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Osaka J. Math.
18 (1981), 257–288

ON THE FUNDAMENTAL SOLUTION FOR A DEGENERATE HYPERBOLIC SYSTEM

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(Received November 2, 1979)

Introduction. Let $S[m_1, m_2]$ denote the set of all C^∞-symbols $a(t, x, \xi)$ on $[0, T]\times R^n_x\times R^n_\xi$ $(0<T\leqq 1)$ such that

$$(0.1)\qquad |D_t^j D_\xi^\alpha D_x^\beta a(t, x, \xi)|\leqq C_{j,\alpha,\beta}\langle\xi\rangle^{m_1-|\alpha|}(t+\langle\xi\rangle^{-\omega})^{m_2-j}$$

for constants $C_{j,\alpha,\beta}$, where $\langle\xi\rangle=(1+|\xi|^2)^{1/2}$ and $\omega=1/(l+1)$ with an integer $l>0$.

Consider a hyperbolic operator of first order:

$$(0.2)\qquad \boldsymbol{L} = \boldsymbol{D}_t - t^l \begin{bmatrix} \mu_1(t, X, D_x) & & 0 \\ & \ddots & \\ 0 & & \mu_m(t, X, D_x) \end{bmatrix} + \boldsymbol{B}(t)\,,$$

where μ_j, $j=1, \cdots, m$ are real valued and satisfy

$$(0.3)\qquad \begin{cases} \text{i)} & \mu_j(t, x, \xi)\in S[1, 0] \\ \text{ii)} & |\mu_j(t, x, \xi)-\mu_k(t, x, \xi)|\geqq c\langle\xi\rangle \qquad (j\neq k) \end{cases}$$

for a constant $c>0$, and the symbol $\sigma(\boldsymbol{B}(t))(x, \xi)$ of the lower order operator $\boldsymbol{B}(t)$ satisfies

$$(0.4)\qquad \sigma(\boldsymbol{B}(t))(x, \xi)\in S[0, -1]\,.$$

The purpose of the present paper is to construct the fundamental solution $\boldsymbol{E}(t, s)$ $(0\leqq s\leqq t\leqq T_0)$ of the Cauchy problem

$$(0.5)\qquad \begin{cases} \boldsymbol{L}U = \Phi(t) & \text{on} \quad [s, T_0]\,, \\ U|_{t=s} = \Psi \end{cases}$$

for a small constant T_0 $(0<T_0\leqq T)$. It should be noted that the operator $\boldsymbol{L}$ is degenerate at $t=0$ and $\boldsymbol{B}(t)$ is not uniformly bounded on $[0, T]$ as a family of pseudo-differential operators with parameter $t\in[0, T]$.

To construct $\boldsymbol{E}(t, s)$, we find first the perfect diagonalizer $\boldsymbol{N}(t)$ such that the symbol $\sigma(\boldsymbol{N}(t))(x, \xi)$ belongs to $S[0, 0]$ and

$$LN(t) \equiv N(t)L_1 \bmod \mathcal{B}_t(S^{-\infty}), \tag{0.6}$$

where L_1 is an operator of the form

$$L_1 = D_t - t^l \begin{bmatrix} \mu_1(t, X, D_x) & & 0 \\ & \ddots & \\ 0 & & \mu_m(t, X, D_x) \end{bmatrix} + \begin{bmatrix} f_1(t, X, D_x) & & 0 \\ & \ddots & \\ 0 & & f_m(t, X, D_x) \end{bmatrix} + R(t) \tag{0.7}$$

such that $f_j(t, x, \xi) \in S[0, -1]$ and $\sigma(R(t))(x, \xi) \in \mathcal{H}^\omega = \bigcap_{\nu=0}^{\infty} S[\omega-\nu\omega, -\nu]$. Then, for L_1 we can construct the fundamental solution $E_1(t, s)$, and, by using $E_1(t, s)$, the fundamental solution $E(t, s)$ for L can be found in the form

$$E(t, s) = N(t)E_1(t, s)N^\sharp(s) + R_{-\infty}(t, s), \tag{0.8}$$

where $N^\sharp(s)$ is a parametrix of $N(s)$ and $\sigma(R_{-\infty}(t, s))(x, \xi) \in \mathcal{B}_{t,s}(S^{-\infty})$.

We note that $E(t, s)$ is represented as the sum of Fourier integral operators which have phase functions $\phi_j(t, s, x, \xi)$ defined as the solutions of eiconal equations:

$$\begin{cases} \partial_t\phi_j - t^l\mu_j(t, x, \nabla_x\phi_j) = 0 & (0 \leqq s \leqq t \leqq T_0), \\ \phi_j(s, s) = x \cdot \xi, \end{cases} \tag{0.9}$$

and have symbols in $\bigcap_{0<\varepsilon<1} S[0, M+\varepsilon, -M-\varepsilon]$. The constant M is defined by

$$M = \max_{1 \leqq i \leqq m} \lim_{R\to\infty} \sup_{\substack{x, t\langle\xi\rangle^\omega \geqq R \\ 0 \leqq t \leqq R^{-1}}} \{t\, \mathcal{Im} f_j(t, x, \xi)\}, \tag{0.10}$$

and indicates the order of regularity-loss of the solution of the Cauchy problem.

Concerning the problem (0.5) Kumano-go [7] constructed the fundamental solution without condition (0.3) ii) by using Fourier integral operators of multi-phase. It should be emphasized that our fundamental solution $E(t, s)$ is represented by Fourier integral operators of single phase, and M is determined explicitly by (0.10). The perfect diagonzalization of (0.6) for L enable us such a construction of $E(t, s)$.

In §1 we define some classes of pseudo-differential operators and Fourier integral operators as variants of classes in Boutet de Monvel [2], and summarize fundamental theorems on operators of these classes. In §2, using a similar method to that of Kumano-go [6], we construct the perfect diagonalizer $N(t)$ such that (0.6) holds. We note that $\sigma(\boldsymbol{R}(t))(x, \xi)\in\mathcal{H}^{\infty}$ and $\in S^{-\infty}$ for any fixed $t>0$, but that $\mathcal{H}^{\infty}\not\subset\mathcal{B}_t(S^{-\infty})$ on $[0, T]$. So we can not apply the method in Kumano-go [6] directly. From this reason, in §3, we first treat a single operator and then construct the fundamental solution $\boldsymbol{E}_2(t, s)$ for a purely diagonal operator $\boldsymbol{L}_2=\boldsymbol{L}_1-\boldsymbol{R}(t)$. In §4 the fundamental solution $\boldsymbol{E}_1(t, s)$ for $\boldsymbol{L}_1$ is constructed in the form

$$\boldsymbol{E}_1(t, s)=\boldsymbol{E}_2(t, s)(I+\boldsymbol{Q}(t, s))+\boldsymbol{Q}_{\infty}(t, s),$$

and by using $\boldsymbol{E}_1(t, s)$ the fundamental solution $\boldsymbol{E}(t, s)$ for the general $\boldsymbol{L}$ can be constructed. The crucial point in the discussions of §4 is in finding $\boldsymbol{Q}(t, s)$. Finally in §5 we consider a higher order operator L of the form:

$$L=D_t^m+\sum_{k=1}^{m}a_k(t, X, D_x)D_t^{m-k}, \tag{0.11}$$

where $a_k(t, x, \xi)$ have the forms

$$a_k(t, x, \xi)=\sum_{j=0}^{k}t^{\sigma(j,k)}a_{k,j}(t, x, \xi) \tag{0.12}$$

with differential polynomials $a_{k,j}(t, x, \xi)$ of order $k-j$ in ξ and $\sigma(j, k)=\max\{0, (k-j)(l+1)-k\}$. We assume that the roots $\mu_1, \cdots, \mu_m$ of the equation

$$\lambda^m+a_{1,0}\lambda^{m-1}+\cdots+a_{m,0}=0 \tag{0.13}$$

are real and satisfy (0.3). Then, we show that the Cauchy problem:

$$\begin{cases} Lu=\varphi(t) & \text{on } [s, T_0] \\ D_t^j|_{t=s}=\psi_j, & j=0, \cdots, m-1 \end{cases} \tag{0.14}$$

is reduced to the system (0.5) by modifying the method in Shinaki [11]. We note that the operator L of this type is a generalization of operators which have been treated by Alinhac [1], Chi Min-You [3], Nakamura [8], Nakamura and Uryu [9], Oleinik [10], Uryu [13] and Yoshikawa [14], [15].

The author would like to express his gratitude to Professor H. Kumano-go, Professor M. Ikawa and Mr. K. Taniguchi for their kind suggestions and a number of stimulating conversations.

1. Preliminaries. For $x\in R_x^n$, $\xi\in R_\xi^n$ and multi-indices α, β we use the following notation:

$$\begin{aligned}
&x\cdot\xi=x_1\xi_1+\cdots+x_n\xi_n, \quad \langle\xi\rangle=(1+|\xi|^2)^{1/2},\\
&|\alpha|=\alpha_1+\cdots+\alpha_n, \qquad \alpha!=\alpha_1!\cdots\alpha_n!.\\
&d\!\!{}^-\xi=(2\pi)^{-n}d\xi, \quad D_t=-i\partial/\partial t, \quad \partial_{\xi_j}=\partial/\partial_{\xi_j},\\
&D_{x_j}=-i\partial/\partial x_j,
\end{aligned}$$

$$\partial_\xi^\alpha = \partial_{\xi_1}^{\alpha_1}\cdots\partial_{\xi_n}^{\alpha_n}, \qquad D_x^\beta = D_{x_1}^{\beta_1}\cdots D_{x_n}^{\beta_n},$$
$$a_{(\beta)}^{(\alpha)}(x, \xi) = \partial_\xi^\alpha D_x^\beta a(x, \xi),$$
$$\nabla_x f(x) = (\partial_{x_1} f(x), \cdots, \partial_{x_n} f(x)).$$

Let $S^\nu (=S_{1,0}^\nu)$ denote Hörmander's class of symbols $a(x, \xi)$ on R^n which satisfy

$$(1.1) \qquad |a_{(\beta)}^{(\alpha)}(x, \xi)| \leqq C_{\alpha\,\beta}\langle\xi\rangle^{\nu-|\alpha|} \quad \text{on} \quad R_x^n \times R_\xi^n,$$

and the associated pseudo-differential operators $a(X, D_x)$ are defined by

$$(1.2) \qquad a(X, D_x)u(x) = 0s-\iint e^{-iy\cdot\xi} a(x, \xi)u(x+y)d\xi dy$$
$$= \lim_{\varepsilon\to 0}\iint e^{-iy\cdot\xi}\chi(\varepsilon\xi, \varepsilon y)a(x, \xi)u(x+y)d\xi dy$$
$$(u\in\mathcal{B}(R^n)),$$

where $\chi(\xi, y)\in\mathcal{S}$ (the Schwartz space of rapidly decreasing functions on R^{2n}) such that $\chi(0, 0)=1$ and $\mathcal{B}(R^n)$ denotes the space of C^∞-functions in R^n whose derivatives of any order are all bounded.

Let $\chi(t)$ be a C^∞-function in R^1 such that

$$(1.3) \qquad \begin{cases} 0\leqq\chi(t)\leqq 1 & \text{on} \quad R^1, \\ \chi(t) = 1 \ (|t|\leqq 1), & = 0 \ (|t|\geqq 2). \end{cases}$$

Set $\omega=1/(l+1)$ for a positive integer l and define a function η by

$$(1.4) \qquad \eta(t) = \eta(t, \xi) = t+\langle\xi\rangle^{-\omega}\chi(t\langle\xi\rangle^\omega).$$

Then we have

$$(1.5) \qquad \begin{cases} (t+\langle\xi\rangle^{-\omega})/2\leqq\eta(t, \xi)\leqq t+\langle\xi\rangle^{-\omega}, \\ \langle\xi\rangle^{-\omega}\leqq\eta(t, \xi)\leqq 2 \qquad (0\leqq t\leqq T\leqq 1) \end{cases}$$

and by easy calculation

$$(1.6) \qquad |D_t^j\partial_\xi^\alpha\eta(t, \xi)| \leqq C_{j,\alpha}\langle\xi\rangle^{-|\alpha|}\eta(t, \xi)^{1-j}.$$

Following Boutet de Monvel [2] we define classes of symbols of pseudo-differential operators.

DEFINITION 1.1. i) For real m_1, m_2 we denote by $S[m_1, m_2]$ the space of all C^∞-symbols $a(t, x, \xi)$ on $[0, T]\times R_x^n\times R_\xi^n$ $(0\leqq T\leqq 1)$ such that for any non-negative integer j and multi-indices α, β we have

$$(1.7) \qquad |D_t^j a_{(\beta)}^{(\alpha)}(t, x, \xi)| \leqq C_{j,\alpha,\beta}\langle\xi\rangle^{m_1-|\alpha|}\eta(t, \xi)^{m_2-j}.$$

ii) For real m_1, m_2, m_3 we denote by $S[m_1, m_2, m_3]$ the space of all C^∞-

symbols $a(t, s, x, \xi)$ on $[0, T]\times[0, T]\times R_x^n\times R_\xi^n$ $(0\leqq T\leqq 1)$ such that for any non-negative integers j, k and multi-indices α, β we have

$$|D_t^j D_s^k a_{(\beta)}^{(\alpha)}(t, s, x, \xi)|\leqq C_{j,k,\alpha\,\beta}\langle\xi\rangle^{m_1-|\alpha|}\eta(t, \xi)^{m_2-j}\eta(s, \xi)^{m_3-k}\,. \tag{1.8}$$

iii) We set

$$\mathcal{B}_t(S^{-\infty}) = \bigcap_\nu S[m_1-\nu, m_2]\,,$$
$$\mathcal{B}_{t,s}(S^{-\infty}) = \bigcap_\nu S[m_1-\nu, m_2, m_3]\,,$$
$$\mathcal{H}^m = \bigcap_\nu S[m-\nu, -\nu(l+1)]\,.$$

REMARK. 1°. From (1.5) and (1.6) we have

$$\eta(t, \xi)^\nu\in S[0, \nu] \qquad \text{for real } \nu \tag{1.9}$$

and

$$\begin{cases} a(t, x, \xi)\in S[m_1, m_2] \Rightarrow a(t, x, \xi)\in S^{m,\omega m_2^-}\,, \\ a(t, s, x, \xi)\in S[m_1, m_2, m_3] \\ \Rightarrow a(t, s, x, \xi)\in S^{m_1+\omega m_2^-+\omega m_3^-}\ (m_j^- = \max\{0, -m_j\}) \\ \text{for any fixed } t \text{ and } s\in[0, T]\,. \end{cases} \tag{1.10}$$

2°. We can consider $a(t, x, \xi)\in S[m_1, m_2]$ as an element of $S[m_1, m_2, 0]$. So by this identification we write $S[m_1, m_2]\subset S[m_1, m_2, 0]$, and the statements for the symbols of $S[m_1, m_2, m_3]$ often hold for symbols of $S[m_1, m_2]$.

3°. It is easily proved that

$$\bigcap_\nu S[m_1-\nu, m_2] = \bigcap_\nu \mathcal{B}_t(S^{-\nu})$$

and

$$\bigcap_\nu S[m_1-\nu, m_2, m_3] = \bigcap_\nu \mathcal{B}_{t,s}(S^{-\nu})\,.$$

Proposition 1.2. i) *$S[m_1, m_2]\cap S[m_1', m_2']$, if $m_1\leqq m_1'$ and $m_1-m_2\omega\leqq m_1'-m_2'\omega$.*

ii) *$S[m_1, m_2, m_3]\subset S[m_1', m_2', m_3']$ if "$m_1\leqq m_1'$, $m_1-m_2\omega\leqq m_1'-m_2'\omega$ and $m_3\geqq m_3'$" or "$m_1\leqq m_1'$, $m_1-m_3\omega\leqq m_1'-m_3'\omega$ and $m_2\geqq m_2'$".*

Proof is omitted.

Proposition 1.3. i) *Let $a(t, s, x, \xi)\in S[m_1, m_2, m_3]$. Then, for any non-negative integers j, k we have*

$$t^j s^k a(t, s, x, \xi)\in S[m_1, m_2+j, m_3+k] \tag{1.11}$$

and

$$D_t^j D_s^k a(t, s, x, \xi) \in S[m_1, m_2-j, m_3-k]\,. \tag{1.12}$$

ii) *Let* $a(t, s, x, \xi) \in S[m_1, m_2, m_3]$ *and* $b(t, s, x, \xi) \in S[m_1', m_2', m_3']$. *Then, we have*

$$a(t, s, x, \xi) b(t, s, x, \xi) \in S[m_1+m_1', m_2+m_2', m_3+m_3']\,. \tag{1.13}$$

iii) *Let* $a(t, x, \xi) \in \mathcal{H}^m$ *and* $b(t, x, \xi) \in S[m_1, m_2]$. *Then, we have*

$$a(t, x, \xi) b(t, x, \xi) \in \mathcal{H}^{m+m_1-m_2\omega}\,. \tag{1.14}$$

Proof. i) and ii) are clear. Writing $\eta(t, \xi)^{m_2} = \langle\xi\rangle^{-m_2\omega}(\langle\xi\rangle^{\omega}\eta(t, \xi))^{m_2}$ we get (1.14).

Lemma 1.4. *Set*

$$h(t, \xi) = \eta(t, \xi)^l \langle\xi\rangle\,. \tag{1.15}$$

Then, we have

i) $h(t, \xi)^\nu \in S[\nu, \nu l]$ *for any real* ν,
ii) $h(t, \xi) - t^l\langle\xi\rangle \in \mathcal{H}^\omega$,
iii) $ih_t(t, \xi)/h(t, \xi) - l/\eta(t, \xi) \in \mathcal{H}^\omega$,

where $h_t(t, \xi) = D_t h(t, \xi)$.

Proof. i) is clear. Since $I(t, \xi) = h(t, \xi)) - t^l\langle\xi\rangle = 0$ when $t\langle\xi\rangle^\omega \geqq 2$, $\eta(t, \xi)\langle\xi\rangle^\omega$ is bounded on supp $I(t, \xi)$. So we have ii). Since $i\eta_t(t, \xi) = 1 + i\chi_t(t\langle\xi\rangle^\omega)$ and $\chi_t(t\langle\xi\rangle^\omega) \in \mathcal{H}^0$, we have by Proposition 1.3-iii)

$$\begin{aligned} &ih_t(t, \xi)/h(t, \xi) - l/\eta(t, \xi) \\ &= il\chi_t(t\langle\xi\rangle^\omega)/\eta(t, \xi) \in \mathcal{H}^\omega\,. \end{aligned}$$

Proposition 1.5. i) *Let* $a_\nu(t, s, x, \xi) \in S[m_1-\nu, m_2, m_3]$ *for* $\nu = 0, 1, \cdots$. *Then, there exists an* $a(t, s, x, \xi) \in S[m_1, m_2, m_3]$ *such that*

$$a \sim a_0 + a_1 + \cdots \qquad \text{mod } \mathcal{B}_{t,s}(S^{-\infty})$$

in the sense

$$a - \sum_{\nu=0}^{N-1} a_\nu \in S[m_1-N, m_2, m_3] \qquad \text{for all } N\,.$$

Two such symbols differ by an element of $\mathcal{B}_{t,s}(S^{-\infty})$.

ii) *Let* $b_\nu(t, x, \xi) \in S[m_1-\nu, m_2-\nu(l+1)]$ *for* $\nu = 0, 1, \cdots$. *Then, there exists a* $b(t, x, \xi) \in S[m_1, m_2]$ *such that*

$$b \sim b_0 + b_1 + \cdots \qquad \text{mod } \mathcal{H}^{m_1-m_2\omega}$$

in the sense

$$b-\sum_{\nu=0}^{N-1} b_\nu \in S[m_1-N,\ m_2-N(l+1)] \qquad \text{for all } N\,.$$

Two such symbols differ by an element of $\mathcal{H}^{m_1-m_2\omega}$.

Proof. Using $\chi(t)$ of (1.3) we set

$$(1.16)\qquad \begin{cases} \psi_\varepsilon(\xi) = 1-\chi(\varepsilon\langle\xi\rangle) \\ \gamma_\varepsilon(t,\xi) = 1-\chi(\varepsilon\eta(t,\xi)^{l+1}\langle\xi\rangle)\,. \end{cases}$$

Then, setting

$$a(t, s, x, \xi) = \sum_{\nu=0}^{\infty} \psi_{\varepsilon_\nu}(\xi) a_\nu(t, s, x, \xi)$$

and

$$b(t, x, \xi) = \sum_{\nu=0}^{\infty} \gamma_{\varepsilon_\nu}(t, \xi) b_\nu(t, x, \xi)$$

for appropriate $1 \geqq \varepsilon_0 > \varepsilon_1 > \cdots > \varepsilon_\nu > \cdots \to 0$, we get i) and ii) by usual method.

Proposition 1.6. *Let* $a(t, s, x, \xi)S\in[m_1, m_2, m_3]$ *and* $b(t, s, x, \xi)\in S[m_1', m_2', m_3']$ *and define* $a\circ b(t, s, x, \xi)$ *by*

$$(1.17)\qquad \begin{aligned} &a\circ b(t, s, x, \xi) \\ &= 0s-\int e^{-iy\cdot\xi'} a(t, s, x, \xi+\xi') b(t, s, x+y, \xi) d\!\!{}^{-}\xi' dy\,. \end{aligned}$$

Then, we have

$$(1.18)\qquad a\circ b(t, s, x, \xi)\in S[m_1+m_1',\ m_2+m_2',\ m_3+m_3']$$

and for $A=a(t, s, X, D_x)$, $B=b(t, s, X, D_x)$ *we have*

$$AB = a\circ b(t, s, X, D_x)\,.$$

Moreover, we have

$$(1.19)\qquad a\circ b(t, s, x, \xi) \sim \sum_\alpha \frac{1}{\alpha!} a^{(\alpha)}(t, s, x, \xi) b_{(\alpha)}(t, s, x, \xi) \qquad \text{mod } \mathcal{B}_{t,s}(S^{-\infty})\,.$$

Proof. If we note Remark 1° of Definition 1.1, the proof is clear.

Corollary 1.7. *Let* $a(t, x, \xi)\in S[m_1, m_2]$ *and* $b(t, x, \xi)\in \mathcal{H}^m$. *Then, both* $a\circ b(t, x, \xi)$ *and* $b\circ a(t, x, \xi)$ *belong to* $\mathcal{H}^{m+m_1-m_2\omega}$.

When $\boldsymbol{A}(t)$ is an $m\times m$ matrix of pseudo-differential operators with symbols in $S[m_1, m_3]$, we also write $\sigma(\boldsymbol{A}(t))\in S[m_1, m_2]$. We denfie $|\sigma(\boldsymbol{A}(t))|$ by

$$|\sigma(\boldsymbol{A}(t))| = \max_{1\leqq j,k\leqq m} |a_{j,k}(t, x, \xi)|\,,$$

where $a_{j,k}(t, x, \xi)$ is the (j, k)-element of $\sigma(A(t))(x, \xi)$.

Lemma 1.8. *Let $\sigma(N^{(\nu)}(t))(x, \xi)\in S[-\nu, -\nu(l+1)]$ $\nu=1, 2, \cdots$, be $m\times m$ matrices. Then, there exists $N(t)$ such that $\sigma(N(t))(x, \xi)\in S[0, 0]$ and*

$$N(t)\sim I+N^{(1)}(t)+N^{(2)}(t)+\cdots \qquad \bmod \mathcal{H}^0 . \tag{1.20}$$

Moreover, $N(t)$ has a parametrix $N(t)^{\#}$ such that $\sigma(N(t)^{\#})(x, \xi)\in S[0, 0]$ and

$$\sigma(N(t)N(t)^{\#}-I), \quad \sigma(N(t)^{\#}N(t)-I)\in \mathcal{B}_t(S^{-\infty}) .$$

Proof. Let $\gamma_\varepsilon(t, \xi)$ be the symbol defined by (1.16). Then, by Proposition 1.5-ii) we see that

$$\sigma(N(t))(x, \xi) = I+ \sum_{\nu=1}^{\infty} \gamma_{\varepsilon_\nu}(t, \xi)\sigma(N^{(\nu)}(t))(x, \xi)$$

belongs to $S[0, 0]$ for appropriate $1\geqq\varepsilon_1>\cdots>\varepsilon_\nu>\cdots\to 0$ and (1.20) holds. Furthermore, noting

$$|D_t^j\gamma_{\varepsilon_\nu}^{(\alpha)}(t, \xi)|\leqq C_{j,\alpha}\varepsilon_\nu\eta(t, \xi)^{l+1}\langle\xi\rangle\langle\xi\rangle^{-|\alpha|}\eta(t, \xi)^{-j}$$

and $\eta(t, \xi)^{l+1}\langle\xi\rangle\sigma(N^{(\nu)}(t))(x, \xi)\in S[-(\nu-1), -(\nu-1)(l+1)]\subset S[0, 0], \nu=1, 2, \cdots$, we get $|\det \sigma(N(t))(x, \xi)|\geqq c$ for a constant $c>0$, if we choose small $\varepsilon_\nu>0$. Noting Remark 1° of Definition 1.1, the parametrix $N(t)^{\#}$ of $N(t)$ can be constructed by usual procedure.

According to Kumano-go [5] we call a real valued C^∞-function $\phi(x, \xi)$ in $R_x^n\times R_\xi^n$ a *phase function*, when it satisfies conditions:

$$\begin{cases} \text{i)} & \phi(x, \xi)-x\cdot\xi\in S^1 \\ \text{ii)} & |\nabla_x\phi(x, \xi)-\xi|\leqq(1-\varepsilon_0)|\xi|+c \\ \text{iii)} & |\nabla_x\nabla_\xi\phi(x, \xi)-I|\leqq 1-\varepsilon_0' \\ & (0<\varepsilon_0\leqq 1,\ 0<\varepsilon_0'\leqq 1,\ c>0) . \end{cases} \tag{1.21}$$

Then the Fourier integral operator $A_\phi=a_\phi(X, D_x)$ with phase function $\phi(x, \xi)$ and symbol $a(x, \xi)\in S^m$ is defined by.

$$A_\phi u(x) = Os-\iint e^{i(\phi(x,\xi)-x'\cdot\xi)}a(x, \xi)u(x')d\xi dx'(u\in \mathcal{B}(R_x^n)) . \tag{1.22}$$

Concerning fundamental theorems on Fourier integral operators, we refer to §2 of [5].

Let $\lambda(t, x, \xi)\in S[1, l]$ be real valued. Consider the Hamilton equation

$$\begin{cases} \dfrac{dt}{dq} = -\nabla_\xi\lambda(t, q, p) , \quad \dfrac{dp}{dt} = \nabla_x\lambda(t, q, p) \qquad \text{on } 0\leqq s,\ t\leqq T_0 , \\ \{q, p\}_{t=s} = \{y, \xi'\} \end{cases} \tag{1.23}$$

and the eiconal equation

$$(1.24)\qquad \begin{cases} \partial_t\phi-\lambda(t, x, \nabla_x\phi)=0 & \text{on } 0\leqq s,\ t\leqq T_0, \\ \phi(s, s, x, \xi)=x\cdot\xi \end{cases}$$

for a small $T_0\,(0<T_0\leqq T)$. Then, we can prove the following statements by the same procedure to §3 in [5].

Lemma 1.9. *For a small $T_1\,(0<T_1\leqq T)$ the initial value problem* (1.23) *has the solution $\{q, p\}$ (t, s, y, ξ') on $0\leqq s,\ t\leqq T_1$ such that*

$$(1.25)\qquad \begin{cases} q(t, s, y, \xi')-y\in S[0, l+1, 0] & (0\leqq s\leqq t\leqq T_1), \\ p(t, s, y, \xi')-\xi'\in S[1, l+1, 0] & (0\leqq s\leqq t\leqq T_1) \end{cases}$$

and

$$(1.25)'\qquad \begin{cases} q(t, s, y, \xi')-y\in S[0, 0, l+1] & (0\leqq t\leqq s\leqq T_1), \\ p(t, s, y, \xi')-\xi'\in S[1, 0, l+1] & (0\leqq t\leqq s\leqq T_1). \end{cases}$$

Lemma 1.10. *Let $T_2\,(0<T_2\leqq T_1)$ and $\varepsilon_1\,(0<\varepsilon_1\leqq 1)$ be constants such that*

$$|\partial q/\partial y-I|\leqq(1-\varepsilon_1)\qquad 0\leqq s,\ t\leqq T_1$$

Then, for the mapping $x=q(t, s, y, \xi)$: $R_y^n\ni y\to x\in R_x^n$ with (t, s, ξ) as parameters, there exists the inverse $y=y(t, s, x, \xi)$ such that

$$(1.26)\qquad \begin{cases} y(t, s, x, \xi)-x\in S[0, l+1, 0] & 0\leqq s\leqq t\leqq T_2 \\ y(t, s, x, \xi)-x\in S[0, 0, l+1] & 0\leqq t\leqq s\leqq T_2 \\ |\partial y/\partial x-I|\leqq(1-\varepsilon_1)/\varepsilon_1. \end{cases}$$

Theorem 1.11. *There exists $T_0\,(0<T_0\leqq T)$ such that the initial value problem* (1.24) *has the unique solution $\phi(t, s)=\phi(t, s, x, \xi)$ on $0\leqq s,\ t\leqq T_0$ which satisfies* (1.21) *and*

$$(1.27)\qquad \begin{cases} \phi(t, s, x, \xi)-x\cdot\xi\in S[1, l+1, 0] & (0\leqq s\leqq t\leqq T_0), \\ \phi(t, s, x, \xi)-x\cdot\xi\in S[1, 0, l+1] & (0\leqq t\leqq s\leqq T_0). \end{cases}$$

Corollary 1.12. *For a C^∞-function $f(t, x, \xi)$ on $[0, T_0]\times R_x^n\times R_\xi^n$ set*

$$\tilde f(t, s, y, \xi)=f(t, q(t, s, y, \xi), \xi).$$

Then, we have

$$(1.28)\qquad \begin{aligned} &D_t\tilde f(t, s, y, \xi) \\ &=\{D_tf(t, x, \xi)-\sum_{|\alpha|=1}\lambda^{(\alpha)}(t, x, \nabla_x\phi(t, s, x, \xi))f_{(\alpha)}(t, x, \xi)\}_{x=q(t, s, y, \xi)}. \end{aligned}$$

The following lemma is important in the proof of Theorem 4.2 in §4.

Lemma 1.13. *Let* $a(t, s, x, \xi)\in S[m_1, m_2, m_3]$ *and* $r(t, s, x, \xi)\in \bigcap_{\nu=0}^{\infty} S[m_1'-\nu, m_2'-\nu(l+1), m_3']$. *Set* $A_\phi=a_\phi(t, s, X, D_x)$ *with* $\phi(t, s, x, \xi)$ *of Theorem* 1.11 *and* $R=r(t, s, X, D_x)$. *Then both* $R_1=A_\phi R$ *and* $R_2=RA_\phi$ *are pseudo-differential operators with symbols*

$$r_j(t, s, x, \xi)\in \bigcap_{\nu=0}^{\infty} S[m_1+m_1'-\nu, m_2+m_2'-\nu(l+1), m_3+m_3'] \qquad (j=1, 2), \tag{1.29}$$

where

$$r_1(t, s, x, \xi') = Os-\iint e^{i\varphi_1}a(t, s, x, \xi)r(t, s, x', \xi')d\!\!{}^-\xi dx' \tag{1.30}$$

with

$$\varphi_1 = \phi(t, s, x, \xi)-x\cdot\xi+(x-x')\cdot(\xi-\xi')$$

and

$$r_2(t, s, x, \xi') = Os-\iint e^{i\varphi_2}r(t, s, x, \xi)a(t, s, x', \xi')d\!\!{}^-\xi dx' \tag{1.31}$$

with

$$\varphi_2 = (x-x')\cdot(\xi-\xi')+\phi(t, s, x', \xi')-x'\cdot\xi' .$$

Moreover, we have

$$r_1(t, s, x, \xi)\sim\sum_{\alpha}\frac{1}{\alpha!}\tilde{a}^{(\alpha)}(t, s, x, \xi)r_{(\alpha)}(t, s, x, \xi) \qquad \mathrm{mod}\ \mathcal{B}_{t,s}(S^{-\infty}) \tag{1.32}$$

and

$$r_2(t, s, x, \xi)\sim\sum_{\alpha}\frac{1}{\alpha!}r^{(\alpha)}(t, s, x, \xi)\tilde{a}_{(\alpha)}(t, s, x, \xi) \qquad \mathrm{mod}\ \mathcal{B}_{t,s}(S^{-\infty}), \tag{1.33}$$

where

$$\tilde{a}(t, s, x, \xi) = e^{i(\phi(t,s,x,\xi)-x\cdot\xi)}a(t, s, x, \xi) . \tag{1.34}$$

Proof. It is clear that r_1 and r_2 are defined by (1.30) and (1.31), respectively. By Theorem 1.11 we have

$$\begin{aligned} &|D_t^j D_s^k \tilde{a}^{(\alpha)}_{(\beta)}(t, s, x, \xi)| \\ &\leqq C_{j,k,\alpha,\beta}\langle\xi\rangle^{m_1-|\alpha|}(\eta(t, \xi)^{l+1}\langle\xi\rangle)^{|\alpha+\beta|+j+k}\eta(t, \xi)^{m_2-j}\eta(s, \xi)^{m_3-k} . \end{aligned} \tag{1.35}$$

On the other hand by the assumption for $r(t, s, x, \xi)$ we have

$$\eta((t, \xi)^{l+1}\langle\xi\rangle)^{\tau}r(t, s, x, \xi)\in \bigcap_{\nu=0}^{\infty} S[m_1'-\nu, m_2'-\nu(l+1), m_3'] \quad \text{for any } \tau . \tag{1.36}$$

Then, from (1.35) and (1.36) we see that

(1.37)
$$\tilde{a}^{(\alpha)}(t, s, x, \xi)r_{(\alpha)}(t, s, x, \xi) \in \bigcap_{\nu=0}^{\infty} S[m_1+m_1'-\nu-|\alpha|, m_2+m_2'-\nu(l+1), m_3+m_3'] .$$

Now we write

$$r_1(t, s, x, \xi')=Os-\iint e^{-iy\cdot\xi''}\tilde{a}(t, s, x, \xi'+\xi'')r(t, s, x+y, \xi')d\xi''dy .$$

Then, by Taylor's expansion

$$\tilde{a}(t, s, x, \xi'+\xi'')=\sum_{|\alpha|<N}\frac{\xi''^{\alpha}}{\alpha!}\tilde{a}^{(\alpha)}(t, s, x, \xi') +N\sum_{|\alpha|=N}\frac{\xi''^{\alpha}}{\alpha!}\int_0^1(1-\theta)^{N-1}\tilde{a}^{(\alpha)}(t, s, x, \xi'+\theta\xi'')d\theta ,$$

we have

(1.38)
$$r_1(t, s, x, \xi')=\sum_{|\alpha|<N}\frac{1}{\alpha!}\tilde{a}^{(\alpha)}(t, s, x, \xi')r_{(\alpha)}(t, s, x, \xi') +N\sum_{|\alpha|=N}\frac{1}{\alpha!}\int_0^1(1-\theta)^{N-1}h_{\alpha\,\theta}(t, s, x, \xi')d\theta ,$$

where

(1.39)
$$\begin{aligned} &h_{\alpha,\theta}(t, s, x, \xi') \\ &=Os-\iint e^{-iy\cdot\xi''}\tilde{a}^{(\alpha)}(t, s, x, \xi'+\theta\xi'')r_{(\alpha)}(t, s, x+y, \xi')\,d\xi''dy \\ &=Os-\iint e^{-iy\cdot\xi'}\langle\xi''\rangle^{-\tilde{n}}\tilde{a}^{(\alpha)}(t, s, x, \xi'+\theta\xi'') \\ &\times\langle D_y\rangle^{\tilde{n}}r_{(\alpha)}(t, x, s+y, \xi')d\xi''dy \end{aligned}$$

for any even integer $\tilde{n}\geqq 0$. Then, noting

$$\begin{cases} \langle\xi'+\theta\xi''\rangle^{\pm1}\leqq 2\langle\xi''\rangle\langle\xi'\rangle^{\pm1} , \\ \eta(t, \xi'+\xi'')^{\pm1}\leqq 2^{\omega}\langle\xi''\rangle^{\omega}\eta(t, \xi')^{\pm1} \end{cases}$$

and using (1.35) we see from the assumption for $r(t, s, x, \xi)$ that

$$\{h_{\alpha,\theta}(t, s, x, \xi)\}_{|\alpha|=N, 0\leqq\theta\leqq1} \text{ is bounded in } \bigcap_{\nu=0}^{\infty} S[m_1+m_1'-N-\nu, m_2+m_2'-\nu(l+1), m_3+m_3'] .$$

Hence, from (1.38), (1.39) we get (1.29) for $j=1$ and (1.32). By the same method we get the statement for $r_2(t, s, x, \xi)$.

2. Diagonalization. In this section we consider a hyperbolic $m\times m$ system

(2.1) $$\boldsymbol{L}_0=\boldsymbol{D}_t-\boldsymbol{A}_1(t)-\boldsymbol{A}_0(t) \quad \text{on } [0, T]$$

of pseudo-differential operators of first order, where

$$\boldsymbol{D}_t=\begin{bmatrix} D_t & & 0 \\ & \ddots & \\ 0 & & D_t \end{bmatrix}$$

and

$$\sigma(\boldsymbol{A}_1(t))(x, \xi)\in S[1, l], \quad \sigma(\boldsymbol{A}_0(t))(x, \xi)\in S[0, -1]$$

for an integer $l>0$. We assume the eigenvalues $\lambda_1(t, x, \xi), \cdots, \lambda_m(t, x, \xi)$ of $\sigma(\boldsymbol{A}_1(t))(x, \xi)$ are all real and belong to $S[1, l]$. Modifying the notion *'prefectly diagonalizable'* in Kumano-go [6] we introduce the following notion.

DEFINITION 2.1. i) For $\eta(t)=\eta(t, \xi)$ defined in (1.4) the operator $\boldsymbol{L}_0$ is said to be $\eta(t)$-diagonalizable, when there exists $\boldsymbol{N}_0(t)$ such that $\sigma(\boldsymbol{N}_0(t))\in S[0, 0]$ and $|\det\sigma(\boldsymbol{N}_0(t))|\geqq c$ on $[0, T]\times R_x^n\times R_\xi^n$ for a constant $c>0$, and we can write

(2.2) $$\boldsymbol{L}_0\boldsymbol{N}_0(t)\equiv\boldsymbol{N}_0(t)\boldsymbol{L} \quad \mod \mathscr{B}_t(S^{-\infty})$$

for some $\boldsymbol{L}$ of the form

(2.3) $$\boldsymbol{L}=\boldsymbol{D}_t-\mathscr{D}(t)+\boldsymbol{B}(t) \quad \text{on } [0, T],$$

where

(2.4) $$\sigma(\mathscr{D}(t))(x, \xi)=\begin{bmatrix} \lambda_1(t, x, \xi) & & 0 \\ & \ddots & \\ 0 & & \lambda_m(t, x, \xi) \end{bmatrix}$$

and $\sigma(\boldsymbol{B}(t))(x, \xi)\in S[0, -1]$.

ii) The operator $\boldsymbol{L}_0$ is said to be *$\eta(t)$-perfectly diagonalizable*, when there exists $\boldsymbol{N}(t)$ such that $\sigma(\boldsymbol{N}(t))\in S[0, 0]$ and $|\det\sigma(\boldsymbol{N}(t))|>c$ on $[0, T]\times R_x^n\times R_\xi^n$ for a constant $c>0$, and we can write

(2.5) $$\boldsymbol{L}_0\boldsymbol{N}(t)\equiv\boldsymbol{N}(t)\boldsymbol{L}_1 \quad \mod \mathscr{B}_t(S^{-\infty})$$

for some $\boldsymbol{L}_1$ of the form

(2.6) $$\boldsymbol{L}_1=\boldsymbol{D}_t-\mathscr{D}(t)+\boldsymbol{F}(t)+\boldsymbol{R}(t) \quad \text{on } [0, T],$$

where $\sigma(\boldsymbol{F}(t))$ is a diagonal matrix of the form

(2.7) $$\sigma(\boldsymbol{F}(t))=\begin{bmatrix} f_1(t, x, \xi) & & 0 \\ & \ddots & \\ 0 & & f_m(t, x, \xi) \end{bmatrix}\in S[0, -1]$$

and $\sigma(\boldsymbol{R}(t))\in\mathscr{H}^\infty$.

$\boldsymbol{N}_0(t)$, $\boldsymbol{N}(t)$ are called the *diagonalizer*, the *perfect diagonalizer* for $\boldsymbol{L}_0$, respectively.

Theorem 2.2. *For $\boldsymbol{L}$ of* (2.3), *assume that there exists a constant $c_0>0$ such that*

$$|\lambda_j(t, x, \xi)-\lambda_k(t, x, \xi)| \geqq c_0\eta(t)^l\langle\xi\rangle \qquad (j \neq k). \tag{2.8}$$

Then, $\boldsymbol{L}$ is $\eta(t)$-perfectly diagonalizable.

Proof. According to Kumano-go [6] we find the perfect diagonalizer $\boldsymbol{N}(t)$ such that

$$\begin{cases} \boldsymbol{N}(t) \sim I+\boldsymbol{N}^{(1)}(t)+\boldsymbol{N}^{(2)}(t)+\cdots & \bmod \mathcal{H}^0 \\ \sigma \boldsymbol{N}^{(\nu)}((t)) \in S[-\nu, -\nu(l+1)] & (\nu = 1, 2, \cdots), \end{cases} \tag{2.9}$$

$$(\boldsymbol{D}_t-\mathcal{D}(t)+\boldsymbol{B}(t))\boldsymbol{N}(t) \equiv \boldsymbol{N}(t)(\boldsymbol{D}_t-\mathcal{D}(t)+\boldsymbol{F}(t)+\boldsymbol{R}(t)) \quad \bmod \mathcal{B}_t(S^{-\infty}), \tag{2.10}$$

and

$$\begin{cases} \boldsymbol{F}(t) \sim \boldsymbol{F}^{(0)}(t)+\boldsymbol{F}^{(1)}(t)+\cdots & \bmod \mathcal{H}^\infty \\ \sigma(\boldsymbol{F}^{(\nu)}(t)) \in S[-\nu, -\nu(l+1)-1] & (\nu = 0, 1, \cdots). \end{cases} \tag{2.11}$$

Let $b_{j,k}(k)$ bet the (j, k)-elements of $\sigma(\boldsymbol{B}(t))$, and set

$$\begin{cases} \boldsymbol{F}^{(0)} = \operatorname{diag}[\boldsymbol{B}] \\ \sigma(\boldsymbol{N}^{(1)}) = (n_{j,k}^{(1)}) \quad \text{by} \\ \qquad n_{j,k}^{(1)} = \begin{cases} b_{j,k}/(\lambda_j-\lambda_k) & (j \neq k) \\ 0 & (j=k) \end{cases} \\ \boldsymbol{B}^{(1)} = (\boldsymbol{D}_t-\mathcal{D}+\boldsymbol{B})(I+\boldsymbol{N}^{(1)})-(I+\boldsymbol{N}^{(1)})(\boldsymbol{D}_t-\mathcal{D}+\boldsymbol{F}^{(0)}), \end{cases} \tag{2.12$_0$}$$

where by diag $[\boldsymbol{B}]$ we denote a diagonal matrix with the same diagonal with $\boldsymbol{B}$'s. Then, we have

$$\boldsymbol{B}^{(1)} = \boldsymbol{B}-[\mathcal{D}, \boldsymbol{N}^{(1)}]-\boldsymbol{F}^{(0)}+\boldsymbol{N}_t^{(1)}+\boldsymbol{B}\boldsymbol{N}^{(1)}-\boldsymbol{N}^{(1)}\boldsymbol{F}^{(0)},$$

where $\quad [\mathcal{D}, \boldsymbol{N}^{(1)}] = \mathcal{D}\boldsymbol{N}^{(1)}-\boldsymbol{N}^{(1)}\mathcal{D}$ and $\sigma(\boldsymbol{N}_t^{(1)}) = D_t\sigma(\boldsymbol{N}^{(1)})$.

Since $\sigma(\boldsymbol{B}-[\mathcal{D}, \boldsymbol{N}^{(1)}]-\boldsymbol{F}^{(0)}) \in S[-1, -1]$, we have

$$\begin{cases} \sigma(\boldsymbol{F}^{(0)}) \in S[0, -1], \\ \sigma(\boldsymbol{N}^{(1)}) \in S[-1, -(l+1)], \\ \sigma(\boldsymbol{B}^{(1)}) \in S[-1, -(l+1)-1]. \end{cases} \tag{2.13$_0$}$$

Now, we assume that $\boldsymbol{F}^{(\mu)}$, $\boldsymbol{N}^{(\mu+1)}$, $\boldsymbol{B}^{(\mu+1)}$, $\mu=0, 1, \cdots, \nu-1$ $(\nu \geqq 1)$ are determined as

$$\begin{cases} \sigma(\boldsymbol{F}^{(\mu)}) \in S[-\mu, -\mu(l+1)-1], \\ \sigma(\boldsymbol{N}^{(\mu+1)}) \in S[-(\mu+1), -(\mu+1)(l+1)], \\ \sigma(\boldsymbol{B}^{(\mu+1)}) \in S[-(\mu+1), -(\mu+1)(l+1)-1], \end{cases} \tag{2.13$_\mu$}$$

and define $\boldsymbol{F}^{(\nu)}$, $\boldsymbol{N}^{(\nu+1)}$, $\boldsymbol{B}^{(\nu+1)}$ by

$$(2.12)_\nu \quad \begin{cases} \boldsymbol{F}^{(\nu)} = \operatorname{diag}[\boldsymbol{B}^{(\nu)}]\,, \\ \sigma(\boldsymbol{N}^{(\nu+1)}) = (n_{j,k}^{(\nu+1)}) \quad \text{by} \\ \qquad n_{j,k}^{(\nu+1)} = \begin{cases} b_{j,k}^{(\nu)}/(\lambda_j-\lambda_k) & (j\neq k) \\ 0 & (j=k) \end{cases} \\ \boldsymbol{B}^{(\nu+1)} = (\boldsymbol{D}_t-\mathcal{D}+\boldsymbol{B})(I+\sum_{\mu=1}^{\nu+1}\boldsymbol{N}^{(\mu)}) \\ \qquad -(I+\sum_{\mu=1}^{\nu+1}\boldsymbol{N}^{(\mu)})(\boldsymbol{D}_t-\mathcal{D}+\sum_{\mu=0}^{\nu}\boldsymbol{F}^{(\mu)})\,, \end{cases}$$

where $b_{j,k}^{(\nu)}$ are the (j, k)-elements of $\sigma(\boldsymbol{B}^{(\nu)})$.

Then, we have

$$\boldsymbol{B}^{(\nu+1)} = (\boldsymbol{B}^{(\nu)}-[\mathcal{D}, \boldsymbol{N}^{(\nu+1)}]-\boldsymbol{F}^{(\nu)})+\boldsymbol{N}_t^{(\nu+1)} \\ +\boldsymbol{B}\boldsymbol{N}^{(\nu+1)}-\sum_{\mu=1}^{\nu}\boldsymbol{N}^{(\mu)}\boldsymbol{F}^{(\nu)}-\boldsymbol{N}^{(\nu+1)}\sum_{\mu=0}^{\nu}\boldsymbol{F}^{(\mu)}\,,$$

and by the definition of $\boldsymbol{F}^{(\nu)}$ and $\boldsymbol{N}^{(\nu+1)}$ we have

$$\sigma(\boldsymbol{B}^{(\nu)}-[\mathcal{D}, \boldsymbol{N}^{(\nu+1)}]-\boldsymbol{F}^{(\nu)})\in S[-(\nu+1), -\nu(l+1)-1]\,.$$

Hence we get $(2.13)_\mu$ for $\mu=\nu$, and by induction, for any $\mu=0, 1, \cdots$.

Now, by Proposition 1.5-ii) there exist $\boldsymbol{N}(t)$ and $\boldsymbol{F}(t)$ such that (2.9) and (2.11) hold. We set

$$\tilde{\boldsymbol{R}} = \boldsymbol{L}\boldsymbol{N}-\boldsymbol{N}(\boldsymbol{D}_t-\mathcal{D}+\boldsymbol{F})\,.$$

Then, we have $\sigma(\tilde{\boldsymbol{R}})\in\mathcal{H}^\infty$. Let $\boldsymbol{N}^\sharp$ be a parametrix of $\boldsymbol{N}$ which exists by Lemma 1.8, and set $\boldsymbol{R}=\boldsymbol{N}^\sharp\tilde{\boldsymbol{R}}$. Then $\sigma(\boldsymbol{R})\in\mathcal{H}^\infty$ and

$$\sigma(\boldsymbol{L}\boldsymbol{N}(t)-\boldsymbol{N}(t)(\boldsymbol{D}_t-\mathcal{D}+\boldsymbol{F}(t)+\boldsymbol{R}(t)))\in\mathcal{B}_t(S^{-\infty})\,.$$

Corollary 2.3. *Let $\boldsymbol{L}_0$ be $\eta(t)$-diagonalizable. Assume that the eigenvalues $\lambda_1(t, x, \xi), \cdots, \lambda_m(t, x, \xi)$ of $\sigma(\boldsymbol{A}_1(t))$ satisfy* (2.8). *Then $\boldsymbol{L}_0$ is perfectly diagonalizable.*

3. Construction of fundamental solution. The first order single operator case. Let L be a single hyperbolic operator of the form

$$(3.1) \qquad L = D_t-\lambda(t, X, D_x)+f(t, X, D_x) \qquad \text{on } [0, T] \quad (0<T\leqq 1)\,,$$

where

$$(3.2) \qquad \begin{cases} \lambda(t, x, \xi)\in S[1, l] & \text{real valued,} \\ f(t, x, \xi)\in S[0, -1]\,. \end{cases}$$

Consider the Cauchy problem

$$(3.3) \qquad \begin{cases} Lu = \varphi(t) & \text{on } [s, T_0] \\ u|_{t=s} = \psi & (0 \leqq s \leqq T_0) \end{cases}$$

for a small T_0 $(0 < T_0 \leqq T)$.

Theorem 3.1. *Set*

$$(3.4) \qquad M = \lim_{R\to\infty} \sup_{\substack{x, t\langle\xi\rangle^{\omega} \geqq R \\ 0 \leqq t \leqq R^{-1}}} \{t\, \mathcal{I}m f(t, x, \xi)\}\,.$$

Then, there exists uniquely a symbol $e(t, s, x, \xi)$ in the class $\bigcap_{0<\varepsilon<1} S[0, M+\varepsilon, -M-\varepsilon]$ on $0 \leqq s \leqq t \leqq T_0$ (with T_0 of Theorem 1.11) such that the Fourier integral operator $E_\phi(t, s) = e_\phi(t, s, X, D_x)$ with phase function $\phi(t, s, x, \xi)$ given by Theorem 1.11 is the fundamental solution of the Cauchy problem(3.1) for L, i.e.,

$$(3.5) \qquad \begin{cases} LE_\phi(t, s) = 0 & \text{on } 0 \leqq s \leqq t \leqq T_0\,, \\ E_\phi(s, s) = I & \text{(identity operator)}\,. \end{cases}$$

REMARK. Since $(t+\langle\xi\rangle^{-\omega})(1-1/(t\langle\xi\rangle^{\omega}+1)) \leqq (t+\langle\xi\rangle^{-\omega}) - \langle\xi\rangle^{-\omega} \leqq \eta(t, \xi) \leqq t+\langle\xi\rangle^{-\omega}$, and $\langle\xi\rangle^{-\omega} \leqq t/R$ when $t\langle\xi\rangle^{\omega} \geqq R$, we have

$$(3.4)' \qquad M = \lim_{R\to\infty} \sup_{\substack{x, t\langle\xi\rangle^{\omega} \geqq R \\ 0 \leqq t \leqq R^{-1}}} \{(t+\langle\xi\rangle^{-\omega})\, \mathcal{I}m f(t, x, \xi)\}\,,$$

$$(3.4)'' \qquad M = \lim_{R\to\infty} \sup_{\substack{x, t\langle\xi\rangle^{\omega} \geqq R \\ 0 \leqq t \leqq R^{-1}}} \{\eta(t, \xi)\, \mathcal{I}m f(t, x, \xi)\}\,.$$

Proof. The uniqueness will be proved after Theorem 3.2. Solving transport equations we first construct an approximate fundamental solution $\tilde{E}_\phi(t, s)$ in the sence

$$(3.6) \qquad \begin{cases} L\tilde{E}_\phi(t, s) \equiv 0 & \text{mod } \mathcal{B}_{t,s}(S^{-\infty}) \text{ on } 0 \leqq s \leqq t \leqq T_0\,, \\ \tilde{E}_\phi(s, s) = I\,. \end{cases}$$

We assume that the symbol $\tilde{e}(t, s, x, \xi)$ of $\tilde{E}_\phi(t, s)$ has the form:

$$(3.7) \qquad \tilde{e}(t, s, x, \xi) \sim \sum_{\nu=0}^{\infty} e_\nu(t, s, x, \xi) \qquad \text{mod } \mathcal{B}_{t,s}(S^{-\infty})$$

and

$$(3.8) \qquad e_\nu(t, s, x, \xi) \in \bigcap_{0<\varepsilon<1} S[-\nu, M+\varepsilon, -M-\varepsilon] \qquad (\nu = 0, 1, 2, \cdots)\,.$$

Set

$$(3.9) \qquad g(t, s, x, \xi) = -i \sum_{|\alpha|=2} \frac{1}{\alpha!} \lambda^{(\alpha)}(t, x, \nabla_x\phi(t, s, x, \xi)) \\ \times \partial_x^\alpha \phi(t, s, x, \xi) + f(t, s, x, \nabla_x\phi(t, s, x, \xi))$$

and consider

$$\mathcal{L}=D_t-\sum_{|\alpha|=1}\lambda^{(\alpha)}(t,x,\nabla_x\phi)D_x^\alpha+g(t,s,x,\xi)\,. \tag{3.10}$$

Then, by the usual expansion formula of Fourier integral operators (See [5]), we have by using (1.24)

$$\sigma(Le_{\nu,\phi}(t,s))(x,\xi)=\mathcal{L}e_{\nu,\phi}+r_\nu(t,s,x,\xi)\,. \tag{3.11}$$

Here

$$r_\nu(t,s,x,\xi)\sim-\sum_{|\alpha|\geqq 2}\frac{1}{\alpha!}\{D_{x'}^\alpha\lambda^{(\alpha)}((t,x,\tilde{\nabla}_x\phi(t,s,x,x',\xi)) \times e_\nu(t,s,x',\xi))\}_{x'=x} \quad \bmod \mathcal{B}_{t,s}(S^{-\infty}) \tag{3.12}$$

and

$$\tilde{\nabla}_x\phi(t,s,x,x',\xi)=\int_0^1\nabla_x\phi(t,s,x'+\theta(x-x'),\xi)d\theta\,. \tag{3.13}$$

Then, from (1.27), (3.2) and (3.8) we see that

$$r_\nu(t,s,x,\xi)\in\bigcap_{0<\varepsilon<1}S[-\nu,M+\varepsilon-1,-M-\varepsilon] \qquad (\nu=0,1,\cdots)\,. \tag{3.14}$$

Hence, if we can determine $e_\nu(t,s)$ as the solution of

$$\begin{cases}\mathcal{L}e_0=0 & \text{on } 0\leqq s\leqq t\leqq T_0\,,\\ e_0(s,s)=1\end{cases} \tag{3.15}$$

and

$$\begin{cases}\mathcal{L}e_\nu+r_{\nu-1}=0 & \text{on } 0\leqq s\leqq t\leqq T_0\,,\\ e_\nu(s,s)=0 & (\nu=1,2,\cdots)\,,\end{cases} \tag{3.16}$$

then we have

$$\begin{aligned}&\sigma(L\sum_{\nu=0}^{N}e_{\nu,\phi}(t,s,X,D_x))\\ &=\sum_{\nu=0}^{N}(\mathcal{L}e_\nu+r_\nu)\\ &=\mathcal{L}e_0+\sum_{\nu=1}^{N}(\mathcal{L}e_\nu+r_{\nu-1})+r_N\\ &=r_N\in\bigcap_{0<\varepsilon<1}S[-N,M+\varepsilon-1,-M-\varepsilon]\,.\end{aligned}$$

Thus, if we determine $\tilde{e}(t,s,x,\xi)$ so that (3.7) holds and $e(s,s)=1$, then we get (3.6).

Now, we solve (3.15) and (3.16) in what follows. Let $q(t,s,y,\xi)$ be the solution of (1.23) given by Lemma 1.9. Then, by Corollary 1.12 the equations (3.15) and (3.16) are reduced, respectively, to

(3.18)
$$\begin{cases} D_t\tilde{e}_0(t,s,y,\xi)+\tilde{g}(t,s,y,\xi)\tilde{e}_0(t,s,y,\xi)=0\,, \\ \tilde{e}_0(s,s,y,\xi)=1 \end{cases}$$

and

(3.19)
$$\begin{cases} D_t\tilde{e}_\nu(t,s,y,\xi) \\ \quad +\tilde{g}(t,s,y,\xi)\tilde{e}_\nu(t,s,y,\xi)+\tilde{r}_{\nu-1}(t,s,y,\xi)=0\,, \\ \tilde{e}_\nu(s,s,y,\xi)=0 \quad (\nu=1,2,\cdots)\,, \end{cases}$$

where

(3.20)
$$\begin{cases} \tilde{e}_\nu(t,s,y,\xi)=e_\nu(t,s,q(t,s,y,\xi),\xi)\,, \\ \tilde{g}(t,s,y,\xi)=g(t,s,q(t,s,y,\xi),\xi)\,, \\ \tilde{r}_\nu(t,s,y,\xi)=r_\nu(t,s,q(t,s,y,\xi),\xi)\,. \end{cases}$$

Hence we have

(3.21)
$$\tilde{e}_0(t,s,y,\xi)=\exp\left[-i\int_s^t \tilde{g}(\sigma,s,y,\xi)d\sigma\right]$$

and

(3.22)
$$\begin{aligned} &\tilde{e}_\nu(t,s,y,\xi) \\ &=-i\int_s^t \tilde{r}_{\nu-1}(\sigma,s,y,\xi)\exp\left[-i\int_\sigma^t \tilde{g}(\sigma',s,y,\xi)d\sigma'\right]d\sigma\,. \end{aligned}$$

Consequently, setting

(3.23)
$$\begin{cases} \tilde{\tilde{g}}(t,\sigma,s,x,\xi)=g(\sigma,s,q(\sigma,s,y(t,s,x,\xi),\xi),\xi) \\ \tilde{\tilde{r}}_\nu(t,\sigma,s,x,\xi)=r_\nu(\sigma,s,q(\sigma,s,y(t,s,x,\xi),\xi),\xi) \end{cases}$$

for the inverse $y=y(t,s,x,\xi)$ of $x=q(t,s,y,\xi)$ given by Lemma 1.10, we have

(3.24)
$$e_0(t,s,x,\xi)=\exp\left[-i\int_s^t \tilde{\tilde{g}}(t,\sigma,s,x,\xi)d\sigma\right]$$

and

(3.25)
$$\begin{aligned} &e_\nu(t,s,x,\xi) \\ &=-i\int_s^t \tilde{\tilde{r}}_{\nu-1}(t,\sigma,s,x,\xi)\exp\left[-i\int_\sigma^t \tilde{\tilde{g}}(t,\sigma',s,x,\xi)d\sigma'\right]d\sigma\,. \end{aligned}$$

Now, we first note that from Remark of Theorem 3.1 we have

(3.26)
$$\begin{aligned} &(t+\langle\xi\rangle^{-\omega})(1-1/(t\langle\xi\rangle^{\omega}+1)) \\ &\leqq \eta(t,\xi)\leqq t+\langle\xi\rangle^{-\omega}\,. \end{aligned}$$

By the definition (3.4)′ of M there exists for any $\varepsilon>0$ a constant C_ε such that

$$\begin{aligned} &\mathcal{I}m\, f(t,s,\xi) \\ &\leqq (M+\varepsilon/2)(t+\langle\xi\rangle^{-\omega})+C_\varepsilon\{1+\langle\xi\rangle^{-\omega}/(t+\langle\xi\rangle^{-\omega})^2\}\,. \end{aligned}$$

Then, using (3.9), (3.23), (3.26), (3.27) and Theorem 1.11, we have

$$\begin{aligned}&\mathcal{R}e(-i\tilde{\tilde{g}}(t,\sigma,s,x,\xi))\\&\leqq(M+\varepsilon)/(\sigma+\langle\xi\rangle^{-\omega})+C'_\varepsilon\{1+\langle\xi\rangle^{-\omega}/(\sigma+\langle\xi\rangle^{-\omega})^2\}\end{aligned}$$

for another constant C'_ε. On the other hand, by (1.5)

$$\log((t+\langle\xi\rangle^{-\omega})/(s+\langle\xi\rangle^{-\omega}))\leqq\log(\eta(t,\xi)/(\eta(s,\xi))+\log 2. \tag{3.29}$$

Hence, using

$$\int_s^t\langle\xi\rangle^{-\omega}(\sigma+\langle\xi\rangle^{-\omega})^2d\sigma\leqq 1$$

and (3.28), we have

$$|\exp[-i\int_s^t\tilde{\tilde{g}}(t,\sigma,s,x,\xi)d\sigma]|\leqq C''_\varepsilon(\eta(t,\xi)/\eta(s,\xi))^{M+\varepsilon} \tag{3.30}$$

for a constant C''_ε. We have

$$\begin{aligned}&|D_t^jD_\sigma^kD_s^{j'}\partial_\xi^\alpha D_x^\beta\tilde{\tilde{g}}(t,\sigma,s,x,\xi)|\\&\leqq C_{j,j',k,\alpha,\beta}\langle\xi\rangle^{-|\alpha|}\eta(t,\xi)^{-j}\eta(\sigma,\xi)^{-1-k}\eta(s,\xi)^{-j'}.\end{aligned}$$

Then, noting

$$\begin{aligned}&\int_s^t\eta(\sigma,\xi)^{-1}d\sigma\leqq 2\log((t+\langle\xi\rangle^{-\omega})/(s+\langle\xi\rangle^{-\omega}))\\&\leqq\tilde{C}_\varepsilon(\eta(t,\xi)/\eta(s,\xi))^\varepsilon\end{aligned}$$

for a constant $\tilde{C}_\varepsilon$, we have

$$\begin{aligned}&|D_t^jD_s^k\partial_\xi^\alpha D_x^\beta\int_s^t\tilde{\tilde{g}}(t,\sigma,s,x,\xi)d\sigma|\\&\leqq C_{\varepsilon,j,k,\alpha,\beta}\langle\xi\rangle^{-|\alpha|}\eta(t,\xi)^{-j}\eta(s,\xi)^{-k}(\eta(t,\xi)/\eta(s,\xi))^\varepsilon.\end{aligned} \tag{3.31}$$

Thus, together with (3.30) we see that

$$e_0(t,s,x,\xi)\in\bigcap_{0<\varepsilon<1}S[0,M+\varepsilon,-M-\varepsilon]. \tag{3.32}$$

We already checked (3.14) for r_ν if (3.8) holds for e_ν. Hence, if we prove (3.8) for e_ν assuming (3.14) for $\nu-1$, then (3.8) holds for any ν. And this fact is clear by (3.25).

Now, from (3.17) we see that there exists $r_\infty(t,s,x,\xi)\in\mathcal{B}_{t,s}(S^{-\infty})$ such that

$$L\tilde{e}_\phi(t,s,X,D_x)=R_\infty(t,s)\,(=r_\infty(t,s,X,D_x)). \tag{3.33}$$

Then, setting

$$\begin{cases}W_1(t,s)=-iR_\infty(t,s),\\ W_{\nu+1}(t,s)=\int_s^tW_1(t,\theta)W_\nu(\theta,s)d\theta\quad(\nu=1,2,\cdots),\end{cases} \tag{3.34}$$

we get the fundamental solution $E(t,s)$ in the form

$$E(t,s) = {}_{\phi}(\tilde{E}t,s)+\int_s^t \tilde{E}_{\phi}(t,\theta)\sum_{\nu=1}^{\infty} W_{\nu}(\theta,s)d\theta\,. \tag{3.35}$$

From the theory of pseudo-differential operators of multiple symbols, there exists a symbol $\tilde{e}_{\infty}(t,s,x,\xi)\in\mathcal{B}_{t,s}(S^{-\infty})$ such that

$$E(t,s) = \tilde{e}_{\phi}(t,s,X,D_x)+\tilde{e}_{\infty,\phi}(t,s,X,D_x)$$

(cf. [5], [12]). Then, setting

$$e(t,s,x,\xi) = \tilde{e}(t,s,x,\xi)+\tilde{e}_{\infty}(t,s,x,\xi)\,,$$

we get the desired result.

Theorem 3.2. *The fundamental solution* $E_{\phi}(t,s)$ $(0\leqq s\leqq t\leqq T_0)$ *given in Theorem* 3.1 *has the meaning even when* $0\leqq t\leqq s\leqq T_0$, *and* $E_{\phi}(t,s)$ $(0\leqq t\leqq s\leqq T_0)$ *is the fundamental solution of the backward initial value problem for L, i.e.,*

$$\begin{cases} LE_{\phi}(t,s) = 0 \quad \text{on } 0\leqq t\leqq s\leqq T_0\,, \\ E_{\phi}(s,s) = I\,. \end{cases} \tag{3.36}$$

Furthermore, we have

$$e(t,s,x,\xi)\in\bigcap_{0<\varepsilon<1} S[0,-M'-\varepsilon,M'+\varepsilon] \quad (0\leqq t\leqq s\leqq T_0)\,, \tag{3.37}$$

where M' *is defined by*

$$\lim_{R\to\infty}\ \sup_{\substack{x,t\langle\xi\rangle^{\omega}\geqq R \\ 0\leqq t\leqq R^{-1}}} \{-t\mathcal{Im} f(t,x,\xi)\}\,. \tag{3.38}$$

Proof. We check the proof of Theorem 3.1. We have by Lemma 1.9, 1.10 and Theorem 1.11 that

$$\begin{cases} q(t,s,y,\xi')-y\in S[0,0,l+1] \\ y(t,s,x,\xi')-x\in S[0,0,l+1] \\ \phi(t,s,x,\xi)-x\cdot\xi\in S[1,0,l+1] \end{cases} \tag{3.39}$$

on $0\leqq t\leqq s\leqq T_0$. Noting (3.24) we write

$$e_0(t,s,x,\xi) = \exp[i\int_t^s \tilde{\tilde{g}}(t,\sigma,s,x,\xi)d\sigma]$$

on $0\leqq t\leqq s\leqq T_0$. Then, from (3.9) and (3.38)

$$e_0(t,s,x,\xi)\in\bigcap_{0<\varepsilon<1} S[0,-M'-\varepsilon,M'+\varepsilon]\,,$$

and, following the similar procedure to the proof of Theorem 3.1 by keeping in mind the fact $0\leqq t\leqq s\leqq T_0$, we complete the proof.

Proof of the uniqness of $E_\phi(t,s)$ in Theorem 3.1. Set

$$L^* = D_t-\lambda^*(t,X,D_x)+f^*(t,X,D_x),$$

where λ^* and f^* are the formal adjoints of λ and f, respectively. Then, L^* is the formal adjoint of L. Since $\lambda(t,x,\xi)$ is real valued, we see that $\lambda^*(t,x,\xi)-\lambda(t,x,\xi)\in S[0,l]$, and, there exists a $\tilde{f}^*(t,x,\xi)\in S[0,-1]$ such that

$$L^* = D_t-\lambda(t,X,D_x)+\tilde{f}^*(t,X,D_x).$$

Therefore, we can apply Theorem 3.2 to L^*. Let $E_\phi^*(t,s)$ $(0\leqq t\leqq s\leqq T_0)$ be the fundamental solution of the backward initial value problem for L^*.

Now, assume that thr there exist two fundamental solutions $E_\phi(t,s)$ and $E'_\phi(t,s)$ $(0\leqq s\leqq t\leqq T_0)$ of L in $\bigcap_{0<\varepsilon<1} S[0,M+\varepsilon, -M-\varepsilon]$. For $w\in\mathcal{S}$ we set

$$u(t,s,x) = (E_\phi(t,s)-E'_\phi(t,s))w \quad (0\leqq s\leqq t\leqq T_0).$$

Then, $u(t,s,x)$ satisfies

$$\begin{cases} Lu(t,s,x) = 0 \text{ on } 0\leqq s\leqq t\leqq T_0, \\ u(s,s,x) = 0. \end{cases}$$

On the other hand, for $b(t,x)\in\mathcal{B}_t(\mathcal{S})$ on $[0,T_0]$ set

$$v(t,x) = i\int_{T_0}^t E^*(t,\sigma)b(\sigma,x)d\sigma.$$

Then, we have

$$L^*v = b(t,x)\ (0\leqq t\leqq T_0),\ v(T_0,x) = 0.$$

Hence, we have

$$\begin{aligned} 0 &= \int_s^{T_0}(Lu, v)d\sigma = \int_s^{T_0}(u, L^*v)d\sigma \\ &= \int_s^{T_0}(u, b)d\sigma \text{ for all } b\in\mathcal{B}_t(\mathcal{S}). \end{aligned}$$

This means that

$$\begin{aligned} &0 = u(t,s,x) = (E_\phi(t,s)-E'_\phi(t,s))w \text{ for all} \\ &w\in\mathcal{S}\ (0\leqq s\leqq t\leqq T_0). \end{aligned}$$

Thus we have $e(t,s,x,\xi)=e'(t,s,x,\xi)$.

Corollary 3.3. i) *The solution $u(t,x)\in\mathcal{B}_t(\mathcal{S})$ on $[0,T_0]$ of the Cauchy problem* (3.3) *for $\varphi(t)\in\mathcal{B}_t(\mathcal{S})$ on $[0,T_0]$ and $\psi\in\mathcal{S}$ exists uniquely and is represented by*

$$u(t,s,x) = E_\phi(t,s)\psi+i\int_s^t E_\phi(t,\theta)\varphi(\theta)d\theta. \tag{3.40}$$

ii) *We have*

$$E_\phi(t,\tau)E_\phi(\tau,s) = E_\phi(t,s)\ (0 \leqq s, \tau, t \leqq T_0)\ . \tag{3.41}$$

Proof. i) It is clear that $u(t,s,x)$ defined by (3.40) is the solution of (3.3). Let $v(t,x) \in \mathcal{B}_t(\mathcal{S})$ on $[0, T_0]$ be the solution of (3.3) for $\varphi(t)=0$ and $\psi=0$, and let $E_\phi^*(t,s)$ be the fundameintal solution for the formal adjoint L^* of L. Set

$$w(t,s) = i\int_{T_0}^{t} E_\phi^*(t,\theta)v(\theta,x)d\theta\ .$$

Then, we have

$$L^*w = v \ \text{ on } [0, T_0],\ w(T_0,x) = 0\ .$$

Hence we have

$$\int_0^{T_0} (v,v)dt = \int_0^{T_0} (v, L^*w)dt = \int_0^{T_0} (Lv, w)dt = 0\ ,$$

and $v(t,x)=0$ on $[0, T_0]$. This proves the uniqueness of the solution of (3.3).

ii) Set $u(t,\tau,x)=E_\phi(t,\tau)E_\phi(\tau,s)\psi$ for $\psi\in\mathcal{S}$. Then, u satisfies

$$\begin{cases} Lu = 0 \ \text{ on } [0, T_0]\ , \\ u(\tau,\tau,x) = E_\phi(\tau,s)\psi\ . \end{cases} \tag{3.42}$$

On the other hand $\tilde{u}(t,x)=E_\phi(t,s)\psi$ also satisfies (3.42). Hence, by i) we have $u=\tilde{u}$ on $[0, T_0]$ which proves (3.41).

Corollary 3.4. *For the operator* $\boldsymbol{L}_1$ *of* (2.6) *let* $\boldsymbol{L}_2$ *be the operator of the form*

$$\boldsymbol{L}_2 = \boldsymbol{D}_t - \mathcal{D}(t) + \boldsymbol{F}(t)\ . \tag{3.43}$$

Let $E_{j,\phi_j}(t,s)$, $(0\leqq s, t \leqq T_0)$ *be the fundamental solution for* $L_j=D_t-\lambda_j(t,X,D_x)+f_j(t,X,D_x)$. *Then, the fundamental solution* $\boldsymbol{E}_2(t,s)\ (0\leqq s, t\leqq T_0)$ *of the Cauchy problem*

$$\begin{cases} \boldsymbol{L}_2 U = \Phi(t) \ \text{ on } [0, T_0]\ , \\ U|_{t=s} = \Psi \ \ (0\leqq s \leqq T_0) \end{cases} \tag{3.44}$$

exists uniquely in the form

$$\boldsymbol{E}_2(t,s) = \begin{bmatrix} E_{1,\phi_1}(t,s) & & 0 \\ & \ddots & \\ 0 & & E_{m,\phi_m}(t,s) \end{bmatrix}. \tag{3.45}$$

4. Construction of fundamental solution. The first order system case. In the first place we prove the fundamental lemma.

Lemma 4.1. *Let L_1 be the operator of the form* (2.6). *Then, the fundamental solution $E_1(t,s)$ of the Cauchy problem*

$$\text{(4.1)} \qquad \begin{cases} L_1U = \Phi(t) \quad \text{on } [s, T_0]\,, \\ U|_{t=s} = \Psi \quad (0 \leqq s \leqq T_0) \end{cases}$$

exists in the form

$$\text{(4.2)} \qquad E_1(t,s) = E_2(t,s)\,(I+Q,(ts))+Q_\infty(t,s)\,,$$

where $E_2(t,s)$ is the fundamental solution for L_2 in Corollary 3.4, *and $Q(t,s)$, $Q_\infty(t,s)$ satisfy*

$$\text{(4.3)} \qquad \begin{cases} Q(s,s) = 0,\ \sigma(Q(t,s))\,(x,\xi) \in S[0,0,0]\,, \\ Q_\infty(s,s) = 0,\ \sigma(Q_\infty(t,s))\,(x,\xi) \in \mathcal{B}_{t,s}(S^{-\infty})\,, \end{cases}$$

$(0 \leqq s \leqq t \leqq T_0)$.

Proof. If we find Q such that $\tilde{E}_1(t,s) = E_2(t,s)(I+Q(t,s))$ satisfies

$$\text{(4.4)} \qquad \sigma(L_1\tilde{E}_1) = \sigma(RE_2 + RE_2Q + E_2D_tQ) \in \mathcal{B}_{t,s}(S^{-\infty})\,,$$

then $\tilde{E}_1(t,s)$ is an approximate fundamental solution for L_1. Hence by the usual procedure, which also used in the proof of Theorem 3.1, we can find $E_1(t,s)$ in the form (4.2).

Set

$$\text{(4.5)} \qquad \tilde{R}(t,s) = E_2(s,t)R(t)E_2(t) \qquad \text{with } R(t) \text{ of (2.6)}.$$

Then, we see that (4.4) is equivalent to

$$\text{(4.6)} \qquad \sigma(D_tQ(t,s) + \tilde{R}(t,s)Q(t,s) + \tilde{R}(t,s)) \in \mathcal{B}_{t,s}(S^{-\infty}) \qquad (0 \leqq s \leqq t \leqq T_0)\,.$$

We find such $Q(t.s) = q(t,s,X,D_x)$ in the form

$$\text{(4.7)} \qquad \begin{cases} q_\nu(t,s) \in S[-\nu,0,0] \quad (0 \leqq s \leqq t \leqq T_0)\,, \\ q_\nu(s,s) = 0 \end{cases}$$

and

$$\text{(4.8)} \qquad q(t,s) \sim q_0(t,s) + q_1(t,s) + \cdots \mod \mathcal{B}_{t,s}(S^{-\infty})\,.$$

We first note that from Theorem 3.1, 3.2 and Corollary 3.4

$$\text{(4.9)} \qquad \begin{cases} \sigma(E_2(t,s))\,(x,\xi) \in \bigcap\limits_{0<\varepsilon<1} S[0,\, M+\varepsilon,\, -M-\varepsilon] \qquad (0 \leqq s \leqq t \leqq T_0)\,, \\ \sigma(E_2(s,t))\,(x,\xi) \in \bigcap\limits_{0<\varepsilon<1} S[0,\, M'+\varepsilon,\, -M'-\varepsilon] \qquad (0 \leqq s \leqq t \leqq T_0)\,, \end{cases}$$

where

$$(4.10)\qquad \begin{cases} \boldsymbol{M} = \max\limits_{1\leq j\leq m} \{M_j\},\ \boldsymbol{M}' = \max\limits_{1\leq j\leq m} \{M'_j\}\ , \\ M_j = \lim\limits_{R\to\infty} \sup\limits_{\substack{x,t\langle\xi\rangle^\omega\geq R \\ 0\leq t\leq R^{-1}}} \{t\mathcal{I}m f_j(t,x,\xi)\}\ , \\ M'_j = \lim\limits_{R\to\infty} \sup\limits_{\substack{x,t\langle\xi\rangle^\omega\geq R \\ 0\leq t\leq R^{-1}}} \{t\mathcal{I}m f_j(t,x,\xi)\}\ . \end{cases}$$

Hence, setting $\tilde{\boldsymbol{r}}(t,s,x,\xi)=\sigma(\tilde{\boldsymbol{R}}(t,s))\,(x,\xi)$, we have by Lemma 1.13

$$(4.11)\qquad \tilde{\boldsymbol{r}}(t,s,x,\xi) \in \bigcap_{0<\varepsilon<1}\bigcap_{\nu\geq 1} S[\omega-j,\ \boldsymbol{M}+\boldsymbol{M}'+\varepsilon-j(l+1),\ -\boldsymbol{M}-\boldsymbol{M}'-\varepsilon]\,.$$

Then, noting

$$(\eta(t,\xi)/\eta(s,\xi))^\nu \leq (\eta(t,\xi)^{l+1}\langle\xi\rangle)^{\nu\omega}(\eta(s,\xi)\langle\xi\rangle^\omega)^{-\nu} \leq (\eta(t,\xi)^{l+1}\langle\xi\rangle)^{\nu\omega}\,,$$

we see that

$$(4.12)\qquad \tilde{\boldsymbol{r}}(t,s,x,\xi)\in \bigcap_{j\geq 1} S[\omega-j,\ -j(l+1),\ 0]\,.$$

If we assume (4.7), we can write for $\boldsymbol{Q}_\nu(t,s)=\boldsymbol{q}_\nu(t,s,X,D_x)$

$$(4.13)\qquad \sigma(D_t\boldsymbol{Q}_\phi+\tilde{\boldsymbol{R}}\boldsymbol{Q}_\nu) = D_t\boldsymbol{q}_\nu+\tilde{\boldsymbol{r}}\boldsymbol{q}_\nu+\boldsymbol{r}_\nu \quad (\nu = 0,1,\cdots)\,,$$

where

$$(4.14)\qquad \boldsymbol{r}_\nu(t,s,x,\xi)\in S[\omega-\nu-1,0,0]\,.$$

Now, using (4.13) we define $\boldsymbol{q}_\nu$ by

$$(4.15)\qquad \begin{cases} D_t\boldsymbol{q}_\nu+\tilde{\boldsymbol{r}}\boldsymbol{q}_\nu+\boldsymbol{r}_{\nu-1} = 0 \quad (\boldsymbol{r}_{-1} = \tilde{\boldsymbol{r}})\,, \\ \boldsymbol{q}_\nu(s,s) = 0 \quad (\nu = 0,1,\cdots) \end{cases}$$

inductively. Then, if we check (4.7) for $\boldsymbol{q}_\nu$, we get $\boldsymbol{q}(t,s,x,\xi)$ by (4.8).

The solution of (4.15) can be written in the form

$$(4.16)\qquad \boldsymbol{q}_\nu(t,s) = -i\int_s^t \boldsymbol{r}_{\nu-1}(s_1,s)ds_1 + \sum_{\mu=2}^{\infty}(-i)^\mu\int_s^t ds_1\int_{s2}^{s_1} ds_2\cdots\int_s^{s_{\mu-1}} \tilde{\boldsymbol{r}}(s_1,s) \cdots\tilde{\boldsymbol{r}}(s_{\mu-1},s)\boldsymbol{r}_{\nu-1}(s_\mu,s)ds_\mu\,.$$

By (4.12) we have

$$(4.17)\qquad |\tilde{\boldsymbol{r}}(t,s)|\leq C\langle\xi\rangle^\omega(t\langle\xi\rangle^\omega+1)^{-2}$$

and get

$$\int_s^t |\tilde{r}(\sigma,s)|\,d\sigma \leqq \int_s^t C\langle\xi\rangle^{\omega}(\sigma\langle\xi\rangle^{\omega}+1)^{-2}d\sigma \tag{4.18}$$
$$= -[C/(\sigma\langle\xi\rangle^{\omega}+1)]_s^t$$
$$\leqq C(t-s)/(t+\langle\xi\rangle^{-\omega})\,.$$

Set

$$I_{\mu}(t,s,x,\xi) \tag{4.19}$$
$$= \int_s^t ds_1 \int_s^{s_1} ds_2 \cdots \int_s^{s_{\mu-1}} |\tilde{r}(s_1,s)| \cdots |\tilde{r}(s_{\mu},s)|\,ds_{\mu}$$

and assume

$$I_{\mu}(t,s,x,\xi) \leqq \frac{C^{\mu}(t-s)^{\mu}}{\mu!\,(t+\langle\xi\rangle^{-\omega})^{\mu}}\,.$$

Then, we have

$$I_{\mu+1}(t,s,x,\xi) \leqq \int_s^t \frac{C\langle\xi\rangle^{\omega}}{(\sigma\langle\xi\rangle^{\omega}+1)^2}\,\frac{C^{\mu}(\sigma-s)^{\mu}}{\mu!(\sigma+\langle\xi\rangle^{-\omega})^{\mu}}\,d\sigma$$
$$= \frac{C^{\mu+1}\langle\xi\rangle^{-\omega}}{\mu!} \int_s^t \frac{(\sigma-s)^{\mu}}{(\sigma+\langle\xi\rangle^{-\omega})^{\mu+2}}\,d\sigma$$

and setting $z=(\sigma-s)/(\sigma+\langle\xi\rangle^{-\omega})$ we have

$$\int_s^t \frac{(\sigma-s)^{\mu}}{(\sigma+\langle\xi\rangle^{-\omega})^{\mu+2}}\,d\sigma = (s+\langle\xi\rangle^{-\omega})^{-1} \int_0^{(t-s)/(t+\langle\xi\rangle^{-\omega})} z^{\mu}dz$$
$$= \frac{(t-s)^{\mu+1}}{(\mu+1)\,(s+\langle\xi\rangle^{-\omega})\,(t+\langle\xi\rangle^{-\omega})^{\mu+1}}\,.$$

So we have

$$I_{\mu+1}(t,s,x,\xi) \leqq \frac{C^{\mu+1}(t-s)^{\mu+1}}{(\mu+1)!\,(t+\langle\xi\rangle^{-\omega})^{\mu+1}}\,.$$

With (4.19) we have

$$I_{\mu}(t,s,x,\xi) \leqq \frac{C^{\mu}(t-s)^{\mu}}{\mu!\,(t+\langle\xi\rangle^{-\omega})^{\mu}} \qquad (\mu = 1,2,\cdots)\,. \tag{4.20}$$

Thus, from (4.16) for $\nu=0$ we have

$$|q_0(t,s)| \leqq \sum_{\mu=1}^{\infty} \frac{C^{\mu}(t-s)^{\mu}}{\mu!\,(t+\langle\xi\rangle^{-\omega})^{\mu}} \tag{4.21}$$
$$\leqq \exp\left[\frac{C(t-s)}{t+\langle\xi\rangle^{-\omega}}\right]-1\,.$$

Differentiate the both sides of (4.16) and estimate similarly. Then, we see that

$$q_0(t,s,x,\xi) \in S[0,0,0],\ q_0(s,s) = 0\,. \tag{4.22}$$

Now, assume that (4.7) holds for some $\nu\geqq 0$. Then, by (4.14)

$$\int_s^t |\boldsymbol{r}_\nu(\sigma,s)|\,d\sigma \leqq C(t-s)\langle\xi\rangle^{\omega-\nu-1}. \tag{4.23}$$

Hence, in (4.16) we use

$$\begin{aligned}\int_s^{s_{\mu-1}} |\boldsymbol{r}_{\nu-1}(s_\mu,s)|\,ds_\mu &\leqq C(s_{\mu-1}-s)\langle\xi\rangle^{\omega-\nu-1}\\ &\leqq C(t-s)\langle\xi\rangle^{\omega-\mu-1}.\end{aligned}$$

Then, by (4.20) we have

$$\begin{aligned}|\boldsymbol{q}_\nu(t,s)| &\leqq C_\nu\langle\xi\rangle^{\omega-\nu-1}\exp\left[\frac{C(t-s)}{t+\langle\xi\rangle^{-\omega}}\right]\\ &\leqq C_\nu e\langle\xi\rangle^{\omega-\nu-1},\end{aligned} \tag{4.24}$$

and finally get (4.7) for all $\nu\geqq 0$. Q.E.D.

Now, we shall state the main theorem of the present paper.

Theorem 4.2. *Let $\boldsymbol{L}$ and $\boldsymbol{L}_0$ be the operators of the form* (2.3) *and* (2.1), *respectively. Let $\boldsymbol{N}(t)$ and $\tilde{\boldsymbol{N}}(t)$ be the perfect diagonalizers for $\boldsymbol{L}$ and $\boldsymbol{L}_0$, respectively, and let $\boldsymbol{E}_1(t,s)$ be the fundamental solution for $\boldsymbol{L}_1$ of* (2.6).

Then, the fundamental solutions $\boldsymbol{E}(t,s)$ and $\boldsymbol{E}_0(t,s)$ for $\boldsymbol{L}$ and $\boldsymbol{L}_0$ can be found in the forms

$$\begin{cases}\boldsymbol{E}(t,s)=\boldsymbol{N}(t)\boldsymbol{E}_1(t,s)\boldsymbol{N}^\sharp(s)+\boldsymbol{R}_\infty(t,s),\\ \sigma(\boldsymbol{R}_\infty(t,s))(x,\xi)\in\mathcal{B}_{t,s}(S^{-\infty})\quad(0\leqq s\leqq t\leqq T_0)\end{cases} \tag{4.25}$$

and

$$\begin{cases}\boldsymbol{E}_0(t,s)=\tilde{\boldsymbol{N}}(t)\boldsymbol{E}_1(t,s)\tilde{\boldsymbol{N}}^\sharp(s)+\tilde{\boldsymbol{R}}_\infty(t,s),\\ \sigma(\tilde{\boldsymbol{R}}_\infty(t,s))(x,\xi)\in\mathcal{B}_{t,s}(S^{-\infty})\quad(0\leqq s\leqq t\leqq T_0),\end{cases} \tag{4.26}$$

respectively, where $\boldsymbol{N}^\sharp(s)$ and $\tilde{\boldsymbol{N}}^\sharp(s)$ are the parametrices of $\boldsymbol{N}(s)$ and $\tilde{\boldsymbol{N}}(s)$, respectively.

Furthermore, both $\boldsymbol{E}(t,s)$ and $\boldsymbol{E}_0(t,s)$ are represented as the sums of Fourier integral operators with phase functions $\phi_j(t,s)$, $j=1,\cdots,m$ and symbols of class

$$\bigcap_{0<\varepsilon<1} S[0,\boldsymbol{M}+\varepsilon,-\boldsymbol{M}-\varepsilon]. \tag{4.27}$$

Proof. It is easy to see that

$$\boldsymbol{N}(t)\boldsymbol{E}_1(t,s)\boldsymbol{N}^\sharp(s)+(I-\boldsymbol{N}(s)\boldsymbol{N}^\sharp(s))$$

is an approximate fundamental solution for $\boldsymbol{L}$. Then, noting $\sigma(I-\boldsymbol{N}(s)\boldsymbol{N}^\sharp(s))\in\mathcal{B}_s(S^{-\infty})$ and solving the integral equation as in (3.35) we get (4.25). Since by Lemma 4.1, $\boldsymbol{E}_1(t,s)$ is the sum of Fourier integral operators with phase functions $\phi_j(t,s)$ and symbols of class stated in (4.27), the rest of the proof for $\boldsymbol{L}$ is clear. Similarly we get for $\boldsymbol{L}_0$.

Theorem 4.3. *The Cauchy problem*

$$\begin{cases} \boldsymbol{L}U = 0 \quad \text{on } [s, T_0] \\ U|_{t=s} = \Psi \in H_\sigma, \ (0 \leqq s \leqq T_0) \end{cases} \tag{4.28}$$

and

$$\begin{cases} \boldsymbol{L}_0 U = 0 \quad \text{on } [s, T_0] \\ U|_{t=s} = \Psi \in H_\sigma, \ (0 \leqq s \leqq T_0) \end{cases} \tag{4.29}$$

have the unique solutions $U(t,s)$ and $U_0(t,s)$ in the form

$$U(t,s) = \boldsymbol{E}(t,s)\Psi, \ U_0(t,s) = \boldsymbol{E}_0(t,s)\Psi, \tag{4.30}$$

respectively, where H_σ is the usual Sobolev space for real σ. Furthermore, for any $\varepsilon > 0$ we have

$$\|\gamma_{M+\varepsilon} U\|_\sigma, \ \|\gamma_{M+\varepsilon} U_0\|_\sigma \leqq C_\varepsilon \|\Psi\|_\sigma \qquad (0 \leqq s \leqq t \leqq T_0) \tag{4.31}$$

and

$$\|U\|_{\sigma-(M+\varepsilon)\omega}, \ \|U_0\|_{\sigma-(M+\varepsilon)\omega} \leqq C'_\varepsilon \|\Psi\|_\sigma \qquad (0 \leqq s \leqq t \leqq T_0) \tag{4.32}$$

where $\gamma_{M+\varepsilon} = \gamma_{M+\varepsilon}(t,s,X,D_x)$ is defined by

$$\gamma_{M+\varepsilon}(t,s,x,\xi) = (\eta(s,\xi)/\eta(t,\xi))^{M+\varepsilon}.$$

REMARK. From (4.32) we see that $\boldsymbol{M}\omega$ denotes the supremum of regularity loss of the solution. It should be noted that in Kumano-go [7] the constant $\boldsymbol{M}$ is determind as a sufficiently large number depending on $\boldsymbol{L}$, and that constants C_ε and C'_ε are independent of t and s for $0 \leqq s \leqq t \leqq T_0$.

Proof. Since $\gamma_{M+\varepsilon}\boldsymbol{E}(t,s)$ is the sum of Fourier integral operators with symbols of class $S[0,0,0]$, we have (4.31) for $0 \leqq s \leqq t \leqq T_0$ and $U(t,s)$. Since

$$\begin{aligned} \|\gamma_{M+\varepsilon} U\|_\sigma &\geqq 2^{-(M+\varepsilon)} \|\eta(s,D_x)^{M+\varepsilon} U\|_\sigma \\ &\geqq 2^{-(M+\varepsilon)} \|U\|_{\sigma-(M+\varepsilon)\omega}, \end{aligned}$$

we get (4.32) for $0 \leqq s \leqq t \leqq T_0$ and $U(t,s)$. The rest of the proof is done similarly.

5. The higher order case. In this section we consider a single higher order operator of the following type:

$$\begin{cases} L = D_t^m + \sum_{k=1}^{m} a_k(t,X,D_x) D_t^{m-k}, \\ a_k(t,x,\xi) = \sum_{j=0}^{k} \eta(t)^{(k-j)(l+1)-k} a_{k,j}(t,x,\xi) \\ a_{k,j}(t,x,\xi) \in S[k-j, 0] \quad \text{on } [0,T] \times R_x^n \times R_\xi^n, \end{cases} \tag{5.1}$$

and consider the Cauchy problem

$$\begin{cases} Lu = \varphi(t) & \text{on } [s, T_0]\,, \\ D_t u|_{t=s} = \psi_j & (j=0, \cdots, m-1)\,. \end{cases} \tag{5.2}$$

Set

$$P(\lambda) = \lambda^m + a_{1,0}\lambda^{m-1} + \cdots + a_{m,0} \tag{5.3}$$

and

$$G(\lambda) = a_{1,1}\lambda^{m-1} + a_{2,1}\lambda^{m-2} + \cdots + a_{m,1}\,. \tag{5.4}$$

Theorem 5.1. *Let the roots $\mu_1(t, x, \xi), \cdots, \mu_m(t, x, \xi)$ of $P(\lambda)=0$ be real valued and satisfy* (0.3) *for a constant $c>0$.*

Then, the equation $Lu=\varphi$ can be reduced to a system $\boldsymbol{L}U={}^t(0, \cdots, 0, \varphi)$, where $\boldsymbol{L}$ has the form (2.3) *with $\lambda_j(t, x, \xi)=\eta(t)^l \mu_j(t, x, \xi)$, $j=1, \cdots, m$. The constant $\boldsymbol{M}$ of* (4.10) *is given by*

$$\boldsymbol{M} = \max_{1\leq k\leq m} \lim_{R\to\infty} \sup_{\substack{x, t\langle\xi\rangle^\omega \geq R \\ 0\leq t\leq R^{-1}}} \{J_k(t, x, \xi)\} + l(m-1) \tag{5.5}$$

where

$$J_k(t, x, \xi) = \frac{\mathcal{Im}\, G(\mu_k) - \eta((t)\partial_t \mu_k + l\mu_k)P''(\mu_k)/2}{P'(\mu_k)}\,,$$

$$P' = \partial_\lambda P \qquad \text{and} \qquad P'' = \partial_\lambda^2 P\,.$$

REMARK. It is easily verified that the differential operator of the form (0.11) satisfies (5.1).

Proof. I (Reduction to first order system). Let

$$\boldsymbol{H}(t) = \begin{bmatrix} h(t, D_x)^{1-m} & & & 0 \\ & h(t, D_x)^{2-m} & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}, \tag{5.6}$$

where $h(t, \xi)$ is the symbol defined in Lemma 1.4, and set

$$U = \boldsymbol{H}(t) \begin{bmatrix} u \\ D_t u \\ \cdots\cdots \\ D_t^{m-1} u \end{bmatrix}. \tag{5.7}$$

Then, $Lu=\varphi$ is reduced to a first order system $\boldsymbol{L}_0 U=\Phi$, where

$$\begin{aligned} \boldsymbol{L}_0 &= \boldsymbol{D}_t - \boldsymbol{A}(t)\,, \\ \sigma(\boldsymbol{A}(t)) &= \begin{bmatrix} (m-1)h_t h^{-1}, & h & & & 0 \\ & (m-2)h_t h^{-1}, & h & & \\ & & \ddots & \ddots & \\ 0 & & & h_t h^{-1}, & h \\ -a_m h^{1-m}, & -a_{m-1} h^{2-m}, & \cdots, & -a_2 h^{-1}, & -a_1 \end{bmatrix} \end{aligned} \tag{5.8}$$

and $\Phi={}^t(0, 0, \cdots, 0, \varphi)$.

By Lemma 1.2, 1.4, Proposition 1.6 and its Corollary we have $\sigma(\boldsymbol{A}(t))\in S[1, l]$.

II (Principal and sub-principal part). We set

$$\sigma(\boldsymbol{A}_1(t))=\begin{bmatrix} 0, & h & & & 0 \\ & \ddots & \ddots & & \\ 0 & & \ddots & 0, & h \\ -\eta^l a_{m,0}\langle\xi\rangle^{1-m}, & -\eta^l a_{m-1,0}\langle\xi\rangle^{2-m}, & \cdots, & & -\eta^l a_{1,0} \end{bmatrix}$$

and

$$\sigma(\boldsymbol{A}_0(t))=\frac{1}{\eta}\begin{bmatrix} -i(m-1)l & & & 0 \\ & -i(m-2)l & & \\ 0 & & \ddots & -il \\ -a_{m,1}\langle\xi\rangle^{1-m}, & -a_{m-1,1}\langle\xi\rangle^{2-m}, & \cdots, & -a_{1,1} \end{bmatrix}$$

Then, we have

$$\left\{\begin{array}{l} \sigma(\boldsymbol{A}_1(t))\in S[1, l] \\ \sigma(\boldsymbol{A}_0(t))\in S[0, -1] \\ \sigma(\boldsymbol{A}(t)-\boldsymbol{A}_1(t)-\boldsymbol{A}_0(t))\in S[-1, -(l+1)-1]\,. \end{array}\right. \tag{5.9}$$

This follows from

$$\begin{aligned} &a_k h^{1-k}-\eta^l a_{k,0}\langle\xi\rangle^{1-k}-\eta^{-1}a_{k,1}\langle\xi\rangle^{1-k} \\ &=\sum_{j=2}^{k}\eta^{(k-j)(l+1)-k}a_{k,j}h^{1-k}\in S[-1, -(l+1)-1] \end{aligned}$$

and

$$-(m-j)h_t h^{-1}-i(m-j)l\eta^{-1}\in\mathcal{H}^\omega\,.$$

III (Diagonalizer). The diagonalizer $\boldsymbol{N}_0(t)$ of $\boldsymbol{L}_0$ is given by

$$\sigma(\boldsymbol{N}_0(t))=\begin{bmatrix} 1 & ,\ \cdots, & 1 \\ \mu_1/\langle\xi\rangle & ,\ \cdots, & \mu_m/\langle\xi\rangle \\ \cdots\cdots & & \\ (\mu_1/\langle\xi\rangle)^{m-1}, & \cdots, & (\mu_m/\langle\xi\rangle)^{m-1} \end{bmatrix}. \tag{5.10}$$

To prove this it is enough to show that

$$\text{the } (j, k)\text{-element of } \sigma(\boldsymbol{N}_0)^{-1}\sigma(\boldsymbol{A}_1)\sigma(\boldsymbol{N}_0)=\delta_{j,k}\eta^l\mu_k\,, \tag{5.11}$$

where $\delta_{j,k}=1$ if $j=k$, $=0$ if $j\neq k$. And, this follows from

$$\text{the } (j, k)\text{-element of } \sigma(\boldsymbol{A}_1)\sigma(\boldsymbol{N}_0)=\eta^l\mu_k(\mu_k/\langle\xi\rangle)^{j-1} \qquad \text{for } j=1, \cdots, m-1\,,$$

$$\begin{aligned}&\text{the } (m, k)\text{-element of } \sigma(A_1)\sigma(N_0)\\&= -\eta^l\langle\xi\rangle^{1-m}\{a_{m,0}+a_{m-1,0}\mu_k+\cdots+a_{1,0}\mu_k^{m-1}\}\\&= \eta^l\mu_k(\mu_k/\langle\xi\rangle)^{m-1}\end{aligned}$$

and

$$\sum_{\nu=1}^{m} q_{j,\nu}(\mu_k/\langle\xi\rangle)^{\nu-1} = \delta_{j,k},$$

where $q_{j,\nu}$ is the (j, ν)-element of $\sigma(N_0)^{-1}$.

IV (Computation of $\boldsymbol{M}$). We have

$$\text{the } (j, k)\text{-element of } \eta\sigma(A_0)\sigma(N_0) = -i(m-j)l(\mu_k/\langle\xi\rangle)^{j-1} \quad \text{for } j=1, \cdots, m-1$$

and

$$\text{the } (m, k)\text{-element of } \eta\sigma(A_0)\sigma(N_0) = -G(\mu_k)\langle\xi\rangle^{1-m}.$$

We define polynomials $Q_j(\lambda)$ of $\lambda(j=1, \cdots, m)$ by

$$Q_j(\lambda) = \sum_{\nu=1}^{m} q_{j,\nu}\langle\xi\rangle^{1-\nu}\lambda^{\nu-1}. \tag{5.12}$$

Then we have

$$Q_j(\mu_k) = \delta_{i,k}.$$

Thus we have

$$Q_j(\lambda) = \prod_{k\neq j}(\lambda-\mu_k)/(\mu_j-\mu_k)$$

and

$$\partial_\lambda Q_k(\mu_k) = \frac{1}{2}P''(\mu_k)/P'(\mu_k).$$

Since $q_{j,m}=\langle\xi\rangle^{m-1}/P'(\mu_k)$, we have

$$\begin{aligned}&\text{the } (k, k)\text{-element of } \eta\sigma(N_0)^{-1}\sigma(A_0)\sigma(N_0)\\&= -i\sum_{\nu=1}^{m-1} q_{k,\nu}(m-\nu)l(\mu_k/\langle\xi\rangle)^{\nu-1}-q_{k,m}G(\mu_k)\langle\xi\rangle^{m-1}\\&= -ilm\sum_{\nu=1}^{m} q_{k,\nu}(\mu_k/\langle\xi\rangle)^{\nu-1}+il\sum_{\nu=1}^{m}\nu q_{k,\nu}(\mu_k/\langle\xi\rangle)^{\nu-1}\\&\quad -G(\mu_k)/P'(\mu_k)\\&= -ilm+il\{\partial_\lambda(\lambda Q_k(\lambda))\}_{\lambda=\mu_k}-G(\mu_k)/P'(\mu_k)\\&= -il(m-1)-(G(\mu_k)-\frac{i}{2}l\mu_kP''(\mu_k))/P'(\mu_k).\end{aligned} \tag{5.13}$$

We also have

$$\text{the } (k, k)\text{-element of } \sigma(N_0)^{-1}\sigma(N_{0,t}) \tag{5.14}$$

$$= \mu_{k,t} \sum_{\nu=2}^{m} q_{k,\nu}(\nu-1)\mu_k^{\nu-2}\langle\xi\rangle^{1-\nu}$$
$$= \mu_{k,t}\partial_\lambda Q_k(\mu_k)$$
$$= -\frac{i}{2}(\partial_t\mu_k)P''(\mu_k)/P'(\mu_k)\,.$$

V (End of the proof of Theorem 5.1). If we set

$$\boldsymbol{L} = \boldsymbol{D}_t - \boldsymbol{N}_0^{\sharp}\boldsymbol{A}\boldsymbol{N}_0 + \boldsymbol{N}_0^{\sharp}\boldsymbol{N}_{0,t}\,, \tag{5.15}$$

then we have

$$\boldsymbol{N}_0\boldsymbol{L} \equiv \boldsymbol{L}_0\boldsymbol{N}_0 \qquad \bmod \mathcal{B}_t(S^{-\infty})\,.$$

By (5.9) and (5.11) we have

$$\boldsymbol{L} = \boldsymbol{D}_t - \mathcal{D} + \boldsymbol{B}\,,$$

where

$$\sigma(\mathcal{D}) = \eta^l \begin{bmatrix} \mu_1(t, x, \xi) & & 0 \\ & \ddots & \\ 0 & & \mu_m(t, x, \xi) \end{bmatrix}$$

and $\sigma(\boldsymbol{B}) \in S[0, -1]$. We set

$$\begin{aligned} & c_k(t, x, \xi) \\ & = \text{the } (k, k)\text{-element of } \sigma(\boldsymbol{B}) \\ & \quad -\{il(m-1)+(G(\mu_k)-\frac{i}{2}(\eta\partial_t\mu_k+l\mu_k)P''(\mu_k))/P'(\mu_k)\}/\eta\,. \end{aligned}$$

Since

$$\sigma(\boldsymbol{N}_0^{\sharp}\boldsymbol{A}_1\boldsymbol{N}_0) - \sigma(\boldsymbol{N}_0)^{-1}\sigma(\boldsymbol{A}_1)\sigma(\boldsymbol{N}_0) \in S[0, 0]\,,$$
$$\sigma(\boldsymbol{N}_0^{\sharp}\boldsymbol{A}_0\boldsymbol{N}_0) - \sigma(\boldsymbol{N}_0)^{-1}\sigma(\boldsymbol{A}_0)\sigma(\boldsymbol{N}_0) \in S[-1, -1]$$

and

$$\sigma(\boldsymbol{N}_0\boldsymbol{N}_{0,t}) - \sigma(\boldsymbol{N}_0)^{-1}\sigma(\boldsymbol{N}_{0,t}) \in S[-1, -1]\,,$$

we have by (5.13) and (5.14)

$$\lim_{R\to\infty} \sup_{\substack{x,t\langle\xi\rangle^{\omega}\geqq R \\ 0\leqq t\leqq R^{-1}}} \{\eta(t, \xi)c_k(t, x, \xi)\} = 0\,.$$

Thus the $\boldsymbol{M}$ of (4.10) is given by (5.5).

Theorem 5.2. *Let L satisfy the condition of Theorem* 5.1. *Then the solution u of the Cauchy problem* (5.2) *with* $\varphi(t) \in \mathcal{B}_t(\mathcal{S})$ *and* $\psi_j \in \mathcal{S}$ $j=0, 1, \cdots, m-1$, *exists uniquely in* $[s, T_0]$ *and it is given by*

$$u(t, x) = \sum_{j=0}^{m-1} E_0^{1,j+1}(t, s, X, D_x)\psi_j + i\int_s^t E_0^{1,m}(t, \sigma, X, D_x)\varphi(\sigma)d\sigma\,, \tag{5.16}$$

where $E_0^{1,j}$ *is the* $(1,j)$*-element of the fundamental solution* $\boldsymbol{E}_0$ *of the operator*

$$\boldsymbol{D}-\begin{bmatrix} 0, & 1 & & & \\ & & & 0 & \\ & 0 & & 0, & 1 \\ -a_m, & -a_{m-1}, & \cdots, & -a_2, & -a_1 \end{bmatrix}. \tag{5.17}$$

The regularity loss caused by $E_0^{1,k}$ *in the sense of Remark to Theorem* 4.3 *is equalt to*

$$m_k=\omega(\boldsymbol{M}-l(m-1)-k+1). \tag{5.18}$$

Proof. An approximate fundamental solution of (5.17) is given by

$$\begin{aligned}\tilde{\boldsymbol{E}}_0=&\boldsymbol{H}(t)\boldsymbol{N}_0(t)\boldsymbol{E}(t,s)\boldsymbol{N}_0^\sharp(s)\boldsymbol{H}^\sharp(s)\\&+(I-\boldsymbol{H}(s)\boldsymbol{N}_0(s)\boldsymbol{N}_0^\sharp(s)\boldsymbol{H}^\sharp(s)),\end{aligned} \tag{5.19}$$

where $\boldsymbol{E}(t,s)$ is the fundamental solution of $\boldsymbol{L}$ of (5.15). Using $\tilde{\boldsymbol{E}}_0$ we can construct the fundamental solution $\boldsymbol{E}_0$ as in the proof of Theorem 3.1. By (5.19) we have

$$\begin{aligned}&\sigma(E_0^{1,k}(t,s))(x,\xi)\\&\in\bigcap_{0<\varepsilon<1}S[1-k,\boldsymbol{M}+\varepsilon-l(m-1),-\boldsymbol{M}-\varepsilon-l(k-m)].\end{aligned} \tag{5.20}$$

Thus, we have (5.18).

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