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RIBBON KNOTS AND RIBBON DISKS

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For a ribbon knot, we will define, in §1, the ribbon disk pair associated with it. On the other hand, J.F.P. Hudson and D.W. Sumners gave a method to construct a disk pair [2], [13]. In §1 and 2, we will generalize their construction and show that a ribbon disk pair is obtained by our construction and vice versa.

In [10], C.D. Papakyriakopoulos proved that the complement of a classical knot is aspherical. As an analogy of this, we will prove, in §3, that the compelment of a ribbon disk is aspherical, and it follows from this fact that the fundamental group of a ribbon knot complement has no element of finite order. In the final section, we will calculate the higher homotopy groups of a higher-dimensional ribbon knot complement, and in Theorem 4.4 we show that *a ribbon n-knot for $n\geqq 3$ is unknotted if the fundamental group of the knot complement is the infinite cyclic group.* This result is proved independently by A. Kawauchi and T. Matumoto [5].

Throughout the paper, we work in the piecewise-linear category although the results remain valid in the smooth category.

1. Preliminaries

1.1. By S^n we denote an n-sphere, and by B^n or D^n an n-disk. By ∂M, int M and cl M we denote the boundary, the interior and the closure of a manifold M respectively. In this paper, every submanifold in a manifold is assumed to be locally flat. If $\partial M\neq\emptyset$, by $\mathcal{D}M$ we mean the *double* of M, i.e. $\mathcal{D}M$ is obtained from the disjoint union of two copies of M by identifying their boundaries via the identity map. For a subcomplex C in a manifold M, $N(C; M)$ is a regular neighbourhood of C in M. By a *pair* (M, W) we denote a manifold M and a *proper* submanifold W in M, i.e. $W\cap\partial M=\partial W$. An *n-disk pair* is a pair (M, W) such that M is a disk and W an n-disk. Two pairs (M_1, W_1) and (M_2, W_2) are *equivalent* if there exists a homeomorphism from M_1 to M_2 which maps W_1 to W_2, and we will identify two equivalent manifold pairs. Let $\mathcal{D}(M, W)=(\mathcal{D}M, \mathcal{D}W)$ and $\partial(M, W)=(\partial M, \partial W)$. We denote the unit interval [0, 1] by I, and the Euclidean n-space by R^n. Let R_t^{n-1} be the hyperplane in R^n whose n-th coordinate is t, R_+^n the half space of R^n whose n-th coordinate is non-negative, and $R_-^n = \mathrm{cl}(R^n - R_+^n)$.

An *n-knot* K^n will mean an embedded n-sphere in an $(n+2)$-sphere S^{n+2}. An n-knot is *unknotted* if it bounds an $(n+1)$-disk in S^{n+2}. For a proper disk D^n in a manifold M, (M, D^n) is *unknotted*, or D^n is *unknotted* in M, if there exists an $(n+1)$-disk D^{n+1} in M such that $D^{n+1} \cap \partial M$ is an n-disk in ∂D^{n+1} and $\mathrm{cl}(\partial D^{n+1} \cap \mathrm{int}\, M) = D^n$. For terminologies in halnde theory, we refer the readers to [11], and for knot theory, to [14].

1.2. Let $S_0^n, S_1^n, \cdots, S_m^n$ be mutually disjoint n-spheres in a q-manifold M^q for $n \geqq 1$, $q \geqq 3$. Suppose that an embedding $\beta: B^n \times I \rightarrow M^q$ satisfies

$$\beta(B^n \times I) \cap (S_0^n \cup \cdots \cup S_m^n) = \beta(B^n \times \partial I).$$

Then we call β or $\beta(B^n \times I)$ a *band compatible with* $S_0^n \cup \cdots \cup S_m^n$.

Let $\beta_1, \cdots, \beta_m$ be bands compatible with $S_0^n \cup \cdots \cup S_m^n$ such that

(1) $\beta_i(B^n \times I) \cap \beta_j(B^n \times I) = \emptyset$ if $i \neq j$, and

(2) $\bigcup \{S_i^n;\ 0 \leqq i \leqq m\} \cup \bigcup \{\beta_j(B^n \times I);\ 1 \leqq j \leqq m\}$ is connected.

Then

$$(\bigcup \{S_i^n; 0 \leqq i \leqq m\} - \cup \{\beta_j(B^n \times \partial I); 1 \leqq j \leqq m\}) \cup \bigcup \{\beta_j(\partial B^n \times I); 1 \leqq j \leqq m\}$$

is an n-sphere, and denoted by

$$\mathcal{F}(S_0^n, \cdots, S_m^n; \beta_1, \cdots, \beta_m).$$

Suppose that $M^q = S^{n+2}$ and there exist mutually disjoint $(n+1)$-disks $B_0^{n+1}, B_1^{n+1}, \cdots, B_m^{n+1}$ with $\partial B_i^{n+1} = S_i^n$ for $0 \leqq i \leqq m$. Then

$$K^n = \mathcal{F}(S_0^n, \cdots, S_m^n; \beta_1, \cdots, \beta_m)$$

is called a *ribbon n-knot of type* $(\beta_1, \cdots, \beta_m)$.

Our definition of a ribbon n-knot is equivalent to that of [19].

REMARK 1.3. In 1.2, it is easily seen that we can deform isotopically each band so that

$$\beta_i(B^n \times I) \cap S_j^n = \begin{cases} \beta_i(B^n \times \{0\}) & \text{if } j = 0, \\ \beta_i(B^n \times \{1\}) & \text{if } j = i, \text{ and} \\ \emptyset & \text{otherwise.} \end{cases}$$

Thus we assume that each band of a ribbon n-knot satisfies this condition.

1.4. Let D^{n+3} be obtained from the disjoint union of $S^{n+2} \times I$ and B^{n+3} by identifying $S^{n+2}\times\{1\}$ and ∂B^{n+3}. Let K^n be a ribbon n-knot of type $(\beta_1, \cdots, \beta_m)$, then we can construct an $(n+1)$-disk L^{n+1} in D^{n+3} which bounds $K^n\times\{0\}$ as follows: Let $D_i^{n+1}=(S_i^n\times[0, 3/4])\cup(B_i^{n+1}\times\{3/4\})$ in $S^{n+2}\times I$ for $0\leqq i\leqq m$, where B_i^{n+1} and S_i^n are as in 1.2. For $1\leqq j\leqq m$, let $\bar\beta_j$: $B_n\times I\times I\rightarrow S^{n+2}\times I$ be the product of β_j and a map from I into I which takes t to $t/2$, i.e.

$$\bar\beta_j(x, y, t) = (\beta_j(x, y), t/2)$$

for $x\in B^n$ and $y, t\in I$. Then

$$L^{n+1} = (\cup\{D_i^{n+1}; 0\leqq i\leqq m\} - \cup\{\bar\beta_j(B^n\times\partial I\times I); 1\leqq j\leqq m\}) \\ \cup \bigcup\{\bar\beta_j(\partial B^n\times I\times I)\cup\bar\beta_j(B^n\times I\times\{1\}); 1\leqq j\leqq m\}$$

is an $(n+1)$-disk and bounds $K^n\times\{0\}$ in D^{n+3}. Note that the section of L^{n+1} by $S^{n+2}\times\{t\}$ is

(1) $K^n\times\{t\}$ if $0\leqq t<1/2$,
(2) $(\bigcup\{S_i^n; 0\leqq i\leqq m\}\cup\bigcup\{\beta_j(B^n\times I); 1\leqq j\leqq m\})\times\{1/2\}$ if $t = 1/2$,
(3) $(S_0^n\cup\cdots\cup S_m^n)\times\{t\}$ if $1/2<t<3/4$,
(4) $(B_0^n\cup\cdots\cup B_m^n)\times\{3/4\}$ if $t = 3/4$,
(5) $\emptyset$ if $3/4<t\leqq 1$. (See Fig. 1.)

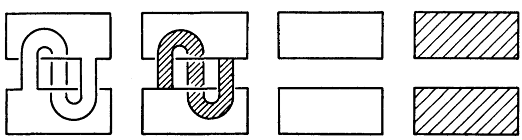

Fig. 1

We call L^{n+1} in D^{n+3} the *ribbon* $(n+1)$-*disk associated with a ribbon n-knot* K^n, or (D^{n+3}, L^{n+1}) the *ribbon* $(n+1)$-*disk pair associated with* K^n.

The double $\mathcal{D}(D^{n+3}, L^{n+1})$ of a ribbon $(n+1)$-disk pair is an $(n+1)$-knot in the $(n+3)$-sphere $\mathcal{D}D^{n+3}$. Since $\mathcal{D}(D_0^{n+1}\cup\cdots\cup D_m^{n+1})$ is a trivial $(n+1)$-link and each $\mathcal{D}(\beta_i(B^n\times I\times I))$ is a band, $\mathcal{D}L^{n+1}$ is a ribbon $(n+1)$-knot. Then we say that $\partial(D^{n+3}, L^{n+1})$ is an *equatorial knot* of $\mathcal{D}(D^{n+3}, L^{n+1})$. (See [19].)

1.5. We will generalize the construction of $(n+1)$-disk pairs in [2] and [13], for $n\geqq 1$. Let D_0^{n+1} be an unknotted $(n+1)$-disk in B^{n+3}. Adding m 1-handles $h_1^1, \cdots, h_m^1$ to B^{n+3} such that $h_i^1\cap D_0^{n+1}=\emptyset$ for each i, we obtain an $(n+3)$-disk with m 1-handles, say V. We take mutually disjoint oriented 1-spheres $\alpha_1, \cdots, \alpha_m$ on ∂V such that α_i intersects the belt sphere of h_i^1 at only one point, $\alpha_i\cap h_j^1=\emptyset$ for $i\neq j$ and that ∂D_0^{n+1} bounds an $(n+2)$-disk in $\partial V - \alpha_1\cup\cdots\cup\alpha_m$. Then we call $\{\alpha_i\}$ a *system of standard curves*, or simply *standard*, on ∂V. Let Δ_0 be a proper

2-disk in $N(\partial D_0^{n+1}; \partial V)$ such that Δ_0 intersects ∂D_0^{n+1} at only one point, then we call Δ_0 a *meridian disk* of ∂D_0^{n+1} in ∂V and $\alpha_0=\partial\Delta_0$ a *meridian* of ∂D_0^{n+1} in ∂V, where we give an orientation to α_0.

Let u_i be a simple closed curve in $\partial V-\partial D_0^{n+1}$ for $1\leqq i\leqq m$ such that there exists an ambient isotopy of ∂V which carries u_i to α_i for all i. Then we add m 2-handles $h_1^2, \cdots, h_m^2$ to V along $u_1, \cdots, u_m$ such that $h_i^2\cap D_0^{n+1}=\emptyset$ for each i. By the handle cancelling theorem, h_i^2 cancels h_i^1 for each i, i.e. $V\cup h_1^2\cup\cdots\cup h_m^2$ is an $(n+3)$-disk D^{n+3}. In general, D_0^{n+1} is not unknotted in D^{n+3}, so we rewrite D_0^{n+1} in D^{n+3} as L^{n+1}. We say that the pair (D^{n+3}, L^{n+1}) is *of S-type*.

Let Δ_{0i}, for $1\leqq i\leqq m$, be mutually disjoint meridian disks of ∂D_0^{n+1} in ∂V, and γ_i a band in ∂V compatible with α_i and $\alpha_{0i}=\partial\Delta_{0i}$ such that

(1) $\gamma_i(B^1\times I)\cap\gamma_j(B^1\times I)=\emptyset$ for $i\neq j$, and

(2) $\gamma_i(B^1\times I)\cap N(\partial D_0^{n+1}; \partial V)=\gamma_i(B^1\times\{0\})$ for $1\leqq i\leqq m$.

Then there exists an ambient isotpoy of ∂V which carries v_i to α_i for $1\leqq i\leqq m$, where $v_i=\mathscr{F}(\alpha_i, \alpha_{0i}; \gamma_i)$ for each i. Thus the $(n+3)$-manifold obtained from V by adding m 2-handles with v_i, for $1\leqq i\leqq m$, as the attaching spheres is an $(n+3)$-disk which contains D_0^{n+1} as a proper $(n+1)$-disk, then this disk pair is said to be *of S^*-type*. Clearly, a disk pair of S^*-type is of S-type.

1.6. Let C_0 be a bouquet of $m+1$ 1-spheres $e_0^1, e_1^1, \cdots, e_m^1$. Let z_i be the element of $\pi_1(C_0)$ represented by e_i^1 for $0\leqq i\leqq m$. By C denote the 2-dimensional cell complex obtained from C_0 by attaching 2-cells $e_1^2, \cdots, e_m^2$ such that ∂e_i^2 is an element $w_i=w_i(z_0, z_1, \cdots, z_m)$ of $\pi_1(C_0)$ with $w_i(1, z_1, \cdots, z_m)=z_i$ for $1\leqq i\leqq m$. Then we call C a *cell complex of S-type*.

In 1.5, $\mathrm{cl}(V-N(D_0^{n+1}; V))$ has a 1-dimensional spine. Hence, by the assumption on the attaching spheres u_i of h_i^2, we have the following:

Proposition 1.7. *Let (D^{n+3}, L^{n+1}) be an $(n+1)$-disk pair of S-type for $n\geqq 1$. Then $\mathrm{cl}(D^{n+3}-N(L^{n+1}; D^{n+3}))$ collapses to a cell complex of S-type.*

1.8. Under the notation in 1.5, for a closed curve c in $\partial V-\partial D_0^{n+1}$, we can choose an element $w\in\pi_1(\partial V-\partial D_0^{n+1})$ such that, by choosing an arc l in $\partial V-\partial D_0^{n+1}$ spanning c and a base point, w is represented by $c\cup l$. Then we say that w is *represented by c*. We remark that the choice of $w\in\pi_1(\partial V-\partial D_0^{n+1})$, represented by c, depends on the choice of l. But, in this paper, our argument does not depend on the choice of l. Let $w\in\pi_1(\partial V-\partial D_0^{n+1})$ be represented by two simple closed curves c_1 and c_2 in $\partial V-D\partial_0^{n+1}$. If $n\geqq 2$, then there exists an ambient isotopy of ∂V which carries c_1 to c_2 and keeps ∂D_0^{n+1} fixed, but this is false for $n=1$.

2. Ribbon disks and disk pairs of S-type

Lemma 2.1. *Let $w=w(z_0, z_1, \cdots, z_m)$ be a word in F, the free group on*

$z_0, z_1, \cdots, z_m$. *Then* $w(1, z_1, \cdots, z_m) = z_i$ *in F if and only if there exist a word* t_j *in F and an integer* ε_j *such that*

$$w = (\prod_j (t_j z_0 t_j^{-1})^{\varepsilon_j}) z_i .$$

Proof. The sufficiency is trivial. To prove the necessity, suppose $w(1, z_1, \cdots, z_m) = z_i$. Then there exists a word w_j in F which does not contain the letter z_0 for $1 \leqq j \leqq r$ such that

$$w = w_1 z_0^{\varepsilon_1} w_2 z_0^{\varepsilon_2} \cdots w_r z_0^{\varepsilon_r} w_{r+1} \text{ and}$$
$$w_1 w_2 \cdots w_r w_{r+1} = z_i \text{ in } F$$

where ε_j is an integer for $1 \leqq j \leqq r$. Let $t_j = w_1 w_2 \cdots w_j$, then it is trivial that the required result holds.

Lemma 2.2. *Let* D_0^{n+1}, V, α_i *and* u_i *be as in* 1.5. *Then there exist mutually disjoint meridian disks* Δ_{0ij} *of* ∂D_0^{n+1} *in* ∂V, *and a band* γ_{ij} *in* ∂V *compatible with* α_i *and* $\tilde{\alpha}_{ij} = \partial \Delta_{0ij}$ *for* $1 \leqq i \leqq m$ *and* $1 \leqq j \leqq r(i)$ *such that*

(1) $\gamma_{ij}(B^1 \times I) \cap \gamma_{kl}(B^1 \times I) = \emptyset$ *if* $(i, j) \neq (k, l)$,

(2) $\gamma_{ij}(B^1 \times I) \cap N(\partial D_0^{n+1}; \partial V) = \gamma_{ij}(B^1 \times \{0\})$,

(3) $\gamma_{ij}(B^1 \times I) \cap \alpha_k = \emptyset$ *if* $i \neq k$, *and*

(4) *there exists an ambient isotopy of* ∂V *which keeps* ∂D_0^{n+1} *fixed and carries* u_i *to the simple closed curve*

$$\mathcal{F}(\alpha_i, \tilde{\alpha}_{i1}, \cdots, \tilde{\alpha}_{ir(i)}; \gamma_{i1}, \cdots, \gamma_{ir(i)})$$

for $1 \leqq i \leqq m$. (See Fig. 2.)

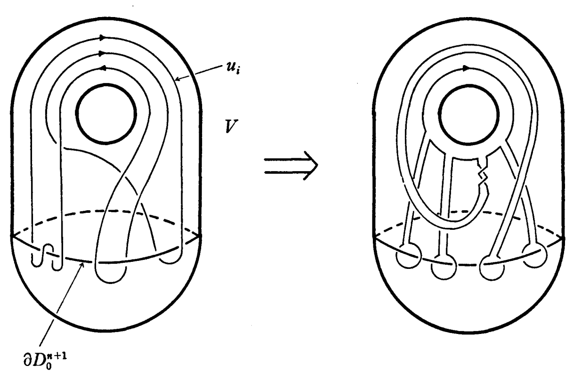

Fig. 2

Proof of Lemma 2.2. For $n=1$, the assertion is easily shown by the modification as in Fig. 3.

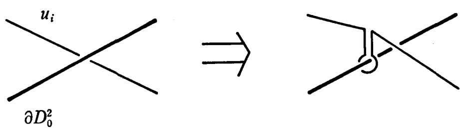

Fig. 3

Suppose $n \geqq 2$. Let $F=\pi_1(\partial V-\partial D_0^{n+1})$, then F is the free group on $z_0, z_1, \cdots, z_m$, where z_i is represented by α_i for $0 \leqq i \leqq m$. Let $w_i=w_i(z_0, z_1, \cdots, z_m)$ be an element in F represented by u_i for $1 \leqq i \leqq m$. Then $w_i(1, z_1, \cdots, z_m)=z_i$ for each i. By Lemma 2.1, there exist a word t_{ij} in F and an integer ε_{ij} for $1 \leqq i \leqq m$ and $1 \leqq j \leqq r(i)$ such that

$$w_i = (\prod_j (t_{ij} z_0 t_{ij}^{-1})^{\varepsilon_{ij}}) z_i \,.$$

We note that w_i is represented by a simple closed curve $\tilde{u}_i$ on $\partial V-\partial D_0^{n+1}$ of the form

$$\mathcal{F}(\alpha_i, \tilde{\alpha}_{i1}, \cdots, \tilde{\alpha}_{ir(i)}; \gamma_{i1}, \cdots, \gamma_{ir(i)})$$

where $\tilde{\alpha}_{ij}$ and γ_{ij} satisfy the required conditions (1), (2) and (3). By 1.8, there exists an ambient isotopy of ∂V which keeps ∂D_0^{n+1} fixed and carries u_i to $\tilde{u}_i$ for all i. This completes the proof.

Using Lemma 2.2, we have the following Proposition 2.3:

Proposition 2.3. *For $n \geqq 1$, an $(n+1)$-disk pair of S-type is of S^*-type.* (The authors should like to thank Prof. F. Hosokawa for pointing out a simpler proof than their original one.)

Proof. Let (D^{n+3}, L^{n+1}) be an $(n+1)$-disk pair of S-type constructed in 1.5. We will use the notations D_0^{n+1}, V, α_i, u_i and h_i^2 in 1.5, and notation in Lemma 2.2. By Lemma 2.2, we may assume that the attaching sphere u_i of h_i^2 is

$$\mathcal{F}(\alpha_i, \tilde{\alpha}_{i1}, \cdots, \tilde{\alpha}_{ir(i)}; \gamma_{i1}, \cdots, \gamma_{ir(i)})$$

for $1 \leqq i \leqq m$. If $r(i)=1$ for all i, then there is nothing to prove. Hence we assume $r(i) \geqq 2$ for some i. The 2-handle h_i^2 can be regarded as an embedding of $B^2 \times B^{n+1}$ in D^{n+3} such that

$$h_i^2(B^2 \times B^{n+1}) \cap V = h_i^2(\partial B^2 \times B^{n+1}) = N(u_i; \partial V-\partial D_0^{n+1}) \,.$$

For some $q \in \operatorname{int} B^{n+1}$, we may assume $h_i^2(\partial B^2 \times q)=u_i$. Then we can define an embedding g_i: $B^2 \to D^{n+3}$ by $g_i(x)=h_i^2(x, q)$ for $x \in B^2$. The number of connected components of $\alpha_i - \cup \{\gamma_{ij}(B^1 \times \{1\}); 1 \leqq j \leqq r(i)\}$ is equal to $r(i)$, and denote the connected components by $U_1, \cdots, U_{r(i)}$. We take a point P_j in ∂B^2 so that $g_i(P_j) \in U_j$ for $2 \leqq j \leqq r(i)$. Then there exist mutually disjoint proper simple arcs Γ_j in B^2 such that one end point of Γ_j is P_j and the other in $g_i^{-1}(U_1)$ for $2 \leqq j \leqq r(i)$.

(See Fig. 4.) Let $W_j=N(\Gamma_j;\, B^2)$ for $2\leqq j\leqq r(i)$, then W_j is a 2-disk. We can regard $h_i^2(W_j\times B^{n+1})$ as a 1-handle on V whose core is $h_i^2(\Gamma_j\times q)$. Let $\tilde{V}$ be obtained from V by attaching 1-handles $h_i^2(W_j\times B^{n+1})$ to V for $2\leqq j\leqq r(i)$, then $\tilde{V}$ is an $(n+3)$-disk with $m+r(i)-1$ 1-handles. Obviously $\mathrm{cl}(B^2-\bigcup\{W_j;\, 2\leqq j\leqq r(i)\})$ has $r(i)$ connected components, say $\tilde{W}_1,\cdots,\tilde{W}_{r(i)}$. (See Fig. 4.) Then

$$h_i^2(B^2\times B^{n+1}) = \bigcup\{h_i^2(W_j\times B^{n+1});\, 2\leqq j\leqq r(i)\}$$
$$\cup \bigcup\{h_i^2(\tilde{W}_k\times B^{n+1});\, 1\leqq k\leqq r(i)\}\,.$$

Hence $h_i^2(\tilde{W}_k\times B^{n+1})$ can be regarded as a 2-handle on $\tilde{V}$, for $1\leqq k\leqq r(i)$, whose core is $h_i^2(\tilde{W}_k\times q)$, thus the attaching sphere is $h_i^2(\partial\tilde{W}_k\times q)$. By choosing a system of standard curves $\{c_k\}$ on $\tilde{V}$ suitably, it follows that $h_i^2(\partial\tilde{W}_k\times q)$ is $\mathcal{F}(c_k, \tilde{\alpha}_{ik};\, \gamma_{ik})$ for $1\leqq k\leqq r(i)$. (See Fig. 5.) For any i with $r(i)\geqq 2$, repeat the above. Then it follows that $(D^{n+3},\, L^{n+1})$ is of S^*-type, and this completes the proof.

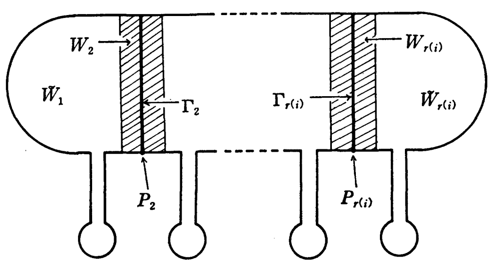

Fig. 4

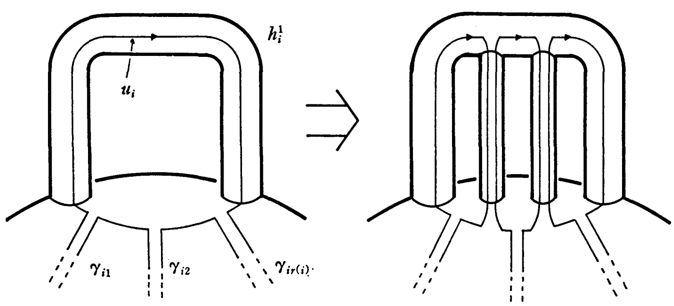

Fig. 5

Theorem 2.4. *Suppose $n\geqq 1$. Then a ribbon $(n+1)$-disk pair is of S-type, and conversely an $(n+1)$-disk pair of S-type is a ribbon disk pair.*

Proof. Suppose that $(D^{n+3},\, L^{n+1})$ is a ribbon $(n+1)$-disk pair, for $n\geqq 1$, constructed in 1.4. In order to prove that $(D^{n+3},\, L^{n+1})$ is of S-type, it suffices to show that $(D^{n+3},\, L^{n+1})$ is obtained from $(V,\, D_0^{n+1})$, as in 1.5, by adding 2-handles on V. We will find V in D^{n+3} such that L^{n+1} is unknotted in V.

Let $(D^{n+3},\, L^{n+1})$ be associated with a ribbon n-knot of type $(\beta_1,\cdots,\beta_m)$. We will use the notation in 1.2 and 1.4. Let Δ^{n+1} be an $(n+1)$-disk, then there exists an embedding $f_i: \Delta^{n+1}\times I\to D^{n+3}$, which is a collaring of D_i^{n+1} in D^{n+3}, i.e. $f_i(\Delta^{n+1}\times I)\cap L^{n+1}\subset f_i(\Delta^{n+1}\times 0)=D_i^{n+1}$, $f_i(\Delta^{n+1}\times I)\cap\partial D^{n+3}=f_i(\partial\Delta^{n+1}\times I)$ and $f_i(\Delta^{n+1}\times I)\cap \mathrm{Im}\,\bar{\beta}_j=\bar{\beta}_j(B^n\times 0\times I)$ for $0\leqq i\leqq m$, $1\leqq j\leqq m$. Let N_i be a regular neighbourhood of $f_i(\Delta^{n+1}\times\{1\})$ in D^{n+3} for $0\leqq i\leqq m$ such that $N_i\cap f_i(\Delta^{n+1}\times I)=f_i(\Delta^{n+1}\times[1/2, 1])$ and $N_i\cap\bar{\beta}_j(B^n\times I\times I)=\emptyset$ for $1\leqq j\leqq m$. We note that there exists a homeomorphism g_i: $\Delta^{n+1}\times D^2\to N_i$ for each i. Let $V=\mathrm{cl}(D^{n+3}-\bigcup\{N_i;\, 0\leqq i\leqq m\})$, then V is homeomorphic to an $(n+3)$-disk with (m+1) 1-handles. Remark that $\{g_i(p\times\partial D^2);\, 0\leqq i\leqq m\}$ is a system of standard curves on V, where $p\in\mathrm{int}\, D^{n+1}$. Then D^{n+3} is obtained from V by adding 2-handles $\{N_i\}$ with the attahcing spheres $\{g_i(p\times\partial D^2)\}$. Let $U=\bigcup\{f_i(\Delta^{n+1}\times[0,1/2]);\, 0\leqq i\leqq m\}\cup\bigcup\{\bar{\beta}_j(B^n\times I\times I);\, 1\leqq j\leqq m\}$, then U is an $(n+2)$-disk in V such that $L^{n+1}\subset\partial U$ and $\mathrm{cl}(\partial U-L^{n+1})$ is an $(n+1)$-disk in ∂V. This implies that L^{n+1} is unknotted in V, hence (D^{n+3}, L^{n+1}) is of S-type.

Let (D^{n+3}, L^{n+1}) be an $(n+1)$-disk pair of S-type. By Proposition 2.3, we may assume that (D^{n+3}, L^{n+1}) is of S^*-type. Suppose that (D^{n+3}, L^{n+1}) is constructed as in 1.5, and we will use the notation in 1.5, i.e. D^{n+3} is obtained from V by adding 2-handles with the attaching spheres $v_i=\mathscr{F}(\alpha_i, \alpha_{0i};\gamma_i)$. Then we can "pull back" v_i along the band γ_i until v_i is deformed to coincide with α_i.

The 1-handle h_i^1, as in 1.5, is homeomorphic to $B^{n+2}\times I$, and we write $h_i^1=(B^{n+2}\times I)_i$ for convenience. We may assume that $h_i^1\cap N(\alpha_i;V)=(B_+^{n+2}\times I)_i$ and $\alpha_i\cap(B_+^{n+2}\times\{1/2\})_i$ is one point for $1\leqq i\leqq m$, where B_+^{n+2} is an $(n+2)$-disk in B^{n+2}. Without loss of generality, we can assume that the band γ_i attaches to α_i in a regular neighbourhood, in α_i, of $(B_+^{n+2}\times\{1/2\})_i\cap\alpha_i$, hence $\mathrm{int}\,(\gamma_i(B^1\times I)\cap\alpha_i)\supset(B_+^{n+2}\times\{1/2\})_i\cap\alpha_i$ for each i. Let θ_i: $B^n\times I\times I\to V$ be an embedding such that $\theta_i(B^n\times I\times I)\cap\partial V=\theta_i(B^n\times I\times 0)$, $\theta_i(B^n\times I\times I)\cap D_0^{n+1}=\theta_i(B^n\times 0\times I)$ and $\theta_i(B^n\times I\times I)\cap(B_+^{n+2}\times\{1/2\})_i=\theta_i(B^n\times\{1\}\times I)\subset(\partial B_+^{n+2}\times\{1/2\})_i$ for $1\leqq i\leqq m$. Let $D_i^{n+1}=\mathrm{cl}((\partial B^{n+2}\times\{1/2\})_i\cap\mathrm{int}\, V)$ for $1\leqq i\leqq m$. By choosing θ_i suitably, we can deform D_0^{n+1} by an ambient isotopy $\{\varphi_t\}$ of V, which is a "pull back" of v_i along the band γ_i, such that φ_0 is the identity map of V and $\varphi_1(D_0^{n+1})$ is

$$(\bigcup\{D_i^{n+1};\, 0\leqq i\leqq m\}-\bigcup\{\theta_j(B^n\times\partial I\times I);\, 1\leqq j\leqq m\})$$
$$\cup\bigcup\{\theta_j(\partial B^n\times I\times I)\cup\theta_j(B^n\times I\times\{1\};\, 1\leqq j\leqq m\}\,.$$

Let $B_-^{n+2}=\mathrm{cl}(B^{n+2}-B_+^{n+2})$, then we can assume that $(B_-^{n+2}\times\{1/2\})_i$ does not intersect 2-handles $\{h_j^2\}$ in V, hence $D_1^{n+1},\cdots,D_m^{n+1}$ are unknotted in $D^{n+3}=V\cup h_1^2\cup\cdots\cup h_m^2$, thus $\varphi_1(D_0^{n+1})$ is a ribbon $(n+1)$-disk in D^{n+3}. This completes the proof.

A. Omae [9] proved that the boundary pair of a 3-disk pair of S-type is a ribbon 2-knot for a special case, and L.R. Hitt [1] announced that he proved that the boundary pair of an $(n+1)$-disk pair of some type is a ribbon n-knot and the converse.

By Proposition 1.7, Lemma 2.1, the proof of Lemma 2.2 and Theorem 2.4, we have the following:

Corollary 2.5. *Let (D^{n+3}, L^{n+1}) be a ribobn $(n+1)$-disk pair for $n\geqq 1$, then* $\mathrm{cl}(D^{n+3}-N(L^{n+1}; D^{n+3}))$ *collapses to a cell complex of S-type. Conversely, let C be a cell complex of S-type. Then there exists a ribbon $(n+1)$-disk pair (D^{n+3}, L^{n+1}) for $n\geqq 1$ such that C is a spine of the exterior of L^{n+1} in D^{n+3}.*

In [19], the third author proved the following Proposition 2.6, and we can give an alternative proof by using Theorem 2.4:

Proposition 2.6. *For $n\geqq 2$, every ribbon n-knot has an equatorial knot.*

Proof. Let K^n be a ribbon n-knot, and (D^{n+3}, L^{n+1}) the ribbon $(n+1)$-disk pair associated with K^n. By Theorem 2.4, (D^{n+3}, L^{n+1}) is of S-type. Hence there exists an unknotted $(n+1)$-disk D_0^{n+1} in V, an $(n+3)$-disk with m 1-handle, such that (D^{n+3}, L^{n+1}) is obtained from (V, D_0^{n+1}) by attaching m 2-handles $h_1^2, \cdots, h_m^2$ to $V-D_0^{n+1}$. (See 1.5.) We can realize V in R^{n+3} so that

(1) V is a regular neighbourhood of W in R^{n+3}, where W is a bouquet of m 1-spheres in R_0^{n+2}, and $V\cap R_0^{n+2}=N(W; R_0^{n+2})$,
(2) the pair (V, D_0^{n+1}) is symmetric with respect to R_0^{n+2}, and
(3) $D_0^{n+1}\cap R_0^{n+2}$ is an n-disk, say $\tilde{D}_0^n$, and $D_0^{n+1}\cap R_\varepsilon^{n+2}$ is an $(n+1)$-disk for $\varepsilon=\pm$.

Let $V_0=V\cap R_0^{n+2}$, then $\partial V_0\subset\partial V$. Hence we can choose a system of standard curves $\{\alpha_i\}$ on ∂V_0 so that it is also standard on ∂V. A meridian α_0 of ∂D_0^n in ∂V_0 is a meridian of ∂D_0^{n+1} in ∂V. For the attaching sphere u_i of a 2-handle h_i^2 on V, let w_i be an element of $\pi_1(\partial V-\partial D_0^{n+1})$ represented by u_i for $1\leqq i\leqq m$. Since $\pi_1(\partial V_0-\partial D_0^n)\cong\pi_1(\partial V-\partial D_0^{n+1})$ by the isomorphism induced by the inclusion, we may regard $w_i\in\pi_1(\partial V_0-\partial D_0^n)$. By Lemma 2.1 and the proof of Lemma 2.2, there exist mutually disjoint simple closed curves $\tilde{u}_1, \cdots, \tilde{u}_m$ in $\partial V_0-\partial D_0^n$ which represent $w_1, \cdots, w_m$, and an ambient isotopy of V_0 which carries $\tilde{u}_i$ to α_i for all i. By 1.8, $\tilde{u}_i$ and u_i are ambient isotopic in $\partial V-\partial D_0^{n+1}$, because $\tilde{u}_i$ and u_i represent the same element w_i in $\pi_1(\partial V-\partial D_0^{n+1})$. This means that we can choose the attaching sphere u_i of h_i^2 in $V_0=V\cap R_0^{n+2}$. Then we can realize each 2-handle h_i^2 in R^{n+3} so that it is symmetric with respect to R_0^{n+2}. Hence it follows that $V\cup\bigcup\{h_i^2; 1\leqq i\leqq m\}$ is symmetric with respect to R_0^{n+2} and

$$(\partial(V\cup\bigcup\{h_i^2; 1\leqq i\leqq m\})\cap R_+^{n+3}, \partial D_0^{n+1}\cap R_+^{n+3}) = (D_+^{n+2}, L_+^n)$$

is an n-disk pair of S-type. Then $(S^{n+2}, K^n)=\mathcal{D}(D_+^{n+2}, L_+^n)$ has the equatorial knot $\partial(D_+^{n+2}, L_+^n)$. This completes the proof.

3. Asphericity of ribbon disks

In this section, we will prove that the complement of a higher dimensional ribbon disk is aspherical which is an analogy to the case of classical knots [10].

3.1. Regarding S^4 as a one point compactification of R^4, we may consider that a 2-knot is in R^4. By Proposition 2.6, we can assume that a ribbon 2-knot K^2 satisfies the followings:

(1) K^2 is symmetric with respect to R_0^3,

(2) $K^2 \cap R_+^4$ has elliptic critical points only in R_2^3, and

(3) $K^2 \cap R_+^4$ has hyperbolic critical points only in R_1^3 (Fig. 6).

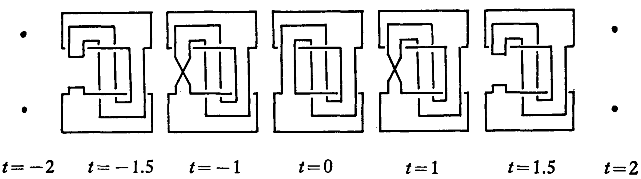

Fig. 6

Deforming the above description, it is easily seen that K^2 can be described as follows:

(1) all elliptic critical points occur at R_2^3 or R_{-2}^3, and

(2) all hyperbolic critical points occur at R_0^3 (Fig. 7).

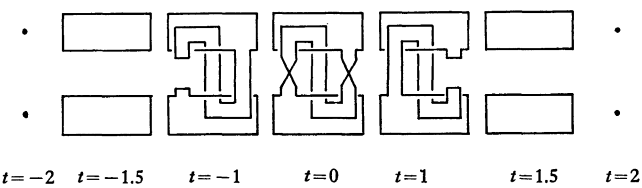

Fig. 7

The latter description of a 2-knot is called a *splitting* by S.J. Lomonaco [7], then using this splitting, he has stated the following in the proof of Theorem 3.2 in [7]:

Proposition 3.2. *Let K^2 be a ribbon 2-knot of type $(\beta_1, \cdots, \beta_m)$, and d_i^3 a proper 3-disk in $N(\beta_i; S^4)$ such that the intersection of d_i^3 and $\beta_i(B^2 \times I)$ is $\beta_i(B^2 \times \{1/2\})$ and $\partial d_i^3 \subset S^4 - K^2$. Let $*$ be a base point in $S^4 - K^2$, and l_i a simple arc in $S^4 - K^2$ which spans the base point $*$ and ∂d_i^3, then we denote by $[\partial d_i^3]$ the element of $\pi_2(S^4 - K^2) = \pi_2(S^4 - K^2, *)$ represented by $l_i \cup \partial d_i^3$. Then $\pi_2(S^4 - K^2)$ is generated by $[\partial d_1^3], \cdots, [\partial d_m^3]$ as a $Z\pi_1$-module, where $Z\pi_1$ is the integral group ring of $\pi_1 = \pi_1(S^4 - K^2)$.* (See Fig. 8.)

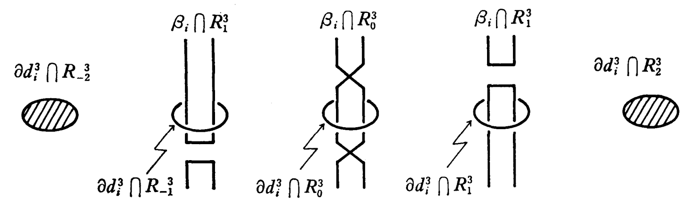

Fig. 8

The following Proposition 3.3 has been proved by the third author [20]:

Proposition 3.3. *Let (D^{n+3}, L^{n+1}) be the ribbon $(n+1)$-disk pair associated with a ribbon n-knot K^n, and $(S^{n+3}, K^{n+1})=\mathcal{D}(D^{n+3}, L^{n+1})$. If $n\geqq 2$, then $\pi_1(S^{n+2}-K^n)\cong\pi_1(D^{n+3}-L^{n+1})\cong\pi_1(S^{n+3}-K^{n+1})$.*

Lemma 3.4. *Let (D^{n+3}, L^{n+1}) be the ribbon $(n+1)$-disk pair associated with a ribbon n-knot K^n. If $n\geqq 2$, then the inclusion from $S^{n+2}-K^n$ into $D^{n+3}-L^{n+1}$ induces an onto-homomorphism $\pi_2(S^{n+2}-K^n)\to\pi_2(D^{n+3}-L^{n+1})$ as $Z\pi_1$-modules.*

Proof. Let N, T be regular neighbourhoods of L^{n+1} in D^{n+3} and K^n in S^{n+2} respectively, then in order to prove Lemma 3.4 it suffices to show the surjectivity of $\pi_2(S^{n+2}-\text{int } T)\to\pi_2(D^{n+3}-\text{int } N)$.

Let Σ^2 be a 2-dimensional polyhedron in $D^{n+3}-\text{int } N$. By Theorem 2.4, $D^{n+3}-\text{int } N$ consists of 0-, 1- and 2-handles. By the general position arguments, we can assume that Σ^2 does not intersect the cores of 0-, 1- and 2-handles. This implies that Σ^2 is in the boundary collar of $D^{n+3}-\text{int } N$. Hence we can move Σ^2 homotopically into $\partial(D^{n+3}-\text{int } N)$, and we denote the image of Σ^2 in $\partial(D^{n+3}-\text{int } N)$ by the same symbol Σ^2. Note that $\partial(D^{n+3}-\text{int } N)=(S^{n+2}-\text{int } T)\bigcup_f B^{n+1}\times S^1$, where f is an identifying map of $\partial(B^{n+1}\times S^1)$ and $\partial(S^{n+2}-\text{int } T)=\partial T$. Again by the general position arguments, Σ^2 does not intersect $p\times S^1$ in $\partial(D^{n+3}-\text{int } N)$, where $p\in \text{int } B^{n+1}$, thus we can push Σ^2 into $S^{n+2}-\text{int } T$. This fact and Proposition 3.3 follow the required result. This completes the proof of Lemma 3.4.

Lemma 3.5. *For a ribbon 3-disk pair (D^5, L^3), we have $\pi_2(D^5-L^3)=0$.*

Proof. Let (D^5, L^3) be associated with a ribbon 2-knot of type $(\beta_1, \cdots, \beta_m)$. Then we will use the notation in 1.4 for $n=2$ and in Proposition 3.2. The 2-sphere ∂d_i^3 bounds the 3-disk $(\partial d_i^3\times[0, 3/4])\cup(d_i^3\times\{3/4\})$ in D^5-L^3 for each i. It follows from this and Lemma 3.4 that $\pi_2(D^5-L^3)=0$.

The following Theorem 3.6 is a generalization of [18]:

Theorem 3.6. *Let (D^{n+3}, L^{n+1}) be a ribbon $(n+1)$-disk pair with $n\geqq 1$, then $D^{n+3}-L^{n+1}$ is aspherical.*

Proof. By Corollary 2.5, there exists a cell complex C of S-type such that $D^{n+3}-L^{n+1}$ is homotopy equivalent to C. Again by Corollary 2.5, there exists a ribbon 3-disk pair (D^5, L^3) such that D^5-L^3 is homotopy equivalent to C. It follows from Lemma 3.5 that $\pi_2(C)=0$. Let $\tilde{C}$ be the universal covering space of C. Then $H_i(\tilde{C})=0$ for $i\geqq 3$, since $\tilde{C}$ is 2-dimensional. Thus, by Hurewicz theorem, $\tilde{C}$ is aspherical, because $\pi_2(\tilde{C})\cong\pi_2(C)=0$. Therefore C is aspherical. This completes the proof.

REMARK 3.7. A cell complex of S-type is a subcomplex of a contractible 2-complex, and it follows from the proof of Theorem 3.6 that a cell complex of S-type is aspherical. This gives a partial answer to a problem of J.H.C. Whitehead: Is any subcomplex of an aspherical 2-complex aspherical?

Corollary 3.8. *Let K^n be a ribbon n-knot for $n\geqq 1$, then $\pi_1(S^{n+2}-K^n)$ has no element of finite order.*

Proof. For $n=1$, the assertion is a special case of [10]. For $n\geqq 2$, this is true by Proposition 3.3, Theorem 3.6 and a result due essentially to P.A. Smith (p. 216 in [3]), namely: The fundamental group of an aspherical polyhedron of finite dimenion has no element of finite order.

T. Yajima characterized the knot groups of ribbon 2-knots in [16], then by Colrollary 3.8 and [16] we have the following:

Corollary 3.9. *Let G be a finitely presented group having a Wirtinger presentation cf deficiency 1 with $G/G'\cong Z$. Then G has no element of finite order.*

4. Unknotting ribbon knots

Theorem 4.1. *Let K^n be a ribbon n-knot for $n\geqq 3$, then we have $\pi_i(S^{n+2}-K^n)=0$ for $2\leqq i\leqq n-1$.*

Proof. By Proposition 2.6, there exists a ribbon n-disk pair (D^{n+2}, L^n) such that $\mathcal{D}(D^{n+2}, L^n)=(S^{n+2}, K^n)$. Let $(D^{n+2}_\varepsilon, L^n_\varepsilon)$ be a copy of (D^{n+2}, L^n) for $\varepsilon=\pm$, then $\mathcal{D}(D^{n+2}, L^n)$ is obtained from the disjoint union of (D^{n+2}_+, L^n_+) and (D^{n+2}_-, L^n_-) by identifying their boundaries via the identity map. Let $(S^{n+1}, K^{n-1}_0)=\partial(D^{n+2}_+, L^n_+)$, i.e. K^{n-1}_0 is an equatorial knot of K^n. Let $\tilde{X}$ be the universal covering space of $S^{n+2}-K^n$, $\tilde{X}_\varepsilon$ the lift of $D^{n+2}_\varepsilon-L^n_\varepsilon$ in $\tilde{X}$ for $\varepsilon=\pm$, and $\tilde{X}_0$ the lift of $S^{n+1}-K^{n-1}_0$ in $\tilde{X}$. By Proposition 3.3, all of $\tilde{X}_+$, $\tilde{X}_-$ and $\tilde{X}_0$ are also universal covering spaces. By the Mayer-Vietoris theorem, we have the following exact sequence:

$$\cdots \rightarrow H_j(\tilde{X}_+)\oplus H_j(\tilde{X}_-) \rightarrow H_j(\tilde{X}) \rightarrow H_{j-1}(\tilde{X}_0) \rightarrow H_{j-1}(\tilde{X}_+)\oplus H_{j-1}(\tilde{X}_-) \rightarrow \cdots .$$

By Theorem 3.6, $H_j(\tilde{X}_\varepsilon)=0$ for $j\geqq 1$ and $\varepsilon=\pm$. Therefore it follows that $H_j(\tilde{X})\cong H_{j-1}(\tilde{X}_0)$ for $j\geqq 2$.

Suppose $n=3$, then $\pi_2(S^5-K^3)\cong H_2(\tilde{X})\cong H_1(\tilde{X}_0)=0$. By induction on the dimension n, it is easily seen that the fact $H_j(\tilde{X})\cong H_{j-1}(\tilde{X}_0)$ and $H_1(\tilde{X}_0)=0$ implies $H_i(\tilde{X}_0)=0$ for $1\leqq i\leqq n-1$, and this implies the required result.

4.2. Addendum to Theorem 4.1. From the proof of Theorem 4.1, it follows that $\pi_n(S^{n+2}-K^n)\cong\pi_{n-1}(S^{n+1}-K^{n-1}_0)$ for $n\geqq 3$. Concerning $\pi_n(S^{n+2}-K^n)$ for a ribbon n-knot K^n with $n\geqq 3$, we can conclude the similar result to that in Proposition 3.2.

The following Proposition 4.3 is due to A. Kawauchi ([4] or p. 331 in [14]):

Proposition 4.3. *For a 2-knot K^2, S^4-K^2 is homotopy equivalent to S^1 if and only if $\pi_1(S^4-K^2)\cong Z$.*

Theorem 4.4. *Let K^n be a ribbon n-knot for $n\geqq 3$. If $\pi_1(S^{n+2}-K^n)\cong Z$, then K^n is unknotted.*

Proof. We can use the notation in the proof of Theorem 4.1. Note that, in the proof of Theorem 4.1, we have $H_j(\tilde{X})\cong H_{j-1}(\tilde{X}_0)$ for $j\geqq 2$.

Suppose $n=3$ and $\pi_1(S^5-K^3)\cong Z$, then by Proposition 3.3 it follows that $\pi_1(S^4-K_0^2)\cong Z$, where K_0^2 is an equatorial knot of K^3. By Proposition 4.3, we have $H_i(\tilde{X}_0)=0$ for all $i\geqq 1$. It follows from this that $H_j(\tilde{X})=0$ for $j\geqq 1$. Therefore S^5-K^3 is homotopy equivalent to S^1, hence by [6], [12] and [15], K^3 is unknotted. Similarly, for $n\geqq 4$, it is easy to see that the assertion is true by induction on the dimension n. This completes the proof.

Recently A. Kawauchi and T. Matumoto [5] have obtained independently the same result as Theorem 4.4.

The following is obtained by Proposition 3.3 and Theorem 4.4:

Corollary 4.5. *Let K^n be a ribbon n-knot for $n\geqq 4$, then any equatorial knot of K^n is unknotted if K^n is unknotted.*

For $n=2$, Corollary 4.5 is false. For example, Kinoshita-Terasaka knot is an equatorial knot of the unknot [8]. The case $n=3$ still remains open.

FINAL REMARK. In 1.2, 1.4 and 1.5, we defined a ribbon knot, a ribbon disk pair and a disk pair of S-type. It is easy to generalize our definition of ribbon knots to the case of links, i.e. *ribbon links*. Then the same generalizations are possible for ribbon disk pairs and "of S-type". In this generalized case, Theorems 2.4 and 3.6 remain valid.

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