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A CHARACTERIZATION OF THE SIMPLE GROUP ON

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The sporadic simple group ON of order $2^9.3^4.5.7^3.11.19.31$ discovered by O'Nan [8] has exactly one conjugacy class of elements of order three and the centralizer of one of its elements of order three is isomorphic to the direct product of the alternating group A_6 of degree six with an elementary abelian group of order nine. The purpose of this paper is to prove the converse under the following

HYPOTHESIS. *The only simple groups containing a standard subgroup A such that $A/Z(A)$ is isomorphic to $PSL(3,4)$ are the sporadic simple groups He, ON and Suz.*

For the definition of a standard subgroup and for the proof of the above hypothesis under some additional conditions see [2].

Our result is the following

Theorem. *Let G be a finite simple group with exactly one conjugacy class of elements of order three. If the centralizer of an element of order three is isomorphic to the direct product of an elementary abelian group of order nine with the alternating group of degree six, then G is isomorphic to ON.*

The notation is hopefully standard. M_{10} stands for the subgroup of Aut A_6 containing A_6 as a subgroup of index two and having semidihedral Sylow 2-subgroups.

In the whole paper G denotes a finite simple group with exactly one conjugacy class of elements of order three. Let a be a fixed element of G of order and let $C_G(a)=R\times K$ where $R=O_3(C_G(a))$ is elementary abelian of order nine and $K=C_G(a)'$ is isomorphic to A_6.

The first lemma follows immediately from the structure of A_6 and is given without proof.

Lemma 1. *The following hold in $C_G(a)$:*

(i) *Every 2-element of $C_G(a)$ is contained in the characteristic subgroup K of $C_G(a)$.*

(ii) *$C_G(a)$ has exactly one conjugacy class of involutions and if $x \in C_G(a)$ is an involution, then $C_G(a,x) = R \times C_K(x)$, where $C_K(x)$ is dihedral of order eight and is a Sylow 2-subgroup of K and hence of $C_G(a)$.*

(iii) *No involution in G centralizes a subgroup of order 27.*

(iv) *$C_G(a)$ has exactly one conjugacy class of elements of order four and such an element normalizes a Sylow 3-subgroup of K.*

(v) *For any subgroup V of order four of $C_G(a)$ we have $C_G(a,V) = R \times V$.*

(vi) *Any 3-subgroup of $C_G(a)$ which is centralized by an involution of $C_G(a)$ is contained in $R = O_3(C_G(a))$.*

(vii) *If y is an element of order three of K, then $C_K(y)$ is a Sylow 3-subgroup of K and is elementary abelian of order nine. $R \times C_K(y)$ is a Sylow 3-subgroup of G.*

(viii) Aut $A_6 \cong P\Gamma L(2,3^2)$ *has no involution which centralizes a Sylow 3-subgroup of it. Aut $A_6/In\,A_6$ is elementary abelian of order four.*

Lemma 2. *$N_G(R)/C_G(R)$ is isomorphic either to the cyclic group of order eight or to the quaternion group. $C_G(K) \cap N_G(R)$ is a Frobenius group of order 36 and $N_G(R)/N_G(R) \cap C_G(K)$ is isomorphic to M_{10}. Furthermore R is conjugate to a Sylow 3-subgroup of K.*

Proof. Since all elements of order 3 are conjugate in G we get that $C_G(r) = C_G(a) = C_G(R)$ for all $r \in R^\#$. This implies that all elements of $R^\#$ are conjugate to a in $N_G(R)$. Thus $|N_G(R) : C_G(a)| = 8$. Since $N_G/(R)/C_G(R)$ acts regularly on R we obtain by [4,5.3.14] that it is isomorphic either to Z_8, the cyclic group of order 8 or to Q_8, the quaternion group.

Since K is a characteristic subgroup of $C_G(R)$ we see that $K \lhd N_G(R)$. Let $N = N_G(R)$. $N/C_N(K)$ is isomorphic to a subgroup of Aut A_6 containing A_6. Since G has exactly one class of elements of order three we see by the structure of a Sylow 2-subgroup (1.ii) of $C_G(a)$, that $(K \times C_N(K))/C_N(K) \cong A_6$ is a proper subgroup of $N/C_N(K)$. Since Aut $A_6/In\,A_6$ is elementary abelian we get that K is centralized by an involution.

Let x be an element of order 3 of K. $C_N(x)/R \times C_K(x) \cong C_N(x)C_G(R)/C_G(R)$ is isomorphic to a subgroup of $N/C_G(R)$. Since Aut A_6 has exactly one class of elements of order three we see that $C_N(x)/R \times C_K(x)$ is cyclic of order four. Here a Sylow 2-subgroup of $C_N(x)$ normalizes both R and $C_K(x)$ and operates regularly on R. By (1.iv) and (1.v) we get that a Sylow 2-subgroup of $C_N(x)$ must centralize $C_K(x)$. In particular $C_K(x) = O_3(C_G(x))$ by (1.vi) and hence $C_K(x)$ is conjugate to R in G. On the other hand we get by (1.viii) that a Sylow 2-subgroup of $C_N(x)$ must induce the trivial automorphism on K, since it centralizes a Sylow 3-subgroup of K. Hence $C_N(x) = C_N(K) \times C_K(x)$ for any element x of order 3 of K, where $C_N(K)$ is a Frobenius group of order 36. This yields that $N/C_N(K)$ is isomorphic to a subgroup of Aut A_6 containing A_6 with index two, in which the centralizer of every element of order three is of order nine. The only such extension is M_{10}, hence $N/C_N(K) \cong M_{10}$.

From now on we let t denote an element of order four in $N_G(R) \cap C_G(K)$. Our aim is to construct $C_G(t)$.

Lemma 3. *For any subgroup V of order four of $C_G(R)$ one of the following holds*:

(i) $C_G(V) \leq N_G(R)$

(ii) $C_G(V)/V \cong PSL(3,4)$

(iii) $C_G(V)/V \cong A_6$ or M_{10}.

Proof. By (1.iii) R is a Sylow 3-subgroup of $C = C_G(V)$. Since $(|r|, |V|) = 1$ for any $r \in R^\#$ we get $C_{\bar{C}}(\bar{r}) = C_C(r)V/V$ where $\bar{C} = C/V$. By (1.v) we get $C_{\bar{C}}(\bar{r}) = \bar{R}$, i.e. $\bar{C}$ is a $3CC$-group. By [1] the result follows.

Lemma 4. *$C_G(t)/\langle t\rangle$ is isomorphic to $PSL(3,4)$ and $C_G(t)$ does not split over $\langle t\rangle$. Furthermore $N_G(\langle t\rangle) = C_G(t)\langle z\rangle$ where z is an involution inducing the unitary automorphism on $C_G(t)'/Z(C_G(t)') \cong PSL(3,4)$.*

(Here by the unitary automorphism α of $PSL(3,4)$ we understand the involutory automorphism of $PSL(3,4)$ which is used to define the unitary group. We have $C_{PSL(3,4)}(\alpha) \cong PSU(3,4)$.)

Proof. Let M be a Sylow 3-subgroup of K. Since M is conjugate to R by (2) we see that the conclusions of (3) also hold for $C_G(t)$ if we replace R by M. Since $K \leq C_G(t)$ we get that $C_G(t)/\langle t\rangle$ is isomorphic to $PSL(3,4)$ or A_6 M_{10}. Furthermore there exists an involution z in $C_G(M)$ which inverts t, hence $N_G(\langle t\rangle) = C_G(t)\langle z\rangle$.

Assume now that $C_G(t)/\langle t\rangle$ is isomorphic to A_6 or M_{10}. Then $\langle t\rangle \times K$ and hence K is normal in $N_G(\langle t\rangle)$. z induces an automorphism on K which centralizes a Sylow 3-subgroup of K. By (1.viii) z must centralize K. Thus $\langle R, t, z\rangle \leq C_G(K)$ and hence $C_G(K) \cong A_6$, since $C_G(K) \leq C_G(M) \cong E_9 \times A_6$.

Let now B_1 be a subgroup of $C_G(K)$ which is isomorphic to the symmetric group S_4 and which contains t and let B_2 be a subgroup of K isomorphic to S_4. Let $T = O_2(B_1 \times B_2)$ and $X = C_G(T)$. Then T is elementary abelian of order 16. In particular $3 \nmid |X|$ by (1.ii). Let m_i be an element of order 3 in B_i, $i = 1, 2$. Then $\langle m_1, m_2\rangle$ is a Sylow 3-subgroup of $B_1 \times B_2$ and is elementary abelian. Since $C_X(m_1) \leq C_G(O_2(B_1)) \cap C_G(m_1) = O_3(C_G(m_1)) \times O_2(B_2)$ by (1.v) we get that $C_X(m_1) = O_2(B_2) = C_T(m_1)$. Similarly $C_X(m_2) = O_2(B_1) = C_T(m_2)$. In particular we see by [4, 10, 2.1] that X/T and hence X is nilpotent. By [4, 6.2.4] we have

$$X=\langle C_X(x)\,|\,1\neq x\in\langle m_1, m_2\rangle\rangle=\langle T, C_X(m_1m_2), C_X(m_1m_2^{-1})\rangle$$

where $C_X(m_1m_2)\sim C_X(m_1m_2^{-1})$ in $N_G(T)$. Since $C_X(m_1m_2)$ is isomorphic to a nilpotent $3'$-subgroup of $C_G(m_1m_2)$ on which m_1 operates fixed-point-freely we see that $C_X(m_1m_2)\cong 1$ or E_4.

Assume that $C_X(m_1m_2)\cong E_4$ and let $Y=C_X(O_2(B_1))$. By the same argument as in (3) we see that $Y/O_2(B_1)$ is a $3CC$-group containing $KO_2(B_1)/O_2(B_1)\cong A_6$. Since $X\leq Y$ we get by the order of a Sylow 2-subgroup of X and by [1] that $Y/O_2(B_1)\cong PSL(3,4)$. $O_2(B_1)$ and hence Y is normalized by $N_{B_1}(\langle m_1\rangle)\cong S_3$ which centralizes the subgroup K of Y. Since $PSL(3,4)$ has no nontrivial automorphism which centralizes a subgroup of it isomorphic to A_6 (See [6, (1.3)]) we see that

$$N_G(O_2(B_1))/O_2(B_1)=B_1/O_2(B_1)\times Y/O_2(B_1)\,.$$

But this contradicts the structure of $C_G(t)$, since $t\in B_1$. Thus $C_X(m_1m_2)=1$ and hence $C_G(T)=T$.

Let $Z=N_G(T)$ and $\bar{Z}=Z/T$. Then $\bar{Z}$ is isomorphic to a subgroup of Aut $T\cong GL(4,2)\cong A_8$. In particular $\langle m_1, m_2\rangle\in Syl_3Z$. By [3] $C_{\bar{Z}}(\bar{x})=\langle\bar{x}\rangle\times\bar{L}$ for any $x\in\langle m_1, m_2\rangle$, where $1\neq\langle\bar{m}_1, \bar{m}_2\rangle\cap\bar{L}=\bar{L}_1$ is a Sylow 3-subgroup of $\bar{L}$. By the structure of A_8 we have $C_{\bar{L}}(\bar{L}_1)=\bar{L}_1$. By [1] and the structure of A_8 we see that $\bar{L}$ is isomorphic to one of the groups Z_3, S_3, A_4, S_4 or A_5. So we get that there exists a four subgroup $\bar{S}$ of $\bar{Z}$ which is normalized by $\langle\bar{m}_1, \bar{m}_2\rangle$ if there exists an element $x\in\langle m_1, m_2\rangle$ such that $C_{\bar{Z}}(\bar{x})$ is not contained in $N_{\bar{Z}}(\langle\bar{m}_1, \bar{m}_2\rangle)$. Then $\langle m_1, m_2\rangle$ normalizes also $C_T(\bar{S})\neq 1$. This implies by [4, 6.2.4] that $C_T(\bar{S})=C_T(m_1)=O_2(B_2)$ or $C_T(\bar{S})=C_T(m_2)=O_2(B_2)$. As above we get then that $C_G(C_T(\bar{S}))/C_T(S)$ is isomorphic to $PSL(3,4)$. On the other hand $M\sim R$ in G implies that $O_2(B_1)\sim O_2(B_2)$ since K has one class of four subgroups under the action of $N_G(R)$ by (2). This yields again a contradiction to the structure of $C_G(t)$. Thus we have that $C_{\bar{Z}}(\bar{x})$ is contained in $N_{\bar{Z}}(\langle\bar{m}_1, \bar{m}_2\rangle)$ for any $x\in\langle m_1, m_2\rangle$).

This yields by [10, lemma 3.1 and lemma 3.2] that either $\langle\bar{m}_1, \bar{m}_2\rangle$ is normal in $\bar{Z}$ or $\bar{Z}$ is isomorphic to S_6. In the first case [9, Proposition] gives that G is of sectional 2-rank at most four. But by [5] it is straight forward to check that no simple group of sectional 2-rank at most four satisfies the assumptions of the theorem. So $N_G(T)/T\cong S_6$. This subgroup structure of a simple group is investigated by Stroth in [11]. In particular there exists an involution $\bar{y}$ in $\bar{Z}$ such that $C_{\bar{Z}}(\bar{y})=\langle\bar{y}\rangle\times\bar{B}$ where $\bar{B}\cong S_4$ and a Sylow 3-subgroup of $\bar{B}$ operates fixed point freely on T. So we can assume that y is an involution in the centralizer of an element m of order 3 of $N_G(T)$ which operates regularly on T. Since $C_T(y)$ is normalized by m, we see that $C_T(y)$ is a four group. Furthermore $\langle C_T(y), y\rangle$ is normalized by the inverse image F of $C_{\bar{Z}}(\bar{y})$ in Z. Since $C_T(C_T(y), y)=C_T(y)$ we see by the structure of $GL(3,2)$ that the elementary abelian group $\langle C_T(y), y\rangle$ of order 8 is centralized by a subgroup A of order 32 of F with $\bar{A}=(O_2(C_{\bar{T}}\bar{y}))$. Since $A/\langle C_T(y), y\rangle$ is a four group we see that A' is cyclic and is contained in $C_T(y)$. Since A is normalized by m which operates regularly on $C_T(y)$ we get that $A'=1$. By the remark following [11, (1.1)] we see that A is the only abelian subgroup of order 32 of a Sylow 2-subgroup of $N_G(T)$ and by [11, (1.5)] that $\Omega_1(A)=\langle C_T(y), y\rangle$. Now all involutions in $C_T(y)y$ are conjugate to y under the action of T. Since all involutions of T are conjugate in $N_G(T)$ we see that all involutions of $\Omega_1(A)$ are involutions which are centralized by some element of order three. Since $C_G(a)$ has only one conjugacy class of involutions by (1.ii) and all elements of order 3 in G are conjugate to a we see that all involutions of $\Omega_1(A)$ are conjugate in G. By [11, (4.1)] we get then that G is isomorphic to HiS. But HiS has Sylow 3-subgroups of order nine. This contradiction shows that $C_G(t)/\langle t\rangle$ is isomorphic to $PSL(3,4)$.

Next we prove the remaining assertions of the lemma.

Assume that $C_G(t)$ splits over $\langle t\rangle$. Then $C_G(t)=\langle t\rangle\times E$ with $E\cong PSL(3,4)$. The involution z normalizes E and centralizes M, which is a Sylow 3-subgroup of E. Then z normalizes $N_F(M)$ which is a Frobenius group of order 72 with quaternion Sylow 2-subgroups. Then $C(M)\cap N_E(M)\langle z\rangle=M\times\langle z\rangle$ and hence $\langle z\rangle$ is centralized by $N_E(M)$. So a Sylow 2-subgroup of $N_G(M)$ is isomorphic to $D_8\times Q_8$ by Lemma 2, since $M\sim R$ in G. But this is not possible since $D_8\times Q_8$ has no factor group isomorphic to the semidihedral group of order 16 which is a Sylow 2-subgroup of $N_G(M)$. Thus $C_G(t)$ does not split over $\langle t\rangle$. In particular $t^2\in Z(C_G(t)')$ and $C_G(t)'/Z(C_G(t)')\cong PSL(3,4)$.

As we have seen in the above paragraph z induces an automorphism on $C_G(t)'/Z(C_G(t)')\cong PSL(3, 4)$ which centralizes the normalizer of a Sylow 3-subgroup of it. So either z induces the unitary automorphism or $[z, C_G(t)']$ is contained in $Z(C_G(t)')$ by [6, (1.3)]. In the second case we have $[C_G(t)', z]=1$ for $C_G(t)'$ is generated by its elements of odd order which are all centralized by z. Then $Z(C_G(t)')=\langle t^2\rangle$. And since $C_G(t^2, z)/\langle t^2, z\rangle$ is a 3CC-group we get by [1] that $C_G(t^2, z)=\langle z\rangle\times C_G(t)'$. But this not possible since $\langle z, t^2\rangle$ is normalized in $C_G(M)$ by an element of order 3 which acts regularly on $\langle z, t^2\rangle$. This completes the proof of the lemma.

Lemma 5. *We have $C_G(t^2)=N_G(\langle t\rangle)$ and hence $G\cong ON$.*

Proof. Let $X=C_G(t^2)$ and $\bar{X}=X/\langle t\rangle^2$. Let M be a Sylow 3-subgroup of K. Then M is also a Sylow 3-subgroup of X by (1.iii) and we have for any $m\in M^\sharp$ that $C_X(m)=M\times\langle t, z\rangle$. Since $O_{3'}(X)=\langle O_{3'}(X)\cap C_X(m)\mid 1\neq m\in M\rangle$ we get by (4) that $O_{3'}(X)\leq\langle t\rangle$. If $O_{3'}(X)=\langle t\rangle$, then $X=N_G(\langle t\rangle)$. So let us assume that $O_{3'}(X)=\langle t^2\rangle$.

By the structure of $C_G(t)$ we see that $O_3(\bar{X})=1$. Since $O_{3'}(\bar{X})=1$ by our assumption we get that $3\mid|\bar{Y}|$ for any minimal normal subgroup $\bar{Y}$ of $\bar{X}$. So

$\bar{Y} \cap \bar{M} \neq 1$. Since $O_3(\bar{X})=1$ this yields that $\overline{\bar{C}_G(t)'} \leq \bar{Y}$. By the structure of the centralizer of an element of order 3 in $\bar{X}$ we obtain then that $\bar{Y}$ is simple. The Frattini argument gives that $\bar{X}=\bar{Y}N_{\bar{X}}(\bar{M})$.

If $\bar{Y}$ is a 3CC-group then $\overline{C_G(t)}=\langle \bar{t} \rangle \times \overline{C_G(t)'} \simeq Z_2 \times PSL(3,4)$ and $\bar{Y}=\overline{C_G(t)'} \simeq PSL(3,4)$ by [1]. But then $C_{\bar{X}}(\bar{Y})=\langle \bar{t} \rangle$ is normal in $\bar{X}$ which contradicts our assumption that $O_{3'}(\bar{X})=1$. So $\bar{Y}$ is not a $3CC$-group. Then one of the following holds:

(i) $\overline{C_G(t)} \leq \bar{Y}$

(ii) $\overline{C_G(t)'} \simeq PSL(3,4)$, $C_{\bar{Y}}(\overline{C_G(t)'})=1$ and $\overline{C_G(t)'}\langle \bar{z} \rangle$ or $\overline{C_G(t)'}\langle \bar{z}\bar{t} \rangle$ is contained in $\bar{Y}$.

In case (i) $\overline{C_G(t)'}$ is a standard subgroup of $\bar{Y}$ and we get a contradiction by the Hypothesis. So we are in case (ii). Let $\bar{A}=\overline{C_G(t)'} \simeq PSL(3,4)$ and $\bar{y} \in \{\bar{z}, \bar{z}\bar{t}\} \cap \bar{Y}$. We have $C_{\bar{A}}(\bar{y})=N_{\bar{A}}(\bar{M})$ by (4). Since $C_{\bar{Y}}(\bar{m}, \bar{y})=\bar{M} \times \langle \bar{y} \rangle$ for each m M we see that $C_{\bar{Y}}(\bar{y})/\langle y \rangle$ is a $3CC$-group. By [1] and the structure of $N_{\bar{Y}}(M)$ we get that either $M \leq C_{\bar{Y}}(y)$ or $C_{\bar{Y}}(\bar{y})/\langle \bar{y} \rangle \simeq M_{10}$ or $C_{\bar{Y}}(\bar{y})/\langle \bar{y} \rangle \simeq PSL(3,4)$. In the last case $C_{\bar{Y}}(\bar{y})'$ is a standard subgroup of $\bar{Y}$ and again the Hypothesis gives a contradiction. If $C_{\bar{Y}}(\bar{y})/\langle \bar{y} \rangle \simeq M_{10}$ then $\bar{t}$ induces an automorphism on $C_{\bar{Y}}(\bar{y})'\langle \bar{y} \rangle/\langle \bar{y} \rangle \simeq A_6$ which centralizes a Sylow 3-subgroup of it. Since this is not possible in Aut A_6, $\bar{t}$ must induce the trivial automorphism on $C_{\bar{Y}}(\bar{y})'\langle \bar{y} \rangle/\langle \bar{y} \rangle$. But this contradicts the structure of the centralizer of z in $C_G(t)$. Thus $C_{\bar{Y}}(\bar{y})=\langle \bar{y} \rangle \times C_{\bar{A}}(\bar{y})$. A Sylow 2-subgroup of $C_{\bar{A}}(\bar{y})$ is isomorphic to Q_8. Therefore $\bar{Y}$ is of sectional 2-rank at most four by [7]. By [6, (1.8)] we obtain again a contradiction.

Thus $O_{3'}(X)=\langle t \rangle$ and $X \leq N_G(\langle t \rangle)$. This implies that $C_G(t)'=A$ is a standard subgroup of G with $2 \mid |Z(A)|$ and $A/Z(A) \simeq PSL(3,4)$. By [2] we get then that $G \simeq ON$.

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