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# ON THE JACOBI DIFFERENTIAL OPERATORS ASSOCIATED TO MINIMAL ISOMETRIC IMMERSIONS OF SYMMETRIC SPACES INTO SPHERES I

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Contents



## Introduction

Let $F: M \to \bar{M}$ be a minimal isometric immersion of a compact Riemannian manifold $M$. For a variation $\{F_t\}$ of $F$ the second variation of the volume $V(t)$ of $F_t(M)$ is described by a differential operator $\tilde{S}$, called the Jacobi differential operator, on the normal bundle as

$$\left.\frac{d^2V(t)}{dt^2}\right|_{t=0} = \int_M \langle \tilde{S}(E^N), E^N \rangle dx,$$

where $E^N$ denotes the infinitesimal normal variation of $\{F_t\}$ (see section 1). The Jacobi differential operator $\tilde{S}$ is self-adjoint and strongly elliptic. Therefore the index and the nullity of $F$ are obtained from the spectra of $\tilde{S}$. Here the index and the nullity are defined as those of the Hessian at $F$ of the volume integral on the space of immersions of $M$ into $\bar{M}$ modulo diffeomorphisms of $M$. For the study of minimal isometric immersions it seems to be important to study $\tilde{S}$ and its spectra. However there have been few studies on these problems except for the recent works of Hasegawa and others. Hasegawa [4] studies the spectral geometry of minimal submanifolds.

Let $M$ be a compact symmetric space, $\bar{M}$ a unit sphere, and $F$ an equivariant

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minimal isometric immersion. Under this situation we study the Jacobi differential operator $\tilde{S}$, applying the representation theory of compact Lie groups. In section 1 we recall some results on minimal isometric immersions. In section 2 we study equivariant isometric immersions of compact homogeneous spaces and their Killing nullities (see Hsiang and Lawson [6] p. 14 for Killing nullities). In section 3 we study equivariant minimal isometric immersions of compact symmetric spaces into unit spheres. And we compute the Jacobi differential operator $\tilde{S}$ in this case (Theorem 1). In section 4, recalling some results on invariant differential operators, we give some propositions, which give criterions in order that our operator $\tilde{S}$ reduces to the Casimir operator. In section 5 the problem of computing the spectra of $\tilde{S}$ is reduced to the eigenvalue problems for certain linear mappings $S_\sigma$ of finite dimensional vector spaces (Theorem 3).

In the forthcoming papers we shall study the linear mappings $S_\sigma$ in detail under certain conditions, and study the index and the nullity of minimally immersed spheres into spheres.

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## 1. Preliminaries

1.1. Let $(M, g)$ be an $n$-dimensional compact connected Riemannian manifold without boundary, and $(\bar{M}, \bar{g})$ an $m$-dimensional Riemannian manifold. Let $F: M \to \bar{M}$ be an isometric immersion of $M$ into $\bar{M}$. We consider the tangent space $T_x(M)$ of $M$ at $x \in M$ as a vector subspace of the tangent space $T_{F(x)}(\bar{M})$ of $\bar{M}$ at $F(x) \in \bar{M}$. We denote by $N_x(M)$ the orthogonal complement of $T_x(M)$ in $T_{F(x)}(\bar{M})$, which is called the *normal space* of the immersed submanifold $M$ of $\bar{M}$ at $x$. Let $T(M)$ (resp. $T(\bar{M})$) be the tangent bundle of $M$(resp. of $\bar{M}$). We denote by $T(\bar{M})|_M$ the bundle induced by $F$ from $T(\bar{M})$. The bundle $N(M) = \bigcup_{x \in M} N_x(M)$ is called the *normal bundle* of $M$. We denote by $\mathfrak{X}(M)$ (resp. $\Gamma(N(M))$) the space of all $C^\infty$ cross-sections of $T(M)$ (resp. of $N(M)$).

Let $B: T_x(M) \times T_x(M) \to N_x(M)$ be the second fundamental form of $M$, and $A: N_x(M) \times T_x(M) \to T_x(M)$ the Weingarten form of $M$. The second fundamental form $B$ is a symmetric bilinear mapping and $A_v$, $v \in N_x(M)$, is a self-adjoint linear mapping of $T_x(M)$. Let $\nabla$(resp. $\bar{\nabla}$) be the Riemannian connection of $M$(resp. $\bar{M}$). Let $D$ be the normal connection of $M$. For any vector fields $X$, $Y \in \mathfrak{X}(M)$ and for any normal vector field $\xi \in \Gamma(N(M))$, we have the following equations (cf. Kobayashi and Nomizu [7] Vol. II Chap. 7 section 3):

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y), \tag{1.1.1}$$

(1.1.2) $\bar{\nabla}_X \xi = -A_\xi X + D_X \xi\,,$

(1.1.3) $\bar{g}(\xi, B(X, Y)) = g(A_\xi X, Y)\,.$

We denote by $H$ the mean curvature of $M$. Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis of $T_x(M)$. Then we have

$$H_x = \sum_{i=1}^{n} B(e_i, e_i)\,.$$

The isometric immersion $F: M \to \bar{M}$ is said to be *minimal*, if the mean curvature $H$ of $M$ vanishes identically.

1.2. Let $\bar{R}$ be the curvature tensor of $\bar{M}$. For $x \in M$ we define linear mappings $\tilde{A}$ and $\tilde{R}$ of $N_x(M)$ as follows:

(1.2.1) $\tilde{A}(v) = \sum_{i,j=1}^{n} \bar{g}(v, B(e_i, e_j)) B(e_i, e_j)\,,$

(1.2.2) $\tilde{R}(v) = \sum_{i=1}^{n} (\bar{R}(e_i, v) e_i)^N \qquad \text{for } v \in N_x(M)\,,$

where $\{e_1, \cdots, e_n\}$ is an orthonormal basis of $T_x(M)$ and $(\bar{R}(*, *)*)^N$ denotes the normal component of $\bar{R}(*, *)*$. The linear mappings $\tilde{A}$ and $\tilde{R}$ are independent of the choice of an orthonormal basis.

If $\bar{M}$ is a space of constant sectional curvature $k$, we have for any vector fields $X$, $Y$ and $Z$ on $\bar{M}$ (cf. Kobayashi and Nomizu [7] Vol. I p. 203):

$$\bar{R}(X, Y)Z = k(\bar{g}(Z, Y)X - \bar{g}(Z, X)Y)\,.$$

Therefore we have

(1.2.3) $\tilde{R}(v) = -nkv \qquad \text{for } v \in N_x(M)\,.$

We denote by $\Delta$ the Laplace operator on $N(M)$ (cf. Simons [10] p. 64). Let $\{E_1, \cdots, E_n\}$ be an orthonormal local basis of $T(M)$ on a neighborhood of $x \in M$. Then we have

(1.2.4) $\Delta f(x) = \sum_{i=1}^{n} (D_{E_i} D_{E_i} f)(x) - \sum_{i=1}^{n} (D_{\nabla_{E_i} E_i} f)(x) \qquad \text{for } f \in \Gamma(N(M))\,.$

We define a differential operator $\tilde{S}$, called the *Jacobi differential operator*, on $N(M)$ as follows:

(1.2.5) $\tilde{S} = -\Delta - \tilde{A} + \tilde{R}\,.$

Let $I$ be an open interval containing $0 \in \boldsymbol{R}$. A 1-parameter family $\{F_t\}_{t \in I}$ of immersions of $M$ into $\bar{M}$ is called a *variation* of $F$, if $F = F_0$ and if the mapping $f: I \times M \to \bar{M}$, defined by $f(t, x) = F_t(x)$, is differentiable. The *variation vector field* $E$ of the variation $\{F_t\}_{t \in I}$ is defined by

$$E_x = df\left(\left(\frac{\partial}{\partial t}\right)_{(0,x)}\right).$$

**Proposition 1.2.1** (cf. Simons [10] p. 73). *Let $F: M \to \bar{M}$ be a minimal isometric immersion, $\{F_t\}_{t \in I}$ a variation of $F$, and $E$ the variation vector field of $\{F_t\}$. We denote by $V(t)$ the volume of $M$ with respect to the Riemannian metric induced by the immersion $F_t$. Let $E^N$ be the normal component of $E$, which is a cross-section of $N(M)$. Then we have*

$$(1.2.6) \qquad \left.\frac{d^2V(t)}{dt^2}\right|_{t=0} = \int_M g(\tilde{S}(E^N), E^N)dx,$$

*where $dx$ is the Riemannian measure of $(M, g)$.*

The vector space $\Gamma(N(M))$ is a pre-Hilbert space with the inner product $(\ ,\ )$:

$$(f, f') = \int_M g(f, f')dx \qquad \text{for } f, f' \in \Gamma(N(M)).$$

We denote by $L^2(N(M))$ the completion of $\Gamma(N(M))$. We consider $\Gamma(N(M))$ as a linear subspace of $L^2(N(M))$. The Jacobi differential operator $\tilde{S}$ is a self-adjoint strongly elliptic operator on $\Gamma(N(M))$. Therefore we have

**Proposition 1.2.2** (cf. Simons [10] p. 74). (1) *The Jacobi differential operator $\tilde{S}$ is diagonalizable in the sense that there exists a complete orthonormal system $\{e_\alpha\}_{\alpha \in A}$ of $L^2(N(M))$ such that each $e_\alpha$ is contained in $\Gamma(N(M))$ and that each $e_\alpha$ is an eigenvector of $\tilde{S}$.*

(2) *Each eigenspace of $\tilde{S}$ is finite dimensional. Let*

$$\lambda_1 < \lambda_2 < \cdots < \lambda_i < \cdots$$

*be the eigenvalues of $\tilde{S}$. Then the sequence $\{\lambda_i\}_{i=1,2\cdots}$ is divergent to $\infty$.*

REMARK 1.2.1. By Proposition 1.2.2 the spectra of $\tilde{S}$ acting on $\Gamma(N(M))$ coincide with ones of $\tilde{S}$ acting on $\Gamma(N(M))^C$, the complexification of $\Gamma(N(M))$.

We define a bilinear form $I(\ ,\ )$ on $\Gamma(N(M))$ as follows:

$$I(V, W) = \int_M g(\tilde{S}(V), W)dx \qquad \text{for } V, W \in \Gamma(N(M)).$$

The *index* and the *nullity* of $F$ are those of the bilinear form $I(\ ,\ )$. By Proposition 1.2.1 and 1.2.2 the index of $F$ is the sum of the dimensions of the eigenspaces corresponding to negative eigenvalues of $\tilde{S}$, and the nullity of $F$ is the dimension of the 0-eigenspace of $\tilde{S}$.

## 2. Equivariant isometric immersions

2.1. In section 2 we assume the followings. Let $G$ be a compact connected Lie group, and $K$ a closed subgroup of $G$. Let $\mathfrak{g}$ be the Lie algebra of $G$, and $\mathfrak{k}$ the Lie subalgebra of $\mathfrak{g}$ corresponding to the Lie subgroup $K$. Let $\langle\ ,\ \rangle$ be an $Ad(G)$-invariant inner product on $\mathfrak{g}$. Then we have an orthogonal decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$, where $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$. We denote by $M$ the quotient space $G/K$. We canonically identify $\mathfrak{p}$ with the tangent space $T_o(M)$ of $M$ at $o=\pi(e)$, where $\pi$ is the natural projection of $G$ onto $M=G/K$. We also denote by $\langle\ ,\ \rangle$ the $G$-invariant Riemannian metric on $M$ which coincides with the inner product $\langle\ ,\ \rangle$ on $\mathfrak{p}=T_o(M)$. Let $F\colon (M, c\langle\ ,\ \rangle)\to\bar{M}$ be an isometric immersion for some $c>0$ which is equivariant in the following sense: There exists a Lie group homomorphism $\rho$ of $G$ into $I(\bar{M})$, the group of all isometries of $\bar{M}$, such that $F(x(yK))=\rho(x)F(yK)$ for $x, y\in G$. We also denote by $\langle\ ,\ \rangle$ the Riemannian metric on $\bar{M}$. Moreover we assume that the image $F(M)$ of $M$ does not coincide with $\bar{M}$.

We define an action $\sigma$ of $G$ on $\Gamma(N(M))$ by

$$(\sigma(x)\tilde{f})(yK)=d(\rho(x))\tilde{f}(x^{-1}y) \qquad \text{for } \tilde{f}\in\Gamma(N(M)) \text{ and } x, y\in G,$$

where $d(\rho(x))$ denotes the differential of the isometry $\rho(x)$. We define an action of $G$ on $\Gamma(T(\bar{M})|_M)$ in the same way as for $\Gamma(N(M))$, where $\Gamma(T(\bar{M})|_M)$ is the space of all $C^\infty$ cross-sections of $T(\bar{M})|_M$. We also denote by $\sigma$ the action of $G$ on $\Gamma(T(\bar{M})|_M)$. Then we have by the equivariance of $F$

$$\begin{cases} \Delta\circ\sigma(x)=\sigma(x)\circ\Delta\,, \\ \tilde{A}\circ\sigma(x)=\sigma(x)\circ\tilde{A}\,, \\ \tilde{R}\circ\sigma(x)=\sigma(x)\circ\tilde{R}\,. \end{cases}$$

Therefore we have

$$\tilde{S}\circ\sigma(x)=\sigma(x)\circ\tilde{S}. \tag{2.1.1}$$

Moreover if $F$ is minimal, each eigenspace of $\tilde{S}$ is $G$-invariant.

Put $U=N_o(M)$. Then $K$ acts on $U$ by the differential of $\rho(k)$, $k\in K$, at $F(o)$. We denote by $\phi$ this action of $K$ on $U$. We denote by $E$ the vector bundle $G\times_K U$ associated with $G$ by $\phi$. Put

$$C^\infty(G;\, U)_K=\left\{f\colon G\to U \quad C^\infty \text{ mapping}; f(xk)=\phi(k)^{-1}f(x) \text{ for } x\in G \text{ and } k\in K\right\}.$$

The space $\Gamma(E)$ of $C^\infty$ cross-sections of $E$ is identified with $C^\infty(G;\, U)_K$ by the following correspondence:

$$C^\infty(G;\, U)_K\ni f\mapsto\tilde{f}\in\Gamma(E),\ \tilde{f}(xK)=x\circ f(x) \qquad \text{for } x\in G\,, \tag{2.1.2}$$

where $x\circ f(x)$ is the image of $(x, f(x))\in G\times U$ by the natural projection $G\times U\to$

$G\times_K U$. We define an action $L$ of $G$ on $C^\infty(G; U)_K$ as follows:

(2.1.3) $\qquad (L_x f)(y) = f(x^{-1}y) \qquad$ for $f\in C^\infty(G; U)_K$ and $x, y\in G$.

Put $V=T_{F(o)}(\bar{M})$ and $W=T_o(M)$. Then $K$ also acts on $V$ (resp. $W$) by the differential of $\rho(k)$(resp. of $k$), $k\in K$, at $F(o)$ (resp. at $o$). We denote by $J$ (resp. $H$) the associated vector bundle $G\times_K V$(resp. $G\times_K W$). We define a space $C^\infty(G; V)_K$(resp. $C^\infty(G; W)_K$) and an action $L$ of $G$ on $C^\infty(G; V)_K$(resp. on $C^\infty(G; W)_K$) in the same way. We can identify $T(\bar{M})|_M$(resp. $N(M)$ and $T(M)$) with $J$(resp. $E$ and $H$) and $\Gamma(T(\bar{M})|_M)$(resp. $\Gamma(N(M))$ and $\mathfrak{X}(M)$) with $C^\infty(G; V)_K$(resp. $C^\infty(G; U)_K$ and $C^\infty(G; W)_K$) in the following way.

**Proposition 2.1.1.** (1) *The vector bundle homomorphism*

$$\iota\colon J\to T(\bar{M})|_M,\ \iota(x\circ v) = d(\rho(x))v \qquad \textit{for } x\in G \textit{ and } v\in V,$$

*is an isomorphism, and $\iota$ induces an isomorphism of $E$(resp. $H$) onto $N(M)$ (resp. $T(M)$).*

(2) *Also denoting by $\iota$ the isomorphism of $C^\infty(G; V)_K$ onto $\Gamma(T(\bar{M})|_M)$ induced from $\iota\colon J\to T(\bar{M})|_M$, the following diagram is commutative:*

$$\begin{array}{ccc} C^\infty(G; V)_K & \xrightarrow{\ \iota\ } & \Gamma(T(\bar{M})|_M) \\ \downarrow L_x & & \downarrow \sigma(x) \\ C^\infty(G; V)_K & \xrightarrow{\ \iota\ } & \Gamma(T(\bar{M})|_M) \end{array} \qquad \textit{for } x\in G.$$

*The isomorphism $\iota\colon C^\infty(G; V)_K\to\Gamma(T(\bar{M})|_M)$ induces an isomorphism of $C^\infty(G; U)_K$ (resp. $C^\infty(G; W)_K$) onto $\Gamma(N(M))$ (resp. $\mathfrak{X}(M)$).*

For $f\in C^\infty(G; V)_K$ we denote by $\tilde{f}$ the image of $f$ by the isomorphism $\iota$.

2.2. For $x\in G$ we define a diffeomorphism $\tau_x$ of $M$ by $\tau_x(yK)=xyK$. Then $\tau_x$ is an isometry of $(M, \langle\ ,\ \rangle)$. For $X\in\mathfrak{g}$ we denote by $X^*$ the infinitesimal transformation on $M$ which generates the 1-parameter group of transformations $\tau_{\exp tX}$ on $M$. We define differential operators $\tilde{A}_0$ and $\Delta_0$ on $N(M)$ as follows:

(2.2.1) $$\tilde{A}_0(\tilde{f}) = \sum_{i=1}^{n+p} B(E_i^*, A_{\tilde{f}}E_i^*),$$

(2.2.2) $$\Delta_0(\tilde{f}) = \sum_{i=1}^{n+p} D_{E_i^*}D_{E_i^*}(\tilde{f}) \qquad \text{for } \tilde{f}\in\Gamma(N(M)),$$

where $\{E_1, \cdots, E_{n+p}\}$ is an orthonormal basis of $\mathfrak{g}$. The operators $\tilde{A}_0$ and $\Delta_0$ are independent of the choice of an orthonormal basis of $\mathfrak{g}$.

**Proposition 2.2.1.** *For the operators $\tilde{A}_0$ and $\tilde{A}$ we have the following equation:*

(2.2.3) $\quad c\tilde{A} = \tilde{A}_0\,.$

Proof. Choose an orthonormal basis $\{E_1, \cdots, E_{n+p}\}$ of $\mathfrak{g}$ with the property that $\{E_1, \cdots, E_n\}$ (resp. $\{E_{n+1}, \cdots, E_{n+p}\}$) is an orthonormal basis of $\mathfrak{p}$ (resp. $\mathfrak{k}$). Then $\left\{\frac{1}{\sqrt{c}}(E_1{}^*)_o, \cdots, \frac{1}{\sqrt{c}}(E_n{}^*)_o\right\}$ is an orthonormal basis of $T_o(M)$ and $(E_{n+1}{}^*)_o = \cdots = (E_{n+p}{}^*)_o = 0$. For $x \in G$ put $F_i = Ad(x)E_i$, $i=1, 2, \cdots, n+p$. Then $\{F_1, \cdots, F_{n+p}\}$ is an orthonormal basis of $\mathfrak{g}$, and we have

$$\begin{aligned}(F_i{}^*)_{xK} &= \left.\frac{d(\exp t(\mathrm{Ad}(x)E_i)\cdot xK)}{dt}\right|_{t=0}\\ &= \left.\frac{d(x(\exp tE_i)\cdot o)}{dt}\right|_{t=0} = d\tau_x(E_i{}^*)_o\,.\end{aligned}$$

Therefore $\left\{\frac{1}{\sqrt{c}}(F_1{}^*)_{xK}, \cdots, \frac{1}{\sqrt{c}}(F_n{}^*)_{xK}\right\}$ is an orthonormal basis of $T_{xK}(M)$ and $(F_{n+1}{}^*)_{xK} = \cdots = (F_{n+p}{}^*)_{xK} = 0$. For $v \in N_{xK}(M)$ we have

$$\begin{aligned}\tilde{A}_0(v) &= \sum_{i=1}^{n+p} B((F_i{}^*)_{xK}, A_v((F_i{}^*)_{xK}))\\ &= c\sum_{i=1}^{n} B\left(\frac{1}{\sqrt{c}}(F_i{}^*)_{xK}, A_v\left(\frac{1}{\sqrt{c}}\left((F_i{}^*)_{xK}\right)\right)\right).\end{aligned}$$

By (1.1.3) we have

$$\begin{aligned}A_v\left(\frac{1}{\sqrt{c}}(F_i{}^*)_{xK}\right) &= \sum_{j=1}^{n}\left\langle A_v\left(\frac{1}{\sqrt{c}}(F_i{}^*)_{xK}\right), \frac{1}{\sqrt{c}}(F_j{}^*)_{xK}\right\rangle \frac{1}{\sqrt{c}}(F_j{}^*)_{xK}\\ &= \sum_{j=1}^{n}\left\langle v, B\left(\frac{1}{\sqrt{c}}(F_i{}^*)_{xK}, \frac{1}{\sqrt{c}}(F_j{}^*)_{xK}\right)\right\rangle \frac{1}{\sqrt{c}}(F_j{}^*)_{xK}\,.\end{aligned}$$

Hence we have by (1.2.1)

$$\begin{aligned}\tilde{A}_0(v) &= c\sum_{i,j=1}^{n}\left\langle v, B\left(\frac{1}{\sqrt{c}}(F_i{}^*)_{xK}, \frac{1}{\sqrt{c}}(F_j{}^*)_{xK}\right)\right\rangle \times\\ &\qquad\qquad B\left(\frac{1}{\sqrt{c}}(F_i{}^*)_{xK}, \frac{1}{\sqrt{c}}(F_j{}^*)_{xK}\right)\\ &= c\tilde{A}(v)\,.\end{aligned}$$

Q.E.D.

**Proposition 2.2.2.** *If the curve $c(t) = exp\, tX\cdot o$ is a geodesic of $M$ for any $X \in \mathfrak{p}$, we have*

(2.2.4) $\quad c\Delta = \Delta_0\,.$

Proof. Fix $x \in G$ and let $\{E_1, \cdots, E_{n+p}\}$ and $\{F_1, \cdots, F_{n+p}\}$ be orthonormal bases in the proof of Proposition 2.2.1. Then we have for $\tilde{f} \in \Gamma(N(M))$

$$(2.2.5) \qquad (\Delta_0 \tilde{f})(xK) = \sum_{i=1}^{n} (D_{F_i^*} D_{F_i^*} \tilde{f})(xK) .$$

We have

$$(F_i^*)_{x(\exp sE_i)\cdot o} = \left.\frac{d\{\exp t(\mathrm{Ad}(x)E_i)\cdot(x(\exp sE_i)\cdot o)\}}{dt}\right|_{t=0}$$
$$= \left.\frac{d\{x(\exp(t+s)E_i)\cdot o\}}{dt}\right|_{t=0} .$$

Hence the curve $x(\exp tE_i)\cdot o$ is an integral curve of $F_i^*$. Since the curves $x(\exp tE_i)\cdot o$, $i=1, \cdots, n$, are geodesics, then

$$(2.2.6) \qquad \nabla_{(F_i^*)_{xK}} F_i^* = 0 .$$

Let $U$ be a normal neighborhood of $xK$. Let $X_i$, $i=1, \cdots, n$, be the vector fields on $U$ adapted to $(F_i^*)_{xK}$, i.e. $(X_i)_q = \tau_{xK}^q (F_i^*)_{xK}$, where $\tau_{xK}^q$ is the parallel translation along the unique geodesic segment in $U$ which joins $xK$ and $q$. Then there exists $\varepsilon > 0$ such that $(X_i)_{x(\exp tE_i)\cdot o} = (F_i^*)_{x(\exp tE_i)\cdot o}$ for $-\varepsilon < t < \varepsilon$. Hence $(D_{X_i}\tilde{f})(x(\exp tE_i)\cdot o) = (D_{F_i^*}\tilde{f})(x(\exp tE_i)\cdot o)$ for $\tilde{f} \in \Gamma(N(M))$ and $-\varepsilon < t < \varepsilon$. Hence we have

$$(2.2.7) \qquad (D_{X_i} D_{X_i} \tilde{f})(xK) = (D_{F_i^*} D_{F_i^*} \tilde{f})(xK) .$$

We have by (1.2.4), (2.5.5), (2.2.6) and (2.2.7)

$$(\Delta \tilde{f})(xK) = \sum_{i=1}^{n} (D_{\frac{1}{\sqrt{c}} X_i} D_{\frac{1}{\sqrt{c}} X_i} \tilde{f})(xK)$$
$$= \frac{1}{c} \sum_{i=1}^{n} (D_{X_i} D_{X_i} \tilde{f})(xK)$$
$$= \frac{1}{c} \sum_{i=1}^{n} (D_{F_i^*} D_{F_i^*} \tilde{f})(xK)$$
$$= \frac{1}{c} (\Delta_0 \tilde{f})(xK) ,$$

which proves (2.2.4). Q.E.D.

REMARK 2.2.1. Suppose that the pair $(G, K)$ is a Riemannian symmetric pair and that the inner product $\langle \ , \ \rangle$ on $\mathfrak{g}$ is invariant under the involutive automorphism of $\mathfrak{g}$ associated to the pair $(G, K)$. Then the condition of Proposition 2.2.2 is satisfied (cf. Helgason [5] pp. 174–177).

In what follows, for a Riemannian symmetric pair $(G, K)$ the inner product $\langle \ , \ \rangle$ on $\mathfrak{g}$ will be always assumed to have the above property.

2.3. In this subsection we moreover assume that the equivariant isometric immersion $F: (M, c\langle \ , \ \rangle) \to \bar{M}$ is minimal and that $\bar{M}$ is compact.

Let $E$ be a Killing vector field on $\bar{M}$ and $E^N$ the normal component of the restriction of $E$ to $M$. The dimension of the space $\{E^N; E$ is a Killing vector field on $\bar{M}\}$ is called the *Killing nullity* of $F$. We have $\tilde{S}(E^N)=0$ (Simons [10] p. 74). Hence the nullity is not less than the Killing nullity. Let $I(\bar{M}, M)$ be the group of isometries of $\bar{M}$ which leave $F(M)$ invariant. Then $I(\bar{M}, M)$ is a closed subgroup of $I(\bar{M})$. Since $\bar{M}$ is compact, the Killing nullity of $F$ is equal to $\dim I(\bar{M})/I(\bar{M}, M)$.

**Proposition 2.3.1.** *Assume that $\bar{M}$ is a compact connected Riemannian homogeneous space and that the equivariant isometric immersion $F: M \to \bar{M}$ is minimal. Then the Killing nullity of $F$ is strictly positive.*

Proof. If the Killing nullity is equal to 0, then $\dim I(\bar{M})=\dim I(\bar{M}, M)$. Since $\bar{M}$ is connected, the group $I(\bar{M}, M)$ is transitive on $\bar{M}$ (cf. Helgason [5] p. 114). Therefore we have $F(M)=I(\bar{M}, M)(F(M))=\bar{M}$, which is a contradiction. Q.E.D.

## 3. Equivariant minimal isometric immersions into spheres

3.1. In section 3 the assumptions and the notation are the same as in subsection 2.1. Moreover we assume that $V$ is a Euclidean vector space with an inner product $\langle\ ,\ \rangle$ and that $\bar{M}$ is the unit sphere $S$ of $V$ with the center 0, the origin of $V$. Since the isometric immersion $F: M \to S$ is equivariant, there exists an orthogonal representation $\rho: G \to GL(V)$ such that $\rho(k)v_0=v_0$ for any $k\in K$, where $v_0=F(o)$.

We identify the tangent space of $V$ with $V$ itself in a canonical way. Then we have $d(\rho(x))=\rho(x)$ for $x\in G$. Since the induced bundle $T(V)|_M$ is trivial, we consider $\Gamma(T(V)|_M)$, the space of all $C^\infty$ cross-sections of $T(V)|_M$, as the space of all $V$-valued $C^\infty$ functions on $M$.

Under the above identification we have an orthogonal decomposition of the tangent space $T_{v_0}(V)$ as follows:

$$T_{v_0}(V) = V^0 + V^T + V^N\,, \tag{3.1.1}$$

where $V^0=\boldsymbol{R}v_0$, $V^T=T_o(M)$ and $V^N=N_o(M)$. By Proposition 2.1.1 we have the following proposition.

**Proposition 3.1.1.** (1) *The vector bundle homomorphism*

$$\iota: G\times_K V \to T(V)|_M\,,\ \iota(x\circ v) = \rho(x)v \qquad \textit{for } x\in G \textit{ and } v\in V\,,$$

*is an isomorphism, and $\iota$ induces an isomorphism of $G\times_K V^N$ (resp. $G\times_K V^T$) onto $N(M)$ (resp. $T(M)$).*

(2) *The following diagram is commutative:*

$$\begin{array}{ccc} C^\infty(G;V)_K & \xrightarrow{\iota} & \Gamma(T(V)|_M) \\ \downarrow L_x & & \downarrow \sigma(x) \\ C^\infty(G;V)_K & \xrightarrow{\iota} & \Gamma(T(V)|_M) \end{array} \qquad \textit{for } x \in G.$$

*The isomorphism* $\iota$: $C^\infty(G; V)_K \to \Gamma(T(V)|_M)$ *induces an isomorphism of* $C^\infty(G; V^N)_K$ *(resp.* $C^\infty(G; V^T)_K$*) onto* $\Gamma(N(M))$ *(resp.* $\mathfrak{X}(M)$*).*

For $f \in C^\infty(G; V)_K$ we denote $\iota(f)$ by $\tilde{f}$. We denote by $S$ the operator of $C^\infty(G; V^N)_K$ corresponding to $\tilde{S}$ by the isomorphism $\iota$.

Let $\bar{\nabla}$ be the connection in $T(V)|_M$ induced from the flat connection in $T(V)$. Then we have for $f \in C^\infty(G; V)_K$ and a vector field $Y \in \mathfrak{X}(M)$

$$\bar{\nabla}_Y \tilde{f} = Y\tilde{f}\,, \tag{3.1.2}$$

where we consider $\tilde{f}$ as a $V$-valued function on $M$. For $X \in \mathfrak{g}$ we denote by $\hat{X}$ the right invariant vector field on $G$ such that $\hat{X}_e = X_e$, where we consider $\mathfrak{g}$ as the Lie algebra of left invariant vector fields on $G$ and $e$ is the unit element of $G$.

**Lemma 3.1.2.** *We have*

$$\bar{\nabla}_{X^*}\tilde{f} = \iota(\hat{X}f + d\rho(Ad(*^{-1})X)f) \qquad \textit{for } f \in C^\infty(G; V)_K \textit{ and } X \in \mathfrak{g}\,. \tag{3.1.3}$$

*Here* $d\rho(Ad(*^{-1})X)f$ *is the* $V$*-valued* $C^\infty$ *function defined by*

$$(d\rho(Ad(*^{-1})X)f)(x) = d\rho(Ad(x^{-1})X)f(x)\,,$$

$d\rho$ *is the differential of the homomorphism* $\rho$, *and* $X^*$ *denotes the infinitesimal transformation which generates the* 1*-parameter group of transformations* $\tau_{\exp tX}$.

Proof. Let $g$ be an element of $C^\infty(G; V)_K$ such that $\tilde{g} = \bar{\nabla}_{X^*}\tilde{f}$. By (2.1.2) and Proposition 3.1.1 we have for $f \in C^\infty(G; V)_K$ and $x \in G$

$$\tilde{f}(xK) = \iota(x \circ f(x)) = \rho(x)f(x)\,.$$

Hence we have by (3.1.2)

$$\begin{aligned} g(x) &= \rho(x)^{-1}(\bar{\nabla}_{X^*}\tilde{f})(xK) = \rho(x)^{-1}X^*_{xK}\tilde{f} \\ &= \rho(x)^{-1}\lim_{t\to 0}\frac{1}{t}(\tilde{f}((\exp tX)xX) - \tilde{f}(xK)) \\ &= \rho(x)^{-1}\lim_{t\to 0}\frac{1}{t}\{\rho((\exp tX)x)f((\exp tX)x) - \rho(x)f(x)\} \\ &= \lim_{t\to 0}\frac{1}{t}\{\rho(\exp t(\mathrm{Ad}(x^{-1})X))f((\exp tX)x) - f((\exp tX)x) \\ &\qquad + f((\exp tX)x) - f(x)\} \\ &= d\rho(\mathrm{Ad}(x^{-1})X)f(x) + (\hat{X}f)(x)\,. \end{aligned}$$

This proves the lemma. Q.E.D.

REMARK 3.1.1. Since left translations of $G$ are commutative with right translations of $G$, we have $\hat{X}f \in C^\infty(G; V)_K$. Therefore we have $d\rho(\mathrm{Ad}(*^{-1})X)f \in C^\infty(G; V)_K$.

**Lemma 3.1.3.** (1) *We have for* $X \in \mathfrak{g}$ *and* $f \in C^\infty(G; V^N)_K$

$$D_{X^*}\tilde{f} = \iota(\hat{X}f + \{d\rho(\mathrm{Ad}(*^{-1})X)f\}^N)\,, \tag{3.1.4}$$

$$-A_{\tilde{f}}X^* = \iota(\{d\rho(\mathrm{Ad}(*^{-1})X)f\}^T)\,, \tag{3.1.5}$$

*where we denote by* $g^N$ *(resp.* $g^T$*) the* $V^N$*-component (resp.* $V^T$*-component) of* $g \in C^\infty(G; V)_K$ *with respect to the decomposition* (3.1.1).

(2) *We have for* $X \in \mathfrak{g}$ *and* $f \in C^\infty(G; V^T)_K$

$$B(X^*, \tilde{f}) = \iota(\{d\rho(\mathrm{Ad}(*^{-1})X)f\}^N)\,. \tag{3.1.6}$$

Proof. The lemma is an easy consequence of (1.1.1), (1.1.2), Proposition 3.1.1 and Lemma 3.1.2. Q.E.D.

For the differential operators $\tilde{A}_0$ and $\Delta_0$ defined in subsection 2.2, we obtain the following two propositions.

**Proposition 3.1.4.** *We have for* $f \in C^\infty(G; V^N)_K$

$$\tilde{A}_0(\tilde{f}) = \iota\,(-\sum_{i=1}^{n+p} \{d\rho(E_i)(d\rho(E_i)f)^T\}^N)\,, \tag{3.1.7}$$

*where* $\{E_1, \cdots, E_{n+p}\}$ *is an orthonormal basis of* $\mathfrak{g}$.

Proof. Applying Lemma 3.1.3, we have

$$\begin{aligned}\tilde{A}_0(\tilde{f}) &= \sum_{i=1}^{n+p} B(E_i^*, A_{\tilde{f}}E_i^*)\\ &= \sum_{i=1}^{n+p} \iota(-(d\rho(\mathrm{Ad}(*^{-1})E_i)\{d\rho(\mathrm{Ad}(*^{-1})E_i)f\}^T)^N)\,.\end{aligned}$$

Put $\mathrm{Ad}(x)E_i = \sum_{i=1}^{n+p} a^j{}_i(x)E_j$ for $x \in G$. Then $(a^i{}_j(x))_{i,j=1,\cdots,n+p}$ is an orthogonal matrix. We have for $x \in G$

$$\begin{aligned}&\sum_{i=1}^{n+p} (d\rho(\mathrm{Ad}(*^{-1})E_i)\{d\rho(\mathrm{Ad}(*^{-1})E_i)f\}^T)^N(x)\\ &= \sum_{i=1}^{n+p} (d\rho(\mathrm{Ad}(x^{-1})E_i)\{d\rho(\mathrm{Ad}(x^{-1})E_i)f(x)\}^T)^N\\ &= \sum_{j,k=1}^{n+p} (\sum_{i=1}^{n+p} a^j{}_i(x^{-1})a^k{}_i(x^{-1})\{d\rho(E_j)(d\rho(E_k)f(x))^T\}^N)\\ &= \sum_{j=1}^{n+p} \{d\rho(E_j)(d\rho(E_j)f)^T\}^N(x)\,.\end{aligned}$$

Therefore

$$\tilde{A}_0(\tilde{f}) = \iota(-\sum_{i=1}^{n+p} \{d\rho(E_i)(d\rho(E_i)f)^T\}^N) .$$

Q.E.D.

**Proposition 3.1.5.** *We have for* $f \in C^\infty(G; V^N)_K$

$$\Delta_0 \tilde{f} = \iota(\sum_{i=1}^{n+p} E_i E_i f + 2\sum_{i=1}^{n+p} (d\rho(E_i)(E_i f))^N + \sum_{i=1}^{n+p} \{d\rho(E_i)(d\rho(E_i)f)^N\}^N) , \tag{3.1.8}$$

*where* $\{E_1, \cdots, E_{n+p}\}$ *is an orthonormal basis of* $\mathfrak{g}$.

Proof. Applying Lemma 3.1.3, we have

$$\begin{aligned}\Delta_0 \tilde{f} &= \sum_{i=1}^{n+p} D_{E_i*} D_{E_i*} \tilde{f} \\ &= \iota(\sum_{i=1}^{n+p} (\hat{E}_i(\hat{E}_i f + \{d\rho(\mathrm{Ad}(*^{-1})E_i f)\}^N) \\ &\qquad + \{d\rho(\mathrm{Ad}(*^{-1})E_i)(\hat{E}_i f + \{d\rho(\mathrm{Ad}(*^{-1})(E_i)f\}^N)\}^N)) \\ &= \iota(\sum_{i=1}^{n+p} \hat{E}_i \hat{E}_i f + \sum_{i=1}^{n+p} \hat{E}_i \{d\rho(\mathrm{Ad}(*^{-1})E_i)f\}^N \\ &\qquad + \sum_{i=1}^{n+p} \{d\rho(\mathrm{Ad}(*^{-1})E_i)(\hat{E}_i f)\}^N \\ &\qquad + \sum_{i=1}^{n+p} \{d\rho(\mathrm{Ad}(*^{-1})E_i)\{d\rho(\mathrm{Ad})*^{-1})E_i)f\}^N\}^N) .\end{aligned}$$

We have (cf. Takeuchi [12] p. 51)

$$\sum_{i=1}^{n+p} \hat{E}_i \hat{E}_i = \sum_{i=1}^{n+p} E_i E_i . \tag{3.1.9}$$

Put $\mathrm{Ad}(x)E_i = \sum_{j=1}^{n+p} a^j{}_i(x) E_j$. Then we have for $x \in G$

$$\begin{aligned}(\hat{E}_i)_x &= dr_x(E_i)_e = dl_x(dl_{x^{-1}} dr_x(E_i)_e) \\ &= dl_x(\mathrm{Ad}(x^{-1})E_i)_e \\ &= \sum_{i=1}^{n+p} a^j{}_i(x^{-1}) dl_x(E_j)_e \\ &= \sum_{i=1}^{n+p} a^i{}_j(x)(E_j)_x ,\end{aligned} \tag{3.1.10}$$

where $r_x$(resp. $l_x$) denotes the right translation (resp. left translation) by $x \in G$. We obtain

$$\{d\rho(\mathrm{Ad}(*^{-1})E_i)f\}(x) = d\rho(\mathrm{Ad}(x^{-1})E_i)f(x) \tag{3.1.11}$$

$$= \sum_{j=1}^{n+p} a^j{}_i(x^{-1}) d\rho(E_j) f(x)$$

$$= \sum_{j=1}^{n+p} a^i{}_j(x) d\rho(E_j) f(x) .$$

By (3.1.11) and (3.1.10) we have

$$\sum_{i=1}^{n+p} \hat{E}_i \{d\rho(\mathrm{Ad}(*^{-1})E_i)f\}^N$$

$$= \sum_{i,j=1}^{n+p} ((\hat{E}_i a^i{}_j)(d\rho(E_j)f)^N + a^i{}_j \{d\rho(E_j)(\hat{E}_i f)\}^N)$$

$$= \sum_{i,j=1}^{n+p} (\hat{E}_i a^i{}_j)(d\rho(E_j)f)^N + \sum_{j,k=1}^{n+p} \sum_{i=1}^{n+p} a^i{}_j a^i{}_k \{d\rho(E_j)(E_k f)\}^N$$

$$= \sum_{i,j=1}^{n+p} (\hat{E}_i a^i{}_j)(d\rho(E_j)f)^N + \sum_{j=1}^{n+p} \{d\rho(E_j)(E_j f)\}^N .$$

Since the inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ is $\mathrm{Ad}(G)$-invariant, we have

$$\begin{aligned}(\hat{E}_i a^i{}_j)(x) &= \lim_{t\to 0} \frac{1}{t} (\langle \mathrm{Ad}((\exp tE_i)x)E_j,\ E_i\rangle - \langle \mathrm{Ad}(x)E_j,\ E_i\rangle) \\ &= \lim_{t\to 0} \frac{1}{t} \langle \mathrm{Ad}(\exp tE_i)\,\mathrm{Ad}(x)E_j - \mathrm{Ad}(x)E_j,\ E_i\rangle \\ &= \langle \mathrm{ad}(E_i)\mathrm{Ad}(x)E_j,\ E_i\rangle \\ &= -\langle \mathrm{Ad}(x)E_j,\ \mathrm{ad}(E_i)E_i\rangle = 0\end{aligned}$$

Therefore we obtain

$$\sum_{i=1}^{n+p} \hat{E}_i \{d\rho(\mathrm{Ad}(*^{-1})E_i)f\}^N = \sum_{i=1}^{n+p} \{d\rho(E_i)(E_i f)\}^N . \tag{3.1.12}$$

We have by (3.1.10) and (3.1.11)

$$\sum_{i=1}^{n+p} \{d\rho(\mathrm{Ad}(*^{-1})E_i)(\hat{E}_i f)\}^N \tag{3.1.13}$$

$$= \sum_{j,k=1}^{n+p} \sum_{i=1}^{n+p} a^i{}_j a^i{}_k \{d\rho(E_j)(E_k f)\}^N$$

$$= \sum_{j=1}^{n+p} \{d\rho(E_j)(E_j f)\}^N .$$

We have by (3.1.11)

$$\sum_{i=1}^{n+p} \{d\rho(\mathrm{Ad}(*^{-1})E_i) \{d\rho(\mathrm{Ad}(*^{-1})E_i)f\}^N\}^N \tag{3.1.14}$$

$$= \sum_{i=1}^{n+p} \{\sum_{j=1}^{n+p} a^i{}_j d\rho(E_j) \{\sum_{k=1}^{n+p} a^i{}_k d\rho(E_k) f\}^N\}^N$$

$$= \sum_{j,k=1}^{n+p} \sum_{i=1}^{n+p} a^i{}_j a^i{}_k \{d\rho(E_j)(d\rho(E_k)f)^N\}^N$$

$$= \sum_{j=1}^{n+p} \{d\rho(E_j)(d\rho(E_j)f)^N\}^N .$$

We obtain (3.1.8) by (3.1.9), (3.1.12), (3.1.13) and (3.1.14). Q.E.D.

3.2. In the rest of this section we moreover assume that the equivariant isometric immersion $F$: $(M, c\langle\ ,\ \rangle) \to S$ is minimal. Let $\Delta_M$ be the Laplace operator of the Riemannian manifold $(M, \langle\ ,\ \rangle)$ acting on functions. Then we have (cf. Wallach [13] p. 20)

$$\Delta_M = \sum_{i=1}^{n+p} (E_i{}^*)^2 ,$$

where $\{E_1, \cdots, E_{n+p}\}$ is an orthonormal basis of $\mathfrak{g}$. Hence the Laplace operator $\Delta_M(c)$ of $(M, c\langle\ ,\ \rangle)$ is given by the following equation:

$$\text{(3.2.1)} \qquad \Delta_M(c) = \frac{1}{c} \sum_{i=1}^{n+p} (E_i{}^*)^2 .$$

Let $\{e_1, \cdots, e_N\}$ be an orthonormal basis of $V$ and $(x_1, \cdots, x_N)$ the coordinate system on $V$ with respect to $\{e_1, \cdots, e_N\}$. Put $F=(f_1, \cdots, f_N)$, i.e. $f_i(xK) = \langle e_i, F(xK)\rangle$. Then it is known (Takahashi [11] p. 383) that

$$\text{(3.2.2)} \qquad \Delta_M(c)f_i = -nf_i , \qquad i=1, \cdots, N .$$

We define an action $L$ of $G$ on $C^\infty(M)$, the space of $C^\infty$ functions on $M$, as follows:

$$(L_x f)(yK) = f(x^{-1}yK) \qquad \text{for } x, y \in G \text{ and } f \in C^\infty(M) .$$

**Proposition 3.2.1.** *Let $\rho$: $G \to GL(V)$ be an orthogonal representation of $G$. Let $F$: $(M, c\langle\ ,\ \rangle) \to S$, $F(xK) = \rho(x)F(o)$, be an equivariant miniaml isometric immersion. If $F$ is full, i.e. if the image $F(M)$ of $M$ is not contained in any great spheres, then the following equation holds*:

$$\text{(3.2.3)} \qquad \sum_{i=1}^{n+p} d\rho(E_i)d\rho(E_i) = -nc1_V ,$$

*where $1_V$ denotes the identity transformation of $V$.*

Proof. Let $\{e_1, \cdots, e_N\}$ be an orthonormal basis of $V$ and put $F=(f_1, \cdots, f_N)$ with respect to this basis. We define a linear mapping $\phi$: $V \to C^\infty(M)$ by $\phi(v)(xK) = \langle v, F(xK)\rangle$ for $v \in V$ and $x \in G$. Then the subspace $\phi(V)$ of $C^\infty(M)$ is spanned by $f_1, \cdots, f_N$. We have for $x, y \in G$ and $v \in V$

$$\begin{aligned} \phi(\rho(x)v)(yK) &= \langle \rho(x)v, F(yK)\rangle = \langle v, \rho(x^{-1})F(yK)\rangle \\ &= \langle v, F(x^{-1}yK)\rangle = \phi(v)(x^{-1}yK) \\ &= (L_x\phi(v))(yK) . \end{aligned}$$

Hence $\phi$ is a $G$-module homomorphism. Let $\psi$: $G\to GL(\phi(V))$ be a representation defined by $\psi(x)=L_x|_{\phi(V)}$. Then we have for $X\in\mathfrak{g}$

(3.2.4) $\quad d\psi(X)=-X^*$ .

We assert that $\dim\phi(V)=N$. If the assertion is not true, there exist real numbers $c_1, \cdots, c_N$, which are not all equal to zero, such that $\sum_{i=1}^{N} c_i f_i=0$. Then the image $F(M)$ is contained in the hyperplane $\sum_{i=1}^{N} c_i x_i=0$, which is a contradiction. Therefore $\phi$: $V\to\phi(V)$ is a $G$-module isomorphism. It follows from (3.2.4), (3.2.1) and (3.2.2) that

$$\begin{aligned}\sum_{i=1}^{n+p} d\psi(E_i)d\psi(E_i)f_k &= \sum_{i=1}^{n+p} E_i^*E_i^*f_k \\ &= c\Delta_M(c)f_k = -ncf_k .\end{aligned}$$

Hence we have $\sum_{i=1}^{n+p} d\psi(E_i)d\psi(E_i)=nc1_{\phi(V)}$, where $1_{\phi(V)}$ denotes the identity transformation of $\phi(V)$. Since $\phi$: $V\to\phi(V)$ is a $G$-module isomorphism, we have

$$\sum_{i=1}^{n+p} d\rho(E_i)d\rho(E_i)=-nc1_V .$$

Q.E.D.

REMARK 3.2.1. Suppose that the linear isotropy representation of $G/K$ is irreducible. Let $\rho$: $G\to GL(V)$ be a real spherical representation of $(G, K)$, i.e. $\rho$ is an irreducible orthogonal representation of $G$ such that there is a unit vector $v\in V$ with the property that $\rho(k)v=v$ for any $k\in K$. Then we can construct a full equivariant minimal isometric immersion of $M=G/K$ in the following way. Let $S$ be the unit sphere of $V$ with the center 0. Define a mapping $F$: $M\to S$ by $F(xK)=\rho(x)v$ for $x\in G$. Then there exists a positive number $c$ such that $F$: $(M, c\langle\ ,\ \rangle)\to S$ is a minimal isometric immersion (cf. Wallach [13] p. 21).

Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$. We denote by $\mathfrak{g}^C$ the complexification of $\mathfrak{g}$. For a linear subspace $\mathfrak{u}$ of $\mathfrak{g}$ we denote by $\mathfrak{u}^C$ the complex linear subspace of $\mathfrak{g}^C$ generated by $\mathfrak{u}$. Let $\mathfrak{r}$ be the root system of $\mathfrak{g}^C$ with respect to $\mathfrak{t}$. A non-zero element $\lambda\in\mathfrak{t}$ is a root, if and only if there exists a non-zero element $X\in\mathfrak{g}^C$ such that $[H, X]=\sqrt{-1}\langle\lambda, H\rangle X$ for any $H\in\mathfrak{t}$. Choosing a linear order in $\mathfrak{t}$, we denote by $\mathfrak{r}^+$ the set of all positive roots. Put $\delta=\frac{1}{2}\sum_{\lambda\in\mathfrak{r}^+}\lambda$.

Let $(G, K)$ be a Riemannian symmetric pair and $D(G, K)$ the set of all equivalence classes of complex spherical representations of $(G, K)$. Recall that an irreducible complex representation $\phi$: $G\to GL(W)$ is called a complex spherical representation of $(G, K)$, if there exists a non-zero vector $w\in W$ such that

$\phi(k)w=w$ for any $k\in K$. For a complex irreducible representation $\phi: G\to GL(W)$, we denote by $[\phi]$ the equivalence class to which $\phi$ belongs. For $[\phi]\in D(G, K)$ we denote by $\mathfrak{o}_{[\phi]}(M)$ the subspace of $C^\infty(M)^C$ generated by $G$-submodules of $C^\infty(M)^C$ which are isomorphic to $\phi$, where $C^\infty(M)^C$ is the complexification of $C^\infty(M)$ (We will not distinguish $G$-modules and representations of $G$). Then $\mathfrak{o}_{[\phi]}(M)$ is isomorphic to $\phi$ as $G$-module and the Laplace operator $\Delta_M$ acts on $\mathfrak{o}_{[\phi]}(M)$ as a scalar operator $c_{[\phi]}$. The scalar $c_{[\phi]}$ is given by $-\langle\Lambda+2\delta, \Lambda\rangle$, where $\Lambda$ is the highest weight of $\phi$ (cf. Takeuchi [12] p. 20, p. 207).

If the Riemannian symmetric pair $(G, K)$ is of rank 1, there exists a dominant integral form $\Lambda_0$ such that the highest weight $\Lambda$ of each complex spherical representation $\phi$ is given by $\Lambda=k\Lambda_0$ for some non-negative integer $k$ (cf. Takeuchi [12] p. 166). Hence the scalar $c_{[\phi]}$ is given by $-\langle k\Lambda_0+2\delta, k\Lambda_0\rangle = -(k^2\langle\Lambda_0, \Lambda_0\rangle+2k\langle\delta, \Lambda_0\rangle)$. Since both $\langle\Lambda_0, \Lambda_0\rangle$ and $\langle\delta, \Lambda_0\rangle$ are positive, it follows that $c_{[\phi]}\neq c_{[\phi']}$ for $[\phi]$, $[\phi']\in D(G, K)$ with $[\phi]\neq[\phi']$. Therefore we have the following lemma.

**Lemma 3.2.2.** *If $(G, K)$ is a Riemannian symmetric pair of rank 1, then each eigenspace of the Laplace operator $\Delta_M$ acting on $C^\infty(M)^C$ is irreducible.*

**Proposition 3.2.3.** *Assume that $(G, K)$ is a Riemannian symmetric pair of rank 1. Let $\rho: G\to GL(V)$ be an orthogonal representation and the mapping $F: (M, c\langle\ ,\ \rangle)\to S$, $F(xK)=\rho(x)F(o)$, an equivariant minimal isometric immersion. If $F$ is full, the complexification $\rho: G\to GL(V^C)$ of $\rho$ is irreducible. Therefore $\rho: G\to GL(V)$ is irreducible.*

Proof. Put $F=(f_1\cdots, f_N)$ as in the proof of Proposition 3.2.1. We also denote by $\langle\ ,\ \rangle$ the Hermitian inner product on $V^C$ which is the extension of the inner product $\langle\ ,\ \rangle$ on $V$. Let $\phi: V^C\to C^\infty(M)^C$ be the $\boldsymbol{C}$-linear mapping defined by $\phi(v)(xK)=\langle v, F(xK)\rangle$ for $v\in V^C$ and $x\in G$. We assert that $\{f_1, \cdots, f_N\}$ is linear independent over $\boldsymbol{C}$. If the assertion is not true, there exist complex numbers $c_1, \cdots, c_N$, which are not all equal to zero, such that $\sum_{i=1}^N c_i f_i=0$. Put $c_i=a_i+\sqrt{-1}\,b_i$, where $a_i$ and $b_i$ are real numbers. Then at least one of the equations $\sum_{i=1}^N a_i x_i=0$ and $\sum_{i=1}^N b_i x_i=0$ defines a hyperplane. Since every $f_i$ is real valued, the image $F(M)$ is contained in this hyperplane. This is a contradiction. Hence by the proof of Proposition 3.2.1 we have that $\phi: V^C\to\phi(V^C)$ is a $G$-module isomorphism and that $\Delta_M f=-ncf$ for $f\in\phi(V^C)$. Therefore it follows from Lemma 3.2.2 that $\phi(V^C)$ is an irreducible $G$-module. Hence $\rho: G\to GL(V^C)$ is irreducible. Q.E.D.

REMARK 3.2.2. Assume that $(G, K)$ is a Riemannian symmetric pair of rank 1. Then full equivariant minimal isometric immersions of $M=G/K$ into spheres are in one-to-one correspondence with complex spherical representations of $(G, K)$. In fact a complex spherical representation of $(G, K)$ corresponds to a full equivariant minimal isometric immersion $F: (M, c\langle\ ,\ \rangle) \to S$ by Proposition 3.2.3. Conversely since $(G, K)$ is of rank 1, every zonal spherical function is real-valued (Do Carmo and Wallach [3] p. 98). Therefore every complex spherical representation of $(G, K)$ is the complexification of a real spherical representation of $(G, K)$. Hence a full equivariant minimal isometric immersion corresponds to a complex spherical representation of $(G, K)$ (Remark 3.2.1).

3.3. In this subsection we assume that $(G, K)$ is a Riemannian symmetric pair.

**Theorem 1.** *Let $\rho: G \to GL(V)$ be an orthogonal representation and $F$: $(M, c\langle\ ,\ \rangle) \to S$, $F(xK) = \rho(x)F(o)$, a full equivariant minimal isometric immersion. Then we have for $f \in C^\infty(G; V^N)_K$*

$$Sf = -\frac{1}{c}\Big(\sum_{i=1}^{n+p} E_i E_i f - 2c_\rho f + 2\sum_{i=1}^{n+p}\{d\rho(E_i)(E_i f)\}^N + 2\sum_{i=1}^{n+p}\{d\rho(E_i)(d\rho(E_i)f)^N\}^N\Big), \tag{3.3.1}$$

*where $c_\rho = -nc$ and $\{E_1, \cdots, E_{n+p}\}$ is an orthonormal basis of* $\mathfrak{g}$.

Proof. Since the condition of Proposition 2.2.2 is satisfied (Remark 2.2.1), it follows from (1.2.5), (1.2.3), (2.2.3) and (2.2.4) that $\tilde{S} = -\frac{1}{c}(\Delta_0 + \tilde{A}_0 + nc1_{\Gamma(N(M))})$, where $1_{\Gamma(N(M))}$ is the identity transformation of $\Gamma(N(M))$. Hence we have by (3.1.7) and (3.1.8)

$$\tilde{S}\tilde{f} = \iota\Big(-\frac{1}{c}\Big(\sum_{i=1}^{n+p} E_i E_i f + 2\sum_{i=1}^{n+p}(d\rho(E_i)(E_i f))^N + \sum_{i=1}^{n+p}\{d\rho(E_i)(d\rho(E_i)f)^N\}^N - \sum_{i=1}^{n+p}\{d\rho(E_i)(d\rho(E_i)f)^T\}^N - c_\rho f\Big)\Big).$$

Applying (3.2.3), we have

$$\begin{aligned}\sum_{i=1}^{n+p}\{d\rho(E_i)(d\rho(E_i)f)^T\}^N &= \sum_{i=1}^{n+p}\{d\rho(E_i)(d\rho(E_i)f)\}^N - \sum_{i=1}^{n+p}\{d\rho(E_i)(d\rho(E_i)f)^N\}^N \\ &\qquad - \sum_{i=1}^{n+p}\{d\rho(E_i)(d\rho(E_i)f)^0\}^N \\ &= -ncf - \sum_{i=1}^{n+p}\{d\rho(E_i)(d\rho(E_i)f)^N\}^N - \sum_{i=1}^{n+p}\{d\rho(E_i)(d\rho(E_i)f)^0\}^N.\end{aligned}$$

In the above equation $(d\rho(E_i)f)^0$ denotes the $V^0$-component of $d\rho(E_i)f$ with respect to the orthogonal decomposition (3.1.1). Since $d\rho(\mathfrak{g})v_0=V^T$, we have $\sum_{i=1}^{n+p}\{d\rho(E_i)(d\rho(E_i)f)^0\}^N=0$. Hence we have

$$Sf=-\frac{1}{c}\Big(\sum_{i=1}^{n+p}E_iE_if+2\sum_{i=1}^{n+p}\{d\rho(E_i)(E_if)\}^N+2\sum_{i=1}^{n+p}\{d\rho(E_i)(d\rho(E_i)f)^N\}^N-2c_\rho f\Big).$$

Q.E.D.

REMARK 3.3.1. It follows from Remark 3.1.1, (3.1.9), (3.1.12) and (3.1.14) that $\sum_{i=1}^{n+p}E_iE_if$, $\sum_{i=1}^{n+p}\{d\rho(E_i)(E_if)\}^N$, $\sum_{i=1}^{n+p}\{d\rho(E_i)(d\rho(E_i)f)^N\}^N\in C^\infty(G;\,V^N)_K$ for $f\in C^\infty(G;\,V^N)_K$. Moreover each of the above three operators is commutative with $L_x$ for all $x\in G$.

We define an operator $S_1$: $C^\infty(G;\,V^N)_K\to C^\infty(G;\,V^N)_K$ by

$$S_1f=\sum_{i=1}^{n+p}\{d\rho(E_i)(E_if)\}^N+\sum_{i=1}^{n+p}\{d\rho(E_i)(d\rho(E_i)f)^N\}^N \quad \text{for } f\in C^\infty(G;\,V^N)_K.$$

By Proposition 3.1.1 the operator $S_1$ corresponds to a first order differential operator on $N(M)$. We denote by $\tilde{S}_1$ the corresponding differential operator on $N(M)$. If $S_1=0$, the operator $S$ reduces to the simple operator

$$-\frac{1}{c}\Big(\sum_{i=1}^{n+p}E_iE_i-2c_\rho 1_{C^\infty(G;\,V^N)_K}\Big),$$

where $1_{C^\infty(G;\,V^N)_K}$ is the identity transformation of $C^\infty(G;\,V^N)_K$. The following lemma provides a sufficient condition for $S_1=0$. In fact this condition is also necessary (see Proposition 4.2.2).

**Lemma 3.3.1.** *If $(d\rho(X)v)^N=0$ for $X\in\mathfrak{p}$ and $v\in V^N$, then we have $S_1=0$.*

Proof. Choose an orthonormal basis $\{E_1,\cdots,E_{n+p}\}$ of $\mathfrak{g}$ such that $\{E_1,\cdots,E_n\}$ (resp. $\{E_{n+1},\cdots,E_{n+p}\}$) is an orthonormal basis of $\mathfrak{p}$ (resp. of $\mathfrak{k}$). We have for $x\in G$, $f\in C^\infty(G;\,V^N)_K$ and $E_i$, $i=n+1,\cdots,n+p$,

$$\begin{aligned}(E_if)(x)&=\lim_{t\to0}\frac{1}{t}(f(x(\exp tE_i))-f(x))\\&=\lim_{t\to0}\frac{1}{t}(\rho(\exp-tE_i)f(x)-f(x))\\&=-d\rho(E_i)f(x).\end{aligned}$$

Hence

$$S_1 f = \sum_{i=1}^{n+p} \{d\rho(E_i)(E_i f)\}^N + \sum_{i=1}^{n+p} \{d\rho(E_i)(d\rho(E_i)f)^N\}^N$$
$$= \sum_{i=1}^{n} \{d\rho(E_i)(E_i f)\}^N - \sum_{i=n+1}^{n+p} \{d\rho(E_i)(d\rho(E_i)f)\}^N$$
$$+ \sum_{i=1}^{n} \{d\rho(E_i)(d\rho(E_i)f)^N\}^N + \sum_{i=n+1}^{n+p} \{d\rho(E_i)(d\rho(E_i)f)^N\}^N .$$

Since $V^N$ is invariant under $\rho(k)$ for $k \in K$, we have $(d\rho(E_i)f)^N = d\rho(E_i)f$, $i = n+1, \cdots, n+p$. Therefore we have

$$S_1 f = \sum_{i=1}^{n} \{d\rho(E_i)(E_i f)\}^N + \sum_{i=1}^{n} \{d\rho(E_i)(d\rho(E_i)f)^N\}^N .$$

Thus we obtain the proposition. Q.E.D.

REMARK 3.3.2. In the following cases the operator $S_1$ vanishes.

(1) The case of the minimal isometric immersion of $S^n$ induced from the representation $\rho_2$, which is defined as follows: When $(G, K) = (SO(n+1), SO(n))$, the highest weight $\phi_1$ of the canonical representation of $SO(n+1)$ has the property of $\Lambda_0$ in the proof of Lemma 3.2.2. Our representation $\rho_2$ is the real spherical representation whose complexification has the highest weight $2\phi_1$ (Remark 3.2.2).

(2) The cases of minimal symmetric $R$-spaces (see Nagura [8]), which include (1) as a special case.

3.4. Let $N$ be a connected Riemannian manifold and $\tilde{N}$ the universal Riemannian covering manifold of $N$. Then we have by the universal property

**Lemma 3.4.1.** *For each isometry $x \in I(N)$ there exists an isometry $\tilde{x} \in I(\tilde{N})$ such that $\pi \circ \tilde{x} = x \circ \pi$, where $\pi: \tilde{N} \to N$ is the covering map.*

In this subsection we assume that $G$ acts on $M$ almost effectively. This means that $\mathfrak{k}$ does not contain any trivial ideals of $\mathfrak{g}$.

**Proposition 3.4.2.** *Let $\tilde{M}$ be the universal Riemannian covering manifold of $M$. If the equivariant minimal isometric immersion $F: (M, c\langle\ ,\ \rangle) \to S$, $F(xK) = \rho(x)F(o)$, is full and if dim $G$ = dim $I(\tilde{M})$, then the Killing nullity of $F$ is equal to $\frac{m(m-1)}{2} - \dim G$. Here $m = \dim V$.*

Proof. Let $I^0(S, M)$ be the identity component of $I(S, M)$. By the argument in subsection 2.3 it is sufficient to show that $\dim I^0(S, M) = \dim G$. It is trivial that $I^0(S, M)$ contains $\rho(G)$. Put $K' = \{x \in G;\ \rho(x)F(o) = F(o)\}$. Since $F$ is an immersion, dim $K'$ = dim $K$ and hence the Lie algebra of $K'$ coincides with $\mathfrak{k}$. Therefore $G$ acts on $V$ almost effectively and we have

$$(3.4.1) \qquad \dim \rho(G) = \dim G\,.$$

Since the image $F(M)$ of $M$ is the orbit of $G$ through $F(o)$, $F(M)$ is a regular submanifold of $S$. Let $I^0(F(M))$ be the identity component of $I(F(M))$, the group of all isometries of the Riemannian manifold $F(M)$. Since $F$ is full, we may consider $\rho(G)$ as a closed subgroup of $I^0(F(M))$. It follows from Lemma 3.4.1, the assumption of the proposition and (3.4.1) that

$$\dim I^0(F(M)) \leqq \dim I(\tilde{M}) = \dim \rho(G)\,.$$

Therefore we have

$$I^0(F(M)) = \rho(G)\,.$$

Let $A$ be an element of $I^0(S, M)$. Since $F(M)$ is a regular submanifold of $S$, $A$ induces an isometry of $F(M)$, which is contained in $I^0(F(M))$. Then there exists an element $x \in G$ such that the actions $\rho(x)$ and $A$ coincide on $F(M)$. Since $F$ is full, we have $A=\rho(x)$. Therefore $I^0(S, M)$ coincides with $\rho(G)$. Thus we obtain the proposition. Q.E.D.

REMARK 3.4.1. The condition $\dim G = \dim I(\tilde{M})$ is satisfied, when the pair $(G, K)$ is an almost effective Riemannian symmetric pair and when $G$ is semi-simple.

## 4. Invariant differential operators

4.1. Let $G$ be a connected Lie group and $K$ a closed subgroup of $G$. We assume that the quotient space $M=G/K$ is reductive, i.e. the Lie algebra $\mathfrak{g}$ of $G$ may be decomposed into a vector space direct sum of the Lie algebra $\mathfrak{k}$ of $K$ and an $\mathrm{Ad}(K)$-invariant subspace $\mathfrak{p}$. We identify $\mathfrak{p}$ with the tangent space $T_o(M)$ at the origin $o \in M$.

Let $\phi$: $K \rightarrow GL(U)$ be a real (or complex) representation and put $\xi = G \times_K U$. For each $x \in G$ we define an automorphism $\alpha_x$: $\xi \rightarrow \xi$ by

$$\alpha_x(y \circ u) = xy \circ u \qquad \text{for } y \in G \text{ and } u \in U\,.$$

We also denote by $\alpha_x$ the automorphism $\alpha_x$ of $\Gamma(\xi)$, the space of all $C^\infty$ cross-sections of $\xi$, defined by $(\alpha_x \tilde{f})(yK) = \alpha_x(\tilde{f}(x^{-1}yK))$ for $\tilde{f} \in \Gamma(\xi)$ and $y \in G$. We have for $\tilde{f} \in \Gamma(\xi)$, $\tilde{a} \in C^\infty(M)$ and $x, y \in G$

$$\begin{aligned}(\alpha_x(\tilde{a}\tilde{f}))(yK) &= \alpha_x(\tilde{a}(x^{-1}yK)\tilde{f}(x^{-1}yK)) \\ &= \tilde{a}(x^{-1}yK)\alpha_x(\tilde{f}(x^{-1}yK)) \\ &= (\tau_{x^{-1}}{}^*\tilde{a})(yK)(\alpha_x\tilde{f})(yK)\,.\end{aligned}$$

Hence we obtain

$$(4.1.1) \qquad \alpha_x(\tilde{a}\tilde{f}) = (\tau_{x^{-1}}{}^*\tilde{a})(\alpha_x\tilde{f})\,.$$

Put

$$C^\infty(G;\, U)_K = \left\{f\colon G\to U,\quad C^\infty \text{ mapping}; f(xK) = \phi(k^{-1})f(x) \atop \text{for } x\in G \text{ and } k\in K\right\}.$$

Then as in subsection 2.1 we have the isomorphism $\iota\colon C^\infty(G;\, U)_K\to\Gamma(\xi)$, $(\iota(f))(xK)=x\circ f(x)$, and the following commutative diagram:

$$\begin{array}{ccc} C^\infty(G;\, U)_K & \xrightarrow{\iota} & \Gamma(\xi) \\ \downarrow L_x & & \downarrow \alpha_x \\ C^\infty(G;\, U)_K & \xrightarrow{\iota} & \Gamma(\xi)\,. \end{array}$$

We denote by $\tilde{f}$ the image $\iota(f)$ of $f$. Put

$$C^\infty(G)_K = \{a\in C^\infty(G)\colon a(xk) = a(x) \qquad \text{for } x\in G \text{ and } k\in K\}\,.$$

Then the pull back $\pi^*\colon C^\infty(M)\to C^\infty(G)_K$ is an isomorphism, where $\pi\colon G\to M=G/K$ is the natural projection. We denote by $\tilde{a}$ the inverse image $\pi^{*-1}(a)$ of $a\in C^\infty(G)_K$. For $f\in C^\infty(G;\, U)_K$ and $a\in C^\infty(G)_K$ we have $af\in C^\infty(G;\, U)_K$ and

$$(4.1.2) \qquad \iota(af) = \tilde{a}\tilde{f}\,.$$

Let $\psi\colon K\to GL(V)$ be a real (or complex) representation and put $\eta=G\times_K V$. We define automorphisms $\beta_x\colon \eta\to\eta$ and $\beta_x\colon \Gamma(\eta)\to\Gamma(\eta)$ in the same manner as for $\xi$. Let $\mathrm{Diff}_h(\xi,\eta)$ be the set of all $h$-th order differential operators from $\xi$ to $\eta$. A differential operator $D\in\mathrm{Diff}_h(\xi,\eta)$ is said to be *invariant,* if $D\circ\alpha_x=\beta_x\circ D$ for every $x\in G$. Let $D$ be an $h$-th order differential operator from $\xi$ to $\eta$. Then for each $p\in M$ the symbol $\sigma_h(D)$ of $D$ defines an $h$-th order homogeneous polynomial mapping from the cotangent space $T_p^*(M)$ to $\mathrm{Hom}(\xi_p, \eta_p)$ (cf. Palais [9] p. 62), where $\mathrm{Hom}(\xi_p, \eta_p)$ denotes the vector space of all linear mappings from $\xi_p$ to $\eta_p$.

Let ${}^t(d\tau_x)$ be the transposed mapping of the differential $d\tau_x$ of $\tau_x$, $x\in G$. Then we have for $\tilde{a}\in C^\infty(M)$ and $x, y\in G$

$$(4.1.3) \qquad d(\tau_{x^{-1}}{}^*\tilde{a})_{xyK} = \tau_{x^{-1}}{}^*(d\tilde{a})_{yK} = {}^t(d\tau_{x^{-1}})(d\tilde{a})_{yK}\,.$$

**Proposition 4.1.1.** *Assume that a differential operator $D\in \mathit{Diff}_h(\xi,\eta)$ is invariant. Then we have for $x, y\in G$, $v\in T_{yK}{}^*(M)$ and $\omega\in\xi_{yK}$*

$$(4.1.4) \qquad \sigma_h(D)({}^t(d\tau_{x^{-1}})v)(\alpha_x(\omega)) = \beta_x(\sigma_h(D)(v)(\omega))\,.$$

Proof. Take $\tilde{a}\in C^\infty(M)$ (resp. $\tilde{f}\in\Gamma(\xi)$) which satisfies $\tilde{a}(yK)=0$ and $d\tilde{a}_{yK}=v$(resp. $\tilde{f}(yK)=\omega$). Then we have

$$(\tau_{x^{-1}}{}^*\tilde{a})(xyK) = \tilde{a}(yK) = 0$$

and

$$(\alpha_x \tilde{f})(xyK) = \alpha_x(\tilde{f}(yK)) = \alpha_x(\omega)\,.$$

By (4.1.3) we have

$$d(\tau_{x^{-1}}{}^*\tilde{a})_{xyK} = {}^t(d\tau_{x^{-1}})(d\tilde{a})_{yK} = {}^t(d\tau_{x^{-1}})v\,.$$

Applying (4.1.1), we have

$$\alpha_x\Big(\frac{1}{h!}\tilde{a}^h\tilde{f}\Big) = \frac{1}{h!}(\tau_{x^{-1}}{}^*\tilde{a})^h(\alpha_x\tilde{f})\,.$$

Hence it follows from the definition of the symbol $\sigma_h(D)$ and the invariance of $D$ that

$$\begin{aligned}\sigma_h(D)({}^t(d\tau_{x^{-1}})v)(\alpha_x(\omega)) &= D\Big(\frac{1}{h!}(\tau_{x^{-1}}{}^*\tilde{a})^h(\alpha_x\tilde{f})\Big)(xyK)\\ &= D\Big(\alpha_x\Big(\frac{1}{h!}\tilde{a}^h\tilde{f}\Big)\Big)(xyK)\\ &= \beta_x\Big(D\Big(\frac{1}{h!}\tilde{a}^h\tilde{f}\Big)(yK)\Big)\\ &= \beta_x(\sigma_h(D)(v)(\omega))\,.\end{aligned}$$

Q.E.D.

**Corollary 1.** *Assume that $D\in Diff_h(\xi, \eta)$ is invariant. If $\sigma_h(D)_o=0$, then $\sigma_h(D)=0$.*

Proof. The corollary is an immediate consequence of the proposition. Q.E.D.

If $D$ is a first order differential operator, the symbol $\sigma_1(D)_p$, $p\in M$, defines a bilinear mapping from $T_p{}^*(M)\times\xi_p$ to $\eta_p$. We also denote by $\sigma_1(D)_p$ the linear mapping from $T_p{}^*(M)\otimes\xi_p$ to $\eta_p$ induced from the bilinear mapping $\sigma_1(D)_p$. We have easily the following corollary.

**Corollary 2.** *If a differential operator $D\in Diff_1(\xi, \eta)$ is invariant, then the linear mapping $\sigma_1(D)_o$: $\mathfrak{p}^*\otimes U=T_o{}^*(M)\otimes\xi_o\rightarrow\eta_o=V$ is a $K$-module homomorphism, i.e. for each $k\in K$*

$$\sigma_1(D)_o\circ{}^t\mathrm{Ad}_{\mathfrak{p}}(k^{-1})\otimes\phi(k) = \psi(k)\circ\sigma_1(D)_o\,,$$

*where the action $Ad_{\mathfrak{p}}(k)$ is the restriction of $Ad(k)$ to $\mathfrak{p}$ and $\mathfrak{p}^*$ denotes the dual space of $\mathfrak{p}$.*

4.2. In this subsection the assumptions and the notation are the same as in subsection 3.3.

The differential operator $\tilde{S}_1$ on $N(M)$ defined in subsection 3.3 is invariant by Remark 3.3.1. Choose an orthonormal basis $\{E_1, \cdots, E_{n+p}\}$ of $\mathfrak{g}$ such that $\{E_1, \cdots, E_n\}$ (resp. $\{E_{n+1}, \cdots, E_{n+p}\}$) is an orthonormal basis of $\mathfrak{p}$ (resp. $\mathfrak{k}$). Let $\{\phi_1, \cdots, \phi_{n+p}\}$ be the basis of the dual space of $\mathfrak{g}$ dual to $\{E_1, \cdots, E_{n+p}\}$. We consider $\{\phi_1, \cdots, \phi_n\}$ as a basis of $T_o^*(M)$. Then we obtain

**Lemma 4.2.1.** *We have for $\phi_i \in T_o^*(M)$, $i=1, \cdots, n$, and $v \in V^N$*

$$\sigma_1(\tilde{S}_1)(\phi_i)(v) = (d\rho(E_i)v)^N . \tag{4.2.1}$$

Proof. Let $N$ be an open neighborhood of $o \in M$ such that $\pi^{-1}(N)$ is diffeomorphic to $N \times K$, where $\pi: G \to G/K$ is the natural projection. Let $(x_1, \cdots, x_n)$ be the local coordinate system on $N$ defined by $x_i(\exp(\sum_{j=1}^n s_j E_j)K) = s_i$ for $-\varepsilon < s_i < \varepsilon$, where $\varepsilon$ is some positive number. For $v \in V^N$ we define a $V^N$-valued $C^\infty$ function $\alpha_v$ on $\pi^{-1}(N)$ by

$$\alpha_v(\exp(\sum_{j=1}^n s_j E_j)k) = \rho(k^{-1})v \qquad \text{for } k \in K .$$

Taking $\varepsilon' > 0$ such that $\varepsilon' < \varepsilon$, put

$$N' = \{\exp(\sum_{j=1}^n s_j E_j)K;\ -\varepsilon' < s_j < \varepsilon'\} .$$

Then there exists a $V^N$-valued $C^\infty$ function $\alpha'_v$ on $G$ such that $\alpha_v = \alpha'_v$ on $\pi^{-1}(N')$. We define a $V^N$-valued $C^\infty$ function $\beta_v$ on $G$ by

$$\beta_v(x) = \int_K \rho(k)\alpha'_v(xk)dk \qquad \text{for } x \in G ,$$

where $dk$ denotes the normalized Haar measure of $K$. Then $\beta_v \in C^\infty(G; V^N)_K$. In fact we have for $x \in G$ and $h \in K$

$$\begin{aligned}
\beta_v(xh) &= \int_K \rho(k)\alpha'_v(xhk)dk \\
&= \int_K \rho(h^{-1}(hk))\alpha'_v(xhk)dk \\
&= \rho(h^{-1})\int_K \rho(hk)\alpha'_v(xhk)dk \\
&= \rho(h^{-1})\beta_v(x) .
\end{aligned}$$

We have for $x = \exp(\sum_{j=1}^n s_j E_j)h(-\varepsilon' < s_j < \varepsilon')$

$$\begin{aligned}
\beta_v(x) &= \rho(h^{-1})\int_K \rho(k)\alpha'_v(\exp(\sum_{j=1}^n s_j E_j)k)dk \\
&= \rho(h^{-1})\int_K v\,dk = \rho(h^{-1})v .
\end{aligned}$$

Therefore $\tilde{\beta}_v(o)=\iota(e\circ\beta_v(e))=v$. Take $\tilde{f}_i\in C^\infty(M)$ such that $\tilde{f}_i=x_i$ on $N'$ and then take $f_i\in C^\infty(G)_K$ such that $\pi^*\tilde{f}_i=f_i$. Then $\tilde{f}_i(o)=0$ and $(d\tilde{f}_i)_o=\phi_i$. We have by (4.1.2)

$$\begin{aligned}\sigma_1(\tilde{S}_1)(\phi_i)(v) &= \tilde{S}_1(\tilde{f}_i\tilde{\beta}_v)(o) = \tilde{S}_1(\iota(f_i\beta_v))(o)\\ &= \iota(S_1(f_i\beta_v))(o) = S_1(f_i\beta_v)(e)\\ &= \sum_{j=1}^{n+p}\{d\rho(E_j)(E_j(f_i\beta_v))(e)\}^N .\end{aligned}$$

We have by (3.1.13)

$$\begin{aligned}&\sum_{j=1}^{n+p}\{d\rho(E_j)(E_j(f_i\beta_v))(e))\}^N\\ &= \sum_{j=1}^{n+p}\{d\rho(\mathrm{Ad}(*^{-1})E_j)(\hat{E}_j(f_i\beta_v))(e)\}^N\\ &= \sum_{j=1}^{n+p}\{d\rho(E_j)\{(\hat{E}_jf_i)(e)\beta_v(e)+f_i(e)(\hat{E}_j\beta_v)(e)\}\}^N\\ &= (d\rho(E_i)v)^N .\end{aligned}$$

This proves (4.2.1). Q.E.D.

**Proposition 4.2.2.** *The following three conditions are equivalent*:
(1) $(d\rho(X)v)^N=0$ *for* $X\in\mathfrak{p}$ *and* $v\in V^N$.
(2) $\tilde{S}_1=0$.
(3) $\sigma_1(\tilde{S}_1)=0$.

Proof. Lemma 3.3.1 shows that (1) implies (2). It is evident that (2) implies (3). Lemma 4.2.1 shows that (3) implies (1). Q.E.D.

The vector spaces $V^N$ and $\mathfrak{p}\otimes V^N$ are $K$-modules in a natural manner. Since $K$ is compact, we may decompose $V^N$(resp. $\mathfrak{p}\otimes V^N$) into a direct sum of irreducible $K$-modules.

**Proposition 4.2.3.** *If any irreducible component of* $\mathfrak{p}\otimes V^N$ *is not isomorphic to any irreducible component of* $V^N$, *then* $S_1=0$.

Proof. Since the representation $\mathrm{Ad}_\mathfrak{p}$: $K\to GL(\mathfrak{p})$ is orthogonal, the contragradient representation of $\mathrm{Ad}_\mathfrak{p}$ coincides with itself. Hence it follows from Corollary 2 for Proposition 4.1.1 and Schur's lemma (cf. Chevalley [2] p. 182) that $\sigma_1(\tilde{S}_1)_o=0$. Therefore we have our proposition by the above proposition. Q.E.D.

## 5. Reduction to the finite dimensional eigenvalue problems

5.1. Let $G$ be a compact connected Lie group and $K$ a closed subgroup of $G$. We denote by $M$ the quotient space $G/K$. The $G$-invariant Riemannian metric $\langle\ ,\ \rangle$ on $M$ is the same as in subsection 2.1. Let $D(G)$ be the set of equivalence classes of complex irreducible representations of $G$. For a complex irreducible representation $\sigma: G \to GL(W)$ we denote by $\sigma^*: G \to GL(W^*)$ the contragradient representation of $\sigma$ on the dual space $W^*$ of $W$. Let $C^\infty(G)^C$ be the space of $C$-valued $C^\infty$ functions on $G$. We define actions $L_x$ and $R_x$ of $G$ on $C^\infty(G)^C$ by the followings:

$$(L_x f)(y) = f(x^{-1}y),\ (R_x f)(y) = f(yx) \qquad \text{for } f \in C^\infty(G)^C .$$

For $[\sigma] \in D(G)$ let $\mathfrak{d}^L{}_{[\sigma]}(G)$ (resp. $\mathfrak{d}^R{}_{[\sigma]}(G)$) be the subspace of $C^\infty(G)^C$ generated by $G$-submodules of $C^\infty(G)^C$ which are isomorphic to $\sigma$ by the $G$-action $L$(resp. by the $G$-action $R$). Then we have $\mathfrak{d}^L{}_{[\sigma]}(G) = \mathfrak{d}^R{}_{[\sigma^*]}(G)$.

Let $U$ be a complex vector space with a Hermitian inner product $\langle\ ,\ \rangle$ and $C^\infty(G;\ U)$ the space of $U$-valued $C^\infty$ functions on $G$. We also denote by $L_x$ (resp. $R_x$) the action of $G$ on $C^\infty(G;\ U)$: $(L_x f)(y) = f(x^{-1}y)$(resp. $(R_x f)(y) = f(yx)$) for $f \in C^\infty(G;\ U)$. Note that our $L_x$(resp. $R_x$) is nothing but the tensor product $L_x \otimes 1_U$ (resp. $R_x \otimes 1_U$) on $C^\infty(G)^C \otimes U = C^\infty(G;\ U)$. Let $\sigma$: $G \to GL(W)$ be a complex irreducible representation. We define a multilinear mapping $\Phi^\sigma$: $W \times W^* \times U \to C^\infty(G;\ U)$ by

$$\Phi^\sigma(w,\ \omega,\ u)(x) = \omega(\sigma^{-1}(x)w)u \qquad \text{for } w \in W,\ \omega \in W^* \text{ and } u \in U .$$

We also denote by $\Phi^\sigma$ the induced linear mapping of $W \otimes W^* \otimes U$ to $C^\infty(G;\ U)$. We define an action $L_\sigma(x)$(resp. $R_{\sigma^*}(x)$) of $G$ on $W \otimes W^* \otimes U$ by $L_\sigma(x) = \sigma(x) \otimes 1_{W^*} \otimes 1_U$(resp. $R_{\sigma^*}(x) = 1_W \otimes \sigma^*(x) \otimes 1_U$). Then we have $\Phi^\sigma \circ L_\sigma(x) = L_x \circ \Phi^\sigma$ and $\Phi^\sigma \circ R_{\sigma^*}(x) = R_x \circ \Phi^\sigma$ for every $x \in G$.

**Theorem 5.1.1** (cf. Takeuchi [12] p. 15). (1) *We consider $W \otimes W^* \otimes U$ (resp. $C^\infty(G;\ U)$) as a G-module with the G-action $L_\sigma$(resp. L). Then $\Phi^\sigma$ is a G-module isomorphism of $W \otimes W^* \otimes U$ onto $\mathfrak{d}^L{}_{[\sigma]}(G) \otimes U$.*

(2) *We consider $W \otimes W^* \otimes U$(resp. $C^\infty(G;\ U)$) as a G-module with the G-action $R_{\sigma^*}$(resp. R). Then $\Phi^\sigma$ is a G-module isomorphism of $W \otimes W^* \otimes U$ onto $\mathfrak{d}^R{}_{[\sigma^*]}(G) \otimes U = \mathfrak{d}^L{}_{[\sigma]}(G) \otimes U$.*

Let $\phi$: $K \to GL(U)$ be a unitary representation and $\langle\ ,\ \rangle$ the Hermitian inner product on $U$. Put $\xi = G \times_K U$. Then $\xi$ has a natural Hermitian fibre metric, which will be also denoted by $\langle\ ,\ \rangle$. We define a subspace $C^\infty(G;\ U)_K$ of $C^\infty(G;\ U)$ by

$$C^\infty(G;\ U)_K = \left\{ f \in C^\infty(G;\ U);\ f(xk) = \phi(k^{-1})f(x) \quad \text{for } x \in G \text{ and } k \in K \right\}.$$

We identify the space $\Gamma(\xi)$ of $C^\infty$ cross-sections of $\xi$ with $C^\infty(G;\ U)_K$. Then $C^\infty(G;\ U)_K$ is a $G$-module with the $G$-action $L$. We define a Hermitian inner product $\langle\ ,\ \rangle$ on $C^\infty(G;\ U)_K$ as follows:

$$\langle f, g\rangle = \int_G \langle f(x), g(x)\rangle dx ,$$

where $dx$ is the normalized Haar measure of $G$. Then we have

$$\langle L_x f, L_x g\rangle = \langle f, g\rangle \qquad \text{for every } x\in G .$$

The space $C^\infty(G; U)_K$ is a pre-Hilbert space. We denote by $L^2(\xi)$ the completion of $C^\infty(G; U)_K$. Identifying as $C^\infty(G; U)=C^\infty(G)^C\otimes U$, we define an action $J$ of $K$ on $C^\infty(G; U)$ by $J(k)=R_k\otimes\phi(k)$ for $k\in K$. Then we have

$$\text{(5.1.1)} \qquad C^\infty(G; U)_K = \{f\in C^\infty(G; U); J(k)f=f \qquad \text{for } k\in K\} .$$

For a complex irreducible representation $\sigma$: $G\to GL(W)$, we define an action $J_\sigma$ of $K$ on $W\otimes W^*\otimes U$ by $J_\sigma(k)=1_W\otimes\sigma^*(k)\otimes\phi(k)$. Then we have

$$\text{(5.1.2)} \qquad \Phi^\sigma\circ J_\sigma(k) = J(k)\circ\Phi^\sigma \qquad \text{for every } k\in K .$$

Let $\mathfrak{o}_{[\sigma]}(\xi)$ be the subspace of $C^\infty(G; U)_K$ generated by all $G$-submodules of $C^\infty(G; U)_K$ which are isomorphic to $W$. Then $\mathfrak{o}_{[\sigma]}(\xi)$ is a $G$-submodule of $\mathfrak{o}^L{}_{[\sigma]}(G)\otimes U$. Put

$$\mathfrak{o}(\xi) = \{f\in C^\infty(G; U)_K; \dim\{L_x f: x\in G\}_C<\infty\} ,$$
$$D(G; K, \phi) = \left\{[\sigma]\in D(G); \sigma^*|_K\otimes\phi \text{ contains a trivial representation}\right\},$$

and

$$(W^*\otimes U)_0 = \{\alpha\in W^*\otimes U; (\sigma^*(k)\otimes\phi(k))(\alpha) = \alpha \qquad \text{for } k\in K\} .$$

Then $W\otimes(W^*\otimes U)_0$ is a $G$-module with the $G$-action $L_\sigma$. We have the following Peter-Weyl theorem for vector bundles.

**Theorem 5.1.2.** (Bott [1] p. 173). (1) *The G-module isomorphism* $\Phi^\sigma$: $W\otimes W^*\otimes U\to\mathfrak{o}^L{}_{[\sigma]}(G)\otimes U$ *in* (1) *of Theorem* 5.1.1 *induces a G-module isomorphism of* $W\otimes(W^*\otimes U)_0$ *onto* $\mathfrak{o}_{[\sigma]}(\xi)$.

(2) *We have the following orthogonal decompositions*:

$$\mathfrak{o}(\xi) = \sum_{[\sigma]\in D(G;K,\phi)} \mathfrak{o}_{[\sigma]}(\xi) \ (\textit{algebraic direct sum}),$$
$$L^2(\xi) = \sum_{[\sigma]\in D(G;K,\phi)} \mathfrak{o}_{[\sigma]}(\xi) \ (\textit{direct sum as Hilbert space}).$$

We have the following theorem for an invariant differential operator.

**Theorem 2.** *Let $D$ be an invariant differential operator on $\xi$ and consider it as an operator on $C^\infty(G; U)_K$(see the commutative diagram in subsection* 4.1). *Let $\sigma$: $G\to GL(W)$ be an irreducible representation with $[\sigma]\in D(G; K, \phi)$. Then $D$ leaves $\mathfrak{o}_{[\sigma]}(\xi)$ invariant and there exists a unique linear mapping $D_\sigma$ of $(W^*\otimes U)_0$ such that*

$$D\circ\Phi^\sigma = \Phi^\sigma\circ(1_W\otimes D_\sigma)\,.$$

Proof. For $f\in\mathfrak{v}(\xi)$ the subspace $\{L_xDf\colon x\in G\}_C=\{DL_xf\colon x\in G\}_C$ of $C^\infty(G, U)$ is finite dimensional, and hence $D$ leaves $\mathfrak{v}(\xi)$ invariant. It follows from Schur's lemma that every $\mathfrak{v}_{[\sigma]}(\xi)$ is invariant under $D$. Let $D'$ be the linear mapping of $W\otimes(W^*\otimes U)_0$ corresponding to $D|\mathfrak{v}_{[\sigma]}(\xi)$ by the $G$-module isomorphism $\Phi^\sigma\colon W\otimes(W^*\otimes U)_0\to\mathfrak{v}_{[\sigma]}(\xi)$. Let $\{\alpha_1, \cdots, \alpha_{m_\sigma}\}$ be a basis of $(W^*\otimes U)_0$. We define linear mappings $f^i{}_j$, $i, j=1, 2, \cdots, m_\sigma$, of $W$ as follows:

$$D'(w\otimes\alpha_j) = \sum_{i=1}^{m_\sigma} f^i{}_j(w)\otimes\alpha_i \qquad \text{for } w\in W\,.$$

Then we have for $x\in G$

$$\begin{aligned} D'(L_\sigma(x)(w\otimes\alpha_j)) &= D'(\sigma(x)w\otimes\alpha_j) \\ &= \sum_{i=1}^{m_\sigma} f^i{}_j(\sigma(x)w)\otimes\alpha_i\,. \end{aligned}$$

On the other hand we have

$$\begin{aligned} D'(L_\sigma(x)(w\otimes\alpha_j)) &= L_\sigma(x)(D'(w\otimes\alpha_j)) \\ &= \sum_{i=1}^{m_\sigma} \sigma(x)f^i{}_j(w)\otimes\alpha_i\,. \end{aligned}$$

Hence

$$f^i{}_j(\sigma(x)w) = \sigma(x)f^i{}_j(w), \qquad i, j = 1, \cdots, m_\sigma\,.$$

It follows from Schur's lemma that there exist complex numbers $c^i{}_j$, $i, j= 1, \cdots, m_\sigma$, such that $f^i{}_j=c^i{}_j1_W$. Hence we have

$$D'(w\otimes\alpha_j) = w\otimes\Big(\sum_{i=1}^{m_\sigma} c^i{}_j\alpha_i\Big)\,.$$

A linear mapping $D_\sigma$ of $(W^*\otimes U)_0$ defined by

$$D_\sigma\alpha_j = \sum_{i=1}^{m_\sigma} c^i{}_j\alpha_i\,, \qquad j = 1, \cdots, m_\sigma\,,$$

is the required one. Q.E.D.

REMARK 5.1.1. If an invariant differential operator $D$ on $\xi$ is self-adjoint with respect to the inner product $\langle\ ,\ \rangle$, each $D|\mathfrak{v}_{[\sigma]}(\xi)$ is diagonalizable. If furthermore $D$ is elliptic, every eigensection of $D$ belongs to $\mathfrak{v}(\xi)$. Thus the problem of computing the spectra of $D$ is reduced to the study of the eigenvalues of $D_\sigma$ for each $[\sigma]\in D(G; K, \phi)$.

5.2. In this subsection the assumptions and the notation are the same as in subsection 3.3. Moreover we assume that the minimal isometric immersion $F\colon (M, c\langle\ ,\ \rangle)\to S$ is full. We also denote by $\langle\ ,\ \rangle$ the Hermitian inner product on $V^C$, the complexification of $V$, which is the extension of the inner product $\langle\,,\,\rangle$ on $V$. Then the orthogonal representation $\rho: G\to GL(V)$ extends to the unitary representation $\rho: G\to GL(V^C)$. Let $(V^N)^C$ be the subspace of $V^C$ generated by $V^N$ and $\rho^N: K\to GL((V^N)^C)$ the unitary representation induced from $\rho: G\to GL(V^C)$. We may identify the complexification $\Gamma(N(M))^C$ of $\Gamma(N(M))$ with $C^\infty(G; (V^N)^C)_K$. Let $(V^T)^C$(resp. $(V^0)^C$) be the complex linear subspace of $V^C$ generated by $V^T$(resp. $V^0$). We have the direct sum decomposition $V^C=(V^0)^C+(V^T)^C+(V^N)^C$. For $v\in V^C$ we denote by $v^N$ the $(V^N)^C$-component of $v$ with respect to this decomposition of $V^C$.

Let $\sigma: G\to GL(W)$ be a complex irreducible representation with $[\sigma]\in D(G; K, \rho^N)$. Put

$$(W^*\otimes(V^N)^C)_0 = \left\{\omega\in W^*\otimes(V^N)^C;\ (\sigma^*(k)\otimes\rho^N(k))(\omega)=\omega \text{ for } k\in K\right\}.$$

Let $S'$ be the linear mapping of $W\otimes(W^*\otimes(V^N)^C)_0$ corresponding to $S|_{\mathfrak{D}_{[\sigma]}(N(M)^C)}$ by the $G$-isomorphism $\Phi^\sigma$: $W\otimes(W^*\otimes(V^N)^C)_0\to\mathfrak{D}_{[\sigma]}(N(M)^C)$, where $N(M)^C$ denotes the complexification of the normal bundle $N(M)$. Then we have by Theorem 1 and (2) of Theorem 5.1.1

$$S' = -\frac{1}{c}(1_W\otimes\{(c_{\sigma^*}-2c_\rho)1_{W^*\otimes(V^N)^C}+2\sum_{i=1}^{n+p}d\sigma^*(E_i)\otimes(d\rho(E_i)*)^N$$
$$+2\sum_{i=1}^{n+p}1_{W^*}\otimes\{d\rho(E_i)(d\rho(E_i)*)^N\}^N\}),$$

where $c_{\sigma^*}$ is the scalar determined by the Casimir operator $\sum_{i=1}^{n+p}d\sigma^*(E_i)d\sigma^*(E_i)$ of $\sigma^*$. Let $c_\sigma$ be the scalar determined by the Casimir operator $\sum_{i=1}^{n+p}d\sigma(E_i)d\sigma(E_i)$ of $\sigma$. Then $c_{\sigma^*}=c_\sigma$. Put

$$S_\sigma = -\frac{1}{c}\{(c_\sigma-2c_\rho)1_{W^*\otimes(V^N)^C}+2\sum_{i=1}^{n+p}d\sigma^*(E_i)\otimes(d\rho(E_i)*)^N$$
$$+2\sum_{i=1}^{n+p}1_{W^*}\otimes\{d\rho(E_i)(d\rho(E_i)*)^N\}^N\}.$$

Then it follows from Remark 5.1.1, Theorem 2 and (2) of Theorem 5.1.2 that the problem of computing the spectra of $\tilde{S}$ is reduced to the eigenvalue problems of the linear mappings $S_\sigma$ of $(W^*\otimes(V^N)^C)_0$ with $[\sigma]\in D(G; K, \rho^N)$.

Summarizing, we get the following theorem.

**Theorem 3.** *Let $F$: $(M, c\langle\,,\,\rangle)\to S$, $F(xK)=\rho(x)F(o)$, be a full equivariant minimal isometric immersion of a compact symmetric space $M=G/K$ into a unit sphere $S$. For a complex irreducible representation $\sigma: G\to GL(W)$ with $[\sigma]\in D(G; K, \rho^N)$, let $\{\lambda_{\sigma;1},\cdots,\lambda_{\sigma;m_\sigma}\}$ be the eigenvalues of $S_\sigma$ on $(W^*\otimes(V^N)^C)_0$. Then the spectra of the Jacobi differential operator $\tilde{S}$ are given by*

$$\bigcup_{[\sigma]\in D(G\,;\,K,\,\rho^N)} \{\underbrace{\lambda_{\sigma\,;\,1}, \cdots, \lambda_{\sigma\,;\,1}}_{d_\sigma}, \cdots, \underbrace{\lambda_{\sigma\,;\,m_\sigma}, \cdots, \lambda_{\sigma\,;\,m_\sigma}}_{d_\sigma}\},$$

*where* $d_\sigma = \dim W$.

For a complex irreducible representation $\sigma: G \to GL(W)$ with $[\sigma] \in D(G; K, \rho^N)$, it follows from Remark 3.3.1 and Theorem 2 that each of the linear mappings $\sum_{i=1}^{n+p} d\sigma^*(E_i) \otimes (d\rho(E_i)*)^N$ and $\sum_{i=1}^{n+p} 1_{W^*} \otimes \{d\rho(E_i)(d\rho(E_i)*)^N\}^N$ leaves $(W^* \otimes (V^N)^C)_0$ invariant. For the study of the linear mapping $S_\sigma$ it is important to study these linear mappings. We shall study these linear mappings.

Let $\mathfrak{g}^C$ be the complexification of $\mathfrak{g}$ and $(\ ,\ )$ the symmetric bilinear form on $\mathfrak{g}^C$ which is the $\boldsymbol{C}$-bilinear extension of the inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$. Choose bases $\{F_1 \cdots, F_{n+p}\}$ and $\{F'_1, \cdots, F'_{n+p}\}$ of $\mathfrak{g}^C$ with the property $(F_i, F'_j) = \delta_{ij}$. Let $\chi: G \to GL(U)$ be an arbitrary unitary representation (not necessarily irreducible). We define a linear mapping $L(\chi, \rho)$ of $U \otimes V^C$ by

$$L(\chi, \rho) = \sum_{i=1}^{n+p} d\chi(F_i) \otimes d\rho(F'_i)\,.$$

The linear mapping $L(\chi, \rho)$ is independent of the choice of bases. In fact let $\{H_1, \cdots, H_{n+p}\}$ and $\{H'_1, \cdots, H'_{n+p}\}$ be bases of $\mathfrak{g}^C$ with $(H_i, H'_j) = \delta_{ij}$. Let $H_i = \sum_{k=1}^{n+p} a^k{}_i F_k$ and $H'_i = \sum_{h=1}^{n+p} b^i{}_h F'_h$, $i = 1, \cdots, n+p$. Then we have

$$\delta_{ij} = (H_i, H'_j) = \sum_{k=1}^{n+p} a^k{}_i b^j{}_k\,.$$

Hence if we put $A = (a^i{}_j)_{i,j=1,\cdots,n+p}$ and $B = (b^i{}_j)_{i,j=1,\cdots,n+p}$, we have $B = A^{-1}$. Therefore we have

$$\sum_{i=1}^{n+p} d\chi(H_i) \otimes d\rho(H'_i) = \sum_{k,h=1}^{n+p} \sum_{i=1}^{n+p} a^k{}_i b^i{}_h d\chi(F_k) \otimes d\rho(F'_h)$$
$$= \sum_{k=1}^{n+p} d\chi(F_k) \otimes d\rho(F'_k)\,.$$

We denote by $C_{\chi\otimes\rho}$(resp. $C_\chi$ and $C_\rho$) the Casimir operator of the representation $\chi \otimes \rho$ (resp. $\chi$ and $\rho$). Since $\sum_{i=1}^{n+p} d\chi(F_i) \otimes d\rho(F'_i) = \sum_{i=1}^{n+p} d\chi(F'_i) \otimes d\rho(F_i)$, we have

$$(5.2.1) \qquad 2L(\chi, \rho) = C_{\chi\otimes\rho} - C_\chi \otimes 1_{V^C} - 1_U \otimes C_\rho\,.$$

We obtain the following lemma by (5.2.1) and the fact that the Casimir operator commutes with the action of $G$.

**Lemma 5.2.1.** *We have*

$$(\chi \otimes \rho)(x) \circ L(\chi, \rho) = L(\chi, \rho) \circ (\chi \otimes \rho)(x) \qquad \textit{for } x \in G\,.$$

Put

$$(U\otimes V^C)_0=\{\omega\in U\otimes V^C;\ (\chi\otimes\rho)(k)\omega=\omega \qquad \text{for } k\in K\}\ .$$

Then we have by the above lemma

$$(5.2.2)\qquad L(\chi,\rho)((U\otimes V^C)_0)\subset(U\otimes V^C)_0\,.$$

Now we come back to our complex irreducible representation $\sigma\colon G\to GL(W)$. We denote by $p_1$ the projection to the first component of the following direct sum decomposition:

$$W^*\otimes V^C=(W^*\otimes(V^N)^C)+(W^*\otimes\{(V^T)^C+(V^0)^C\})\,.$$

Then we have

**Lemma 5.2.2.**

$$(5.2.3)\qquad \sum_{i=1}^{n+p}d\sigma^*(E_i)\otimes(d\rho(E_i)*)^N=p_1\sum_{i=1}^{n+p}d\sigma^*(E_i)\otimes d\rho(E_i)\qquad \textit{on } W^*\otimes V^C\,,$$

$$(5.2.4)\qquad \sum_{i=1}^{n+p}d\sigma^*(E_i)\otimes d\rho(E_i)((W^*\otimes V^C)_0)\subset(W^*\otimes V^C)_0\,,$$

*where* $(W^*\otimes V^C)_0=\{\omega\in W^*\otimes V^C,\ (\sigma^*(k)\otimes\rho(k))\omega=\omega\qquad \textit{for } k\in K\}$ .

Proof. The first equality is trivial. Since $\sum_{i=1}^{n+p}d\sigma^*(E_i)\otimes d\rho(E_i)=L(\sigma^*,\rho)$, we have (5.2.4) by (5.2.2). Q.E.D.

**Lemma 5.2.3.** *We have*

$$(5.2.5)\qquad \rho(k)\sum_{i=1}^{n+p}\{d\rho(E_i)(d\rho(E_i)v)^N\}^N$$
$$=\sum_{i=1}^{n+p}(d\rho(E_i)\{d\rho(E_i)\rho(k)v\}^N)^N\qquad \textit{for } k\in K \textit{ and } v\in V^C\,.$$

Proof. For $k\in K$ the linear mapping $\rho(k)$ leaves $(V^N)^C$, $(V^T)^C$ and $(V^0)^C$ invariant respectively. Therefore we have

$$\rho(k)\sum_{i=1}^{n+p}\{d\rho(E_i)(d\rho(E_i)v)^N\}^N$$
$$=\sum_{i=1}^{n+p}(\{\rho(k)d\rho(E_i)\rho(k^{-1})\}[\{\rho(k)d\rho(E_i)\rho(k^{-1})\}(\rho(k)v)]^N)^N$$
$$=\sum_{i=1}^{n+p}(d\rho(\mathrm{Ad}(k)E_i)\{d\rho(\mathrm{Ad}(k)E_i)(\rho(k)v)\}^N)^N\,.$$

Since $\{\mathrm{Ad}(k)E_1,\cdots,\mathrm{Ad}(k)E_{n+p}\}$ is an orthonormal basis of $\mathfrak{g}$, we have

$$\sum_{i=1}^{n+p}(d\rho(\mathrm{Ad}(k)E_i)\{d\rho(\mathrm{Ad}(k)E_i)(\rho(k)v)\}^N)^N$$
$$=\sum_{i=1}^{n+p}(d\rho(E_i)\{d\rho(E_i)(\rho(k)v)\}^N)^N\,.$$

Q.E.D.

In the forthcoming papers we shall study the linear mappings

$$\sum_{i=1}^{n+p} d\sigma^*(E_i)\otimes d\rho(E_i)\colon (W^*\otimes V^C)_0 \to (W^*\otimes V^C)_0$$

and

$$\sum_{i=1}^{n+p} \{d\rho(E_i)(d\rho(E_i)*)^N\}^N\colon (V^N)^C \to (V^N)^C\,.$$

These studies, together with Lemma 5.2.2 and Lemma 5.2.3, will give us informations on the linear mapping $S_\sigma$.

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