Umeda, T.
Osaka J. Math.
18 (1981), 361–392

SCATTERING THEORY FOR PSEUDO-DIFFERENTIAL OPERATORS I. THE EXISTENCE OF WAVE OPERATORS

Tomio UMEDA

(Received November 28, 1979)

1. Introduction

This paper and the following paper are concerned with scattering theory for pseudo-differential operators. The operators we consider are of the form

$$P(D)+A(X, D) \tag{1.1}$$

in $\boldsymbol{R}^n$, where the unperturbed operator $P(D)$ and the perturbation $A(X, D)$ are pseudo-differential operators. We examine in the present paper the existence of wave operators, while we shall prove the completeness of wave operators in the succeeding paper.

We briefly recall the definition of wave operators $W_\pm$. Let H and H_0 be the self-adjoint realizations of $P(D)+A(X, D)$ and $P(D)$ in $L^2(\boldsymbol{R}^n)$, respectively. Then $W_\pm$ are defined by the limits

$$W_\pm = s\text{-}\lim_{t\to\pm\infty} e^{itH}e^{-itH_0}P_{ac}(H_0)\,. \tag{1.2}$$

Here $P_{ac}(H_0)$ denotes the orthogonal projection onto the subspace of absolutely continuity with respect to H_0. (We refer to Kato [7, Chapter X] for definitions and results from spectral theory.)

Some authors have studied scattering theory for pseudo-differential operators. Among others, recently Simon [10] has considered operators $H=H_0+V$ where H_0 is a pseudo-differential operator and showed that the main conclusions of scattering theory hold (namely the wave operators exist and are complete etc.). He does not necessarily require that V be a differential operator, or even a pseudo-differential operator. In fact, he only needs that V be a symmetric operator with some falloff at infinity. He used the methods which have been originally found by Enss [5]. The condition that Simon calls the Enss condition (see [10]) plays an important role in proving the completeness of wave operators. Enss and Simon used purely time-dependent methods.

Schechter [9] considered the operators of the form (1.1) and proved the main theorems of scattering theory. For the perturbations he took the specific operators of the form

$$\Sigma q_k(X)\sigma_k(D)$$

or of the form

$$\Sigma \tau_k(D)q_k(X)\sigma_k(D)\,.$$

Here $q_k(X)$ is an operator of multiplication by a function $q_k(x)$; $\sigma_k(D)$ and $\tau_k(D)$ are pseudo-differential operators with symbols $\sigma_k(\xi)$ and $\tau_k(\xi)$ respectively. He exploited time-independent methods.

The main purpose of the present paper is to give a sufficient condition for wave operators to exist. It is easy to find examples which are covered by the present paper but which are not included in the results of [9], [10]. Such an example is given in Section 2. However, the hypotheses of both [9] and [10] assure the completeness of wave operators, which suggests that our hypotheses are too weak to assure the completeness of wave operators. In addition, we prove the symmetry and the self-adjointness of pseudo-differential operators under suitable hypotheses.

We make assumptions directly on the symbols of $P(D)$ and $A(X, D)$. But Schechter [9] and Simon [10] did not do so.

We use Cook's method, which is the main time-dependent technique, to show the existence of wave operators. Our proof is similar to that of Hörmander [6] which is based on the method of stationary phase, though he treated differential operators only.

Finally, we sketch the contents of this paper. Section 2 contains the main theorems and some examples. The proofs of the main theorems are given in Sections 3 and 4 after we prove several lemmas except a key lemma. We prove the key lemma in Section 5. In Section 6, we give a necessary and sufficient condition for pseudo-differential operators to be symmetric. In Section 7, we give sufficient conditions for pseudo-differential operators to have self-adjoint extentions.

The writer would like to express his sincere gratitude to Professors Y. Saito and S. Ukai for their valuable advices and encouragements.

2. Existence of wave operators

In this section, we shall mention two basic theorems and some examples. One of the theorems is a characterization of the subspace of absolute continuity with respect to $H_0=P(D)$. The other theorem gurantees the existence of wave operators.

Before giving the assumption of $P(D)$ and $A(X, D)$ we shall list the notations which will be employed in the sequel without further reference.

$\boldsymbol{R}^n$; product of n copies of the real line $\boldsymbol{R}$.

$\langle x, \xi\rangle = x_1\xi_1 + \cdots + x_n\xi_n$, $(x, \xi \in \boldsymbol{R}^n)$.

$\boldsymbol{Z}_+^n$; product of n copies of the set $\mathbf{Z}_+$ of nonnegative integers.

$|\alpha| = \alpha_1 + \cdots + \alpha_n$; the *length* of the multi-index $\alpha \in \mathbf{Z}_+^n$. $\alpha! = \alpha_1! \cdots \alpha_n!$.

supp u; the *support* of a function u.

$D_j = -i\partial/\partial x_j$, $(i = \sqrt{-1})$.

$(\partial/\partial x)^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$, $\quad (\alpha \in \mathbf{Z}_+^n)$.

$f' = (\partial f/\partial x_1, \cdots, \partial f/\partial x_n)$; the *gradient* of a function f.

$f'' = (\partial^2 f/\partial x_j \partial x_k)_{j,k=1,\cdots,n}$; the *Hesse matrix* of f.

$\mathcal{F}u(\xi) = \hat{u}(\xi) = \int e^{-i\langle x,\xi\rangle} u(x)dx$; *Fourier transform* of $u = u(x)$.

$\bar{\mathcal{F}}v(x) = \int e^{i\langle x,\xi\rangle} v(\xi)đ\xi$; *inverse Fourier transform* of $v = v(\xi)$, $(đ\xi = (2\pi)^{-n}d\xi)$.

$\Delta = (\partial/\partial x_1)^2 + \cdots + (\partial/\partial x_n)^2$; the Laplace operator.

$C^N(\Omega)$; the space of complex-valued functions, defined and N times continuously differentiable in an open set Ω, equipped with the topology of uniform convergence on every compact subset of Ω, of the functions and of each one of their derivatives of order $< N+1 (N \in \mathbf{Z}_+$ or $N = +\infty)$.

$C_0^N(\Omega)$; the subspace of $C^N(\Omega)$ consisting of the functions having a compact support; if $u \in C_0^N(\Omega)$ we write

$$|u|_N = \max_{|\alpha| \leqslant N} \sup_x |(\partial/\partial x)^\alpha u(x)|.$$

$L^2(\boldsymbol{R}^n)$; the Hilbert space of measurable functions u square integrable over $\boldsymbol{R}^n$, equipped with the norm

$$||u||_{L^2} = \left(\int |u(x)|^2 \mathrm{d}x\right)^{1/2},$$

(if there are no risks of confusion we will omit the subscript L^2 and write as $||u||$).

$\mathcal{S}(\boldsymbol{R}^n)$; the space of C^∞ functions u in $\boldsymbol{R}^n$ such that, for any nonnegative integer N,

$$|u|_{N,\mathcal{S}} \equiv \max_{|\alpha|+k \leqslant N} \sup_x (1+|x|^2)^{k/2} |(\partial/\partial x)^\alpha u(x)| < \infty,$$

equipped with the topology defined by the seminorms $|\ |_{N,\mathcal{S}}$.

$\mathcal{S}'(\boldsymbol{R}^n)$; the dual space of $\mathcal{S}(\boldsymbol{R}^n)$, also the space of tempered distributions in $\boldsymbol{R}^n$.

$H_s(\boldsymbol{R}^n)$; the *Sobolev space* of order $s \in \boldsymbol{R}$ in $\boldsymbol{R}^n$, i.e., the space of tempered distributions u in $\boldsymbol{R}^n$ whose Fourier transform $\hat{u}$ is a measurable function such that

$$||u||_s = \left(\int (1+|\xi|^2)^s |\hat{u}(\xi)|^2 đ\xi\right)^{1/2} < +\infty,$$

equipped with the Hilbert space structure defined by the norm $||\ ||_s$.

meas$_k$; the *Lebesgue measure* on $\boldsymbol{R}^k$.
$[s]$; the integral part of $s\in\boldsymbol{R}$.
$\mathcal{H}_{\mathrm{p}}(H)$; the closed linear manifold spanned by all eigenvectors of an operator H in $L^2(\boldsymbol{R}^n)$.
$\mathcal{H}_{\mathrm{ac}}(H)$; the *subspace of absolute continuity* with respect to a self-adjoint operator H in $L^2(\boldsymbol{R}^n)$.
$P_{\mathrm{ac}}(H)$; the orthogonal projection onto $\mathcal{H}_{\mathrm{ac}}(H)$.
$\mathcal{H}_{\mathrm{s}}(H)$; the *subspace of singularity* with respect to a self-adjoint operator H in $L^2(\boldsymbol{R}^n)$.

Now let us make the following assumption on symbols $p(\xi)$ of $P(D)$ and $a(x, \xi)$ of $A(X, D)$. The set of all critical points of $p(\xi)$ will be denoted by Σ:

$$\Sigma = \{\xi\in\boldsymbol{R}^n \,|\, p'(\xi) = 0\} .$$

Assumption 2.1.

(A) $a(x, \xi)$ is a complex-valued C^∞ function on $\boldsymbol{R}^n_x\times\boldsymbol{R}^n_\xi$ such that for all multi-indicies α, β the estimate

$$\begin{aligned}&|(\partial/\partial\xi)^\alpha(\partial/\partial x)^\beta a(x, \xi)|\\ &\leqslant C_{\alpha\beta}(1+|x|)^{l+\tau|\alpha|}(1+|\xi|)^{m+\delta|\beta|}, \qquad x\in\boldsymbol{R}^n,\ \xi\in\boldsymbol{R}^n,\end{aligned}$$

is valid for some constant $C_{\alpha\beta}$, where l, m, δ and τ are constants with $l,\ m\geqslant 0$, $0\leqslant\delta$, $\tau<1$

$(P.\ 1)$ $p(\xi)$ is a real-valued C^∞ function on $\boldsymbol{R}^n_\xi$ such that for every multi-index α, the estimate

$$|(\partial/\partial\xi)^\alpha p(\xi)|\leqslant C_\alpha(1+|\xi|)^{N_\alpha}$$

is valid for some constants C_α and N_α.

$(P.\ 2)$ There is a closed set $\Xi\subset\boldsymbol{R}^n$ with the following properties:
  (a) meas$_n(\Xi\backslash\Sigma)=0$;
  (b) every point $\xi_0\in\boldsymbol{R}^n\backslash\Xi$ has a neighborhood where rank $p''(\xi)$ is constant;
  (c) rank $p''(\xi)\geqslant 1$, $\xi\in\boldsymbol{R}^n\backslash\Xi$.

(H) $P(D)+A(X,D)$ with domain $\mathcal{S}(\boldsymbol{R}^n)$ has a self-adjoint extention in $L^2(\boldsymbol{R}^n)$.

Let H be a self-adjoint extention in $L^2(\boldsymbol{R}^n)$ of $P(D)+A(X, D)$ with domain $\mathcal{S}(\boldsymbol{R}^n)$ and let H_0 be the closure in $L^2(\boldsymbol{R}^n)$ of $P(D)$ with domain $\mathcal{S}(\boldsymbol{R}^n)$. Our main results are:

Theorem 1. *Let hypothesis* $(P.\ 1)$ *be fulfilled. Then we have*

$$\mathcal{H}_{\mathrm{ac}}(H_0) = \{u\in L^2(\boldsymbol{R}^n)\,|\,\hat{u}(\xi) = 0 \textit{ for almost every } \xi\in\Sigma\}$$
$$\mathcal{H}_{\mathrm{s}}(H_0) = \{u\in L^2(\boldsymbol{R}^n)\,|\,\hat{u}(\xi) = 0 \textit{ for almost every } \xi\in\boldsymbol{R}^n\backslash\Sigma\} .$$

Theorem 2. *Let Assumption* 2.1 *be fulfilled and put*

$$\Omega_k = \{\xi\in \boldsymbol{R}^n\backslash\Xi \mid p'(\xi)\neq 0,\ \operatorname{rank} p''(\xi)=k\} \qquad (k=1, \cdots, n)\,.$$

For every k such that $\Omega_k\neq\phi$ *assume that,*

(c_k) *for any* $r>0$ *and any compact set* $K\subset\Omega_k$ *there is an integer* $N_k\geqslant[k/2]+2$ *such that*

$$\int_1^\infty t^{(n-k)/2}\Big[\int_{r\leqslant|y|\leqslant 2r}(1+td_K(y))^{-N_k} \times\{\max_{|\alpha|\leqslant 2N_k}(\sup_{\xi\in K}|(\partial/\partial\xi)^\alpha a(\pm ty,\xi)|)^2\}\,dy\Big]^{1/2} dt<\infty.$$

Then the wave operators (1.2) *exist. Here* $[k/2]$ *is the integral part of* $k/2$ *and* $d_K(y)$ *is the distance from* y *to the set*

$$p'(K) = \{p'(\xi)\mid\xi\in K\}\,.$$

Theorems 1 and 2 will be proven in Sections 3 and 4, respectively.

REMARK 2.1. Since $p(\xi)$ is a real-valued C^∞ function, $P(D)$ with domain $\mathcal{S}(\boldsymbol{R}^n)$ is essentially self-adjoint. Its unique self-adjoint extention H_0 is given by its closure. The domain $\mathcal{D}(H_0)$ consists of those u in $L^2(\boldsymbol{R}^n)$ such that $p\hat{u}$ is also in $L^2(\boldsymbol{R}^n)$.

In the rest of this section we illustrate some applications of Theorems 1 and 2.

EXAMPLE 2.2. Let $n=1$ and let $p(\xi)=\varphi(\xi)\xi^2$ $(\xi\in\boldsymbol{R})$ where $\varphi\in C^\infty(\boldsymbol{R})$ with $\varphi(\xi)=0$ (resp. 1) for $|\xi|\leqslant 1$, (resp. $\geqslant 2$). Suppose that

$$\xi\varphi'(\xi)+2\varphi(\xi)\neq 0\,, \qquad |\xi|>1\,.$$

Then it follows immediatly from Theorem 1 that

$$\mathcal{H}_{ac}(H_0) = \{u\in L^2(\boldsymbol{R})\mid\hat{u}(\xi)=0 \text{ for almost every } |\xi|\leqslant 1\}\,.$$

It is easy to see that H_0 has a single eigenvalue which is equal to zero (see the proof of Proposition 3.5).

Let A be a multiplication operator by a real-valued C^∞ function $a(x)$ such that

$$(2.1) \qquad |a(x)|\leqslant C(1+|x|)^{-1-\varepsilon}\,, \qquad x\in\boldsymbol{R},\ \varepsilon>0\,.$$

Put $H=H_0+A$. Then H is self-adjoint since A is bounded. Applying Theorem 2, we will show the existence of wave operators (1.2). Suppose that $p''(\xi)\neq 0$ for $|\xi|>1$ except finitely many points $\xi_1, \cdots, \xi_N$. Put

$$\Xi = \{\xi\in\boldsymbol{R}\mid -1\leqslant\xi\leqslant 1\}\cup\{\xi_1, \cdots, \xi_N\}\,.$$

Then Assumption 2.1 is fulfilled. Since

$$\int_1^\infty \left(\int_{r\leqslant|y|\leqslant 2r} |a(\pm ty)|^2 dy\right)^{1/2} dt \leqslant C\int_1^\infty t^{-1-\varepsilon}dt$$

for every $r>0$, the condition (c_1) holds. Hence, from Theorem 2, it follows that the wave operators (1.2) exist.

On the other hand, the proper wave operators $W_\pm = s-\lim_{t\to\pm\infty} e^{itH}e^{-itH_0}$ do not necessarily exist. For example, let us assume, in addition to (2.1), that $a(x)$ does not vanish almost everywhere and suppose that the proper wave operators exist. Let u be an eigenvector of H_0 corresponding to the eigenvalue zero. Then $e^{-itH_0}u=u$ for all $t\in \boldsymbol{R}$. Since the proper wave operators exist,

$$||e^{itH}u-u|| = ||e^{i(t+s)H}e^{-i(t+s)H_0}u - e^{isH}e^{-isH_0}u||$$

converges to zero as s tends to ∞. Therefore

$$e^{itH}u = u\,. \tag{2.2}$$

It follows immediately from (2.2) that $u\in\mathcal{D}(H)$ and $Hu=0$. Since $H=H_0+A$ and since $H_0u=0$, we have $au=0$. Thus $u=0$ because $a\neq 0$ a.e.. This contradicts the fact that u is an eigenvector. Hence the proper wave operators do not exist.

Next, we shall give an example which satisfies the conditions of Theorem 2 but which does not satisfy the Enss condition [10]. Let us first recall the definition of the Enss condition. Let A be a symmetric operator in $L^2(\boldsymbol{R}^n)$ such that $\mathcal{D}(A)\supset H_{2N}$ for some N. Put

$$h(R) = ||A((-\Delta)^N+1)^{-1}F(|x|\geqslant R)||, \tag{2.3}$$

where

$$F(|x|\geqslant R)u(x) = \begin{cases} 0 & (|x|<R) \\ u(x) & (|x|\geqslant R)\,. \end{cases}$$

The norm in (2.3) is the operator norm. Simon [10] calls the condition

$$h(0)<\infty,\ \int_0^\infty h(R)dR<\infty \tag{2.4}$$

the *Enss condition*. Roughly speaking, the Enss condition implies the existence and the completeness of the wave operators.

EXAMPLE 2.3. Let $n=2$ and let A be a multiplication operator by a real-valued C^∞ function $a(x)$ on $\boldsymbol{R}^2$ such that

$$(2.5)\qquad \begin{cases} |a(x)|\leqslant C,\ x=(x_1, x_2)\in \boldsymbol{R}^2 \\ |a(x)|\geqslant C(1+|x|)^{-\varepsilon},\ x_2^4>x_1^2+1,\ 0\leqslant\varepsilon\leqslant 1 \\ a(x)=0,\ x_2^4<x_1^2\,. \end{cases}$$

Then A is a bounded operator on $L^2(\boldsymbol{R}^2)$. But the integral in (2.4) diverges. We show this by using the fact that when $h(0)<\infty$,

$$\int_0^\infty h(R)dR<\infty$$

if and only if

$$(2.6)\qquad \int_0^\infty \|AJ_R((-\Delta)^N+1)^{-1}\|dR<\infty\,.$$

Here J_R is the multiplication operator by the function $J_R(x)=\varphi(x/R)$ with $\varphi\in C^\infty(\boldsymbol{R}^2)$ and $\varphi(x)=0$ (resp. 1) for $|x|\leqslant 1$, (resp. $\geqslant 2$). (The details can be found on p. 124 of Simon [10].)

Now, let N be an integer. Choose $v\in C_0^\infty$ such that

$$\operatorname{supp} v\subset\{x\in\boldsymbol{R}^2|\,|x|<1\},\ \|((-\Delta)^N+1)v\|_{L^2}=1$$

and put

$$v_R(x)=v(x-3Re_2),\quad e_2=(0,1)$$
$$u_R(x)=((-\Delta)^N+1)v_R(x)\,.$$

Then it follows that

$$\|u_R\|^2=\int|(|\xi|^{2N}+1)\hat{v}(\xi)|^2d\xi=1\,.$$

Noting that

$$\operatorname{supp} v_R\subset\{x\in\boldsymbol{R}^2|\,|x|\geqslant 2R\}\,,$$

we have

$$\begin{aligned}\|AJ_R((-\Delta)^N+1)^{-1}\|&\geqslant\|AJ_R((-\Delta)^N+1)^{-1}u_R\|\\&=\|AJ_Rv_R\|\\&=\|Av_R\|\\&\geqslant C(1+R)^{-\varepsilon}\|v\|\,.\end{aligned}$$

Since $\varepsilon\leqslant 1$, the integral in (2.6) diverges. Thus $\int_0^\infty h(R)dR$ diverges and the Enss condition is not satisfied.

On the other hand, if we regard A as a perturbation of $P(D)=(1+D_1^2)^{1/2}$, then the existence of wave operators follows from Theorem 2. In fact, Assumption 2.1 is clearly fulfilled (with $\Xi=\{\xi\in\boldsymbol{R}^2|\,\xi_1=0\}$). Since the symbol of $P(D)$ is $(1+\xi_1^2)^{1/2}$, we have

$$p'(\xi) = (\xi_1(1+\xi_1^2)^{-1/2},\ 0)$$
$$p''(\xi) = \begin{bmatrix} (1+\xi_1^2)^{-3/2} & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$\Omega_1 = \{\xi \in \boldsymbol{R}^2 \mid \xi_1 \neq 0\}$$
$$\Omega_2 = \phi$$

(see Theorem 2 for the definition of Ω_k). We examine that the integral in (c_1) converges for any $r>0$ and any compact set $K \subset \Omega_1$. Set

$$\mathcal{O}_t = \{y \in \boldsymbol{R}^2 \mid r \leqslant |y| \leqslant 2r,\ t^4 y_2^4 \geqslant t^2 y_1^2\}\ .$$

Noting that the set $p'(K)$ is in the y_1-axis, we have

$$d_K(y) \geqslant Ct^{-1/2}, \qquad y \in \mathcal{O}_t\,. \tag{2.7}$$

By (2.5) and (2.7)

$$\int_1^\infty t^{1/2}\Big(\int_{r \leqslant |y| \leqslant 2r} (1+td_K(y))^{-N_1} |a(\pm ty)|^2 dy\Big)^{1/2} dt$$
$$\leqslant C\int_1^\infty t^{1/2}\Big(\int_{\mathcal{O}_t} (1+t^{1/2})^{-N_1} dy\Big)^{1/2} dt$$
$$\leqslant C\int_1^\infty (1+t)^{(2-N_1)/4} dt\,.$$

Hence the condition (c_1) holds if we take $N_1=7$. Thus the wave operators exist. Incidentally it follows from Theorem 1 that $\mathcal{H}_{ac}(H_0)=L^2(\boldsymbol{R}^2)$. We should note that if we take $P(D)=\Delta$ then (c_1) does not hold.

EXAMPLE 2.4 (Higher order perturbations). Let us consider two self-adjoint operators

$$H_0 = (1-\Delta)^{1/2} \tag{2.8}$$
$$H = (1-\Delta)^{1/2} + (1+|x|^2)^{-\varepsilon/4}(1-\Delta)^{3/2}(1+|x|^2)^{-\varepsilon/4} \tag{2.9}$$

in $L^2(\boldsymbol{R}^3)$, where $\varepsilon>1$. Since the right side of (2.9) with domain $\mathcal{S}(\boldsymbol{R}^3)$ is a real operator, it admits a self-adjoint extention. We want to check the hypotheses of Theorem 2. The symbol of the perturbation

$$(1+|x|^2)^{-\varepsilon/4}(1-\Delta)^{3/2}(1+|x|^2)^{-\varepsilon/4}$$

is given by the following oscillatory integral:

$$a(x,\xi) = \mathrm{O_s}\!-\!\iint e^{-i\langle y,\eta\rangle}(1+|x|^2)^{-\varepsilon/4}(1+|\xi+\eta|^2)^{3/2}(1+|x+y|^2)^{-\varepsilon/4} dy\, d\eta$$

(see Kumano-go [8]). By repeated integration by parts, we have

$$(2.10)\qquad a(x,\xi)$$
$$=(1+|x|^2)^{-\varepsilon/4}\iint e^{-i\langle y,\eta\rangle}(1+|y|^2)^{-2}(1-\Delta_\eta)^2(1+|\eta|^2)^{-4}(1+|\xi+\eta|^2)^{3/2}$$
$$\times(1-\Delta_y)^4(1+|x+y|^2)^{-\varepsilon/4}dyd\eta\,.$$
$$\equiv(1+|x^2|)^{-\varepsilon/4}\iint e^{-i\langle y,\eta\rangle}I\,dyd\eta\,.$$

Since, as can be easily verified,

$$|(\partial/\partial\xi)^\alpha(\partial/\partial x)^\beta I|$$
$$\leqslant C^1_{\alpha\beta}(1+|y|)^{-4+\varepsilon/2}(1+|\eta|)^{-5}(1+|x|)^{-\varepsilon/2}(1+|\xi|)^{3-|\alpha|}$$

with a constant $C^1_{\alpha\beta}$ for all multi-indices α, β, it follows by differentiation under the integral sign that

$$a(x,\xi)\in C^\infty(\boldsymbol{R}^3\times\boldsymbol{R}^3)$$

and

$$(2.11)\qquad |(\partial/\partial\xi)^\alpha(\partial/\partial x)^\beta a(x,\xi)|\leqslant C^2_{\alpha\beta}(1+|x|)^{-\varepsilon}(1+|\xi|)^{3-|\alpha|}$$

with a constant $C^2_{\alpha\beta}$ for all α, β. Hence the hypothesis (A) is satisfied. The symbol $p(\xi)$ of H_0 is $(1+|\xi|^2)^{1/2}$, so it follows that $\Sigma=\{0\}$ and $\operatorname{rank}p''(\xi)=3$ at every $\xi\in\boldsymbol{R}^3$. Thus Assumption 2.1 is fulfilled (with $\Xi=\{0\}$). Finally, we check the condition (c_k). In this case, we have $\Omega_1=\Omega_2=\phi$ and $\Omega_3=\boldsymbol{R}^3\setminus\{0\}$. Notice that, by (2.11),

$$\int_{r\leqslant|y|\leqslant 2r}(\sup_{\xi\in K}|(\partial/\partial\xi)^\alpha a(\pm ty,\xi)|)^2dy\leqslant C(1+|t|)^{-2\varepsilon}$$

for any $r>0$, any compact set $K\subset\Omega_3$ and any multi-index α. Then it is obvious that (c_3) holds. Thus the wave operators exist. By Theorem 1, we have also $\mathcal{H}_{\mathrm{ac}}(H_0)=L^2(\boldsymbol{R}^3)$.

3. Proof of Theorem 1

In the present section we shall prove Theorem 1 and some propositions on the spectral structure of H_0. First we should note that the hypothesis $(P.\ 1)$ can be further relaxed. We only need that $p(\xi)$ be n times continuously differentiable.

As a preliminary to the proof, we note that since the Fourier transformation is unitary, it is sufficient to consider the self-adjoint operator $\hat{H}_0$ defined by

$$\hat{H}_0=\mathcal{F}H_0\overline{\mathcal{F}}$$

instead of H_0. It is easy to see that

$$u\in\mathcal{H}_{\mathrm{ac}}(H_0)\Leftrightarrow\hat{u}\in\mathcal{H}_{\mathrm{ac}}(\hat{H}_0)\,. \tag{3.1}$$

This implies that

$$u\in\mathcal{H}_{\mathrm{s}}(H_0)\Leftrightarrow\hat{u}\in\mathcal{H}_{\mathrm{s}}(\hat{H}_0)\,. \tag{3.2}$$

The operator $\hat{H}_0$ coincides with the multiplication operator defined by

$$\begin{cases} \mathcal{D}(\hat{H}_0)=\{\hat{u}\in L^2(\boldsymbol{R}^n)\,|\,p\hat{u}\in L^2(\boldsymbol{R}^n)\} \\ \hat{H}_0\hat{u}(\xi)=p(\xi)\hat{u}(\xi)\,. \end{cases}$$

Let $\hat{E}(B)$ be the spectral measure associated with $\hat{H}_0$, where B varies over all Borel sets of real line. Then

$$\|\hat{E}(B)\hat{u}\|^2=\int_{p^{-1}(B)}|\hat{u}(\xi)|^2d\xi,\qquad \hat{u}\in L^2(\boldsymbol{R}^n) \tag{3.3}$$

(see Kato [7], p. 520).

We prepare a lemma which will be used later.

Lemma 3.1. *Let $\varphi\in C^1(\Omega)$ be a real-valued function in an open set $\Omega\subset\boldsymbol{R}^n$ and assume that $\varphi'(\xi)\neq 0$ for every $\xi\in\Omega$. If B is a Borel set of $\boldsymbol{R}$ with* $\mathrm{meas}_1(B)=0$, *then $\varphi^{-1}(B)$ is also a Borel set of $\boldsymbol{R}^n$, and* $\mathrm{meas}_n(\varphi^{-1}(B))=0$.

Proof. It is easily seen that $\varphi^{-1}(B)$ is a Borel set of $\boldsymbol{R}^n$. We have only to show that $\mathrm{meas}_n(\varphi^{-1}(B))=0$. Note that if

$$\mathrm{meas}_n(\{\xi\in K\,|\,\varphi(\xi)\in B\})=0$$

for every compact set $K\subset\Omega$ then

$$\mathrm{meas}_n(\varphi^{-1}(B))=0\,.$$

So it suffices to show that to every $\xi_0\in\Omega$ there corresponds a neighborhood U of ξ_0 such that

$$\mathrm{meas}_n(\{\xi\in U\,|\,\varphi(\xi)\in B\})=0\,.$$

Suppose $\xi_0\in\Omega$. Since $\varphi'(\xi_0)\neq 0$, we may assume, without loss of generality, that $\partial\varphi(\xi_0)/\partial\xi_1\neq 0$. The Jacobian matrix of the map

$$f:\,\xi\to(\varphi(\xi),\,\xi_2,\,\cdots,\,\xi_n)$$

is non-singular at ξ_0. By the inverse function theorem, there is a neighborhood U of ξ_0 such that f is a diffeomorphism of class C^1 between U and $f(U)$. Since

$$f(\{\xi\in U\,|\,\varphi(\xi)\in B\})\subset B\times\boldsymbol{R}^{n-1}\,,$$

we have

$$\text{meas}_n(f(\{\xi\in U\,|\,\varphi(\xi)\in B\}))\leqslant\text{meas}_n(B\times\boldsymbol{R}^{n-1})=0\,.$$

Thus

$$\text{meas}_n(\{\xi\in U\,|\,\varphi(\xi)\in B\})=\text{meas}_n(f^{-1}(f(\{\xi\in U\,|\,\varphi(\xi)\in B\})))=0$$

where we use the fact that if g is a C^1 map of a neighborhood of a null set $N\subset\boldsymbol{R}^n$ into $\boldsymbol{R}^n$, then $\text{meas}_n(g(N))=0$. Q.E.D.

Throughout this section, we assume that p is real-valued.

Proposition 3.2. *If* $p\in C^n(\boldsymbol{R}^n)$, *then*

(3.4) $\mathcal{H}_{\text{ac}}(H_0)=\{u\in L^2(\boldsymbol{R}^n)\,|\,\hat{u}(\xi)=0$ *for almost every* $\xi\in\Sigma\}$

(3.5) $\mathcal{H}_{\text{s}}(H_0)=\{u\in L^2(\boldsymbol{R}^n)\,|\,\hat{u}(\xi)=0$ *for almost every* $\xi\in\boldsymbol{R}^n\backslash\Sigma\}$.

REMARK 3.3. To prove Proposition 3.2, we apply Sard's theorem (see Sternberg [11], p. 47):

Let M_1 and M_2 be C^k manifolds of dimension n_1 and n_2 respectively. Let f be a map of class C^k of $M_1\to M_2$. The critical values of f form a set of measure zero if $k-1\geqslant\max(n_1-n_2,\,0)$.

Proof. As remarked before, we consider $\hat{H}_0$ instead of H_0. Define

$$\mathcal{L}=\{\hat{u}\in L^2(\boldsymbol{R}^n)\,|\,\hat{u}(\xi)=0\text{ for almost every }\xi\in\Sigma\}$$
$$\mathcal{M}=\{\hat{u}\in L^2(\boldsymbol{R}^n)\,|\,\hat{u}(\xi)=0\text{ for almost every }\xi\in\boldsymbol{R}^n\backslash\Sigma\}\,.$$

Let $\hat{u}\in\mathcal{L}$, and let B be a Borel set of the real line with $\text{meas}_1(B)=0$. Then by (3.3) we have

$$||\hat{E}(B)\hat{u}||^2=\int_{p^{-1}(B)\cap(\boldsymbol{R}^n\backslash\Sigma)}|\hat{u}(\xi)|^2d\xi\,.$$

Since $p'(\xi)\neq0$ for every $\xi\in\boldsymbol{R}^n\backslash\Sigma$, we have

$$\text{meas}_n(p^{-1}(B)\cap(\boldsymbol{R}^n\backslash\Sigma))=0$$

by Lemma 3.1. Thus

$$||\hat{E}(B)\hat{u}||^2=0$$

which means that $\hat{u}\in\mathcal{H}_{\text{ac}}(\hat{H}_0)$. Thus we have $\mathcal{L}\subset\mathcal{H}_{\text{ac}}(\hat{H}_0)$.

Similarly, we show that $\mathcal{M}\subset\mathcal{H}_{\text{s}}(\hat{H}_0)$. Let $\hat{u}\in\mathcal{M}$, and let B_0 be the set of critical values of p. Then, by Sard's theorem

$$\text{meas}_1(B_0)=0\,.$$

Moreover, B_0 is a Borel set. In fact, with $B_j=\{\xi\in\boldsymbol{R}^n\,|\,|\xi|\leqslant j\}$

$$B_0=p(\Sigma)=\bigcup_{j=1}^{\infty}p(\Sigma\cap B_j)$$

decomposes B_0 as the union of a countable collection of closed sets. If $B\subset R\backslash B_0$, then $p^{-1}(B)\cap\Sigma=\phi$, and

$$\|\hat{E}(B)\hat{u}\|^2=\int_{p^{-1}(B)\cap\Sigma}|\hat{u}(\xi)|^2d\xi=0\,.$$

Thus the measure $\|\hat{E}(B)\hat{u}\|^2$ is singular with respect to the Lebesgue measure, so $\hat{u}\in\mathcal{H}_s(\hat{H}_0)$. Therefore we have $\mathcal{M}\subset\mathcal{H}_s(\hat{H}_0)$.

It is easy to see that $\mathcal{L}$ and $\mathcal{M}$ are closed linear subspaces of $L^2(\boldsymbol{R}^n)$, are orthogonal complements to each other. This implies that $\mathcal{L}=\mathcal{H}_{ac}(\hat{H}_0)$ and $\mathcal{M}=\mathcal{H}_s(\hat{H}_0)$. Thus, by (3.1) and (3.2), we obtain the conclusions. Q.E.D.

Proof of Theorem 1. It is an immediate consequence of Proposition 3.2. Q.E.D.

REMARK 3.4. As mentioned before, the hypothesis (P. 1) can be further relaxed. Simon [10] allows the possibility of singular points to include an example like $p(\xi)=|\xi|$. In such cases we are also able to show a result similar to Theorem 1:

$$\mathcal{H}_{ac}(H_0)=\{u\in L^2(\boldsymbol{R}^n)\,|\,\hat{u}(\xi)=0 \text{ for almost every } \xi\in\overline{C_p\cup S_p}\}$$
$$\mathcal{H}_s(H_0)=\{u\in L^2(\boldsymbol{R}^n)\,|\,\hat{u}(\xi)=0 \text{ for almost every } \xi\notin\overline{C_p\cup S_p}\}\,.$$

Here C_p (S_p) is the set of critical (singular) points of p. (See [10] for the definition.) The proof exactly follows from that of Proposition 3.2 with a minor change.

If $\text{meas}_n(\Sigma)>0$ and the boundary $\partial\Sigma=\Sigma\backslash\text{Int}\,\Sigma$ is of measure zero, then we can improve Proposition 3.2. It should be noted that that $p\in C^1(\boldsymbol{R}^n)$ does not imply that $\partial\Sigma$ is of measure zero.

Proposition 3.5. *Let* $\text{meas}_n(\Sigma)>0$. *If* $p\in C^1(\boldsymbol{R}^n)$ *and* $\text{meas}_n(\partial\Sigma)=0$, *then we have*

$$\mathcal{H}_{ac}(H_0)=\{u\in L^2(\boldsymbol{R}^n)\,|\,\hat{u}(\xi)=0 \textit{ for almost every } \xi\in\Sigma\}$$
$$\mathcal{H}_s(H_0)=\{u\in L^2(\boldsymbol{R}^n)\,|\,\hat{u}(\xi)=0 \textit{ for almost every } \xi\in\boldsymbol{R}^n\backslash\Sigma\}\,.$$

Moreover,

$$\mathcal{H}_p(H_0)=\mathcal{H}_s(H_0)\,.$$

Proof. We also consider the operator $\hat{H}_0$ and define $\mathcal{L}$ and $\mathcal{M}$ as in the proof of Proposition 3.2. It is obvious that $\mathcal{L}\subset\mathcal{H}_{ac}(\hat{H}_0)$. We show that $\mathcal{M}=\mathcal{H}_p(\hat{H}_0)$ which implies that $\mathcal{M}\subset\mathcal{H}_s(\hat{H}_0)$.

Let $\{O_j\}$ be the collection of connected components of $\text{Int}\,\Sigma$ (note that $\text{Int}\,\Sigma\neq\phi$). Then, as can be easily verified, there exists $\lambda_j\in\boldsymbol{R}$ corresponding to O_j such that

$$p(\xi) = \lambda_j\,, \qquad \xi \in O_j\,.$$

Since the characteristic function $1_{O_j}(\xi)$ satisfies that

$$p(\xi)1_{O_j}(\xi) = \lambda_j 1_{O_j}(\xi)\,, \qquad \xi \in \boldsymbol{R}^n\,,$$

we have $\{\lambda_j\} \subset \sigma_p(\hat{H}_0)$, where $\sigma_p(\hat{H}_0)$ denotes the set of eigenvalues of $\hat{H}_0$. Furthermore, if $\hat{H}_0\hat{u} = \lambda\hat{u}$ with $\lambda \in \boldsymbol{R}\setminus\{\lambda_j\}$, then $\hat{u}=0$ by

$$\text{(3.6)} \qquad \text{meas}_n(\{\xi \in \boldsymbol{R}^n \mid p(\xi) = \lambda\}) = 0\,.$$

In (3.6) we used the hypothesis

$$\text{meas}_n(\partial\Sigma) = 0$$

and the fact

$$\text{meas}_n(\{\xi \in \boldsymbol{R}^n\setminus\Sigma \mid p(\xi) = \lambda\}) = 0$$

which follows from Lemma 3.1. Thus we have $\sigma_p(H_0) = \{\lambda_j\}$. Finally, writing

$$\mathcal{M}_j = \{\hat{u} \in L^2(\boldsymbol{R}^n) \mid \hat{u}(\xi)=0 \text{ for almost every } \xi \in \boldsymbol{R}^n\setminus O_j\}\,,$$

we see that $\mathcal{M}_j$ is the eigenspace of $\hat{H}_0$ corresponding to the eigenvalue λ_j and that the closed linear manifold spanned by all $\mathcal{M}_j$ coincides with $\mathcal{M}$. Hence, by the definition of $\mathcal{H}_p(\hat{H}_0)$, $\mathcal{M}$ coincides with $\mathcal{H}_p(\hat{H}_0)$.

Thus we have shown that $\mathcal{L} \subset \mathcal{H}_{ac}(\hat{H}_0)$ and $\mathcal{M} = \mathcal{H}_p(\hat{H}_0)$. By the arguments in the last step of the proof of Proposition 3.2, the result follows. Q.E.D.

It is well known that when $p \in C^1(\boldsymbol{R}^n)$, that $\text{meas}_n(\Sigma)=0$ implies that $\mathcal{H}_{ac}(H_0) = L^2(\boldsymbol{R}^n)$. The converse is also true when $p \in C^n(\boldsymbol{R}^n)$.

Proposition 3.6. *Suppose $p \in C^n(\boldsymbol{R}^n)$. If $\mathcal{H}_{ac}(H_0) = L^2(\boldsymbol{R}^n)$, then* $\text{meas}_n(\Sigma) = 0$.

Proof. We consider $\hat{H}_0$ instead of H_0. Let B_0 be the set of critical values of p. Then, by Sard's theorem, $\text{meas}_1(B_0)=0$. Recall (3.1). Since $\mathcal{H}_{ac}(\hat{H}_0) = L^2(\boldsymbol{R}^n)$, the measure $\|\hat{E}(B)\hat{u}\|^2$ is absolutely continuous with respect to the Lebesgue measure for all $\hat{u} \in L^2(\boldsymbol{R}^n)$, and so $\|\hat{E}(B_0)\hat{u}\|^2=0$ for all $\hat{u} \in L^2(\boldsymbol{R}^n)$. Noting (3.3), we have $\text{meas}_n(p^{-1}(B_0))=0$. Thus we obtain

$$\text{meas}_n(\Sigma) = 0$$

since $\Sigma \subset p^{-1}(B_0)$. Q.E.D.

4. Proof of Theorem 2

In the present section we shall prove Theorem 2. For the proof we need several lemmas.

Lemma 4.1. *Under hypothesis (A) the psuedo-differential operator $A=A(X, D)$ is continuous from $\mathcal{S}(\boldsymbol{R}^n)$ to $\mathcal{S}(\boldsymbol{R}^n)$.*

Proof. Suppose $u\in\mathcal{S}(\boldsymbol{R}^n)$. Then

$$Au(x)=\int e^{i\langle x,\xi\rangle}a(x, \xi)\hat{u}(\xi)d\xi .$$

Since $\hat{u}\in\mathcal{S}(\boldsymbol{R}^n)$, we can differentiate with respect to x under the integral sign as often as we like. Therefore, it follows that $Au\in C^\infty(\boldsymbol{R}^n)$. Moreover, for every multi-index α and every integer $k\geqslant 0$ we obtain

$$(4.1) \qquad (1+|x|)^{-l+2(1-\tau)k}|(\partial/\partial x)^\alpha Au(x)|\leqslant C_{k,\alpha}|u|_{N,\mathcal{S}}$$

where $C_{k,\alpha}$ is a constant and $N=|\alpha|+[m]+2(n+k)+3$. Indeed, by differentiation under the integral sign and integration by parts, we see that

$$\begin{aligned}&(1+|x|^2)^k(\partial/\partial x)^\alpha Au(x)\\&=\int e^{i\langle x,\xi\rangle}(1-\Delta_\xi)^k\left\{\sum_{\beta\leqslant\alpha}\frac{\alpha!}{\beta!(\alpha-\beta)!}(\partial/\partial x)^{\alpha-\beta}a(x, \xi)(i\xi)^\beta\hat{u}(\xi)\right\}d\xi ,\end{aligned}$$

where the integrand can be estimated by

$$(1+|x|)^{l+2\tau k}(1+|\xi|)^{-n-1}|\hat{u}|_{|\alpha|+[m]+n+2k+2,\mathcal{S}}\cdot$$

Noting that for every integer $j\geqslant 0$

$$|\hat{u}|_{j,\mathcal{S}}\leqslant C_j|u|_{j+n+1,\mathcal{S}}$$

with a constant C_j, we obtain (4.1). Since $\tau<1$, $-l+2(1-\tau)k\to\infty$ as $k\to\infty$. Hence (4.1) means that $Au\in\mathcal{S}(\boldsymbol{R}^n)$ and A is a continuous map of $\mathcal{S}(\boldsymbol{R}^n)$ into $\mathcal{S}(\boldsymbol{R}^n)$. Q.E.D.

Lemma 4.2. *Under hypothesis (P. 1) we have*

$$(4.2) \qquad (e^{-itH_0}u)(x)=\int e^{i(\langle x,\xi\rangle-tp(\xi))}\hat{u}(\xi)d\xi$$

for all $u\in\mathcal{S}(\boldsymbol{R}^n)$.

Proof. Put $U_t=\bar{\mathcal{F}}e^{-itp(\xi)}\mathcal{F}$. Then U_t, $-\infty<t<\infty$, is a group of unitary operators in $L^2(\boldsymbol{R}^n)$. Let K be the infinitesimal generator of U_t. For $u\in\mathcal{D}(H_0)$

$$\|t^{-1}(U_t-I)u-(-iH_0)u\|^2=\int|t^{-1}(e^{-itp(\xi)}-1)-(-ip(\xi))|^2|\hat{u}(\xi)|^2d\xi\to 0$$

as $t\to 0$, where we use the Lebesgue dominated convergence theorem. Thus we have

$$-iH_0 \subset K. \tag{4.3}$$

According to the theory of semi-groups, K has the resolvent $(K-\lambda I)^{-1}$ for every $\lambda>0$. Since H_0 is self-adjoint, $-iH_0$ also has the resolvent $(-iH_0-\lambda I)^{-1}$ for every $\lambda>0$. Therefore, both

$$K-\lambda I:\ \mathcal{D}(K)\to L^2(\boldsymbol{R}^n)$$

and

$$-iH_0-\lambda I:\ \mathcal{D}(H_0)\to L^2(\boldsymbol{R}^n)$$

are bijective, provided that $\lambda>0$. Combining this fact and (4.3), we see that $\mathcal{D}(K)=\mathcal{D}(H_0)$. Thus

$$-iH_0=K$$

which implies that $U_t=e^{-itH_0}$. Q.E.D.

REMARK 4.3. The hypothesis (P. 1) in Lemma 4.2 is not essential; by the arguments in the proof of Lemma 4.2, one can show that $e^{-itH_0}=\overline{\mathcal{F}}e^{-itp(\xi)}\mathcal{F}$ even if p is a real-valued continuous function.

Lemma 4.4. *Let hypothesis* (P. 1) *be fulfilled. If* $u\in\mathcal{S}(\boldsymbol{R}^n)$ *then*

$$t\to e^{-itH_0}u$$

is a continuous function with values in $\mathcal{S}(\boldsymbol{R}^n)$.

Proof. By Lemma 4.2, it is easy to see that $(e^{-itH_0}u)(x)$ is a complex-valued C^∞ function. Integrating by parts after differentiating under the integral sign, we have

$$(1+|x|^2)^k(\partial/\partial x)^\alpha(e^{-itH_0}u)(x)=\int e^{i\langle x,\xi\rangle}(1-\Delta_\xi)^k\{e^{-itp(\xi)}(i\xi)^\alpha\hat{u}(\xi)\}\,d\xi \tag{4.4}$$

for every multi-index α and every integer $k\geqslant 0$. The integrand is integrable by (P. 1), thus

$$\sup_x (1+|x|^2)^k|(\partial/\partial x)^\alpha(e^{-itH_0}u)(x)|<\infty.$$

Thus $e^{-itH_0}u\in\mathcal{S}(\boldsymbol{R}^n)$. It follows from (4.4) that for every integer $N\geqslant 0$

$$\begin{aligned} &|e^{-itH_0}u-e^{-isH_0}u|_{N,\mathcal{S}} \\ &\leqslant C_N\max_{k+|\alpha|\leqslant N}\int|(1-\Delta_\xi)^k\{(e^{-itp(\xi)}-e^{-isp(\xi)})(i\xi)^\alpha\hat{u}(\xi)\}|\,d\xi \end{aligned} \tag{4.5}$$

with a constant C_N. Letting $|t-s|\to 0$, we see that the right side of (4.5) tends to zero, by the Lebesgue dominated convergence theorem, so that the left side of (4.5) tends to zero, which proves the lemma. Q.E.D.

Lemma 4.5. *Let hypotheses (A), (P. 1) and (H) be fulfilled. If* $u \in \mathcal{S}(\boldsymbol{R}^n)$, *then*

$$t \to e^{itH} e^{-itH_0} u$$

is a C^1 *function with values in* $L^2(\boldsymbol{R}^n)$ *and*

$$\frac{d}{dt}(e^{itH} e^{-itH_0} u) = ie^{itH} A e^{-itH_0} u \,. \tag{4.6}$$

Proof. First note that $\mathcal{S}(\boldsymbol{R}^n) \subset \mathcal{D}(H_0)$ and $\mathcal{S}(\boldsymbol{R}^n) \subset \mathcal{D}(H)$. Since, by Lemma 4.4, $e^{-itH_0} u \in \mathcal{S}(\boldsymbol{R}^n)$, we have

$$He^{-itH_0}u - H_0 e^{-itH_0} u = Ae^{-itH_0} u \,. \tag{4.7}$$

Using (4.7), we write

$$\begin{aligned} & h^{-1}(e^{i(t+h)H} e^{-i(t+h)H_0} u - e^{itH} e^{-itH_0} u) - ie^{itH} A e^{-itH_0} u \\ & = e^{i(t+h)H} \{h^{-1}(e^{-i(t+h)H_0} - e^{-itH_0})u - (-iH_0)e^{-itH_0} u\} \\ & \quad + (e^{i(t+h)H} - e^{itH})(-iH_0)e^{-itH_0} u \\ & \quad + \{h^{-1}(e^{i(t+h)H} - e^{itH})e^{-itH_0} u - (iH)e^{itH} e^{-itH_0} u\} \,. \end{aligned} \tag{4.8}$$

By the unitarity of e^{itH} and the fact that $u \in \mathcal{D}(H_0)$, the first term in the right side of (4.8) tends to zero in the L^2 norm as $h \to 0$. Note that for every $v \in L^2(\boldsymbol{R}^n)$

$$t \to e^{itH} v$$

is a continuous function with values in $L^2(\boldsymbol{R}^n)$. Then it is clear that the second term tends to zero in the L^2 norm as $h \to 0$. The third term goes to zero in the L^2 norm as $h \to 0$, since $e^{-itH_0} u \in \mathcal{D}(H)$. Thus we obtain (4.6). It follows immediately from Lemmas 4.1 and 4.4 that

$$t \to Ae^{-itH_0} u$$

is a continuous function with values in $\mathcal{S}(\boldsymbol{R}^n)$, so that it is a continuous function with values in $L^2(\boldsymbol{R}^n)$. Thus,

$$t \to ie^{itH} A e^{-itH_0} u$$

is an L^2-valued continuous function. Q.E.D.

The following is a key to the proof of Theorem 2. The proof will be given in Section 5, as it is fairly long.

Lemma 4.6. *Let all of the hypotheses of Theorem 2 be fulfilled. Then for every non-empty* Ω_k, *we have*

$$(4.9)\qquad \int_{-\infty}^{\infty} \|Ae^{-itH_0}u\|dt < \infty\,, \qquad \hat{u}\in C_0^\infty(\Omega_k)\,.$$

Proof of Theorem 2. First let $\mathcal{Y}$ be the subspace of $L^2(\boldsymbol{R}_\xi^n)$ defined by

$$\mathcal{Y} = C_1^\infty(\Omega_1)\oplus\cdots\oplus C_0^\infty(\Omega_n)\,.$$

If Ω_k is empty, we interpret $C_0^\infty(\Omega_k)$ as $\{0\}$, i.e., the space consisting of the only function which is identically zero in Ω_k. Since

$$\bigcup_{k=1}^{n}\Omega_k = \boldsymbol{R}^n\backslash(\Xi\cup\Sigma)$$

and since

$$\mathrm{meas}_n((\boldsymbol{R}^n\backslash\Sigma)\backslash\bigcup_{k=1}^{n}\Omega_k) = 0\,,$$

$\mathcal{Y}$ is dense in the subspace

$$\{\hat{u}\in L^2(\boldsymbol{R}^n)\,|\,\hat{u}(\xi) = 0 \text{ for almost every } \xi\in\Sigma\}$$

of $L^2(\boldsymbol{R}_\xi^n)$. Writing $\mathcal{X}=\bar{\mathcal{F}}\mathcal{Y}$, it follows from Theorem 1 that $\mathcal{X}$ is dense in $\mathcal{H}_{\mathrm{ac}}(H_0)$.

Now, let $u\in\mathcal{X}$. Then, by Lemmas 4.5 and 4.6,

$$\int_{-\infty}^{\infty}\left\|\frac{d}{dt}(e^{itH}e^{-itH_0}u)\right\|dt<\infty.$$

This implies that $e^{itH}e^{-itH_0}u$ converges in the L^2 norm (i.e., converges strongly) as $t\rightarrow\pm\infty$. Since $\mathcal{X}$ is dense in $\mathcal{H}_{\mathrm{ac}}(H_0)$ and since $\|e^{itH}e^{-itH_0}\|=1$, we see that the limits

$$s\text{—}\lim_{t\to\pm\infty} e^{itH}e^{-itH_0}u$$

exist for every $u\in\mathcal{H}_{\mathrm{ac}}(H_0)$. Thus, we have proved the existence of the wave operators (1.2). Q.E.D.

5. Proof of the key lemma 4.6

We now turn to the proof of Lemma 4.6. We shall divide the proof into two cases: (i) $k=n$ (ii) $k<n$.

Proof of Lemma 4.6. It suffices to show that to each $\xi_0\in\Omega_k$ there corresponds a neighborhood U of ξ_0 such that

$$(5.1)\qquad \int_{1\leq|t|<\infty}\|Ae^{-itH_0}u\|dt<\infty, \qquad \hat{u}\in C_0^\infty(U)\,.$$

In fact, by a partition of unity we then obtain (4.9).

(i) Assume that $\xi_0 \in \Omega_n$. Then $\det p''(\xi_0) \neq 0$ which means that the Jacobian of the map

$$p' : \xi \to p'(\xi)$$

is non-zero at ξ_0. By the inverse function theorem there exists a neighborhood U of ξ_0 such that p' is a diffeomorphism between U and $p'(U)$. Here we may assume that $U \subset \Omega_n$ and

$$p'(U) \subset \{r \leqslant |y| \leqslant 2r\}$$

for some positive number r.

Now, let $\hat{u} \in C_0^\infty(U)$. Then, by Lemma 4.2,

$$(Ae^{-itH_0}u)(ty) = \int e^{it(\langle y,\xi\rangle - p(\xi))} a(ty, \xi)\hat{u}(\xi)d\xi\,. \tag{5.2}$$

Let $y_* \in p'(\operatorname{supp} \hat{u})$. Then we can find a unique critical point $\eta_* \in U$ of the phase function

$$\xi \to \langle y_*, \xi\rangle - p(\xi)$$

with $\det p''(\eta_*) \neq 0$. By Lemma II of Appendix and the remarks following it, there is a sufficiently small number $\varepsilon_* > 0$ such that for y with $|y - y_*| < \varepsilon_*$

$$\begin{aligned} &\Big| \int e^{it(\langle y,\xi\rangle - p(\xi))} a(ty, \xi)\hat{u}(\xi)d\xi \\ &\qquad - |\det p''(\eta)/2\pi|^{-1/2} e^{\pi i\sigma/4} e^{itp(\eta)} L_y a(ty, \eta)u(\eta)|t|^{-n/2} \Big| \\ &\leqslant C|a(ty, \cdot)\hat{u}|_{[n/2]+3}|t|^{-n/2-1}, \qquad t \geqslant 1 \end{aligned} \tag{5.3}$$

where L_y is a constant depending on y and $\sigma = \operatorname{sgn} p''(\eta_*)$ (i.e., the signature of the symmetric matrix $p''(\eta_*)$), $y = p'(\eta)$ $(\eta \in U)$. From (5.2) and (5.3) we have

$$|(Ae^{-itH_0}u)(ty)| \leqslant C\{\max_{|\alpha| \leqslant [n/2]+3} (\sup_{\xi \in \operatorname{supp} \hat{u}} |(\partial/\partial\xi)^\alpha a(ty, \xi)|)\}|t|^{-n/2} \tag{5.4}$$

when $|y - y_*| < \varepsilon_*$, $|t| \geqslant 1$. Since $p'(\operatorname{supp} \hat{u})$ is compact, we can choose a neighborhood V of $p'(\operatorname{supp} \hat{u})$ so that $\bar{V} \subset p'(U)$ and so that (5.4) remains valid for $y \in V$. Furthermore, we choose a compact set $K \subset U$ so that $p'(K) = \bar{V}$. Thus we have

$$\begin{aligned} &\int_{1 \leqslant |t| < \infty} \left(\int_{y \in V} |Ae^{-itH_0}u(ty)|^2 |t|^n dy \right)^{1/2} dt \\ &\leqslant C \int_{1 \leqslant |t| < \infty} \Big[\int_{r \leqslant |y| \leqslant 2r} (1 + |t|d_K(y))^{-N_n} \\ &\qquad \times \{\max_{|\alpha| \leqslant 2N_n} (\sup_{\xi \in K} |(\partial/\partial\xi)^\alpha a(ty, \xi)|)^2\} dy \Big]^{1/2} dt\,. \end{aligned} \tag{5.5}$$

Here we have used the facts that $d_K(y)=0$ when $y\in V$ and

$$V\subset\{r\leqslant|y|\leqslant 2r\}\ .$$

By condition (c_n) the right side of (5.5) is convergent. Thus we have shown that

$$\int_{1<|t|<\infty}\Big(\int_{x/t\in V}|Ae^{-itH_0}u(x)|^2dx\Big)^{1/2}dt<\infty. \tag{5.6}$$

Now, we shall show that

$$\int_{1<|t|<\infty}\Big(\int_{x/t\notin V}|Ae^{-itH_0}u(x)|^2dx\Big)^{1/2}dt<\infty. \tag{5.7}$$

To do so we shall apply Lemma A.1 of Hörmander [6] to the integral

$$(Ae^{-itH_0}u)(x)=\int e^{i(|x|+|t|)(\langle x,\xi\rangle-tp(\xi))/(|x|+|t|)}a(x,\xi)\hat{u}(\xi)d\xi\ .$$

Let Φ be the set of phase functions given by

$$\Phi=\overline{\{\xi\to(\langle x,\xi\rangle-tp(\xi))/(|x|+|t|)\,|\,x/t\in \boldsymbol{R}^n\backslash V\}}\qquad\text{in } C^{N+1}(\boldsymbol{R}^n,\boldsymbol{R})\ .$$

Then Φ is a compact subset of $C^{N+1}(\boldsymbol{R}^n,\boldsymbol{R})$ such that for every $f\in\Phi$

$$f'(\xi)\neq 0\qquad\text{if } \xi\in\operatorname{supp}\hat{u}\ .$$

In fact, writing

$$Z=\overline{\{(z,s)\in\boldsymbol{R}^{n+1}\,|\,(z,s)=(x,t)/(|x|+|t|)\ \text{for some}\ x/t\in\boldsymbol{R}^n\backslash V\}}$$

we have

$$\Phi=\{\xi\to\langle z,\xi\rangle-sp(\xi)\,|\,(z,s)\in Z\}\ .$$

It is easily seen that

$$(z,s)\to\langle z,\xi\rangle-sp(\xi)\in C^{N+1}(\boldsymbol{R}^n,\boldsymbol{R})$$

is continuous. Since Z is compact, it follows immediately that Φ is a compact subset of $C^{N+1}(\boldsymbol{R}^n,\boldsymbol{R})$. Thus Lemma A.1 of [6] shows that for every N

$$|(Ae^{-itH_0}u)(x)|\leqslant C_N(1+|x|+|t|)^{-N}|a(x,\cdot)\hat{u}|_N \tag{5.8}$$

if $x/t\in\boldsymbol{R}^n\backslash V$. By Assumption (2.1) (A) the right side of (5.8) is bounded by

$$C'_N(1+|x|+|t|)^{l+(\tau-1)N}\ .$$

Thus we see that

$$\int_{x/t\notin V}|Ae^{-itH_0}u(x)|^2dx\leqslant C'_N\Big(\int(1+|x|)^{-n-1}dx\Big)(1+|t|)^{n+1+2l-2(1-\tau)N}$$

for every integer N with $(n+1)/2+l-(1-\tau)N<-1$. This proves (5.7). Thus we have proved (5.1) in the case $k=n$.

(ii) Assume that $\xi_0\in\Omega_k$. Since the rank of the symmetric matrix $p''(\xi_0)$ is k, we can find a principal minor of order k which does not vanish. So we may assume that

$$\det(\partial^2 p(\xi_0)/\partial\xi_i\partial\xi_j)_{i,j=1,\cdots,k}\neq 0 .$$

Let

$$\begin{aligned} y'_0 &= p_{\xi'}(\xi_0) \\ &\equiv (\partial p(\xi_0)/\partial\xi_1, \cdots, \partial p(\xi_0)/\partial\xi_k) . \end{aligned}$$

Then, by the implicit function theorem, there is a neighborhood W of y'_0, a neighborhood U' of ξ'_0, a neighborhood U'' of ξ''_0 and a unique function

$$\varphi=(\varphi_1, \cdots, \varphi_k)\in C^\infty(W\times U'')$$

such that

$$\begin{aligned} &\{(y', \xi)\in W\times U'\times U''\,|\,y'=p_{\xi'}(\xi)\} \\ &=\{(y', \xi)\in \boldsymbol{R}^k\times\boldsymbol{R}^n\,|\,\xi'=\varphi(y', \xi'') \text{ for some } (y', \xi'')\in W\times U''\} . \end{aligned}$$

Here $\xi'=(\xi_1, \cdots, \xi_k)$, $\xi''=(\xi_{k+1}, \cdots, \xi_n)$ and $y'=(y_1, \cdots, y_k)$. We now set

$$U=\{\xi\in\boldsymbol{R}^n\,|\,\xi'=\varphi(y', \xi'') \text{ for some } (y', \xi'')\in W\times U''\} .$$

Obviously U is a neighborhood of ξ_0. We may assume that $U\subset\Omega_k$ and that

$$p'(U)\subset\{y\in\boldsymbol{R}^n\,|\,5r/4\leqslant|y|\leqslant 7r/4\} \tag{5.9}$$

for some $r>0$. Using Assumption (2.1) (P. 2) (b) we shall show that $p_{\xi''}(\varphi(y', \xi''), \xi'')$ is independent of ξ''. Put

$$g_j(y', \xi'')=\partial p(\varphi(y', \xi''), \xi'')/\partial\xi_j \qquad (j=k+1, \cdots, n) .$$

Then we have the differential

$$dg_j=\sum_{i=1}^{k}(\partial^2 p/\partial\xi_i\partial\xi_j)d\varphi_i+\sum_{i=k+1}^{n}(\partial^2 p/\partial\xi_i\partial\xi_j)d\xi_i . \tag{5.10}$$

On the other hand, since $y_j=\partial p(\varphi(y', \xi''), \xi'')/\partial\xi_j$ ($j=1, \cdots, k$) we have

$$dy_j=\sum_{i=1}^{k}(\partial^2 p/\partial\xi_i\partial\xi_j)d\varphi_i+\sum_{i=k+1}^{n}(\partial^2 p/\partial\xi_i\partial\xi_j)d\xi_i . \tag{5.11}$$

Since the rank of $p''(\varphi(y', \xi''), \xi'')$ is k and since

$$\det p_{\xi'\xi'}(\varphi(y', \xi''), \xi'')\neq 0 ,$$

(5.10) and (5.11) imply that

$$dg_j = \sum_{i=1}^{k} c_{ji}(y', \xi'')dy_i \qquad (j=k+1, \cdots, n)$$

where $c_{ji}(y', \xi'')$ are suitable functions. Hence

$$\partial g_j(y', \xi'')/\partial \xi_i = 0 \qquad (i, j = k+1, \cdots, n)$$

which means that $p_{\xi''}(\varphi(y', \xi''), \xi'')$ is independent of ξ''. Hence there is a function $g \in C^\infty(W)$ such that

$$p'(U) = \{y \in \boldsymbol{R}^n \,|\, y' \in W, y'' = g(y')\} .$$

Let us now estimate the L^2 norm of $Ae^{-itH_0}u$. Let $u \in C_0^\infty(U)$. We write

$$Ae^{-itH_0}u(ty) = \int d\xi'' \int e^{it(\langle y, \xi\rangle - p(\xi))} a(ty, \xi)\hat{u}(\xi)d\xi' .$$

Set

$$\Gamma = \{\xi'' \in \boldsymbol{R}^{n-k} \,|\, (\xi', \xi'') \in \operatorname{supp} \hat{u} \text{ for some } \xi' \in \boldsymbol{R}^k\} .$$

Let $y_* \in p'$ (supp $\hat{u}$) and let $\xi''_* \in \Gamma$. We consider the phase function

$$\xi' \to \langle y'_*, \xi'\rangle + \langle y''_*, \xi''_*\rangle - p(\xi', \xi''_*) .$$

Since $y'_* = p_{\xi'}(\varphi(y'_*, \xi''_*), \xi''_*)$, this phase function has a unique critical point $\varphi(y'_*, \xi''_*) \in U'$. By Lemma II and the remarks following it, for every integer $N>0$ there is a positive number ε_* such that when $|y-y_*| < \varepsilon_*$ and $|\xi''-\xi''_*| < \varepsilon_*$,

$$\Big| \int e^{it(\langle y, \xi\rangle - p(\xi))} a(ty, \xi)\hat{u}(\xi)d\xi' - \sum_{j=0}^{N-1} |\det p_{\xi'\xi'}(\eta', \xi'')/2\pi|^{-1/2} \tag{5.12}$$
$$\times e^{\pi i\sigma/4} e^{it(\langle y', \eta'\rangle + \langle y'', \xi''\rangle - p(\eta', \xi''))} L_{y\xi''j}(a(ty, \xi)\hat{u}(\xi))\,|_{\xi'=\eta'} |t|^{-j-k/2} \Big|$$
$$\leqslant C_N |a(ty, \cdot, \xi'')\hat{u}(\cdot, \xi'')|_s |t|^{-k/2-N}, \qquad |t| \geqslant 1 .$$

Here $\eta' = \varphi(y', \xi'')$, $s = 2N + [k/2] + 1$ and $\sigma = \operatorname{sgn} p_{\xi'\xi'}(\varphi(y'_*, \xi''_*), \xi''_*)$. Differential operators $L_{y\xi''j}$ depend on y, ξ'' and by virtue of Lemma A.5 of [6] their coefficients are C^∞ functions of y, ξ''. Since p'(supp $\hat{u}$) and Γ are compact, we can find a neighborhood V of p'(supp $\hat{u}$) such that (5.12) remains valid for $y \in V$ and $\xi'' \in \Gamma$. Here we may assume that

$$p'(\operatorname{supp} \hat{u}) \subset V \cap p'(U)$$

and

$$V \subset \{y \in \boldsymbol{R}^n \,|\, r \leqslant |y| \leqslant 2r\} \tag{5.13}$$

with the same r as in (5.9). At the critical point $\eta' = \varphi(y', \xi'')$ the phase

$$\xi' \to \langle y', \xi'\rangle + \langle y'', \xi''\rangle - p(\xi', \xi'')$$

is of the form

$$f(y)+\langle y''-g(y'),\ \xi''\rangle .$$

Indeed, writing

$$\tilde{f}(y,\ \xi'')=\langle y',\ \varphi(y',\ \xi'')\rangle+\langle y'',\ \xi''\rangle-p(\varphi(y',\ \xi''),\ \xi'')$$

and using the fact that $y'=p_{\xi'}(\varphi(y',\ \xi''),\ \xi'')$, we see that

$$d\tilde{f}=\langle dy',\ \varphi\rangle+\langle dy'',\ \xi''\rangle+\langle y''-g(y'),\ d\xi''\rangle .$$

Therefore $\tilde{f}_{\xi''}=y''-g(y')$.

Summing up, we have for every $N>0$ and every $y\in V$

$$\begin{aligned}(5.14)\qquad & |Ae^{-itH_0}u(ty)| \\ & \leqslant C\sum_{j=0}^{N-1}\left|\int e^{it\langle y''-g(y'),\xi''\rangle}|\det p_{\xi'\xi'}(\varphi(y',\ \xi''),\ \xi'')|^{-1/2}\right. \\ & \quad \left.\times L_{y\xi'' j}(a(ty,\ \xi)\hat{u}(\xi))|_{\xi'=\varphi(y',\xi'')}d\xi''\right|\times|t|^{-j-k/2} \\ & \quad +C_N|t|^{-k/2-N}\int|a(ty,\ \cdot,\ \xi'')\hat{u}(\cdot,\ \xi'')|_s d\xi'' \\ & \equiv C\sum_{j=0}^{N-1}I_j|t|^{-j-k/2}+C_N J|t|^{-k/2-N} .\end{aligned}$$

Applying Lemma I to I_j, we see that for every integer L

$$I_j\leqslant C_L(1+|t|\,|y''-g(y')|)^{-L}\max_{|\alpha|\leqslant 2j+L}(\sup_{\xi\in\operatorname{supp}\hat{u}}|(\partial/\partial\xi)^{\alpha}a(ty,\ \xi)|) .$$

It is obvious that

$$J\leqslant C_N'\max_{|\alpha|\leqslant s}(\sup_{\xi\in\operatorname{supp}\hat{u}}|(\partial/\partial\xi)^{\alpha}a(ty,\ \xi)|) .$$

Choose a compact set K so that supp $\hat{u}\subset K$ and so that $V\cap p'(U)\subset p'(K)$. Then

$$1+|t|d_K(y)\leqslant 1+|t|\,|y''-g(y')|,\qquad y\in V$$

and

$$1+|t|d_K(y)\leqslant C|t|,\qquad y\in V,\ |t|\geqslant 1 .$$

Thus we have

$$\begin{aligned}(5.15)\qquad & \int_{1\leqslant|t|<\infty}\Big(\int_{x/t\in V}|Ae^{-itH_0}u(x)|^2dx\Big)^{1/2}dt \\ & \leqslant C_{N,L}\int_{1\leqslant|t|<\infty}|t|^{(n-k)/2}\Big[\int_{r\leqslant|y|\leqslant 2r}(1+|t|d_K(y))^{-2L} \\ & \quad \times\max_{|\alpha|\leqslant 2N+L-2}(\sup_{\xi\in K}|(\partial/\partial\xi)^{\alpha}a(ty,\ \xi)|)^2dy\Big]^{1/2}dt\end{aligned}$$

$$+C_N\int_{1\leqslant|t|<\infty}|t|^{(n-k)/2}\Big[\iint_{r\leqslant|y|\leqslant 2r}(1+|t|d_K(y))^{-2N}$$
$$\times\max_{|\alpha|\leqslant s}(\sup_{\xi\in K}|(\partial/\partial\xi)^\alpha a(ty,\xi)|)^2 dy\Big]^{1/2}dt$$

where we use (5.13). If we choose N, L so that $N_k\leqslant 2N=2L\leqslant N_k+1$, then $2N+L-2\leqslant 2N_k$ and $s\leqslant 2N_k$. Thus, by condition (c_k), the integrals in the right side of (5.15) are convergent.

Finally, (5.7) is shown in quite the same way as in (i). Thus we have proved (5.1) in the case $k<n$. Q.E.D.

6. Symmetry of $A(X, D)$

In this section we shall determine completely when the pseudo-differential operator $A=A(X,D)$ is symmetric on $\mathcal{S}(\boldsymbol{R}^n)$. To avoid confusion, we denote $A\restriction\mathcal{S}(\boldsymbol{R}^n)$ by A_0.

Lemma 6.1. *Let hypothesis* (*A*) *be fulfilled. Then the domain* $\mathcal{D}(A_0^*)$ *of the adjoint* A_0^* *contains* $\mathcal{S}(\boldsymbol{R}^n)$ *and*

$$(A_0^*v)(x)=\int e^{i\langle x,\xi\rangle}\Big(\int e^{-i\langle x',\xi\rangle}\bar{a}(x',\xi)v(x')dx'\Big)d\xi\,,\qquad v\in\mathcal{S}(\boldsymbol{R}^n).$$

Here $\bar{a}(x,\xi)=\overline{a(x,\xi)}$.

Proof. See [8], p.61. Q.E.D.

We wish to get the representation of the form

$$(A_0^*v)(x)=\int e^{i\langle x,\xi\rangle}b(x,\xi)\hat{v}(\xi)d\xi\,.$$

Following ideas in [8], we shall consider the symbol $b(x,\xi)$ defined by the oscillatory integral

$$b(x,\xi)=\mathrm{O_s}\text{-}\iint e^{-i\langle y,\eta\rangle}\bar{a}(x+y,\xi+\eta)dyd\eta\,. \tag{6.1}$$

For oscillatory integrals we refer to [8, Chapter I].

Lemma 6.2. *Let hypothesis* (*A*) *be fulfilled and let* $b(x,\xi)$ *be as in* (6.1). *Then* $b\in C^\infty(\boldsymbol{R}^n\times\boldsymbol{R}^n)$ *and for all multi-indices* α, β *the estimate*

$$\begin{aligned}&|(\partial/\partial\xi)^\alpha(\partial/\partial x)^\beta b(x,\xi)|\\&\leqslant C_{\alpha\beta}(1+|x|)^{l+\tau(|\alpha|+2l(\alpha))}(1+|\xi|)^{m+\delta(|\beta|+2m(\beta))}\end{aligned} \tag{6.2}$$

is valid for some constant $C_{\alpha\beta}$, *where* $l(\alpha)=[(l+n+\tau|\alpha|)/2(1-\tau)]+1$, $m(\beta)=[(m+n+\delta|\beta|)/2(1-\delta)]+1$.

Proof. First put

$$(6.3)\qquad b_\varepsilon(x, \xi)=\iint e^{-i\langle y,\eta\rangle}\chi(\varepsilon y, \varepsilon\eta)a(x+y, \xi+\eta)dy\,d\eta\,, \qquad \varepsilon>0\,,$$

where $\chi(y, \eta)\in\mathcal{S}(\boldsymbol{R}^n\times\boldsymbol{R}^n)$ and $\chi(0, 0)=1$. Since we can differentiate under the sign of integration as often as we like, we conclude that $b_\varepsilon\in C^\infty$. Furthermore, by repeated integration by parts, we have

$$(6.4)\qquad \begin{aligned}&(\partial/\partial\xi)^\alpha(\partial/\partial x)^\beta b_\varepsilon(x, \xi)\\ &=\iint e^{-i\langle y,\eta\rangle}(1+|y|^2)^{-l(\alpha)}(1-\Delta_\eta)^{l(\alpha)}\{(1+|\eta|^2)^{-m(\beta)}\\ &\quad\times(1-\Delta_y)^{m(\beta)}(\chi(\varepsilon y, \varepsilon\eta)a_{(\beta)}^{(\alpha)}(x+y, \xi+\eta))dy\,d\eta\\ &\equiv\iint e^{-i\langle y,\eta\rangle}I_\varepsilon(x, \xi, y, \eta)dy\,d\eta\end{aligned}$$

with the notation $a_{(\beta)}^{(\alpha)}=(\partial/\partial\xi)^\alpha(\partial/\partial x)^\beta a$. By using hypothesis (A) we have

$$(6.5)\qquad \begin{aligned}|I_\varepsilon|\leqslant{}&C'_{\alpha\beta}(1+|x|)^{l+\tau|\alpha|+2\tau l(\alpha)}(1+|\xi|)^{m+\delta|\beta|+2\delta m(\beta)}\\ &\times(1+|y|)^{l+\tau|\alpha|+2(\tau-1)l(\alpha)}(1+|\eta|)^{m+\delta|\beta|+2(\delta-1)m(\beta)}\end{aligned}$$

where $C'_{\alpha\beta}$ is independent of ε. Since $l+\tau|\alpha|+2(\tau-1)l(\alpha)<-n$ and since $m+\delta|\beta|+2(\delta-1)m(\beta)<-n$, we can derive from (6.4) and (6.5)

$$(6.6)\qquad \begin{aligned}&|(\partial/\partial\xi)^\alpha(\partial/\partial x)^\beta b_\varepsilon(x, \xi)|\\ &\leqslant C''_{\alpha\beta}(1+|x|)^{l+\tau|\alpha|+2\tau l(\alpha)}(1+|\xi|)^{m+\delta|\beta|+2\delta m(\beta)}\end{aligned}$$

with a constant $C''_{\alpha\beta}$ independent of ε.

Now, let

$$b_{\alpha\beta}(x, \xi)=\mathrm{O_s}-\iint e^{-i\langle y,\eta\rangle}a_{(\beta)}^{(\alpha)}(x+y, \xi+\eta)dy\,d\eta\,.$$

Then, by repeated integration by parts,

$$(6.7)\qquad \begin{aligned}b_{\alpha\beta}(x, \xi)=\iint e^{-i\langle y,\eta\rangle}&(1+|y|^2)^{-l(\alpha)}(1-\Delta_\eta)^{l(\alpha)}\{(1+|\eta|^2)^{-m(\beta)}\\ &\times(1-\Delta_y)^{m(\beta)}a_{(\beta)}^{(\alpha)}(x+y, \xi+\eta)\}dy\,d\eta\,.\end{aligned}$$

From (6.4) and (6.7) we see that for every compact set $K\subset\boldsymbol{R}^n\times\boldsymbol{R}^n$ and all multi-indices α, β

$$(6.8)\qquad \begin{aligned}&\sup_{(x,\xi)\in K}|b_{\alpha\beta}(x, \xi)-(\partial/\partial\xi)^\alpha(\partial/\partial x)^\beta b_\varepsilon(x, \xi)|\\ &\leqslant C_{K\alpha\beta}\int\Big(\max_{|\mu|+|\nu|\leqslant N_{\alpha\beta}}|(\partial/\partial\eta)^\mu(\partial/\partial y)^\nu(1-\chi(\varepsilon y, \varepsilon\eta))|\Big)\\ &\quad\times(1+|y|)^{l+\tau|\alpha|+2(\tau-1)l(\alpha)}(1+|\eta|)^{m+\delta|\beta|+2(\delta-1)m(\beta)}dy\,d\eta\end{aligned}$$

is valid for some constant $C_{K\alpha\beta}$, where $N_{\alpha\beta}=2(l(\alpha)+m(\beta))$. Since $(\partial/\partial\eta)^{\mu}\times(\partial/\partial y)^{\nu}(1-\chi(\varepsilon y, \varepsilon\eta))\to 0$ as $\varepsilon\to 0$, (6.8) implies that for all multi-indices α, β $(\partial/\partial\xi)^{\alpha}(\partial/\partial x)^{\beta}b_{\varepsilon}$ converges uniformly on every compact set to $b_{\alpha\beta}$ as $\varepsilon\to 0$. In particular, b_{ε} converges uniformly on every compact set to b as $\varepsilon\to 0$. Hence $b\in C^{\infty}$ and

$$(\partial/\partial\xi)^{\alpha}(\partial/\partial x)^{\beta}b(x, \xi) = b_{\alpha\beta}(x, \xi)\,. \tag{6.9}$$

Thus (6.2) follows from (6.6) and (6.9). Q.E.D.

The proof of the following lemma is virtually identical to some arguments of [8, chapter 2].

Lemma 6.3. *Let hypothesis* (*A*) *be fulfilled and let* $b(x, \xi)$ *be as in* (6.1). *Then*

$$(A_0^* v)(x) = \int e^{i\langle x,\xi\rangle}b(x, \xi)\hat{v}(\xi)d\xi\,, \qquad v\in\mathcal{S}(\boldsymbol{R}^n)\,. \tag{6.10}$$

Proof. Let $b_{\varepsilon}(x, \xi)$ be as in the proof of Lemma 6.2. Then the Fubini theorem and a change of variables give

$$\begin{aligned}
&\int e^{i\langle x,\xi\rangle}b(x, \xi)\hat{v}(\xi)d\xi \\
&=\iiint e^{i\langle x,\xi\rangle-i\langle y,\eta\rangle}\chi(\varepsilon y, \varepsilon\eta)\bar{a}(x+y, \xi+\eta)\hat{v}(\xi)dy\,d\eta\,d\xi \\
&=\int e^{i\langle x,\xi'\rangle}\Big\{\int e^{-i\langle x',\xi'\rangle}\bar{a}(x', \xi') \\
&\qquad\times\Big(\int e^{i\langle x',\xi\rangle}\chi(\varepsilon(x-x'), \varepsilon(\xi'-\xi))\hat{v}(\xi)d\xi\Big)dx'\Big\}d\xi' \\
&\equiv\int e^{i\langle x,\xi'\rangle}I_{\varepsilon}(x, \xi')d\xi'\,.
\end{aligned} \tag{6.11}$$

By (6.6), (6.11) and the Lebesgue dominated convergence theorem

$$\lim_{\varepsilon\to 0}\int e^{i\langle x,\xi'\rangle}I_{\varepsilon}(x, \xi')d\xi' = \int e^{i\langle x,\xi\rangle}b(x, \xi)\hat{v}(\xi)\,d\xi\,. \tag{6.12}$$

By repeated integration by parts it follows that for every integer $k>0$

$$\Big|\int e^{i\langle x',\xi\rangle}\chi(\varepsilon(x'-x), \varepsilon(\xi'-\xi))\hat{v}(\xi)\,d\xi\Big|\leqslant C_k(1+|x'|)^{-k} \tag{6.13}$$

with C_k independent of ε. Therefore, by the Lebesgue dominated convergence theorem,

$$\lim_{\varepsilon\to 0} I_{\varepsilon}(x, \xi') = \int e^{-i\langle x',\xi'\rangle}\bar{a}(x', \xi')v(x')dx'\,. \tag{6.14}$$

Here we have used Fourier's inversion formula

$$v(x')=\int e^{i\langle x',\xi\rangle}\hat{v}(\xi)d\xi .$$

Note that for every integer $j\geqslant 0$

$$(6.15)\qquad I_\varepsilon(x,\xi')$$
$$=(1+|\xi'|^2)^{-j}\int e^{-i\langle x',\xi'\rangle}(1-\Delta_{x'})^j\Big\{\bar{a}(x',\xi')$$
$$\times\Big(\int e^{i\langle x',\xi\rangle}\chi(\varepsilon(x'-x),\ \varepsilon(\xi'-\xi))\hat{v}(\xi)d\xi\Big)\Big\}dx' .$$

Since, by (6.13) and hypothesis (A), the integrand in (6.15) can be estimated by

$$(1+|\xi'|)^{m+2j\delta}(1+|x'|)^{l-2k}$$

for every integer $k\geqslant 0$, it follows that for every integer $j\geqslant 0$

$$|I_\varepsilon(x,\xi')|\leqslant C_j'(1+|\xi'|)^{m+2(\delta-1)j}$$

with a constant C_j' independent of ε. Choosing j so that $m+2(\delta-1)j<-n$ and using the Lebesgue dominated convergence theorem, we see that

$$(6.16)\qquad \lim_{\varepsilon\to\infty}\int e^{i\langle x,\xi'\rangle}I_\varepsilon(x,\xi')dx'$$
$$=\int e^{i\langle x,\xi'\rangle}\Big(\int e^{-i\langle x',\xi'\rangle}\bar{a}(x',\xi')v(x')dx'\Big)d\xi' .$$

(Recall (6.14).) By Lemma 6.1, the right side of (6.16) equals $(A_0^*v)(x)$. Thus, combining (6.16) with (6.12), we obtain (6.10). Q.E.D.

Now we give a theorem.

Theorem 3. *Let hypothesis (A) be fulfilled. The operator A_0 is symmetric if and only if*

$$(6.17)\qquad \mathcal{F}_x[a](\eta,\xi)=\mathcal{F}_x[\bar{a}](\eta,\xi+\eta)\qquad in\ \mathcal{S}'(\boldsymbol{R}^n_\eta\times\boldsymbol{R}^n_\xi) .$$

Here $\mathcal{F}_x[a]$ is the Fourier transform of a with respect to x.

Proof. Let b be as in (6.1). If

$$(6.18)\qquad \mathcal{F}_x[b](\eta,\xi)=\mathcal{F}_x[\bar{a}](\eta,\xi+\eta)\qquad \text{in }\mathcal{S}'(\boldsymbol{R}^n_\eta\times\boldsymbol{R}^n_\xi) ,$$

then (6.17) is equivalent to

$$(6.19)\qquad \mathcal{F}_x[a](\eta,\xi)=\mathcal{F}_x[b](\eta,\xi)\qquad \text{in }\mathcal{S}'(\boldsymbol{R}^n_\eta\times\boldsymbol{R}^n_\xi) ,$$

or what is the same thing,

$$a(x, \xi) = b(x, \xi) . \tag{6.20}$$

Hence, by Lemma 6.3, (6.17) is equivalent to the fact that A_0 is symmetric. Thus, it suffices to show (6.18)

(6.18) means that for every $\varphi(\eta, \xi) \in \mathcal{S}(\boldsymbol{R}^n_\eta \times \boldsymbol{R}^n_\xi)$

$$\begin{aligned} \iint b(x, \xi)\mathcal{F}_\eta[\varphi](x, \xi)dxd\xi \\ = \iint \bar{a}(x, \xi)\Big(\int e^{-i\langle x,\eta\rangle}\varphi(\eta, \xi-\eta)d\eta\Big)dxd\xi . \end{aligned} \tag{6.21}$$

Choose $\chi_0 \in \mathcal{S}(\boldsymbol{R}^n)$ so that $\chi_0(0)=1$ and replace $\chi(\varepsilon y, \varepsilon\eta)$ in (6.3) by $\chi_0(\varepsilon y)\chi_0(\varepsilon\eta)$. Then b_ε has the same properties as before. By (6.6) and the fact that $b_\varepsilon \to b$ as $\varepsilon \to 0$, the left-hand side of (6.21) is equal to the limit of

$$\iint b_\varepsilon(x, \xi)\mathcal{F}_\eta[\varphi](x, \xi)dxd\xi \tag{6.22}$$

which, by the definition of b_ε, equals

$$\iiiint e^{-i\langle y,\eta'\rangle}\chi_0(\varepsilon y)\chi_0(\varepsilon\eta')\bar{a}(x+y, \xi+\eta')\mathcal{F}_\eta[\varphi](x, \xi)dy\,d\eta'dxd\xi . \tag{6.23}$$

By a change of variables, (6.23) equals

$$\begin{aligned} \iint \bar{a}(x, \xi)\Big(\iint e^{-i\langle y,\eta'\rangle}\chi_0(\varepsilon y)\chi_0(\varepsilon\eta')\mathcal{F}_\eta[\varphi](x-y, \xi-\eta')dy\,d\eta'\Big)dxd\xi \\ \equiv \iint \bar{a}(x, \xi)J_\varepsilon(x, \xi)dxd\xi . \end{aligned} \tag{6.24}$$

Finally we should examine J_ε. Using the Fubini theorem, we see that

$$J_\varepsilon(x, \xi) = \iint e^{-i\langle x,\eta\rangle}\varphi(\eta, \xi-\eta')\chi_0(\varepsilon\eta')\Big(\int e^{-i\langle y,\eta'-\eta\rangle}\chi_0(\varepsilon y)dy\Big)d\eta\, d\eta' . \tag{6.25}$$

Writing $y' = \varepsilon y$, we see that the right side of (6.25) is equal to

$$\iint e^{-i\langle x,\eta\rangle}\varphi(\eta, \xi-\eta')\chi_0(\varepsilon\eta')\hat{\chi}_0(\varepsilon^{-1}(\eta'-\eta))\varepsilon^{-n}d\eta\, d\eta' .$$

Thus a change of variables gives

$$J_\varepsilon(x, \xi) = \iint e^{-i\langle x,\eta'-\varepsilon\theta\rangle}\varphi(\eta'-\varepsilon\theta, \xi-\eta')\chi_0(\varepsilon\eta')\hat{\chi}_0(\theta)\, d\theta\, d\eta' .$$

Hence

(6.26) $$\lim_{\varepsilon\to 0} J_\varepsilon(x, \xi) = \int e^{-i\langle x,\eta'\rangle}\varphi(\eta', \xi-\eta')\Big(\int \hat{\chi}_0(\theta)d\theta\Big)d\eta'$$
$$= \int e^{-i\langle x,\eta'\rangle}\varphi(\eta', \xi-\eta')d\eta'$$

where we use $\chi_0(0)=1$. On the other hand, since $\mathcal{F}_\eta[\varphi]\in\mathcal{S}(\boldsymbol{R}^n\times\boldsymbol{R}^n)$, it follows that for every integer $k\geqslant 0$

(6.27) $$|J_\varepsilon(x, \xi)|\leqslant C_k(1+|x|)^{-k}(1+|\xi|)^{-k}$$

with a constant C_k independent of ε. Hence, by (6.22)–(6.24), (6.26), (6.27) and the Lebesgue dominated convergence theorem, (6.21) now follows. Q.E.D.

7. Self-adjoint extensions of $P(D)+A(X, D)$

In this section we give sufficient conditions for $P(D)+A(X,D)$ with domain $\mathcal{S}(\boldsymbol{R}^n)$ to have self-adjoint extensions. Let A_0 be as in Section 6 and let $P_0=P(D)\restriction\mathcal{S}(\boldsymbol{R}^n)$. Throughout this section, we assume that A_0 is symmetric. The following is an extension of a proposition of [1].

Theorem 4. *Let hypothesis (A) be fulfilled and assume that p is a real-valued continuous function so that*

$$|p(\xi)|\leqslant C(1+|\xi|)^j$$

for some real numbers C and j. If there is a symmetric and orthogonal matrix M of type $n\times n$ such that

(7.1) $$p(-M\xi) = p(\xi)\,,\quad \bar{a}(Mx, -M\xi) = a(x, \xi)$$

then P_0+A_0 has a self-adjoint extension.

Proof. For $u\in L^2(\boldsymbol{R}^n)$, define

$$(Uu)(x) = \overline{u(Mx)}\,.$$

Since M is symmetric and orthogonal, U is a conjugation in the sense of [4, Definition 17, p. 1231]. Using (7.1), we shall show that P_0+A_0 commutes with U. Let $u\in\mathcal{S}(\boldsymbol{R}^n)$. Since $|Mx|=|x|$, we see that $Uu\in\mathcal{S}(\boldsymbol{R}^n)$. By (7.1) and the fact that $\mathcal{F}[Uu](\xi)=\overline{\mathcal{F}[u](-M\xi)}$,

$$(P_0+A_0)\,Uu(x) = \int e^{i\langle x,\xi\rangle}(p(-M\xi)+\bar{a}(Mx, -M\xi))\overline{\mathcal{F}[u](-M\xi)}d\xi\,.$$

A change of variables gives

(7.2) $$(P_0+A_0)Uu(x) = \int e^{i\langle -Mx,\eta\rangle}(p(\eta)+\bar{a}(Mx, \eta))\overline{\hat{u}(\eta)}d\eta\,.$$

Thus, by Theorem 18, p. 1231 of [4], P_0+A_0 has equal deficiency indices, which implies that P_0+A_0 admits a self-adjoint extension. Q.E.D.

Next, we shall give a sufficient condition for P_0+A_0 to be essentially self-adjoint. We will need a stronger condition on $A(X, D)$.

Theorem 5. *Let m, ρ, δ be real numbers with $0\leqslant\delta\leqslant\rho\leqslant 1$, $\delta<1$ and assume that a is a C^∞ function so that*

$$(7.3)\qquad |(\partial/\partial\xi)^\alpha(\partial/\partial x)^\beta a(x, \xi)|\leqslant C_{\alpha\beta}(|\xi|)(1+|\xi|)^{m+\delta|\beta|-\rho|\alpha|}$$

for all multi-indices α, β. Here $C_{\alpha\beta}(|\xi|)$ is a continuous function on $\boldsymbol{R}^n$ such that $C_{\alpha\beta}(|\xi|)\to 0$ as $|\xi|\to\infty$. Moreover, assume that p is a real-valued continuous function so that

$$(7.4)\qquad C_1(1+|\xi|)^m\leqslant|p(\xi)|\leqslant C_2(1+|\xi|)^{\bar{m}}$$

for some C_1, C_2 and $\bar{m}$. Then P_0+A_0 is essentially self-adjoint.

Proof. For every $R>0$, choose a function $\varphi_R\in C^\infty(\boldsymbol{R}^n)$ with $\varphi_R(\xi)=1$ when $|\xi|<R$ and $\varphi_R(\xi)=0$ when $|\xi|>R+1$. Let $\Phi_R(D)$ be a pseudo-differential operator with the symbol $\varphi_R(\xi)$. We make a decomposition

$$(7.5)\qquad \begin{aligned} &A(X, D)(1-\Delta)^{-m/2}\\ &=A(X, D)(1-\Phi_R(D))(1-\Delta)^{-m/2}+A(X, D)\Phi_R(D)(1-\Delta)^{-m/2}. \end{aligned}$$

Note that the symbol of the first term on the right side is

$$a(x, \xi)(1-\varphi_R(\xi))(1+|\xi|^2)^{-m/2}.$$

Since $1-\varphi_R(\xi)=0$ for $|\xi|<R$, we obtain for all multi-indices α, β

$$\begin{aligned} &|(\partial/\partial\xi)^\alpha(\partial/\partial x)^\beta\{a(x, \xi)(1-\varphi_R(\xi))(1+|\xi|^2)^{-m/2}\}|\\ &\leqslant M_{\alpha\beta}(\max_{\substack{\mu\leqslant\alpha\\ \nu\leqslant\beta}}\sup_{|\xi|\geqslant R} C_{\mu\nu}(|\xi|))(1+|\xi|)^{\delta|\beta|-\rho|\alpha|} \end{aligned}$$

with a constant $M_{\alpha\beta}$. Here we have used (7.3). From a theorem due to Calderon and Vaillancourt ([8], p. 215) it follows that

$$(7.6)\qquad ||A(X, D)(1-\Phi_R(D))(1-\Delta)^{-m/2}v||_{L^2}\leqslant C_R||v||_{L^2}, \qquad v\in L^2(\boldsymbol{R}^n),$$

where

$$(7.7)\qquad \begin{aligned} C_R = C\max_{\substack{|\alpha|\leqslant M\\ |\beta|\leqslant N}}\sup_{x,\xi}[|(\partial/\partial\xi)^\alpha(\partial/\partial x)^\beta\{a(x, \xi)\\ \times(1-\varphi_R(\xi))(1+|\xi|^2)^{(-m-\delta|\beta|+\rho|\alpha|)/2}\}|] \end{aligned}$$

and $M=2[n/2+1]$, $N=2[n/(2(1-\delta))+1]$. The constant C_R can be estimated by

$$\max_{\substack{|\alpha|\leq M \\ |\beta|\leq N}} (\sup_{|\xi|\geq R} C_{\alpha\beta}(|\xi|)) . \tag{7.8}$$

Similarly, we note that the symbol of the second term on the right side of (7.5) is

$$a(x, \xi)\varphi_R(\xi)(1+|\xi|^2)^{-m/2}$$

which vanishes when $|\xi|>R$. Again we can apply a theorem due to Calderon and Vaillancourt which shows that

$$||A(X, D)\Phi_R(D)(1-\Delta)^{-m/2}v||_{L^2} \leq C'_R||v||_{-m}, \qquad v\in H_{-m} \tag{7.9}$$

with C'_R depending on R.

Now, let $u\in\mathcal{S}(\boldsymbol{R}^n)$ and put $v=(1-\Delta)^{m/2}u$. Then,

$$||v||_{-m} = ||u||_{L^2} \tag{7.10}$$

and

$$\begin{aligned} ||v||^2_{L^2} &= \left(\int_{|\xi|<R} + \int_{|\xi|>R}\right)(1+|\xi|^2)^m |\hat{u}(\xi)|^2 d\xi \\ &\leq (1+R^2)^{|m|}||u||^2_{L^2} + C||P_0u||^2_{L^2} \end{aligned} \tag{7.11}$$

where we use (7.4). Hence, by (7.5)–(7.11),

$$\begin{aligned} ||A_0u||_{L^2} &= ||A(X, D)(1-\Delta)^{-m/2}v||_{L^2} \\ &\leq C(\max_{\substack{|\alpha|\leq M \\ |\beta|\leq N}} \sup_{|\xi|\geq R} C_{\alpha\beta}(|\xi|))||P_0u||_{L^2} + C''_R||u||_{L^2} \end{aligned} \tag{7.12}$$

with C''_R depending on R and C independent of R. Since $C_{\alpha\beta}(|\xi|)\to 0$ as $|\xi|\to\infty$, (7.12) means that A_0 is P_0-bounded with relative bound 0. Note that P_0 is essentially self-adjoint. Thus, by Theorem 4.4, p. 288 of [7], we conclude that P_0+A_0 is also essentially self-adjoint. Q.E.D.

Appendix

In [6], Hörmander systematized the method of stationary phase. In this appendix, we shall reproduce his results in a somewhat different form.

Lemma I. *Let $K\subset\boldsymbol{R}^n$ be a compact set, Ω a neighborhood of K and let Φ be a subset of $C^{k+1}(\Omega, \boldsymbol{R})$ with the following properties:*

(*a*) *There is a constant C_1 such that for every $f\in\Phi$ and every multi-index α with $1\leq|\alpha|\leq k+1$ the estimate*

$$|(\partial/\partial x)^\alpha f(x)|\leq C_1, \qquad x\in\Omega$$

is valid.

(*b*) *There is a positive constant C_2 such that for every $f\in\Phi$*

$$|f'(x)|\geqslant C_2, \qquad x\in K.$$

Then we have for $\omega\in\boldsymbol{R}$ *and all* $f\in\Phi$

$$\left|\int e^{i\omega f(x)}u(x)dx\right|\leqslant C(1+|\omega|)^{-k}|u|_k, \qquad u\in C_0^k(K).$$

If one looks carefully into the proof of Lemma A.1 of [6] one finds easily that the conclusion of Lemma I remains valid. So we omit the proof.

Lemma II. *Let* $f\in C^\infty(\Omega)$ *be a real-valued function in a neighborhood* Ω *of* 0 *in* $\boldsymbol{R}^n$. *Assume that* $f'(0)=0$ *and that* $f''(0)$ *is non-singular. Moreover, assume that* $f'(x)\neq 0$ *for every* $x\in\Omega\backslash\{0\}$. *Then there exist differential operators* $L_{f,j}$ *of order* $2j$ *such that for every integer* $k>0$ *and every* $s>2k+n/2$, *we have when* $u\in C_0^\infty(K)$, K *compact* $\subset\Omega$,

$$\text{(A.1)}\quad \left|\int u(x)e^{i\omega f(x)}dx-\sum_{j=0}^{k-1}|\det f''(0)/2\pi|^{-1/2}e^{\pi i\sigma/4}e^{i\omega f(0)}L_{f,j}u(0)\omega^{-j-n/2}\right|$$
$$\leqslant C\omega^{-n/2-k}|u|_s, \qquad \omega>1.$$

Here $\sigma=\operatorname{sgn}f''(0)$, *i.e.*,

$$\sigma=\sum\operatorname{sgn}\lambda_j$$

in which λ_j, $j=1,\cdots,n$, *are the repeated eigenvalues of the symmetric matrix* $f''(0)$.

Since we assume that $f'(x)\neq 0$ for every $x\in\Omega\backslash\{0\}$, this lemma follows immediately from Lemma A.1 and Lemma A.4 of [6]. In fact, choosing $\alpha\in C_0^\infty(\Omega)$ so that supp α is contained in a small neighborhood of the origin and using Lemma A.4 of [6], we find that there exist differential operators $L_{f,j}$ of order $2j$ such that

$$\text{(A.2)}\quad \left|\int u(x)\alpha(x)e^{i\omega f(x)}dx-\sum_{j=0}^{k-1}|\det f''(0)/2\pi|^{-1/2}e^{\pi i\sigma/4}e^{i\omega f(0)}L_{f,j}(u\alpha)(0)\omega^{-j-n/2}\right|$$
$$\leqslant C\omega^{-n/2-k}|u\alpha|_s.$$

If $r_0>0$ is sufficiently small and $\alpha(x)=1$ when $|x|\leqslant r_0$, then f has no critical point in supp $u(1-\alpha)$. Applying Lemma A.1 of [6] to the integral

$$\int u(x)(1-\alpha(x))e^{i\omega f(x)}dx$$

we have

$$\text{(A.3)}\quad \left|\int u(x)(1-\alpha(x))e^{i\omega f(x)}dx\right|$$
$$\leqslant C(1+|\omega|)^{-k-[n/2]-1}|(1-\alpha)u|_{k+[n/2]+1}.$$

Combining (A.2) and (A.3), we obtain the conclusion of Lemma II.

We remark that Lemma II is applicable uniformly to all functions in a small neighborhood of f in $C^\infty(\Omega)$. Indeed, Hörmander [6] showed that (A.2) is applicable uniformly to all functions in a small neighborhood of f in $C^\infty(\Omega)$. Remembering that $f'(x)\neq 0$ for every $x\in\Omega\setminus\{0\}$, we find that for any compact set $K\subset\Omega\setminus\{0\}$ and any integer $k\geqslant 0$ the hypotheses of Lemma I are satisfied, provided that we take Φ to be a small neighborhood of f in $C^\infty(\Omega)$. Thus (A.3) is also applicable uniformly to all functions in a small neighborhood of f in $C^\infty(\Omega)$.

References

[1] A.M. Berthier and P. Collet: *Existence and completeness of the wave operators in scattering theory with momentum-dependent potentials*, J. Func Anal. **26** (1977), 1–15.
[2] M.S. Birman: *Scattering problems for differential operators with constant coefficients*, Functional Anal. Appl. **3** (1969), 167–180.
[3] J.M. Cook: *Convergence of the Møller wave-matrix*, J. Math. Phys. **36** (1957), 82–87.
[4] N. Dunford and J.T. Schwartz: Linear operators II, Interscience, New York, 1963.
[5] V. Enss: *Asymptotic completeness for quantum mechanical potential scattering*, Comm. Math. Phys. **61** (1978), 285–291.
[6] L. Hörmander: *The existence of wave operators in scattering theory*, Math. Z. **146** (1976), 69–91.
[7] T. Kato: Perturbation theory for linear operators, Springer, Berlin, 1966.
[8] H. Kumano-go: Pseudo-differential operators, Iwanami, Tokyo, 1974 (in Japanese).
[9] M. Schechter: *Scattering theory for pseudodifferential operators*, Quart. J. Math. Oxford Ser. (2) **27** (1976), 111–121.
[10] B. Simon: *Phase space analysis of simple scattering systems: Extentions of some work of Enss*, Duke Math. J. **46** (1979), 119–168.
[11] S. Sternberg: Lectures on differential geometry, Prentice-Hall, New Jersey, 1965.

Department of Mathematics
Osaka City University
Osaka 558
Japan