfrak{g}(\mathcal{D})$, so that $W_{00} = 0$ by Cartan's principle: $\mathfrak{g}(\mathcal{D}) \cap \sqrt{-1}\mathfrak{g}(\mathcal{D}) = \{0\}$. Thus, our assertion is really true for $\mu = -1$. Supposing that our assertion holds for $\mu \geq -1$, we take an arbitrary vector field X on \mathcal{D} belonging to $\mathfrak{g}'_{\mu+1}$. By (2.13) X may be written in the form

$$(2.14) \quad X = Z_{(\mu+1)_1} + W_{(\mu+2)_0} + W_{\mu_2}.$$

Then, since $[\partial/\partial z_k, X] \in \mathfrak{g}'_\mu$, $[\partial/\partial z_k, Z_{(\mu+1)_1}] \in \mathfrak{Z}_{\mu_1}$, $[\partial/\partial z_k, W_{(\mu+2)_0}] \in \mathfrak{W}_{(\mu+1)_0}$, $[\partial/\partial z_k, W_{\mu_2}] \in \mathfrak{W}_{(\mu-1)_2}$ for every $k=1, 2, \dots, n$ and the $\mathfrak{W}_{(\mu+1)_0}$ -component of any vector field belonging to \mathfrak{g}'_μ does not appear by the induction assumption, we conclude that

$$(2.15) \quad [\partial/\partial z_k, W_{(\mu+2)_0}] = 0 \quad \text{for } k = 1, 2, \dots, n,$$

which implies that $W_{(\mu+2)_0} = 0$. We have thus proved that $X = Z_{(\mu+1)_1} + W_{\mu_2}$, and so $\mathfrak{g}'_\mu = \mathfrak{g}(\mathcal{D}) \cap (\mathfrak{Z}_{\mu_1} + \mathfrak{W}_{(\mu-1)_2})$ for every μ . As a consequence of this fact, we also see that $\mathfrak{g}_{-1} = \mathfrak{g}(\mathcal{D}) \cap \mathfrak{Z}_{00}$. Once it is shown that the coefficient of every vector field on \mathcal{D} belonging to \mathfrak{g}_{-1} are real, our proof is completed. But this follows from the proof of Theorem 3 in [3]. q.e.d.

Lemma 2.3. *Let \mathfrak{r} be the radical of $\mathfrak{g}(\mathcal{D})$. Then we have*

$$\mathfrak{r} = \sum_{\mu \geq -1} \mathfrak{r}_\mu, \quad \text{where } \mathfrak{r}_\mu = \mathfrak{r} \cap \mathfrak{g}_\mu.$$

Moreover, $\mathfrak{r}_\mu = \mathfrak{g}_\mu$ for $\mu \geq 2$.

Proof. This can be proved in exactly the same way as Lemma 4.1 in [3]. q.e.d.

Now, let $A = \sum_{k=1}^n a_k \frac{\partial}{\partial z_k}$ ($a_k \in \mathbf{R}$) be an element of \mathfrak{g}_{-1} . According to Kaup, Matsushima and Ochiai [3], we shall define the linear mapping $\Phi_A: \mathfrak{g}_1 \rightarrow \mathfrak{g}_{-1}$ by

$$(2.16) \quad \Phi_A(X) = (1/2)(adA)^2 \cdot X \quad \text{for } X \in \mathfrak{g}_1.$$

Then, using the concrete expression of X as in (2.6), we can show by a straightforward computation that

$$(2.17) \quad X(\sqrt{-1}a, 0) = -\Phi_A(X)(\sqrt{-1}a, 0) \quad \text{for all } X \in \mathfrak{g}_1,$$

where $a = (a_1, \dots, a_n)$. From this we can verify easily the following lemmas with the same arguments as in the proofs of Lemmas 4.2 and 4.3 in [3]. So we will omit the proofs.

Lemma 2.4. $\mathfrak{r} \cap \mathfrak{g}_1 = \{0\}$.

Lemma 2.5. $\mathfrak{g}_\mu = \{0\}$ for $\mu = 2, 3, \dots$

Thus, summing up we have the following

Proposition 2.6. *Let \mathcal{D} be a generalized Siegel domain in $\mathbf{C}^n \times \mathbf{C}^m$ with exponent $c=1$. For each $\mu \geq -1$, let \mathfrak{g}_μ be the subspace of $\mathfrak{g}(\mathcal{D})$ as defined in Proposition 2.1. Then we have*

$$(2.18) \quad \mathfrak{g}(\mathcal{D}) = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1, \quad [\mathfrak{g}_\mu, \mathfrak{g}_\nu] \subset \mathfrak{g}_{\mu+\nu},$$

where

$$(2.19) \quad \mathfrak{g}_{-1} = \left\{ \sum_{k=1}^n a_k \frac{\partial}{\partial z_k} \mid (a_1, \dots, a_n) \in \mathbf{R}^n \right\},$$

(2.20) for $\mu=0, 1$, $\mathfrak{g}_\mu = \mathfrak{g}'_\mu + \mathfrak{g}''_\mu$ (direct sum), where

$$\begin{cases} \mathfrak{g}'_\mu = \mathfrak{g}(\mathcal{D}) \cap (\mathfrak{B}_{(\mu+1)0} + \mathfrak{B}_{\mu 1}); \\ \mathfrak{g}''_\mu = \mathfrak{g}(\mathcal{D}) \cap (\mathfrak{B}_{\mu 1} + \mathfrak{B}_{(\mu-1)2}). \end{cases}$$

3. Proof of Theorem 1

Throughout this section we denote by \mathcal{D} a generalized Siegel domain in $\mathbf{C} \times \mathbf{C}^m$ with exponent c , unless otherwise stated. By change of linear coordinates if necessary, we may assume without loss of generality that $(\sqrt{-1}, 0) \in \mathcal{D}$.

Lemma 3.1. *If $(z, w) \in \mathcal{D}$, then $Im.z > 0$.*

Proof. In the case where the exponent c of \mathcal{D} is non-zero, this can be verified in the same way as in the proof of Lemma 1 in [6].

We next consider the case $c=0$. Suppose that there exists a point

$(z_0, w_0) \in \mathcal{D}$ such that $\text{Im}.z_0 \leq 0$. Then, by the same reasoning as in the proof of Lemma 1 in [6], \mathcal{D} contains a point of the form $(0, \tilde{w}_0)$. Then, by the definition 1, \mathcal{D} also contains the set $\{(a, \tilde{w}_0) \in \mathbf{C} \times \mathbf{C}^m \mid a \in \mathbf{R}\}$. Moreover, since \mathcal{D} is open in $\mathbf{C} \times \mathbf{C}^m$, we can choose a positive number r_0 in such a way that the points $-(\sqrt{-1}r_0, \tilde{w}_0)$ and $(\sqrt{-1}r_0, \tilde{w}_0)$ are contained in \mathcal{D} . Then \mathcal{D} also contains the set $\{(z, \tilde{w}_0) \in \mathbf{C} \times \mathbf{C}^m \mid \text{Im}.z \neq 0\}$. As a result, we conclude that \mathcal{D} contains the set $\{(z, \tilde{w}_0) \in \mathbf{C} \times \mathbf{C}^m \mid z \in \mathbf{C}\}$, which is naturally identified with \mathbf{C} . But, since \mathcal{D} is holomorphically equivalent to a bounded domain in \mathbf{C}^{m+1} , this is a contradiction. q.e.d.

Lemma 3.2. *We put $\mathcal{D}_{\sqrt{-1}} = \{w \in \mathbf{C}^m \mid (\sqrt{-1}, w) \in \mathcal{D}\}$. Then*

- (3.1) $\mathcal{D}_{\sqrt{-1}}$ is a circular domain in \mathbf{C}^m containing the origin o ;
 (3.2) $\mathcal{D} = \{(z, w) \in \mathbf{C} \times \mathbf{C}^m \mid \text{Im}.z > 0, w / (\text{Im}.z)^c \in \mathcal{D}_{\sqrt{-1}}\}$.

Proof. This is immediate from the definition of \mathcal{D} and Lemma 3.1. q.e.d.

Proof of Theorem 1. The second statement (2) of the theorem is nothing but a result due to Kaup, Matsushima and Ochiai [3]. Moreover, combining Theorem 3.2 in Kaup and Upmeyer [4] and Proposition 7.1 in Vey [13] with Proposition 2.6 in section 2, we obtain the first assertion of (1).

In the following part of the proof, we denote by \mathcal{D} a generalized Siegel domain in $\mathbf{C} \times \mathbf{C}^m$ with exponent c . We have now two cases to consider. Consider first the case $c=0$. Then, by Lemma 3.2, \mathcal{D} is the direct product $\mathfrak{H} \times \mathcal{D}_{\sqrt{-1}}$, where \mathfrak{H} is the upper half plane $\{z \in \mathbf{C} \mid \text{Im}.z > 0\}$ and $\mathcal{D}_{\sqrt{-1}}$ is the circular domain defined in the same Lemma 3.2. Combining this fact with Propositions 7.1 and 8.1 in Vey [13], we can see that $\mathfrak{g}(\mathcal{D})$ has the following structure:

$$(3.3) \quad \mathfrak{g}(\mathcal{D}) = \mathfrak{g}(\mathfrak{H}) + \mathfrak{g}(\mathcal{D}_{\sqrt{-1}}) \quad (\text{direct sum of ideals}) ;$$

$$(3.4) \quad \mathfrak{g}(\mathfrak{H}) = \mathfrak{g}_{-1} + \mathfrak{g}'_0 + \mathfrak{g}_1 ;$$

$$(3.5) \quad \mathfrak{g}(\mathcal{D}_{\sqrt{-1}}) = \mathfrak{g}(\mathcal{D}) \cap (\mathfrak{B}_{00} + \mathfrak{B}_{01} + \mathfrak{B}_{02}) \subset \mathfrak{g}_0 ,$$

where

$$(3.6) \quad \mathfrak{g}_{-1} = \left\{ a \frac{\partial}{\partial z} \mid a \in \mathbf{R} \right\} ;$$

$$(3.7) \quad \mathfrak{g}'_0 = \mathfrak{g}(\mathcal{D}) \cap \mathfrak{B}_{10} \subset \mathfrak{g}_0 ;$$

$$(3.8) \quad \mathfrak{g}_1 = \mathfrak{g}(\mathcal{D}) \cap \mathfrak{B}_{20} .$$

Therefore, putting $\mathfrak{g}'_0' = \mathfrak{g}(\mathcal{D}_{\sqrt{-1}})$, we have our assertion.

Consider next the case $c \neq 0$. By Theorem 3.2 in Kaup and Upmeyer [4], the linear mapping $(ad(\partial/\partial z))^2: \mathfrak{g}_1 \rightarrow \mathfrak{g}_{-1}$ is injective in the case $c \neq 1$. We also claim that this is true for the case $c=1$. Indeed, using the equality (2.17), this can be verified with the same arguments as in the proof of Lemme 6.4 in

Vey [13]. Consequently, we have $\dim \mathfrak{g}_1 \leq 1$, because $\dim \mathfrak{g}_{-1} = 1$ by Proposition 2.6. We want to show that $\dim \mathfrak{g}_1 = 1$. For this it is sufficient to prove that $\mathfrak{g}_1 \neq \{0\}$. We put $\mathcal{D}_0 = \{(z, w) \in \mathcal{D} \mid w = 0\}$. Then, \mathcal{D}_0 is identified with the upper half plane \mathfrak{H} by Lemma 3.1. Now, it is well-known that each element $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{R})$ acts on \mathfrak{H} by a holomorphic transformation

$$(3.9) \quad l_\gamma(z) = (az + b) \cdot (cz + d)^{-1},$$

and conversely each biholomorphic transformation of \mathfrak{H} onto itself is obtained in the manner described in (3.9). For each element $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{R})$, we here define a mapping $\tilde{l}_\gamma: \mathfrak{H} \times \mathbf{C}^m \rightarrow \mathfrak{H} \times \mathbf{C}^m$ by

$$(3.10) \quad \tilde{l}_\gamma(z, w) = (l_\gamma(z), (cz + d)^{-2c}w).$$

It is then checked easily that \tilde{l}_γ is a holomorphic mapping and $\tilde{l}_\gamma(\mathcal{D}) \subset \mathcal{D}$, so that \tilde{l}_γ induces a biholomorphic transformation of \mathcal{D} onto itself (cf. [6], Corollary 3). By the construction of \tilde{l}_γ , it is obvious that $\tilde{l}_\gamma = l_\gamma$ on \mathcal{D}_0 . Therefore, the group $\text{Aut}_0(\mathcal{D}_0)$ can be identified with a subgroup of $\text{Aut}_0(\mathcal{D})$ via the correspondence $l_\gamma \mapsto \tilde{l}_\gamma$. Finally, consider the global one-parameter subgroup

$$(3.11) \quad \tilde{l}_{\gamma_t}: (z, w) \mapsto (l_{\gamma_t}(z), (tz + 1)^{-2c}w), \quad t \in \mathbf{R},$$

of $\text{Aut}_0(\mathcal{D})$ defined by the one-parameter subgroup

$$(3.12) \quad \gamma_t = \exp t \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \quad t \in \mathbf{R},$$

of $SL(2, \mathbf{R})$. Then, we can see by a direct computation that $\{\tilde{l}_{\gamma_t}\}_{t \in \mathbf{R}}$ defines a non-zero vector field on \mathcal{D} belonging to \mathfrak{g}_1 . Consequently, we have $\mathfrak{g}_1 \neq \{0\}$, as desired.

Now, noting that $\mathfrak{g}_0 = \{X \in \mathfrak{g}(\mathcal{D}) \mid [E, X] = 0\}$ and the group $\text{Aut}_0(\mathcal{D}_0)$ can be considered as a subgroup of $\text{Aut}_0(\mathcal{D})$ as above, we can show that $\text{Aut}_0(\mathcal{D})$ leaves invariant the complex submanifold \mathcal{D}_0 of \mathcal{D} , and in fact \mathcal{D}_0 coincides with the $\text{Aut}_0(\mathcal{D})$ -orbit passing through the point $(\sqrt{-1}, 0)$: $\mathcal{D}_0 = \text{Aut}_0(\mathcal{D}) \cdot (\sqrt{-1}, 0)$. Hence, there is a natural homomorphism $\pi: \mathfrak{g}(\mathcal{D}) \rightarrow \mathfrak{g}(\mathcal{D}_0)$ induced by the Lie group homomorphism of $\text{Aut}_0(\mathcal{D})$ to $\text{Aut}_0(\mathcal{D}_0)$ defined by $g \mapsto g|_{\mathcal{D}_0}$, where $g|_{\mathcal{D}_0}$ denotes the restriction of $g \in \text{Aut}_0(\mathcal{D})$ to \mathcal{D}_0 . Let $\mathfrak{g}(\mathcal{D}_0) = \mathfrak{g}^0_{-1} + \mathfrak{g}^0_0 + \mathfrak{g}^0_1$ be the decomposition of $\mathfrak{g}(\mathcal{D}_0)$ as in Kaup, Matsushima and Ochiai [3]. Then, since $\pi(E) = z \frac{\partial}{\partial z}$, π preserves the gradation, i.e., $\pi(\mathfrak{g}_\lambda) \subset \mathfrak{g}_\lambda^0$. Moreover, it is clear that $\pi(\mathfrak{g}_\lambda) = \mathfrak{g}_\lambda^0$ for $\lambda = -1$ and 1 . On the other hand, since $\mathfrak{g}(\mathcal{D}_0)$ is a simple Lie algebra isomorphic to $sl(2, \mathbf{R})$, we have $\mathfrak{g}_0^0 = [\mathfrak{g}_{-1}^0, \mathfrak{g}_1^0]$, so that $\mathfrak{g}_0^0 = \pi([\mathfrak{g}_{-1}, \mathfrak{g}_1]) \subset \pi(\mathfrak{g}_0)$. Therefore, π is surjective. Put $\mathfrak{g}'_0 = \text{Ker } \pi$ and $\mathfrak{g}'_0 =$

$[\mathfrak{g}_{-1}, \mathfrak{g}_1] \subset \mathfrak{g}_0$. Since π is injective on \mathfrak{g}_λ for $\lambda = -1$ and 1 , we see $\mathfrak{g}'_0 \subset \mathfrak{g}_0$. From this we conclude that $[\mathfrak{g}'_0, \mathfrak{g}_\lambda] = \{0\}$ for $\lambda = -1$ and 1 , and hence $[\mathfrak{g}'_0, \mathfrak{g}'_0] = \{0\}$ by the Jacobi identity. Finally it is an easy matter to see that $\mathfrak{g}_{-1} + \mathfrak{g}'_0 + \mathfrak{g}_1$ and \mathfrak{g}'_0 are ideals of $\mathfrak{g}(\mathcal{D})$ satisfying the condition: $\mathfrak{g}(\mathcal{D}) = (\mathfrak{g}_{-1} + \mathfrak{g}'_0 + \mathfrak{g}_1) + \mathfrak{g}'_0$ (direct sum), completing the proof.

4. Proof of Theorem 2

Throughout this section we denote by \mathcal{D} (resp. \mathcal{D}') a generalized Siegel domain in $\mathbb{C} \times \mathbb{C}^m$ with exponent c (resp. c'). In general, for given two domains S and S' we employ the notation A' for denoting the object for S' corresponding to an object A for S .

Now, we begin with the following

Lemma 4.1. *Let B and B' be two hyperbolic circular domains in \mathbb{C}^N containing the origin o . Suppose that the following two conditions are satisfied:*

(4.1) *The Lie algebra $\mathfrak{g}(B)$ (resp. $\mathfrak{g}(B')$) contains the element \bar{I} (resp. \bar{I}') of the form*

$$\bar{I} = \sqrt{-1} \sum_{k=2}^N z_k \frac{\partial}{\partial z_k} \quad \left(\text{resp. } \bar{I}' = \sqrt{-1} \sum_{k=2}^N z'_k \frac{\partial}{\partial z'_k} \right);$$

(4.2) *There exists a non-singular linear mapping $\Phi: \mathbb{C}^N \rightarrow \mathbb{C}^N$ of the form*

$$\Phi : \begin{pmatrix} z'_1 \\ z'_2 \\ \vdots \\ z'_N \end{pmatrix} = \begin{pmatrix} 1 & \Phi_2^1 & \dots & \Phi_N^1 \\ 0 & \Phi_2^2 & \dots & \Phi_N^2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \Phi_2^N & \dots & \Phi_N^N \end{pmatrix} \cdot \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{pmatrix}$$

such that $\Phi(B) = B'$.

Then $\Phi_k^1 = 0$ for $k = 2, 3, \dots, N$.

Proof. Let $\Lambda: B' \rightarrow B$ be the inverse mapping of Φ and put

$$(4.3) \quad \Lambda : \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{pmatrix} = \begin{pmatrix} 1 & \Lambda_2^1 & \dots & \Lambda_N^1 \\ 0 & \Lambda_2^2 & \dots & \Lambda_N^2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \Lambda_2^N & \dots & \Lambda_N^N \end{pmatrix} \cdot \begin{pmatrix} z'_1 \\ z'_2 \\ \vdots \\ z'_N \end{pmatrix}$$

Denoting by $\Lambda_*: \mathfrak{g}(B') \rightarrow \mathfrak{g}(B)$ the differential of Λ , we have by routine calculation that

$$(4.4) \quad \begin{aligned} \Lambda_* \bar{I}' &= \sqrt{-1} \sum_{s=1}^N \left(\sum_{k,l=2}^N \Lambda_j^s \Phi_l^k z_l \right) \frac{\partial}{\partial z_s} \\ &= (-\sqrt{-1} \sum_{l=2}^N \Phi_l^1 z_l) \frac{\partial}{\partial z_1} + \sqrt{-1} \sum_{k=2}^N z_k \frac{\partial}{\partial z_k} \end{aligned}$$

$$= (-\sqrt{-1} \sum_{i=2}^N \Phi_i^1 z_i) \frac{\partial}{\partial z_1} + \bar{I}.$$

Consequently, the vector field

$$(4.5) \quad X = (-\sqrt{-1} \sum_{i=2}^N \Phi_i^1 z_i) \frac{\partial}{\partial z_1}$$

also belongs to $\mathfrak{g}(B)$. Then, as we can see easily, the global one-parameter group $\{\phi_t\}_{t \in \mathbf{R}}$ generated by X is given by

$$(4.6) \quad \phi_t = \begin{pmatrix} 1 & -\sqrt{-1}\Phi_2^1 t & \cdots & -\sqrt{-1}\Phi_N^1 t \\ 0 & 1 & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & 1 \end{pmatrix}, \quad t \in \mathbf{R},$$

so that ϕ_t acts on B by the following transformation

$$(4.7) \quad \begin{cases} z_1 \mapsto z_1 - \sqrt{-1} (\sum_{i=2}^N \Phi_i^1 z_i) t \\ z_k \mapsto z_k \quad \text{for } k = 2, 3, \dots, N. \end{cases}$$

Here we notice that the group $\text{Aut}_0(B)$ contains the global one-parameter subgroup

$$(4.8) \quad \Psi_\theta: (z_1, z_2, \dots, z_N) \mapsto (e^{\sqrt{-1}\theta} z_1, z_2, \dots, z_N), \quad \theta \in \mathbf{R}.$$

In fact, $\{\Psi_\theta\}_{\theta \in \mathbf{R}}$ is the one-parameter subgroup of $\text{Aut}_0(B)$ generated by the holomorphic vector field $\sqrt{-1} z_1 \frac{\partial}{\partial z_1} = \sqrt{-1} \sum_{k=1}^N z_k \frac{\partial}{\partial z_k} - \bar{I}$ belonging to $\mathfrak{g}(B)$.

Now, suppose that $(\Phi_2^1, \Phi_3^1, \dots, \Phi_N^1) \neq (0, 0, \dots, 0)$. Then, choosing a point $p_0 = (0, z_2^0, \dots, z_N^0)$ of B such that $\sum_{i=2}^N \Phi_i^1 z_i^0 \neq 0$, we see that B contains the set

$$\{\Psi_\theta \cdot \phi_t \cdot p_0 \mid \theta, t \in \mathbf{R}\} = \{(z_1, z_2^0, \dots, z_N^0 \in \mathbf{C}^N \mid z_1 \in \mathbf{C}\},$$

which is canonically identified with the complex plane \mathbf{C} . But this is impossible, because B is hyperbolic in the sense of Kobayashi [5]. Thus we have proved that $(\Phi_2^1, \Phi_3^1, \dots, \Phi_N^1) = (0, 0, \dots, 0)$. q.e.d.

We now consider a mapping $\varphi; \{z \in \mathbf{C} \mid \text{Im}.z > 0\} \times \mathbf{C}^m \rightarrow \mathbf{C}^{m+1}$ defined by

$$(4.9) \quad \varphi: \begin{cases} z_1 = (z - \sqrt{-1}) \cdot (z + \sqrt{-1})^{-1} \\ z_k = \frac{4^c w_{k-1}}{(z - \sqrt{-1})^{2c}} \quad \text{for } k = 2, 3, \dots, m+1, \end{cases}$$

where c is the exponent of \mathcal{D} . As we can see easily, φ defines a biholomorphic

isomorphism of \mathcal{D} onto the image domain $\mathcal{B}=\varphi(\mathcal{D})$ in \mathbf{C}^{m+1} .

In the case $c \neq 0, 1/2$, we know already from the proof of Theorem 1 that

$$\text{Aut}_0(\mathcal{D}) \cdot (\sqrt{-1}, 0) = \{(z, 0) \in \mathbf{C} \times \mathbf{C}^m \mid \text{Im. } z > 0\}$$

and hence

$$\text{Aut}_0(\mathcal{B}) \cdot o = \{(z_1, 0, \dots, 0) \in \mathbf{C}^{m+1} \mid |z_1| < 1\}$$

by (4.9). Moreover, by direct computations as in the proof of Theorem 2 in [6], the structure of $\text{Aut}_0(\mathcal{B})$ is explicitly determined as follows. Let $SU(1, 1)$ be the matrix group defined by

$$(4.10) \quad SU(1, 1) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{C}) \left| \begin{array}{l} |a|^2 - |c|^2 = 1 \\ |b|^2 - |d|^2 = -1 \\ \bar{b}a - \bar{d}c = 0 \end{array} \right. \right\}$$

and $K_{\sqrt{-1}}^0 \subset GL(m, \mathbf{C})$ the identity component of the isotropy subgroup of $\text{Aut}(\mathcal{D}_{\sqrt{-1}})$ at the origin o of \mathbf{C}^m , where $\mathcal{D}_{\sqrt{-1}}$ is the circular domain defined in Lemma 3.2. Then we can verify that the group $\text{Aut}_0(\mathcal{B})$ consists of all transformations of the following type (cf. [6], REMARK 3):

$$(4.11) \quad \begin{cases} \mathfrak{z} \mapsto (a\mathfrak{z} + b) \cdot (c\mathfrak{z} + d)^{-1} \\ \mathfrak{z}' \mapsto K \cdot (c\mathfrak{z} + d)^{-2c} \cdot \mathfrak{z}' \end{cases}$$

where $\mathfrak{z} = z_1, \mathfrak{z}' = (z_2, \dots, z_{m+1}), \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(1, 1)$ and $K \in K_{\sqrt{-1}}^0 \subset GL(m, \mathbf{C})$.

The following lemma is essential to the proof of Theorem 2.

Lemma 4.2. *With the notation as above, the domain \mathcal{B} is a hyperbolic circular domain in \mathbf{C}^{m+1} containing the origin o . Moreover $\mathfrak{g}(\mathcal{B})$ contains the element \bar{I} of the form $\bar{I} = \sqrt{-1} \sum_{k=2}^{m+1} z_k \frac{\partial}{\partial z_k}$.*

Proof. By using (4.11), the first statement can be verified in exactly the same way as in the proof of Lemma 1 in [8].

For the second assertion, we recall that $\text{Aut}_0(\mathcal{D})$ contains the following global one-parameter subgroup

$$(4.12) \quad l_\theta : (z, w) \mapsto (z, e^{\sqrt{-1}\theta} w), \quad \theta \in \mathbf{R}.$$

By way of (4.9), $\{l_\theta\}_{\theta \in \mathbf{R}}$ induces the global one-parameter subgroup

$$(4.13) \quad \bar{I}_\theta : (z_1, z_2, \dots, z_{m+1}) \mapsto (z_1, e^{\sqrt{-1}\theta} z_2, \dots, e^{\sqrt{-1}\theta} z_{m+1}), \quad \theta \in \mathbf{R},$$

of $\text{Aut}_0(\mathcal{B})$, which defines the desired element \bar{I} . q.e.d.

We are now prepared to prove Theorem 2.

Proof of Theorem 2. Since it is trivial that \mathcal{D} and \mathcal{D}' are holomorphically equivalent if they are linearly equivalent, we have only to prove the converse.

Let $\varphi: \mathcal{D} \rightarrow \mathcal{B}$ be the biholomorphic isomorphism of \mathcal{D} onto \mathcal{B} defined in (4.9) and $\varphi': \mathcal{D}' \rightarrow \mathcal{B}'$ the corresponding isomorphism of \mathcal{D}' onto the image domain \mathcal{B}' . Suppose that there exists a biholomorphic isomorphism $\Phi: \mathcal{D} \rightarrow \mathcal{D}'$ of \mathcal{D} onto \mathcal{D}' . We put $\tilde{\Phi} = \varphi' \cdot \Phi \cdot \varphi^{-1}$. Then $\tilde{\Phi}$ gives rise to a biholomorphic isomorphism of \mathcal{B} onto \mathcal{B}' . Now, we know already by Lemma 4.2 that \mathcal{B} and \mathcal{B}' are hyperbolic circular domains in \mathbf{C}^{m+1} containing the origin o . Moreover, since \mathcal{B} (resp. \mathcal{B}') is holomorphically equivalent to a bounded domain, \mathcal{B} (resp. \mathcal{B}') has the Bergman metric $ds_{\mathcal{B}}^2$ (resp. $ds_{\mathcal{B}'}^2$), which is $\text{Aut}(\mathcal{B})$ (resp. $\text{Aut}(\mathcal{B}')$)-invariant Kähler metric. Hence, it follows immediately from Theorem 3 that there exists a non-singular linear mapping $\tilde{\mathcal{L}}: \mathbf{C}^{m+1} \rightarrow \mathbf{C}^{m+1}$ such that $\tilde{\mathcal{L}}(\mathcal{B}) = \mathcal{B}'$. We shall prove that this isomorphism $\tilde{\mathcal{L}}$ induces a linear isomorphism $\mathcal{L}: \mathbf{C} \times \mathbf{C}^m \rightarrow \mathbf{C} \times \mathbf{C}^m$ such that $\mathcal{L}(\mathcal{D}) = \mathcal{D}'$. The proof is divided into three cases as follows.

Case I: $c \neq 0, 1/2$.

In this case we know from the proof of Theorem 1 and (4.9) that

$$(4.14) \quad \text{Aut}_0(\mathcal{B}) \cdot o = \{(z_1, 0, \dots, 0) \in \mathbf{C}^{m+1} \mid |z_1| < 1\}.$$

Since $\dim_{\mathbf{C}}(\text{Aut}_0(\mathcal{B}') \cdot o) = \dim_{\mathbf{C}}(\text{Aut}_0(\mathcal{B}) \cdot o) = 1$, we obtain that

$$(4.15) \quad \tilde{\mathcal{L}}(\text{Aut}_0(\mathcal{B}) \cdot o) = \text{Aut}_0(\mathcal{B}') \cdot o = \{(z'_1, 0, \dots, 0) \in \mathbf{C}^{m+1} \mid |z'_1| < 1\},$$

from which we conclude that $\tilde{\mathcal{L}}: \mathbf{C}^{m+1} \rightarrow \mathbf{C}^{m+1}$ is of the form

$$(4.16) \quad \tilde{\mathcal{L}}: \begin{pmatrix} z'_1 \\ z'_2 \\ \vdots \\ z'_{m+1} \end{pmatrix} = \begin{pmatrix} a & \vdots & * \\ \vdots & \ddots & \vdots \\ 0 & \vdots & A \\ \vdots & \vdots & \vdots \\ 0 & \vdots & \vdots \end{pmatrix} \cdot \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{m+1} \end{pmatrix}, \quad |a| = 1,$$

where A is an $m \times m$ non-singular matrix. Since the group $\text{Aut}_0(\mathcal{B})$ contains the linear transformations

$$\tilde{l}_t: (z_1, \dots, z_{m+1}) \mapsto (e^{\sqrt{-1}t}z_1, \dots, e^{\sqrt{-1}t}z_{m+1}), \quad t \in \mathbf{R},$$

changing $\tilde{\mathcal{L}}$ by a suitable linear transformation $\tilde{\mathcal{L}} \cdot \tilde{l}_t$ if necessary, we may assume that $a = 1$. Then, as a consequence of Lemmas 4.2 and 4.1, $\tilde{\mathcal{L}}$ is reduced to the following form

$$(4.17) \quad \tilde{\mathcal{L}}: \begin{pmatrix} z'_1 \\ z'_2 \\ \vdots \\ z'_{m+1} \end{pmatrix} = \begin{pmatrix} 1 & \vdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \vdots & A \\ \vdots & \vdots & \vdots \end{pmatrix} \cdot \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{m+1} \end{pmatrix}.$$

Recalling the definitions of the isomorphisms $\varphi: \mathcal{D} \rightarrow \mathcal{B}$ and $\varphi': \mathcal{D}' \rightarrow \mathcal{B}'$, we put $\mathcal{L} = \varphi'^{-1} \cdot \tilde{\mathcal{L}} \cdot \varphi$. Then it is easily checked that \mathcal{L} is a biholomorphic isomorphism of \mathcal{D} onto \mathcal{D}' of the following form

$$(4.18) \quad \mathcal{L} : \begin{cases} z' = z \\ w'_\alpha = \frac{4^{c-c'}}{(z + \sqrt{-1})^{2(c-c')}} \sum_{\beta=1}^m A_{\alpha\beta} w_\beta \end{cases} \quad (1 \leq \alpha \leq m),$$

where we put $A = (A_{\alpha\beta})_{1 \leq \alpha, \beta \leq m}$. Once it is shown that the exponents c and c' are identical, we may conclude from (4.18) that the mapping $\mathcal{L}: \mathcal{D} \rightarrow \mathcal{D}'$ gives a desired linear equivalence between \mathcal{D} and \mathcal{D}' . Now, we start out to prove $c = c'$. Let $\{\varphi'_t\}_{t \in \mathbf{R}}$ be the global one-parameter subgroup

$$(4.19) \quad \varphi'_t : (z', w') \mapsto (e^t z', e^{c't} w'), \quad t \in \mathbf{R}$$

of $\text{Aut}(\mathcal{D}')$. By direct computations, we can show that the global one-parameter subgroup $\{\tilde{\varphi}_t\}_{t \in \mathbf{R}}$ of $\text{Aut}(\mathcal{D})$ defined by $\tilde{\varphi}_t = \mathcal{L}^{-1} \cdot \varphi'_t \cdot \mathcal{L}$ is given by

$$(4.20) \quad \tilde{\varphi}_t : (z, w) \mapsto \left(e^t z, \frac{e^{c't}}{(e^t z + \sqrt{-1})^{2(c'-c)} (z + \sqrt{-1})^{2(c-c')}} \cdot w \right),$$

so that the complete holomorphic vector field X on \mathcal{D} induced by $\{\tilde{\varphi}_t\}_{t \in \mathbf{R}}$ is of the following form

$$(4.21) \quad X = z \frac{\partial}{\partial z} + \sum_{\alpha=1}^m \left(c' - \frac{2(c'-c)z}{z + \sqrt{-1}} \right) w_\alpha \frac{\partial}{\partial w_\alpha}.$$

On the other hand, we know from [3] that every complete holomorphic vector field on \mathcal{D} is a polynomial vector field. By (4.21), it is clear that X is a polynomial vector field only if $c = c'$, as desired.

Case II: $c = 0$.

By Lemma 3.2 \mathcal{D} is the direct product $\mathcal{D} = \mathfrak{H} \times \mathcal{D}_{\sqrt{-1}}$ so that $\mathcal{B} = U \times \mathcal{D}_{\sqrt{-1}}$, where \mathfrak{H} is the upper half plane and U is the unit disk $\{z_1 \in \mathbf{C} \mid |z_1| < 1\}$. We have two cases to consider. Consider first the case where $\dim_{\mathbf{C}}(\text{Aut}_0(\mathcal{B}) \cdot o) = 1$. In this case we have

$$(4.22) \quad \text{Aut}_0(\mathcal{B}) \cdot o = \{(z_1, 0, \dots, 0) \in \mathbf{C}^{m+1} \mid |z_1| < 1\}$$

and

$$(2.23) \quad \tilde{\mathcal{L}}(\text{Aut}_0(\mathcal{B}) \cdot o) = \text{Aut}_0(\mathcal{B}') \cdot o = \{(z'_1, 0, \dots, 0) \in \mathbf{C}^{m+1} \mid |z'_1| < 1\}.$$

From this, repeating the same arguments as in the Case I, we can see that \mathcal{D} and \mathcal{D}' are linearly equivalent and $c = c'$. Consider next the case where $\dim_{\mathbf{C}}(\text{Aut}_0(\mathcal{B}) \cdot o) > 1$. We first claim that the exponent c' is also zero. It is

evident that $c'=0$ or $1/2$, since $\dim_{\mathbb{C}}(\text{Aut}_0(\mathcal{B}')\cdot o)=1$ in the case $c'\neq 0, 1/2$. Suppose that $c'=1/2$. Then, as we have observed in the previous paper [6], the orbit $\text{Aut}_0(\mathcal{B}')\cdot o$ is a unit ball. In particular, $\text{Aut}_0(\mathcal{B}')\cdot o$ is irreducible in the sense of Kähler geometry. On the other hand, by Theorem 3 and the fact that \mathcal{B} is the direct product $\mathcal{B}=U\times\mathcal{D}_{\sqrt{-1}}$, we see that the orbit $\text{Aut}_0(\mathcal{B})\cdot o$ is also the direct product $\text{Aut}_0(\mathcal{B})\cdot o=U\times S$, where S is a positive dimensional Hermitian symmetric space of non-compact type. Since $\text{Aut}_0(\mathcal{B})\cdot o$ and $\text{Aut}_0(\mathcal{B}')\cdot o$ are holomorphically equivalent, this is a contradiction. Thus we have proved that $c'=0$, and hence \mathcal{D}' is also the direct product $\mathcal{D}'=\mathfrak{H}\times\mathcal{D}'_{\sqrt{-1}}$ by Lemma 3.2. Since $\Phi:\mathcal{D}=\mathfrak{H}\times\mathcal{D}_{\sqrt{-1}}\rightarrow\mathcal{D}'=\mathfrak{H}\times\mathcal{D}'_{\sqrt{-1}}$ is a biholomorphic isomorphism and the upper half plane \mathfrak{H} is of course a hyperbolic complex manifold in the sense of Kobayashi [5], it follows immediately from Theorem *U* in section 1 that $\mathcal{D}_{\sqrt{-1}}$ and $\mathcal{D}'_{\sqrt{-1}}$ are also holomorphically equivalent. Now, being isomorphic to a complex submanifold of \mathcal{D} (resp. \mathcal{D}'), the domain $\mathcal{D}'_{\sqrt{-1}}$ (resp. $\mathcal{D}_{\sqrt{-1}}$) is a hyperbolic circular domain in \mathbb{C}^m containing the origin. Moreover, noting the fact $\mathcal{D}_{\sqrt{-1}}=\mathfrak{H}\times\mathcal{D}_{\sqrt{-1}}$ (resp. $\mathcal{D}'=\mathfrak{H}\times\mathcal{D}'_{\sqrt{-1}}$) in our case, the domain $\mathcal{D}_{\sqrt{-1}}$ (resp. $\mathcal{D}'_{\sqrt{-1}}$) has the $\text{Aut}(\mathcal{D}_{\sqrt{-1}})$ (resp. $\text{Aut}(\mathcal{D}'_{\sqrt{-1}})$)-invariant Kähler metric induced from the Bergman metric of \mathcal{D} (resp. \mathcal{D}'). Hence, it follows from Theorem 3 that $\mathcal{D}_{\sqrt{-1}}$ and $\mathcal{D}'_{\sqrt{-1}}$ are linearly equivalent. It is now trivial that $\mathcal{D}=\mathfrak{H}\times\mathcal{D}_{\sqrt{-1}}$ and $\mathcal{D}'=\mathfrak{H}\times\mathcal{D}'_{\sqrt{-1}}$ are linearly equivalent.

Case III: $c = 1/2$.

In the case where $\dim_{\mathbb{C}}(\text{Aut}_0(\mathcal{B})\cdot o)=1$, our assertion can be proved in the same way as Case I. Next, consider the case where $\dim_{\mathbb{C}}(\text{Aut}_0(\mathcal{B})\cdot o)>1$. We assert that the exponent c' of \mathcal{D}' is also $1/2$. In fact, replacing \mathcal{D} by \mathcal{D}' in the second case of the Case II, this can be verified easily. As a result, two domains \mathcal{D} and \mathcal{D}' are generalized Siegel domains in $\mathbb{C}\times\mathbb{C}^m$ with exponent $1/2$. Therefore, our assertion follows from the previous paper [8]. q.e.d.

References

- [1] E. Cartan: *Groupes simples clos et ouverts et géométrie riemannienne*, J. Math. Pures Appl. **8** (1929), 1–33.
- [2] H. Cartan: *Les fonctions de deux variables complexes et le problème de la représentation analytique*, J. Math. Pures Appl. **10** (1931), 1–114.
- [3] W. Kaup, Y. Matsushima and T. Ochiai: *On the automorphisms and equivalences of generalized Siegel domains*, Amer. J. Math. **92** (1970), 475–498.
- [4] W. Kaup and H. Upmeyer: *An infinitesimal version of Cartan's uniqueness theorem*, Manuscripta Math. **22** (1977), 381–401.
- [5] S. Kobayashi: *Hyperbolic manifolds and holomorphic mappings*, Pure and Appl. Math., 2, Dekker, New York 1970.
- [6] A. Kodama: *On generalized Siegel domains*, Osaka J. Math. **14** (1977), 227–252.

- [7] A. Kodama: *On the equivalence problem for a certain class of generalized Siegel domains*, Mem. Fac. Ed. Akita Univ. **28** (1978), 45–54.
- [8] A. Kodama: *On the equivalence problem for a certain class of generalized Siegel domains*, II. Osaka J. Math. **16** (1979), 133–142.
- [9] A. Kodama: *On generalized Siegel domains with exponent (c_1, c_2, \dots, c_s)* , J. Math. Soc. Japan **33** (1981), 247–259.
- [10] R. Narasimhan: *Several complex variables*, Univ. Chicago Press, Chicago and London, 1971.
- [11] T. Sunada: *Holomorphic equivalence problem for bounded Reinhardt domains*, Math. Ann. **235** (1978), 111–128.
- [12] T. Urata: *Holomorphic automorphisms and cancellation theorems*, Nagoya Math. J. **81** (1981), 91–103.
- [13] J. Vey: *Sur la division des domaines de Siegel*, Ann. Sci. Ecole Norm. Sup., 4^e série, **3** (1970), 479–506.

Department of Mathematics
Faculty of Education
Akita University
Akita 010
Japan

