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FREQUENCY OF EXCEPTIONAL GROWTH OF THE *N*-PARAMETER WIENER PROCESS

KEIZO TAKASHIMA

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1. Introduction

Orey and Taylor [5] and Kôno [3] studied the set of times where the local growth rate of a standard Brownian motion is higher than a given function. In this paper we shall discuss such a problem for an N-parameter Wiener process.

Let $(\Omega, \mathcal{B}, P)$ be a complete probability space and R_+^N be the set of points of R^N with all components nonnegative. We shall write $t=\langle t_1, \cdots, t_N\rangle$ or simply $t=\langle t_\mu\rangle$ for a point t of R_+^N. An N-parameter Wiener process $\{w(t): t\in R_+^N\}$ is to be a separable real valued Gaussian process with mean 0 and covariance

$$E[w(s)w(t)] = \prod_{\mu=1}^{N} s_\mu \wedge t_\mu\,, \quad s=\langle s_\mu\rangle\,, \quad t=\langle t_\mu\rangle\,.$$

We consider $\{w^d(t): t\in R_+^N\}$; the process with values in R^d determined by making each component an N-parameter Wiener process, the components being independent. For $s=\langle s_\mu\rangle$, $t=\langle t_\mu\rangle$ of R_+^N with $s_\mu\leq t_\mu$, increments are defined as follows: for $u^d(t)=(w_1(t), \cdots, w_d(t))$

$$\begin{aligned} w_i\Delta((s, t)) &= w_i(t)-\textstyle\sum_{1\leq\mu\leq N} w_i(\langle t_1, \cdots, s_\mu, \cdots, t_N\rangle) \\ &\quad +\textstyle\sum_{1\leq\mu_1<\mu_2\leq N} w_i(\langle t_1, \cdots, s_{\mu_1}, \cdots, s_{\mu_2}, \cdots, t_N\rangle)-\cdots \\ &\quad +(-1)^N w_i(s)\,, \qquad i=1, \cdots, d\,, \end{aligned}$$

and

$$w^d(\Delta(s, t)) = (w_1(\Delta(s, t)), \cdots, w_d(\Delta(s, t)))\,,$$

where $\Delta(s, t)$ denotes the product of N one-dimensional intervals (s_μ, t_μ). We call such a set an *"interval"*. For a given constant $\alpha>1$, we consider a class Q of intervals $\Delta(s, t)$ in $(0, 1)^N$ with

$$0<\max_{1\leq\mu\leq N}(t_\mu-s_\mu)\leq\alpha\min_{1\leq\mu\leq N}(t_\mu-s_\mu)\,.$$

Let ϕ be a positive, non-increasing, continuous function defined on (0, 1]. Our subject is the random time set

$$E(\phi, \omega) = \{t\in(0, 1)^N: \exists\Delta_n\in \underset{\sim}{Q},\ t\in\Delta_n,\ |\Delta_n| \downarrow 0 \quad \text{as} \quad n\uparrow\infty \\ \|w^d(\Delta_n, \omega)\| > |\Delta_n|^{1/2}\phi(|\Delta_n|)\}$$

where $\|\cdot\|$ denotes the d-dimensional Euclidean norm and $|\cdot|$ denotes the N-dimensional Lebesgue measure. The aim of this paper is to obtain information about the size of $E(\phi, \omega)$ by examining its Hausdorff measure. For this sake, we consider a nonnegative, non-decreasing, continuous function h defined on $[0, 1]$ with $h(0)=0$. The Hausdorff h-measure of a subset A of R^N is defined by

$$h\text{-}m(A) = \lim_{\delta\downarrow 0}\inf_{\mathfrak{U}_\delta} \textstyle\sum_{U\in\mathfrak{U}_\delta} h(d(U)) \tag{1.1}$$

where the infimum extends over all countable covers $\mathfrak{U}_\delta$ of A by open balls U of diameter $d(U)\leq\delta$. Our result is the following.

Theorem. *Let ϕ be a positive, non-increasing, continuous function defined on* $(0, 1]$ *satisfying*

$$\int_{+0} x^{-2}\phi^{4N+d-2}(x)\exp\{-\phi^2(x)/2\}dx = \infty\,, \tag{1.2}$$

$$\int_{+0} x^{-1}\phi^{4N+d-2}(x)\exp\{-\phi^2(x)/2\}dx < \infty \tag{1.3}$$

and h be a nonnegative, non-decreasing, continuous function defined on $[0, 1]$ *satisfying* $h(0)=0$ *and*

$$h(x)/x^N\uparrow\infty \qquad as \quad x\downarrow 0\,. \tag{1.4}$$

Then

$$h\text{-}m(E(\phi, \omega)) = 0 \qquad or \quad \infty \quad a.s.$$

according as the integral

$$\int_{+0} x^{-2}\phi^{4N+d-2}(x)\exp\{-\phi^2(x)/2\}h(x^{1/N})dx \tag{1.5}$$

converges or diverges.

Kôno [3] obtained this result in the one-parameter case under an additional condition on ϕ ([3] p. 259, (1.8)), which is, in this paper, removed by Lemma 4.1 and Lemma 4.2.

The paper is arranged as follows. In Section 2 we collect general lemmas that we need. Section 3 deals with the proof for the case that the integral (1.5) converges. Our arguements go similarly as in [3]. Sections 4, 5 and 6 deal with the proof for the divergent case. In Section 4 we prepare some lemmas relating to ϕ and h. In Section 5 we make an arguement similar to that in [3] to make a preparation for a method of Jarnik [2]. In Section 6 we estimate the Hausdorff measure of $E(\phi, \omega)$ by the method of Jarnik and complete the proof.

Finally the author wishes to express his gratitude to Prof. T. Sirao for suggesting that the result might be obtained without the condition of [3] and to Prof. N. Kôno for his advice on the whole of the paper.

2. Preliminary lemmas

In this section we shall state some results that we need to prove the theorem.

Lemma 2.1. *Let U be a normal random variable in R^d with mean* 0 *and identity covariance matrix. Then*

$$P(||U|| \geq a) \sim c_d a^{d-2} \exp(-a^2/2).$$

This estimate is well known and we do not prove it (see Orey-Pruitt [4] p. 141).

Lemma 2.2. *Let (U, V) be a normal random variable in R^{2d} with mean* 0. *Assume that*

$$\begin{aligned} &E[U_i U_j] = E[V_i V_j] = \delta_{ij}\,, \\ &E[U_i V_j] = \rho\delta_{ij}\,, \qquad\qquad i, j = 1, 2, \cdots, d\,, \end{aligned}$$

where ρ is a constant and δ_{ij} is the Kronecker symbol.
(i) *There exists a positive constant c_1, independent of ρ, such that if $|\rho| < (ab)^{-1}$, then*

$$P(||U|| \geq a,\ ||V|| \geq b) \leq c_1 P(||U|| \geq a) P(||V|| \geq b).$$

(ii) *There eixsts a positive constant c_2, independent of ρ, such that*

$$P(||U|| \geq a,\ ||V|| \geq a) \leq c_2 \exp\{-(1-\rho^2)a^2/8\} P(||U|| \geq a)$$

for all $a \geq 0$.
(iii) *There exists a positive constant c_3, independent of ρ, such that if $a \geq b \geq \gamma^{-1}$ and $(1-2\gamma)b \geq |\rho| a$ for some $0 < \gamma < 1/4$, then*

$$P(||U|| \geq a,\ ||V|| \geq b) \leq c_3 \exp(-\gamma^2 b^2/4) P(||U|| \geq a).$$

Proof. The estimates (i), (ii) are due to Orey-Pruitt [4], so we prove only (iii). In case $\rho = 0$, U and V are independent of each other, so the estimate (iii) is easily derived from Lemma 2.1. In case $|\rho| = 1$, the condition $(1-2\gamma)b \geq |\rho| a$ does not hold for any $a \geq b \geq \gamma^{-1}$. Thus it suffices to show (iii) for $0 < |\rho| < 1$.

Now

$$P(\|U\|\geq a,\ \|V\|\geq b)\leq P(a\leq\|U\|\leq(1-\gamma)b/|\rho|,\ \|V\|\geq b)$$
$$+P(\|U\|\geq(1-\gamma)b/|\rho|).$$

As for the first term of the right-hand side, if $a\leq\|x\|\leq(1-\gamma)b/|\rho|$ and $\|y\|\geq b$, then $\|y-\rho x\|\geq\gamma b$. Therefore

$$\begin{aligned}&P(a\leq\|U\|\leq(1-\gamma)b/|\rho|,\ \|V\|\geq b)\\&\quad\leq(2\pi)^{-d}(1-\rho^2)^{-d/2}\int_{a\leq\|x\|\leq(1-\gamma)b/|\rho|}\exp(-\|x\|^2/2)dx\\&\quad\times\int_{\|y\|\geq\gamma b}\exp\{-(1-\rho^2)^{-1}\|y\|^2/2\}dy\,.\end{aligned}$$

Since $\gamma b\geq 1$, by Lemma 2.1, there esists a positive constant K_1, independent of ρ, such that

$$\begin{aligned}&\int_{\|y\|\geq\gamma b}(2\pi)^{-d/2}(1-\rho^2)^{-d/2}\exp\{-(1-\rho^2)^{-1}\|y\|^2/2\}dy\\&\quad\leq K_1(\gamma b)^{d-2}\exp\{-(1-\rho^2)^{-1}\gamma^2b^2/2\}\\&\quad\leq K_1\exp(-\gamma^2b^2/4)(\gamma b)^{d-2}\exp(-\gamma^2b^2/4)\,.\end{aligned}$$

Again by $\gamma b\geq 1$, it is easily seen that $K_1\exp(-\gamma^2b^2/4)(\gamma b)^{d-2}$ is bounded by a constant K_2, independent of ρ, a, b and γ. Therefore

$$\begin{aligned}&P(a\leq\|U\|\leq(1-\gamma)b/|\rho|,\ \|V\|\geq b)\\&\quad\leq K_2\exp(-\gamma^2b^2/4)(2\pi)^{-d/2}\int_{\|x\|\geq a}\exp(-\|x\|^2/2)dx\\&\quad=K_2\exp(-\gamma^2b^2/4)P(\|U\|\geq a)\,.\end{aligned}$$

On the other hand, since $0<\gamma<1/4$, $\gamma b\geq 1$, $(1-\gamma)b/|\rho|\geq 1$,

$$\begin{aligned}&P(\|U\|\geq(1-\gamma)b/|\rho|)\\&\quad\leq K_3\{(1-\gamma)b/|\rho|\}^{d-2}\exp\{-|\rho|^{-2}(1-\gamma)^2b^2/2\}\\&\quad\leq K_3\{(1-\gamma)/(1-2\gamma)\}^{d-2}\{(1-2\gamma)b/|\rho|\}^{d-2}\exp\{-|\rho|^2(1-2\gamma)^2b^2/2\}\\&\qquad\qquad\times\exp(-\gamma^2b^2/4)\\&\quad\leq K_4\exp(-\gamma^2b^2/4)P(\|U\|\geq(1-2\gamma)b/|\rho|)\\&\quad\leq K_4\exp(-\gamma^2b^2/4)P(\|U\|\geq a)\,,\end{aligned}$$

where K_3 and K_4 are constants independent of ρ, a, b, γ. Putting these estimates together, we have the estimate of $P(\|U\|\geq a,\ \|V\|\geq b)$, and the proof is completed.

The following lemma is due to Kôno [3] and we state it here in a form convenient for our use. Now we begin with some preparations for the lemma. Let (S, λ) be a compact metric space and $\{X(t): t\in S\}$ be a real valued continuous Gaussian process. Assume that

$$(2.1) \qquad E[X(t)]=0\,,\quad E[X(t)^2]=1\,,\quad t\in S\,,$$

and that there exists a positive constant η such that

$$(2.2) \qquad E[(X(s)-X(t))^2]\leq \eta^2\lambda(s, t)\,,\quad s, t\in S\,.$$

We denote $\{X^d(t): t\in S\}$ as the stochastic process in R^d whose components are independent copies of $\{X(t): t\in S\}$. Now assume that there exist a positive constant c_4 and a positive integer ν such that

$$(2.3) \qquad N(\varepsilon; B, \lambda)\leq c_4(d(B)/\varepsilon)^\nu\,,\quad 0<\varepsilon\leq d(B)\,,$$

holds for all closed balls B of S, where $d(B)$ denotes the diameter of B and $N(\varepsilon; B, \lambda)$ denotes the minimal number of sets of diameter at most 2ε which cover B. Under these assumptions we have the following estimate.

Lemma 2.3. *There exist two positive constants c_5, c_6 such that*

$$(2.4) \qquad \mathrm{P}(\sup_{t\in S}||X^d(t)||\geq a)\leq c_4c_5N((2\eta^2a^2)^{-1}; S, \lambda)a^{d-2}\exp(-a^2/2)$$

holds for all $a\geq 1+c_6$, where constants c_5, c_6 depend only on ν.

Next we state two lemmas relating to Hausdorff measures. We give another definition of h-measure. For a subset A of R^N, let us consider countable covers $\mathfrak{V}$ of A by cubes V. Let $d'(V)$ denote the length of side of cube V. For a function h satisfying (1.4) we define

$$(2.5) \qquad h\text{-}m'(A)=\lim_{\delta\downarrow 0}\inf_{\mathfrak{V}_\delta}\textstyle\sum_{V\in\mathfrak{V}_\delta}h(d'(V))$$

where the infimum extends over all countable covers $\mathfrak{V}_\delta$ of A by open cubes V with $d'(V)\leq\delta$.

Lemma 2.4. *Let h be a nonnegative, non-decreasing, continuous function defined on* [0, 1], *satisfying $h(0)=0$ and* (1.4). *For a subset A of R^N*

$$N^{-N/2}h\text{-}m(A)\leq h\text{-}m'(A)\leq h\text{-}m(A)\,.$$

Proof. This follows easily from the facts that if h satisfies (1.4), then for any $0\leq x\leq 1$

$$N^{-N/2}h(N^{1/2}x)\leq h(x)\leq h(N^{1/2}x)$$

and that for any cube V of $d'(V)=\delta$ there exist two balls U and U' with $d(U)=\delta$, $d(U')=N^{1/2}\delta$, $U\subset V\subset U'$.

Finally we give a well-known condition for a set A to have zero h-measure.

Lemma 2.5 ([6], Theorem 32, p. 59).

$$h\text{-}m'(A)=0$$

if and only if there exists a sequence $U_i, i=1,2,\cdots$, *of cubes with* $\sum_{i=1}^{\infty}h(d'(U_i))<\infty$, *such that any point of* A *belongs to infinitely many of* U_i.

It follows from Lemma 2.4 that in order to prove the theorem it is sufficient to show

$$h\text{-}m'(E(\phi,\omega))=0 \quad (or \quad h\text{-}m'(E(\phi,\omega))=\infty) \quad a.s.$$

if the integral (1.5) converges (or diverges). Thus, in the following, we take the definition (2.5) as the definition of h-measure and we write simply $h\text{-}m(A)$, $d(V)$ for $h\text{-}m'(A)$ and $d'(V)$.

3. Proof (I)

In this section we shall assume that the integral (1.5) converges. In this case our arguements closely follow Kôno [3].

Let $i=(i_1,\cdots,i_N)$. Define the time sets

$$\begin{aligned}K_j(n;i)=\{(s,t)\in R_+^N\times R_+^N\colon\ & 2^{-n-1}\leq t_j-s_j\leq 2^{-n},\\ & 2^{-n-1}\leq t_\mu-s_\mu\leq\alpha 2^{-n},\ \mu\neq j,\\ & i_\mu 2^{-n-1}\leq t_\mu\leq(i_\mu+1)2^{-n-1},\ \mu=1,\cdots,N\},\end{aligned}$$

the covering cubes

$$I(n;i)=\{t\in R_+^N\colon (i_\mu-2\alpha)2^{-n-1}\leq t_\mu\leq(i_\mu+1)2^{-n-1},\ \mu=1,\cdots,N\}$$

and the events

$$E_j(n;i)=\{\omega\colon \sup_{(s,t)\in K_j(n;i)}\|w^d(\Delta(s,t),\omega)\|\,|\Delta(s,t)|^{-1/2}\geq\phi(\alpha^{N-1}2^{-nN})\}\,.$$

The parameters will be restricted to the following ranges:

$$(3.1)\qquad 0\leq i_\mu\leq 2^{n+1}-1\,,\quad \mu=1,\cdots,N,\quad j=1,\cdots,N\,,\quad n\geq 3\,.$$

Furthermore let

$$I(\omega) = \bigcap_{m=3}^{\infty} \bigcup_{n\geq m} \bigcup_{j=1}^{N} \bigcup_{i} I(n; i)\chi(n; i, j, \omega)$$

where $\chi(n; i, j, \omega)$ denotes the indicator function of $E_j(n; i)$ and for a set I

$$\xi I = \begin{cases} \text{the empty set}, & \text{if } \xi = 0, \\ I, & \text{if } \xi = 1. \end{cases}$$

We shall show that

$$h\text{-}m(I(\omega)) = 0 \qquad \text{with probability 1.} \tag{3.2}$$

This suffices to prove the theorem for the case that the integral (1.5) converges, since for all ω,

$$E(\phi, \omega) \subset I(\omega).$$

This fact is proved in the same way as in [3], so we do not repeat it. From Lemma 2.5, in order to verify (3.2), it is sufficient to show that the sum

$$\sum E[h(d(I(n; i))\chi(n; i, j, \omega,)] \quad (= \sum \mathrm{P}(E_j(n; i))h(d(I(n; i)))) \tag{3.3}$$

over all i, j and n satisfying (3.1) converges. Now we estimate $\mathrm{P}(E_j(n; i))$, using Lemma 2.3. By definition it holds for all intervals Δ, Δ' of R_+^N that

$$E[w(\Delta)w(\Delta')] = |\Delta \cap \Delta'|. \tag{3.4}$$

It is easily seen from this that

$$E[\{w(\Delta(s, t))|\Delta(s, t)|^{-1/2} - w(\Delta(s', t'))|\Delta(s', t')|^{-1/2}\}^2] \\ \leq \alpha^{N-1}2^{N+n+3/2}N^{1/2}\|(s, t)-(s', t')\|_{2N}$$

holds for all (s, t), (s', t') of $K_j(n; i)$, where $\|\cdot\|_{2N}$ denotes the $2N$-dimensional Euclidean norm. Thus applying Lemma 2.3 to $\{w(\Delta(s, t))|\Delta(s, t)|^{-1/2}: (s, t) \in K_j(n; i)\}$ with $c_4=1$, $\nu=2N$, $\eta^2=\alpha^{N-1}2^{N+n+3/2}N^{1/2}$, we have

$$\mathrm{P}(E_j(n; i)) \leq K_1\phi^{4N+d-2}(\alpha^{N-1}2^{-nN}) \exp\{-\phi^2(\alpha^{N-1}2^{-nN})/2\},$$

since

$$N((2\eta^2a^2)^{-1}; K_j(n; i), \ \|\cdot\|_{2N}) \leq K_2a^{4N}.$$

Here K_1 and K_2 are positive constants independent of i, j and n. Therefore we get the bound

$$\sum \mathrm{P}(E_j(n; i))h(d(I(n; i))) \\ \leq K_3 \sum_{n\geq 3} 2^{nN}\phi^{4N+d-2}(\alpha^{N-1}2^{-nN}) \exp\{-\phi^2(\alpha^{N-1}2^{-nN})/2\}h((2\alpha+1)2^{-n-1}),$$

where K_3 is a positive constant. This sum is seen to converge by comparison with the integral (1.5). Thus $E(\phi, \omega)$ has zero h-measure a.s. by Lemma 2.5.

4. Proof (II)

Now we start the proof of the theorem for the case that the integral (1.5) diverges. The main part of the proof is how to construct a subset of $E(\varphi, \omega)$ which has infinite h-measure. This part will be stated in Sections 5 and 6. In this section we prepare some lemmas, first the next trapping lemma.

Lemma 4.1. *It is sufficient to prove the theorem for ϕ satisfying*

$$(4.1)\qquad (2\log H(x))^{1/2} \leq\phi(x)\leq(2\log H(x)+(4N+d+1)\log\log H(x))^{1/2},$$

where

$$H(x)=\int_x^1 h(y^{1/N})y^{-2}dy\,.$$

Proof. Set $\phi_1(x)=(2\log H(x))^{1/2}$, $\phi_2(x)=(2\log H(x)+(4N+d+1)\log\log H(x))^{1/2}$, and $\phi^*(x)=(\phi(x)\vee\phi_1(x))\wedge\phi_2(x)$. Then ϕ^* is a positive, non-increasing, continuous function satisfying (4.1). Since $H(x)\geq 3\log 1/x$ for small x, ϕ_1 satisfies (1.3), which implies that ϕ^* also satisfies (1.3). It is easily derived from (4.2) below that ϕ^* satisfies (1.2). Now we show that

$$(4.2)\qquad \int_{+0} x^{-2}\phi^{*4N+d-2}(x)\exp(-\phi^{*2}(x)/2)h(x^{1/N})dx=\infty\,,$$

and furthermore that if $h\text{-}m(E(\phi^*,\omega))=\infty$ a.s., then $h\text{-}m(E(\phi,\omega))=\infty$ a.s. As for (4.2), since we assume that (1.5) diverges, if $\phi^*\leq\phi$ near 0, then (4.2) holds. On the other hand, if there exists a sequence $x_n\downarrow 0$ such that $\phi(x_n)<\phi^*(x_n)$, then $\phi^*(x_n)=\phi_1(x_n)$ and

$$\begin{aligned}&\int_{x_n}^1 y^{-2}\phi^{*4N+d-2}(y)\exp(-\phi^{*2}(y)/2)h(y^{1/N})dy\\&\geq\phi^{*4N+d-2}(x_n)\exp(-\phi^{*2}(x_n)/2)\int_{x_n}^1 h(y^{1/N})y^{-2}dy\\&\geq\phi_1^{4N+d-2}(x_n)\exp(-\phi_1^2(x_n)/2)H(x_n)\,.\end{aligned}$$

The right-hand side tends to ∞ as $x_n\downarrow 0$, and it follows again that (4.2) holds.

Next we verify that $h\text{-}m(E(\phi,\omega))=\infty$ is derived from $h\text{-}m(E(\phi^*,\omega))=\infty$ with probability 1. Let $\phi'(x)=\phi(x)\vee\phi_1(x)$. Then $\phi\leq\phi'$, and

$$(4.3)\qquad E(\phi',\omega)\subset E(\phi,\omega)\qquad\text{for all}\quad\omega\,.$$

While $E(\phi_2, \omega) \subset E(\phi^*, \omega)$ for all ω, it is derived that

$$(4.4) \qquad E(\phi^*, \omega) - E(\phi_2, \omega) \subset E(\phi', \omega) \qquad \text{for all} \quad \omega .$$

In fact, for any t of $E(\phi^*, \omega) - E(\phi_2, \omega)$, there exists a sequence Δ_n of Q such that $t \in \Delta_n$, $|\Delta_n| \downarrow 0$ as $n \uparrow \infty$, and

$$\|w^d(\Delta_n, \omega)\| > |\Delta_n|^{1/2} \phi^*(|\Delta_n|) .$$

Since t does not belong to $E(\phi_2, \omega)$, $|\Delta_n|$ must belong to $\{x: \phi'(x) \leq \phi_2(x)\}$ for sufficiently large n. Then $\phi^*(|\Delta_n|) = \phi'(|\Delta_n|)$, and this means that t belongs to $E(\phi^*, \omega)$. Thus (4.4) has been verified. Now ϕ_2 is easily seen to satisfy

$$\int_{+0} x^{-2} \phi_2^{4N+d-2}(x) \exp(-\phi_2^2(x)/2) h(x^{1/N}) dx < \infty ,$$

so that the first part of the theorem shows that

$$h\text{-}m(E(\phi_2, \omega)) = 0 , \qquad \text{with probability 1.}$$

By (4.3) and (4.4), $h\text{-}m(E(\phi, \omega)) = \infty$ is derived from $h\text{-}m(E(\phi^*, \omega)) = \infty$ for almost all ω. This completes the proof of the lemma.

REMARK. From (4.1), particularly, we have

$$(4.5) \qquad (2 \log \log 1/x)^{1/2} \leq \phi(x) \leq (3 \log 1/x)^{1/2} .$$

Lemma 4.2. *A function ϕ which satisfies (4.1) is slowly varying at 0, that is, for any $\beta > 0$,*

$$\lim_{x \downarrow 0} \phi(\beta x)/\phi(x) = 1 .$$

Proof. For a fixed $\beta > 1$,

$$H(\beta x) = \beta^{-1} \int_x^{\beta^{-1}} h(\beta^{1/N} y^{1/N}) y^{-2} dy \geq \beta^{-1}(H(x) - H(\beta^{-1})) .$$

It is derived from this that $\log H(x)$ is slowly varying. From (4.1), this fact implies that ϕ is slowly varying at 0, and the proof is completed.

The following lemma is a simple variant of Lemma 5 in Kôno [3].

Lemma 4.3. *For the proof of the theorem, it is sufficient to consider h satisfying the following:*

$$(4.6) \qquad x^{-1} \phi^{4N+d-2}(x) \exp(-\phi^2(x)/2) h(x^{1/N}) \qquad \textit{is bounded for } 0 < x < 1 .$$

5. Proof (III)

In this section we shall construct, for almost all ω, a family $\{I\}$ of cubes and families $\mathfrak{J}(I)$ of subcubes of I satisfying the following three conditions:
(i) for every $J\in\mathfrak{J}(I)$, $\bar{J}\subset I$ and

$$||w^d(J, \omega)||>|J|^{1/2}\phi(|J|),$$

where is $\bar{J}$ the closure of J.
(ii) Any two cubes J_1, J_2 of $\mathfrak{J}(I)$ are disjoint; furthermore

$$\max_{1\leq\mu\leq N}\inf\{|t_\mu-s_\mu|: \langle t_\mu\rangle\in J_1,\ \langle s_\mu\rangle\in J_2\}$$
$$\geq 2^{-12-15/N}d(I)h^{-1/N}(d(I))\{h^{1/N}(d(J_1))+h^{1/N}(d(J_2))\}.$$

(iii) $$\sum_{J\in\mathfrak{J}(I)} h(d(J))\geq 2^{6N+8}h(d(I)).$$

In the following we shall use the next notations:

$$\varepsilon_n = 2^{-n},\quad \delta_n = a_1\varepsilon_{n+2}\phi^{-2}(\varepsilon_{n+2}{}^N),\quad d_n = [\varepsilon_{n+2}\delta_n^{-1}],$$

where $[x]$ denotes the integral part of x and a_1 is a positive constant such that

$$4^N c_2\sum_{r\geq 1} r^{2N}\exp(-a_1 r/72)<1/2. \tag{5.1}$$

Let $i=(i_1, \cdots, i_N)$, $j=(j_1, \cdots, j_N)$ and $k=(k_1, \cdots, k_N)$. Define the events

$$A(n; k, i, j)\qquad (= A(n; k_\mu, i_\mu, j_\mu))$$
$$= \{\omega: ||w^d(\Delta(s, t), \omega)||>|\Delta(s, t)|^{1/2}\phi(\varepsilon_{n+2}{}^N)\},$$

where $s=\langle k_\mu\varepsilon_n+i_\mu\delta_n\rangle$, $t=\langle k_\mu\varepsilon_n+\varepsilon_{n+1}+j_\mu\delta_n\rangle$. The parameters will be restricted to the following ranges:

$$1\leq i_\mu, j_\mu\leq 2(\alpha-1)/(1+\alpha)d_n,\quad \mu = 1, \cdots, N, \tag{5.2}$$

$$0\leq k_\mu\leq 2^n-1,\quad \mu = 1, \cdots, N. \tag{5.3}$$

Let $X(n; k)$ $(=X(n; k_\mu))$ denote the indicator function of $\bigcup_{i,j} A(n; k, i, j)$, where i and j run over the above range (5.2). Since the N-parameter Wiener process has stationary increments, $\mathrm{P}(X(n; k)=1)$ does not depend on k, so we denote it by p_n. The next lemma gives information about the magnitude of p_n.

Lemma 5.1.

$$2^{-1}\sum_{i,j}\mathrm{P}(A(n; k, i, j))\leq p_n\leq \sum_{i,j}\mathrm{P}(A(n; k, i, j)), \tag{5.4}$$

where $\sum_{i,j}$ *means the summation over all* i, j *satisfying* (5.2). *In particular, there* *exist two positive constants c, c' such that for sufficiently large n*

$$c\phi^{4N+d-2}(\varepsilon_{n+2}{}^N)\exp(-\phi^2(\varepsilon_{n+2}{}^N)/2)$$
$$\leq p_n \leq c'\phi^{4N+d-2}(\varepsilon_{n+2}{}^N)\exp(-\phi^2(\varepsilon_{n+2}{}^N)/2).$$

Proof. It is clear that $p_n \leq \sum_{i,j}(A(n; k, i, j))$ and

$$p_n \geq \sum_{i,j}(A(n; k, i, j)) - \sum_{i,j,i',j'} \mathrm{P}(A(n; k, i, j) \cap A(n; k, i', j')), \tag{5.5}$$

where $i'=(i'_1, \cdots, i'_N)$, $j'=(j'_1, \cdots, j'_N)$ and $\sum_{i,j,i',j'}$ means the summation over i', j', satisfying (5.2) with $i'_\mu \neq i_\mu$, or $j'_\mu \neq j_\mu$, for some μ. To estimate the second term of the right-hand side of (5.5), we put

$$X = w_1(\Delta(s, t))\,|\Delta(s, t)|^{-1/2},$$
$$Y = w_1\Delta((s', t'))\,|\Delta(s', t')|^{-1/2},$$

where $s=\langle k_\mu\varepsilon_n + i_\mu\delta_n\rangle$, $t=\langle k_\mu\varepsilon_n+\varepsilon_{n+1}+j_\mu\delta_n\rangle$, $s'=\langle k_\mu\varepsilon_n+i'_\mu\delta_n\rangle$ and $t'=\langle k_\mu\varepsilon_n+\varepsilon_{n+1}+j'_\mu\delta_n\rangle$. Then by (3.4)

$$1-E[XY] \geq 9^{-1}a_1\phi^{-2}(\varepsilon_{n+2}{}^N)\sum_{\mu=1}^N(|i_\mu - i'_\mu| + |j_\mu - j'_\mu|).$$

Using Lemma 2.2, (ii), we obtain

$$\mathrm{P}(A(n; k, i, j)A \cap (n; k, i', j'))$$
$$\leq c_2 \exp\{-(1-E[XY]^2)\phi^2(\varepsilon_{n+2}{}^N)/8\}\mathrm{P}(A(n; k, i, j))$$
$$\leq c_2 \exp\{-a_1\sum_{\mu=1}^N(|i_\mu - i'_\mu| + |j_\mu - j'_\mu|)/72\}\mathrm{P}(A(n; k, i, j)).$$

Now let $r=\sum_{\mu=1}^N(|i_\mu - i'_\mu| + |j_\mu - j'_\mu|)$, then there are no more than $(2r)^{2N}$ ways of choosing i' and j' to accomplish this. Thus we have

$$\sum_{i,j,i',j'} \mathrm{P}(A(n; k, i, j) \cap A(n; k, i', j'))$$
$$\leq 4^N c_2 \sum_{i,j}\{\sum_{r=1}^\infty r^{2N}\exp(-a_1 r/72)\}\mathrm{P}A(n; k, i, j)).$$

Therefore by (5.1)

$$\sum_{i,j,j',j'} \mathrm{P}(A(n; k, i, j) \cap A(n; k, 'i, j'))$$
$$\leq 2^{-1}\sum_{i,j}\mathrm{P}(A(n; k, i, j)).$$

This and (5.5) yield (5.4).

The latter part of the lemma is easily derived from (5.4) by Lemma 2.1 and the proof is completed.

In the following we consider sufficiently large n and choose n_1, n_2 for each

n, such that $n_2>n_1>n$ and

$$(5.6)\qquad h(\varepsilon_{n_1+2})\leq n^{-2}h(\varepsilon_{n+2})\varepsilon_{n+2}{}^{N}$$

$$(5.7)\qquad \varepsilon_{n+2}\varepsilon_{n_1+2}{}^{-1}h^{-1/N}(\varepsilon_{n+2})h^{1/N}(\varepsilon_{n_1+2})\geq 2^{12+15N}$$

$$(5.8)\qquad 2^{8N+11}h(\varepsilon_{n+2})\varepsilon_{n+2}{}^{N}\leq \textstyle\sum_{m=n_1}^{n_2} h(\varepsilon_{n+2})\varepsilon_{m+2}{}^{-N}p_m\leq 2^{8N+13}h(\varepsilon_{n+2})\varepsilon_{n+2}{}^{-N}\,.$$

In fact, since $h(x)\downarrow 0$ and $h(x)/x^N\uparrow\infty$ as $x\downarrow 0$, we can choose n_1 so that (5.6) and (5.7) hold. On the other hand, Lemma 5.1 and Lemma 4.3 show that $h(\varepsilon_{m+2})\varepsilon_{m+2}{}^{-N}p_m$ are bounded and that $\sum_{m=n_1}^{\infty} h(\varepsilon_{m+2})\varepsilon_{m+2}{}^{-N}p_m$ is seen to diverge by comparison with the integral (1.5). This ensures us the existence of n_2 satisfying (5.8). Set

$$b_{n,m}=[2^{-12-15/N}\varepsilon_{n+2}\varepsilon_{m+2}{}^{-1}h^{-1/N}(\varepsilon_{n+2})h^{1/N}(\varepsilon_{m+2})]+1\,,\quad n_1\leq m\leq n_2\,,$$

where $[x]$ denotes the integral part of x. For k' $(=(k'_1,\cdots,k'_N))$ satisfying (5.3), we define the random variables

$$Y(n,m;k')=\textstyle\prod_{\nu=n_1}^{m}\prod_{(q)}(1-X(\nu;[k'_\mu 2^{\nu-m}]+q_\mu))$$
$$Z(n,m;k')=\textstyle\prod_{\nu=n_1}^{m-1}(1-X(\nu;[k'_\mu 2^{\nu-m}]))\,,$$

where $\prod_{(q)}$ denotes the product over $q(=(q_1,\cdots,q_N))$ satisfying

$$(5.9)\qquad q_\mu \text{ are integers with } |q_\mu|\leq b_{n,\nu} \text{ and } \textstyle\sum_\mu |q_\mu|\geq 1\,.$$

Now for an open cube $I_{n,k}(=\prod_{\mu=1}^{N}(k_\mu\varepsilon_n,(k_\mu+1)\varepsilon_n))$, we define the families of random subcubes of $I_{n,k}$

$$\begin{aligned}\mathfrak{I}_m(I_{n,k})=\{&X(m;k')Y(n,m;k')Z(n,m;k')I(m;k')\\ &:k_\mu\varepsilon_n\leq k'_\mu\varepsilon_m<(k_\mu+1)\varepsilon_n,\ \mu=1,\cdots,N\}\\ &n_1\leq m\leq n_2\,,\end{aligned}$$

and

$$\mathfrak{I}(I_{n,k})=\bigcup_{n_1\leq m\leq n_2}\mathfrak{I}_m(I_{n,k})$$

where $I(m;k')=\prod_{\mu=1}^{N}(k'_\mu\varepsilon_m+\varepsilon_{m+2},k'_\mu\varepsilon_m+\varepsilon_{m+1})$ and for a cube I

$$\xi I=\begin{cases}\text{the empty set}, & \text{if }\ \xi=0\,,\\ I\qquad\qquad, & \text{if }\ \xi=1\,.\end{cases}$$

The aim of this section is to show that for almost all ω there exists an integer $n(\omega)$ such that for all $n\geq n(\omega)$ and k satisfying (5.3), $\{I_{n,k}\}$ and $\mathfrak{I}(I_{n,k})$ satisfy the conditions (i), (ii), and (iii). By the definition of $\mathfrak{I}_m(I_{n,k})$, (i) is clear. As for (ii), if $J_1, J_2\in\mathfrak{I}(I_{n,k})$ and $J_1\in\mathfrak{I}_m(I_{n,k})$, $J_2\in\mathfrak{I}_{m'}(I_{n,k})$, $m\leq m'$, then the definitions of Y and Z imply that

$$\max_{1\le\mu\le N} \inf \{|t_\mu - s_\mu| : t = \langle t_\mu\rangle \in J_1,\ s = \langle s_\mu\rangle \in J_2\}$$
$$\ge b_{n,m}\varepsilon_m$$
$$\ge 4^{-1}(b_{n,m}\varepsilon_m + b_{n,m'}\varepsilon_{m'})$$
$$\ge 2^{-12-15/N} d(I_{n,k}) h^{-1/N}(d(I_{n,k}))\{h^{1/N}(d(J_1)) + h^{1/N}(d(J_2))\} .$$

It remains to verify (iii) and to show the existence of $n(\omega)$. For this sake, we consider the radom variables

$$H(n; k) = \textstyle\sum_{J\in\mathfrak{I}(I_{n,k})} h(d(J)) .$$

If n is so large that

$$\phi(\varepsilon_{n+2}{}^N) \ge 10^2 , \tag{5.10}$$

$$20c_3 a_1{}^{-2N}\phi^{4N+2}(\varepsilon_{n+2}{}^N) \exp\{-\phi^2(\varepsilon_{n+2}{}^N)/(4\cdot 10^4)\} < 1/4 , \tag{5.11}$$

$$\phi(\varepsilon_m)/\phi(\varepsilon_{m+1}) \ge (\sqrt{3}/2)^{1/2}(1-2\cdot 10^{-2})^{-1} , \quad \text{for} \quad m \ge nN , \tag{5.12}$$

$$4c_1 \textstyle\sum_{m\ge n_1} p_m < 1/4 , \tag{5.13}$$

then the following estimates hold.

Lemma 5.2.

$$E[H(n; k)] \le 2^{8N+13} h(\varepsilon_{n+2}) , \tag{5.14}$$

$$E[H(n; k)] \ge]2^{8N+9} h(\varepsilon_{n+2}) . \tag{5.15}$$

Lemma 5.3. *There exists a positive constant M, independent of n, k such that*

$$E[(H(n; k) - E[H(n; k)])^2] \le M n^{-2} \varepsilon_{n+2} h^2(\varepsilon_{n+2}) . \tag{5.16}$$

Assuming these lemmas for a moment, we shall complete the proof of (iii). By (5.14), (5.15) and (5.16),

$$\mathrm{P}(|H(n; k) - E[H(n; k)]| \ge 2^{-1}E[H(n; k)] \text{ for some } k) \le M' n^{-2} ,$$

for some positive constant M'. Then by the Borel-Cantelli lemma, for almost all ω there exists $n(\omega)$ such that for any $n \ge n(\omega)$ and k satisfying (5.3),

$$|H(n; k) - E[H(n; k)]| \le 2^{-1}E[H(n; k)] .$$

Thus by (5.15) and (1.7)

$$H(n; k) \ge 2^{-1}E[H(n; k)] \ge 2^{8N+8} h(\varepsilon_{n+2})$$
$$\ge 2^{6N+8} h(\varepsilon_n) = 2^{6N+8} h(d(I_{n,k})) .$$

This verifies (iii). We shall denote, by Ω_0, the set of ω for which there exists $n(\omega)$ such that $I_{n,k}$ and $\mathfrak{J}(I_{n,k})$ satisfy the conditions (i), (ii) and (iii) for all $n \geq n(\omega)$, k satisfying (5.3).

Now we return back to the proofs of Lemma 5.2 and Lemma 5.3.

Proof of Lemma 5.2. First we prove (5.14). It is easily seen that for $H(n, m; k) = \sum_{J \in \mathfrak{J}_m(I_{n,k})} h(d(J))$,

$$\begin{aligned} E[H(n, m; k)] &\leq \textstyle\sum_{k'} E[X(m; k')]h(\varepsilon_{m+2}) \\ &\leq \varepsilon_{n+2}{}^N \varepsilon_{m+2}{}^{-N} h(\varepsilon_{m+2}) p_m \end{aligned}$$

where $\sum_{k'}$ denotes the summation over k' satisfying

$$k_\mu \varepsilon_n \leq k'_\mu \varepsilon_m < (k_\mu + 1)\varepsilon_n\,, \quad \mu = 1, \cdots, N\,. \tag{5.17}$$

Thus by (5.8)

$$\begin{aligned} E[H(n; k)] &\leq \textstyle\sum_{m=n_1}^{n_2} \varepsilon_{n+2}{}^N \varepsilon_{m+2}{}^{-N} h(\varepsilon_{m+2}) p_m \\ &\leq 2^{8N+13} h(\varepsilon_{n+2})\,. \end{aligned}$$

Next we verify (5.15). A simple calculation shows that

$$\begin{aligned} E[H(n, m; k)] &\geq \textstyle\sum_{k'} h(\varepsilon_{m+2}) E[X(m; k')] \{1 - \sum_{\nu=n_1}^{m-1} X(\nu; [k'_\mu 2^{\nu-m}]) \\ &\qquad - \textstyle\sum_{\nu=n_1}^{m} \sum_{(q)} X(\nu; [k'_\mu 2^{\nu-m}] + q_\mu)\}] \\ &= \textstyle\sum_{k'} h(\varepsilon_{m+2}) \{p_m - \sum_{\nu=n_1}^{m} E[X(m; k') X(\nu; [k'_\mu 2^{\nu-m}]) \\ &\qquad - \textstyle\sum_{\nu=n_1}^{m} \sum_{(q)} p_\nu p_m\}\,, \end{aligned} \tag{5.18}$$

where $\sum_{k'}$ denotes the summation over k' satisfying (5.17) and $\sum_{(q)}$ denotes the summation over q satisfying (5.9). As for

$$\textstyle\sum_{\nu=n_1}^{m} \sum_{(q)} p_\nu p_m\,,$$

by (5.8), this sum is less than

$$2^N p_m \textstyle\sum_{\nu=n_1}^{m} b_{n,\nu}{}^N p_\nu \leq 4^{-1} p_m\,. \tag{5.19}$$

Now we estimate

$$\textstyle\sum_{\nu=n_1}^{m-1} E[X(m; k') X(\nu; [k'_\mu 2^{\nu-m}])]\,,$$

using Lemma 2.2, (iii) and Lemma 4.2. Put

$$\begin{aligned} X &= w_1(\Delta(s, t)) \,|\, \Delta(s, t) \,|^{-1/2}\,, \\ Y &= w_1(\Delta(s, t')) \,|\, \Delta(s', t') \,|^{-1/2} \end{aligned}$$

where $s = \langle k'_\mu \varepsilon_m + i_\mu \delta_m \rangle$, $t = \langle k'_\mu \varepsilon_m + \varepsilon_{m+1} + j_\mu \delta_m \rangle$, $s' = \langle [k'_\mu 2^{\nu-m}] \varepsilon_\nu + i'_\mu \delta_\nu \rangle$, $t' = \langle [k'_\mu 2^{\nu-m}] \varepsilon_\nu + \varepsilon_{\nu+1} + j'_\mu \delta_\nu \rangle$, $n_1 \leq \nu \leq m-1$. Then

(5.20) $$0\leq E[XY]\leq(\sqrt{3}/2)^N, \qquad \text{if} \quad \nu=m-1,$$
$$0\leq E[XY]\leq(\sqrt{3}\,2^{(\nu-m)/2})^N, \qquad \text{if} \quad \nu\leq m-2.$$

We consider the next two cases:

(A) $$n_1\leq\nu\leq\bar{m}$$

(B) $$\bar{m}\leq\nu\leq m-1$$

where $\bar{m}=m-10\log\phi(\varepsilon_{m+2}{}^N)$. In the case (A), since

$$E[XY]\phi(\varepsilon_{m+2}{}^N)\phi(\varepsilon_{\nu+2}{}^N)<1$$

by (4.5) and (5.20), an application of Lemma 2.2, (i) shows

$$\mathrm{P}(A(m;\,k',\,i,\,j)\cap A(\nu;\,k^*,\,i',\,j'))\leq c_1\mathrm{P}(A(m;\,k',\,i,\,j))\mathrm{P}(A(\nu;\,k^*,\,i',\,j'))$$

where $k^*=([k_1'2^{\nu-m}],\,\cdots,\,[k_N'2^{\nu-m}])$. Thus by Lemma 5.1,

$$\begin{aligned}E[X(m;\,k')X(\nu;\,k^*)]&\leq \textstyle\sum_{i,j}\sum_{i',j'}\mathrm{P}(A(m;\,k',\,i,\,j)\cap A(\nu;\,k^*,\,i',\,j'))\\ &\leq c_1\textstyle\sum_{i,j}\mathrm{P}(A(m;\,k',\,i,\,j))\sum_{i',j'}\mathrm{P}(A(\nu;\,k^*,\,i',\,j'))\\ &\leq 4c_1p_mp_\nu\,,\end{aligned}$$

where $\sum_{i,j}$ and $\sum_{i',i'}$ denote the summations over i,j and i',j' satisfying (5.2) respectively. Therefore by (5.13), we have

(5.21) $$\begin{aligned}&\textstyle\sum_{\nu=n_1}^{\bar{m}}E[X(m;\,k')X(\nu;\,k^*)]\\ &\qquad\leq 4c_1p_m\textstyle\sum_{\nu=n_1}^{\infty}p_\nu\leq 4^{-1}p_m\,.\end{aligned}$$

In the case (B), since it is derived from (5.12) and (5.20) that

$$(1-2\cdot 10^{-2})\phi(\varepsilon_{\nu+2}{}^N)\geq E[XY]\phi(\varepsilon_{m+2}{}^N)\,,$$

an application of Lemma 2.2, (iii) to X and Y with $\gamma=10^{-2}$, $a=\phi(\varepsilon_{m+2}{}^N)$, $b=\phi(\varepsilon_{\nu+2}{}^N)$ shows

$$\begin{aligned}&\mathrm{P}(A(m;\,k',\,i,\,j))\cap A(\nu;\,k^*,\,i',\,j'))\\ &\qquad\leq c_3\exp\{-\phi^2(\varepsilon_{\nu+2}{}^N)/(4\cdot 10^4)\}\mathrm{P}(A(m;\,k',\,i,\,j))\,.\end{aligned}$$

Thus

$$\begin{aligned}&E[X(m;\,k')X(\nu;\,k^*)]\\ &\qquad\leq \textstyle\sum_{i,j}\sum_{i',j'}\mathrm{P}(A(m;\,k',\,i,\,j)\cap A(\nu;\,k^*,\,i',\,j'))\\ &\qquad\leq 2c_3a_1{}^{-2N}\phi^{4N}(\varepsilon_{\nu+2}{}^N)\exp\{-\phi^2(\varepsilon_{\nu+2}{}^N)/(4\cdot 10^4)\}\,p_m\,.\end{aligned}$$

Since we may assume that m is so large that

$$\log \phi(\varepsilon_{m+2}{}^N) \leq \phi^2(\varepsilon_{\bar{m}+2}{}^N),$$

we have

$$(5.22) \qquad \sum_{\bar{m}<\nu<m-1} E[X(m;k')X(\nu;k^*)]$$
$$\leq 20c_3a_1{}^{-2N}\phi^{4N+2}(\varepsilon_{\bar{m}+2}{}^N)\exp\{-\phi^2(\varepsilon_{\bar{m}+2}{}^N)/(4\cdot 10^4)\}p_m$$
$$\leq 4^{-1}p_m, \qquad \text{(by (5.11))}.$$

Putting (5.18), (5.19), (5.21) and (5.22) together, we obtain

$$E[H(n,m;k)] \geq 4^{-1}\varepsilon_{n+2}{}^N\varepsilon_{m+2}{}^{-N}h(\varepsilon_{m+2})p_m.$$

Hence

$$E[H(n;k)] \geq 4^{-1}\varepsilon_{n+2}{}^N \sum_{m=n_1}^{n_2} \varepsilon_{m+2}{}^N h(\varepsilon_{m+2})p_m \geq 2^{8N+9}h(\varepsilon_{n+2}), \qquad \text{(by (5.8))}.$$

This completes the proof.

Proof of Lemma 5.3. The outline of the proof is similar to that of Kôno's lemma (Lemma 8 in [3]).

Now put

$$X^*(n,m;k') = X(m;k')Y(n,m;k')Z(n,m;k')$$
$$-E[X(m;k')Y(n,m;k')Z(n,m;k')].$$

Then it is clear that

$$(5.23) \quad E[(H(n;k)-E[H(n;k)])^2] = \sum_{m=n_1}^{n_2}\sum_{k'} h^2(\varepsilon_{m+2})E[X^*(n,m;k')^2]$$
$$+\sum_{m=n_1}^{n_2}\sum_{k',k''} h^2(\varepsilon_{m+2})E[X^*(n,m;k')X^*(n,m;k'')]$$
$$+2\sum_{n_1\leq m<m'\leq n_2}\sum_{k'}\sum_{k''} h(\varepsilon_{m+2})h(\varepsilon_{m'+2})E[X^*(n,m;k')X^*(n,m';k'')]$$

where $\sum_{k'}$ and $\sum_{k''}$ denote the summations over k' and k'' satisfying (5.17) respectively and $\sum_{k',k''}$ denotes the summation over k', k'' satisfying (5.17) and $k'_\mu \neq k''_\mu$ for some μ. Using (5.6) and (5.8), we have

$$(5.24) \qquad \sum_{m=n_1}^{n_2}\sum_{k'} h^2(\varepsilon_{m+2})E[X^*(n,m;k')^2]$$
$$\leq \sum_{m=n_1}^{n_2}\sum_{k'} h^2(\varepsilon_{m+2})E[X(m;k')]$$
$$\leq \sum_{m=n_1}^{n_2} \varepsilon_{n+2}{}^N\varepsilon_{m+2}{}^{-N}h^2(\varepsilon_{m+2})p_m$$
$$\leq 2^{8N+13}n^{-2}h^2(\varepsilon_{n+2})\varepsilon_{n+2}{}^N.$$

As for the second term in the right-hand side of (5.23), note that $X^*(n,m;k')$ and $X^*(n,m;k'')$ are independent if $|k'_\mu - k''_\mu| > 4b_{n,n_1}\varepsilon_{n_1}\varepsilon_m^{-1}$ foı some μ. Therefore

$$\begin{aligned}&\textstyle\sum_{k',k''} h^2(\varepsilon_{m+2})E[X^*(n, m; k')X^*(n, m; k'')]\\ &\quad\le \textstyle\sum' h^2(\varepsilon_{m+2})E[X(m; k')X(m; k'')]\\ &\quad\le 8^N b_{n,n_1}{}^N\varepsilon_n{}^N\varepsilon_{n_1}{}^N\varepsilon_m{}^{-N}h^2(\varepsilon_{m+2})p_m{}^2,\end{aligned}$$

where $\sum'$ denotes the summation over k' and k'' satisfying (5.17) such that $|k'_\mu-k''_\mu|\le 4b_{n\ n}\varepsilon_{n_1}\varepsilon_m{}^{-1}$, $\mu=1,\cdots,N$ and $k'_\nu\ne k''_\nu$ for some ν. Thus there exists a positive constant K, independent of n, such that

$$\begin{aligned}(5.25)\qquad &\textstyle\sum_{m=n_1}^{n_2}\sum_{k',k''} h^2(\varepsilon_{m+2})E[X^*(n, m; k')X^*(n, m; k'')]\\ &\le \textstyle\sum_{m=n_1}^{n_2} 8^N b_{n,n_1}{}^N\varepsilon_n{}^N\varepsilon_{n_1}{}^N\varepsilon_m{}^{-2N}h^2(\varepsilon_{m+2})p_m{}^2\\ &\le 8^N b_{n,n_1}{}^N\varepsilon_{n_1+2}{}^N\varepsilon_{n+2}{}^N\left(\textstyle\sum_{m=n_1}^{n_2}\varepsilon_{m+2}{}^{-N}h(\varepsilon_{m+2})p_m\right)^2\\ &\le Kn^{-2}h^2(\varepsilon_{n+2})\varepsilon_{n+2}{}^N.\end{aligned}$$

It remains to estimate the third term in the right-hand side of (5.23). We do this, by considering the following three cases:

(i) for some μ,

$$k''_\mu\varepsilon_{m'}-(k'_\mu+1)\varepsilon_m>4b_{n,n_1}\varepsilon_{n_1}$$

or

$$k'_\mu\varepsilon_m-(k''_\mu+1)\varepsilon_{m'}>4_{n,n_1}b\varepsilon_{n_1}.$$

(ii) The condition of (i) does not hold but for some μ,

$$k''_\mu\varepsilon_{m'}-(k'_\mu+1)\varepsilon_m\ge 0$$

or

$$k'_\mu\varepsilon_m-(k''_\mu+1)\varepsilon_{m'}\ge 0.$$

(iii) Neither condition of (i) nor of (ii) hodls, that is,

$$k'_\mu\varepsilon_m\le k''_\mu\varepsilon_{m'}<(k''_\mu+1)\varepsilon_{m'}\le(k'_\mu+1)\varepsilon_m,\qquad \mu=1,\cdots,N.$$

In the case (i), $X^*(n, m; k')$ and $X^*(n, m'; k'')$ are independent, so we have $E[X^*(n, m; k')X^*(n, m'; k'')]=0$. In the case (ii), $X(m; k')$ and $X(m'; k'')$ are independent, so we have

$$E[X^*(n, m; k')X^*(n, m'; k'')]\le E[X(m, k')X(m'; k'')]=p_mp_{m'}.$$

In the case (iii), we further subdivide the case as follows:

$$\text{(A)}\qquad m'-m>10\log\phi(\varepsilon_{m'+2}{}^N),$$

$$\text{(B)}\qquad m'-10\log\phi(\varepsilon_{m'+2}{}^N)\le m\le m'-1.$$

The same arguements employed in the proof of Lemma 5.2 show that

$$E[X^*(n, m; k')X^*(n, m'; k'')] \leq E([X(m; k')X(m'; k'')] \leq 4c_1 p_m p_{m'}$$

in the case (A) and

$$\begin{aligned}&E[X^*(n, m; k')X^*(n, m'; k'')]\\ &\leq E[X(m; k')X(m'; k'')]\\ &\leq 2c_3 a_1^{-2N}\phi^{4N}(\varepsilon_{m+2}{}^N) \exp\{-\phi^2(\varepsilon_{m+2}{}^N)/(4\cdot 10^2)\} p_{m'},\end{aligned}$$

in the case (B). Putting these estimates together, in the case (A), we have

$$\begin{aligned}&\sum_{k'}\sum_{k''} h(\varepsilon_{m+2})h(\varepsilon_{m'+2})E[X^*(n, m; k')X^*(n, m'; k'')]\\ &\leq 8^N b_{n,n_1}{}^N \varepsilon_n{}^N \varepsilon_{n_1}{}^N \varepsilon_m{}^{-N}\varepsilon_{m'}{}^{-N} h(\varepsilon_{m+2})h(\varepsilon_{m'+2})p_m p_{m'}\\ &\quad +4c_1\varepsilon_n{}^N\varepsilon_{m'}{}^{-N}h(\varepsilon_{m+2})h(\varepsilon_{m'+2})p_m p_{m'}\\ &\leq (8^N b_{n,n_1}{}^N\varepsilon_n{}^N\varepsilon_{n_1}{}^N + 4c_1\varepsilon_n{}^N\varepsilon_{n_1}{}^N)\varepsilon_m{}^{-N}\varepsilon_{m'}{}^{-N}h(\varepsilon_{m+2})h(\varepsilon_{m'+2})\\ &\quad \times \varepsilon_n{}^N\varepsilon_{m'}{}^N h(\varepsilon_{m+2})h(\varepsilon_{m+2})p_{m'}.\end{aligned}$$

and by (5.6), (5.8),

$$\begin{aligned}(5.26)\qquad &2\sum_{(A)}\sum_{k'}\sum_{k''} h(\varepsilon_{m+2})h(\varepsilon_{m'+2})E[X^*(n, m; k')X^*(n, m'; k'')]\\ &\leq K' n^{-2}h^2(\varepsilon_{n+2})\varepsilon_{n+2}{}^N,\end{aligned}$$

where K' is a positive constant independent of n, k and $\sum_{(A)}$ denotes the summation over m and m' satisfying the condition (A). In the case (B),

$$\begin{aligned}&\sum_{k'}\sum_{k''} h(\varepsilon_{m+2})h(\varepsilon_{m'+2})E[X^*(n, m; k')X^*(n, m'; k'')]\\ &\leq 8^N b_{n,n_1}{}^N\varepsilon_n{}^N\varepsilon_{n_1}{}^N\varepsilon_m{}^{-N}\varepsilon_{m'}{}^{-N}h(\varepsilon_{m+2})h(\varepsilon_{m'+2})p_m p_{m'}\\ &\quad +2c_3 a_1^{-2N}\phi^{4N}(\varepsilon_{m+2}{}^N)\exp\{-\phi^2(\varepsilon_{m+2}{}^N)/(4\cdot 10^2)\}\varepsilon_n{}^N\varepsilon_{m'}{}^N h(\varepsilon_{m+2})h(\varepsilon_{m+2})p_{m'}.\end{aligned}$$

Since $\log \phi(\varepsilon_{m'+2}{}^N) \leq \phi^2(\varepsilon_{m+2})$, we have by (5.6), (5.8) and (5.11)

$$\begin{aligned}(5.27)\qquad &2\sum_{(B)}\sum_{k'}\sum_{k''} h(\varepsilon_{m+2})h(\varepsilon_{m'+2})E[X^*(n, m; k')]X^*(n, m'; k'')]\\ &\leq 8^N b_{n,n_1}{}^N\varepsilon_n{}^N\varepsilon_{n_1}{}^N\Big(\sum_{m=n_1}^{n_2}\varepsilon_{m+2}{}^{-N}h(\varepsilon_{m+2})p_m\Big)^2\\ &\quad +2^{-1}\varepsilon_{n+2}{}^N h(\varepsilon_{n_1+2})\sum_{m=n_1}^{n_2}\varepsilon_{m+2}{}^{-N}h(\varepsilon_{m+2})p_m\\ &\leq K'' n^{-2}h^2(\varepsilon_{n+2})\varepsilon_{n+2}{}^N,\end{aligned}$$

where $\sum_{(B)}$ denotes the summation over m and m' satisfying the condition (B) and K'' is a positive constant independent of n, k.

Putting (5.23), (5.24), (5.25), (5.26) and (5.27) together, we have the bound for the variance of $H(n; k)$ and the proof has been completed.

6. Proof (IV): The method of Jarnik

We shall complete the proof of the theorem by showing that for almost all ω and any fixed $M>0$, there exists a subset of $E(\phi, \omega)$ having more than M h-measure; our arguements follow Jarnik [2] (see also Kôno [3]).

In the following we shall consider $\omega \in \Omega_0$ fixed. Let n_0 be an integer sufficiently large such that $n_0 \geq n(\omega)$ and

$$h(\varepsilon_{n_0})\varepsilon_{n_0}{}^{-N} \geq 2^{6N+7}M . \tag{6.1}$$

Define the systems of random cubes

$$\mathfrak{J}_1 = \bigcup_k \mathfrak{J}(I_{n_0,k})$$

where the union extends over all k satisfying (5.3), and inductively

$$\mathfrak{J}_m = \bigcup_{I \in \mathfrak{J}_{m-1}} \mathfrak{J}(I) , \qquad m = 2, 3, \cdots .$$

Set $F = \bigcap_{m\geq 1} \bigcup_{I\in \mathfrak{J}_m} I$. Then F is easily seen to be included in $E(\phi, \omega)$ and by the condition (i) in Section 5, F is compact. The aim of this section is to show that F has more than M h-measure. Since M is taken arbitrarily, this suffices to prove the theorem. For this sake, we consider a covering $\mathfrak{U}_\delta$ of F by cubes U of $d(U)<\delta$. We may assume that $\mathfrak{U}_\delta$ is finite, since F is compact. Moreover if δ is less than the minimum of distances between cubes of $\mathfrak{J}_1$, it is sufficient to consider only the coverings, every cube of which intersects F. Since

$$\max_{I\in\mathfrak{J}_m} d(I) \to 0 , \qquad \text{as} \quad m \uparrow \infty ,$$

for any W of $\mathfrak{U}_\delta$, there exists an integer $\nu \geq 1$ such that W intersects a cube J of $\mathfrak{J}_\nu$ and I_1, I_2 of $\mathfrak{J}(J)$. Let ν be the minimum of such integers. Then there exists a cube W' such that $d(W') \leq d(W)$ and $W \cap J \subset W' \subset J$. By replacing W with W', we obtain a covering $\mathfrak{U}'$ of F. Now we prepare some terminologies after Jarnik [2] and Kôno [3].

DEFINITION 6.1. An open cube W is called to be *of degree* $\nu (\nu \geq 1)$, if and only if there exists a cube J of $\mathfrak{J}_\nu$ such that J includes W and W intersects at least two cubes of $\mathfrak{J}(J)$.

DEFINITION 6.2. An open cube W is called *normal* if and only if the degree of W is determined.

REMARK. The degree of a cube is uniquely determined if it can be determined.

DEFINITION 6.3. A point p is said to *attach to a normal cube W of degree* ν if and only if there exists a cube I of $\mathfrak{I}_{\nu+1}$ such that p belongs to I and I intersects W.

DEFINITION 6.4. A system $\mathfrak{U}$ of normal cubes is called a *normal estimating system* if and only if any point of F attaches to some cube of $\mathfrak{U}$.

DEFINITION 6.5. *The degree of a normal estimating system* is the maximum degree of its cubes.

DEFINITION 6.6. A normal estimating system $\mathfrak{U}$ is called *irreducible* if and only if $\mathfrak{U}$ does not contain any proper normal estimating subsystem.

Now, for a normal estimating system $\mathfrak{U}$ of degree ν, set

$$\Lambda^*(\mathfrak{U}) = \begin{cases} \sum_{(1)} h(d(W)) + 2^{-6N-8} \sum_{(2)} h(d(W)), & \text{if } \nu > 1, \\ 2^{-6N-8} \sum_{W \in \mathfrak{U}} h(d(W)), & \text{if } \nu = 1, \end{cases}$$

where $\sum_{(1)}$ denotes the summation over all W of $\mathfrak{U}$ of degree less than ν, and $\sum_{(2)}$ denotes the summation over all W of $\mathfrak{U}$ of degree ν. Since any covering of F by normal cubes is a normal estimating system, it is derived from the definition of $h\text{-}m(F)$ that

$$h\text{-}m(F) \geq \liminf_{\delta \downarrow 0} \inf_{\mathfrak{U}} \Lambda^*(\mathfrak{U}) , \tag{6.2}$$

where the infimum extends over all irreducible estimating systems $\mathfrak{U}$ of F by cubes W of $d(W) < \delta$. We prepare the next two key lemmas in the method of Jarnik.

Lemma 6.1. *For a normal cube W of degree ν which is included in a cube J of $\mathfrak{I}_\nu$,*

$$d(W) \geq 2^{-12-15/N} d(J) h^{-1/N}(d(J)) \{\textstyle\sum' h(d(I))\}^{1/N} ,$$

where $\sum'$ denotes the summation over all cubes I of $\mathfrak{I}(J)$ which intersect W.

Proof. For any I and I' neighboring with each other, by the condition (ii) in Section 5, we can construct two cubes between I and I', contained in W, with sides longer than

$$2^{-12-15/N} d(J) h^{-1/N}(d(J)) h^{1/N}(d(I))$$

and

$$2^{-12-15/N} d(J) h^{-1/N}(d(J)) h^{1/N}(d(I'))$$

respectively. This means that the volume of W is more than

$$2^{-12N-15}d(J)^N h^{-1}(d(J))\sum' h(d(I))$$

Since W is a cube, its side is longer than

$$2^{-12-15/N}d(J)h^{-1/N}(d(J))\{\sum' h(d(I))\}^{1/N}$$

and this completes the proof.

Lemma 6.2. *For an irreducible normal estimating system* $\mathfrak{U}$ *of degree* ν $(\nu>1)$*, there exists an irreducible normal estimating system* $\mathfrak{U}'$ *of degree less than* ν*, such that*

$$\Lambda^*(\mathfrak{U}')\leq\Lambda^*(\mathfrak{U})\,. \tag{6.3}$$

Proof. The proof of this lemma goes exactly as in Jarnik [2], but we state its outline for completeness.

It is sufficient to show the existence of a normal estimating system of degree less than ν which satisfies (6.3).

Each cube of degree ν of $\mathfrak{U}$ is included in a cube of $\mathfrak{J}_\nu$, so included in one cube (uniquely determined) of $\mathfrak{J}_{\nu-1}$. Let $J_1, \cdots, J_r$ be the totality of such cubes of $\mathfrak{J}_{\nu-1}$, and set

$$\mathfrak{U}' = [\mathfrak{U}-\{W\in\mathfrak{U}\colon \text{ of degree } (\nu-1) \text{ or } \nu,\ W\subset J_i \text{ for some } J_i\}] \cup \{J_1, \cdots, J_r\}\,.$$

Then $\mathfrak{U}'$ is a normal estimating system of degree $(\nu-1)$. It remains to show that $\mathfrak{U}'$ satisfies (6.3). Before doing this, note that a cube W of $\mathfrak{U}$ to which a point p of $F\cap J_i$ attaches is of degree $(\nu-1)$ or ν. In fact, if W is of degree m $(<\nu-1)$, then there exists J' of $\mathfrak{J}_{m+1}$ which includes J_i. Thus any point of $F\cap J_i(\subset F\cap J')$ attaches to W. This implies that

$$\mathfrak{U}-\{W'\subset\mathfrak{U};\ \text{of degree } \nu,\ W'\subset J_i\}$$

is a normal estimating system. This contradicts the irreducibility of $\mathfrak{U}$. Therefore W must be of degree $(\nu-1)$ or ν.

Now we shall estimate the contribution of cubes of degree $(\nu-1)$ or ν, included in J_i, to $\Lambda^*(\mathfrak{U})$. Let $W_1, \cdots, W_m$ be the totality of cubes of $\mathfrak{U}$, of degree $(\nu-1)$, included in J_i. Suppose that $\mathfrak{J}(J_i)=\{U_1, \cdots, U_k, U_{k+1}, \cdots, U_a\}$ and U_j intersects some W_n if $1\leq j\leq k$, does no W_n if $k+1\leq j\leq a$. By the condition (iii) of Section 5,

$$\textstyle\sum_{j=1}^a h(d(U_j))\geq 2^{6N+8}h(d(J_i))\,.$$

Now we consider the next two cases:

(1) $$\sum_{j=1}^{k} h(d(U_j)) \geq 2^{6N+7} h(d(J_i)),$$

(2) $$\sum_{j=k+1}^{a} h(d(U_j)) \geq 2^{6N+7} h(d(J_i)).$$

In the case (1), by Lemma 6.1

$$d(W_n) \geq 2^{-12-15/N} d(J_i) h^{-1/N}(d(J_i)) \{\sum_{(n)} h(d(U_j))\}^{1/N},$$

where $\sum_{(n)}$ denotes the summation over all U_j which intersect W_n. On the other hand, it is easily derived from (1.5) that

$$\sum_n h(x_n^{1/N}) \geq h((\sum_n x_n)^{1/N}) \qquad \text{for} \quad x_n \geq 0.$$

Using this, we have

(6.4) $$\sum_{n=1}^{m} h(d(W_n)) \geq h(\{\sum_{n=1}^{m} d(W_n)^N\}^{1/N}) \geq 2^{-6N-8} h(d(J_i)).$$

In the case (2), any point of $F \cap U_j$ $(k+1 \leq j \leq a)$ attaches to a cube V of degree ν, included in U_j. Let $V_1, \cdots, V_b$ be the totality of cubes of $\mathfrak{U}$, of degree ν, included in U_j. Again by Lemma 6.1

$$d(V_q) \geq 2^{-12-15/N} d(U_j) h^{-1/N}(d(U_j)) \{\sum_{(q)} h(d(I))\}^{1/N},$$

where $\sum_{(q)}$ denotes the summation over all I of $\mathfrak{Z}(U_j)$ which intersect V_q. Thus

$$\sum_{q=1}^{b} h(d(V_q)) \geq h(\{\sum_{q=1}^{b} d(V_q))^N\}^{1/N} \geq 2^{-6N-7} h(d(U_j)).$$

Summing these estimates over j, $k+1 \leq j \leq a$, we have

(6.5) $$\sum h(d(V)) \geq h(d(J_i))$$

where the summation in the left-hand side extends over all cubes of $\mathfrak{U}$, of degree ν, included in J_i. Putting (6.4) and (6.5) together, we obtain (6.3) and the proof of the lemma has been completed.

Now we are on the last stage in the proof of the theorem. Lemma 6.2 and (6.2) tell us that

(6.6) $\Lambda^*(\mathfrak{U}) \geq M$, for any irreducible normal estimating system of degree 1

implies h-$m(F) \geq M$. For an irreducible normal estimating system of degree 1, by Lemma 6.1 and the condition (iii) in Section 5, we have

$$\begin{aligned}
\Lambda^*(\mathfrak{U}) &\geq 2^{-6N-8} \sum_{W \in \mathfrak{U}} h(d(W)) \\
&\geq 2^{-6N-8} \sum_{W \subset J \in \mathfrak{Z}_1} h(2^{-12-15/N} d(J) h^{-1/N}(d(J)) \{\sum' h(d(I))\}^{1/N}) \\
&\geq 2^{-6N-8} \sum_{J \in \mathfrak{Z}_1} h(2^{-12-15/N} d(J) h^{-1/N}(d(J)) \{\sum_{I \in \mathfrak{Z}(J)} h(d(I))\}^{1/N}) \\
&\geq 2^{-12N-15} \sum_{J \in \mathfrak{Z}_1} h(d(J)) \\
&\geq 2^{-6N-7} \varepsilon_{n_0}^{N} h(\varepsilon_{n_0})
\end{aligned}$$

where $\sum'$ denotes the summation over all I of $\mathfrak{J}(J)$ which intersect W. Since we have chosen n_0 so large that (6.1) holds, from the above we can derive (6.6). Thus we have verified the theorem.

REMARK. With respect to the conditions (1.2) and (1.3), note the following. If ϕ satisfies (1.2), then ϕ is a lower function for the uniform modulus of continuity in the sense of Orey-Taylor [5] ([7]). This implies that $E(\phi, \omega)$ is not empty a.s. On the other hand, if ϕ satisfies (1.3), then ϕ is an upper function for the local two-sided growth in the sense of Jain-Taylor [1] ([7]). An application of the Fubini theorem shows that $E(\phi, \omega)$ has zero Lebesgue measure a.s. Thus the size of $E(\phi, \omega)$ comes into question.

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Department of Mathematics
Osaka City University
Sugimoto-cho, Sumiyoshi-ku
Osaka 558, Japan