Miyanishi, M. and Sugie, T.
Osaka J. Math.
**18** (1981), 1–11

# ON A PROJECTIVE PLANE CURVE WHOSE COMPLEMENT HAS LOGARITHMIC KODAIRA DIMENSION $-\infty$

MASAYOSHI MIYANISHI AND TOHRU SUGIE

(Received September 14, 1979)

**Introduction.** Let $k$ be an algebraically closed field of characteristic zero and let $D$ be a reduced curve on the projective plane $\boldsymbol{P}_k^2$ such that its complement $X:=\boldsymbol{P}_k^2-D$ has logarithmic Kodaira dimension $-\infty$ (cf. Iitaka [3, 4] for the definition and relevant results). We call such a curve $D$ *a projective plane curve with complementary logarithmic Kodaira dimension* $-\infty$, a plane curve with *CLKD* $-\infty$, for short. The purpose of the present article is to give a characterization of plane curves with *CLKD* $-\infty$. Namely, we shall prove the following:

**Theorem.** *Let $D$ be a reduced plane curve with CLKD $-\infty$. Then the following assertions hold true*:

(1) *There exists an irreducible linear pencil $\Lambda$ on $\boldsymbol{P}_k^2$ such that $D$ is a union of (irreducible components of) members of $\Lambda$, where $\Lambda$ satisfies the following properties*:

1° *$\Lambda$ has only one base point $P_0$, and, for a general member $C$ of $\Lambda$, $C-\{P_0\}$ is isomorphic to the affine line $\boldsymbol{A}_k^1$; the point $P_0$ is, therefore, a one-place point of $C$.*

2° *All members of $\Lambda$ are irreducible, and $\Lambda$ has at most two multiple members.*

3° *Let $d$ be the degree of a general member $C$ of $\Lambda$; if $d=1$ then $\Lambda$ is the pencil of lines through $P_0$; if $d>1$, let $l_0$ be the tangent line of a general member $C$, i.e., a line of maximal contact with $C$ at $P_0$; if $dl_0\in\Lambda$ then $dl_0$ is the unique multiple member of $\Lambda$; if $dl_0\notin\Lambda$ then $\Lambda$ has two multiple members $aF$ and $bG$, where $F$ and $G$ are irreducible and $a$, $b$ are integers $\geqq 2$ such that $a=\deg G$, $b=\deg F$, $d=ab$ and $G.C.D.(a,b)=1$; $D$ is said to be of the first kind or of the second kind according as $dl_0\in\Lambda$ or $dl_0\notin\Lambda$, respectively.*

(2) *There exists a Cremona transformation $\varphi$: $\boldsymbol{P}_k^2\rightarrow\boldsymbol{P}_k^2$ of degree $d$ such that*:

1° *The pencil $\Lambda$ corresponds (under $\varphi$) to the pencil of lines through a point $Q_0$;*

2° *if $D$ is of the first kind, the line $l_0$ is the unique exceptional curve for $\varphi$; if*

*$L_0$ is the line through $Q_0$ corresponding to $dl_0$ then $L_0$ is the unique exceptional curve for $\varphi^{-1}$;*

*3° if $D$ is of the second kind, the curves $F$ and $G$ exhaust the exceptional curves for $\varphi$; if $L_1$ and $L_2$ are lines through $Q_0$ corresponding to $aF$ and $bG$ under $\varphi$, then $L_1$ and $L_2$ exhaust the exceptional curves for $\varphi^{-1}$.*

*Conversely, let $\Lambda$ be an irreducible linear pencil on $\boldsymbol{P}_k^2$ satisfying the property 1° of the assertion* (1), *and let $D$ be the union of* (*irreducible components of*) *members of $\Lambda$. Then $D$ is a plane curve with CLKD $-\infty$.*

If $D$ is of the first kind, the theorem implies that the Cremona transformation $\varphi$ induces a biregular automorphism of the affine plane $\boldsymbol{P}_k^2 - l_0 \xrightarrow{\sim} \boldsymbol{P}_k^2 - L_0$ and that $X := \boldsymbol{P}_k^2 - D$ contains an open set which is isomorphic (under $\varphi$) to the affine plane with finitely many parallel lines deleted off. So far, we have only one example for $D$ of the second kind. That is, a quintic rational curve with only one cusp of multiplicity 2, which was obtained by Yoshihara [8]. In the final section, we discuss Yoshihara's example together with some detailed observations on plane curves $D$ of the second kind with $CLKD$ $-\infty$.

Our termiology and notation conform with those of Miyanishi [5] and Miyanishi-Sugie [7]. We only modify some of the definitions given in [5] so as to suit the present situation.

(1) Let $V_0$ be a nonsingular projective surface, let $P_0$ be a point on $V_0$ and let $l_0$ be an irreducible curve on $V_0$ such that $P_0$ is a simple point of $l_0$. Let $d_0$ and $d_1$ be positive integers such that $d_1 < d_0$. Define positive integers $d_2, \cdots, d_\alpha$ and $q_1, \cdots, q_\alpha$ by the following Euclidean algorithm:

$$\begin{array}{ll} d_0 = d_1 q_1 + d_2 & 0 < d_2 < d_1 \\ d_1 = d_2 q_2 + d_3 & 0 < d_3 < d_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ d_{\alpha-2} = d_{\alpha-1} q_{\alpha-1} + d_\alpha & 0 < d_\alpha < d_{\alpha-1} \\ d_{\alpha-1} = d_\alpha q_\alpha & 1 < q_\alpha . \end{array}$$

Let $N = q_1 + \cdots + q_\alpha$. Define the infinitely near points $P_i$'s of $P_0$ (for $1 \leqq i < N$) and the quadratic transformation $\sigma_i$: $V_i \to V_{i-1}$ with center $P_{i-1}$ (for $1 \leqq i \leqq N$) inductively in the following fashion:

(i) $P_i$ is an infinitely near point of order one of $P_{i-1}$ for $1 \leqq i < N$;

(ii) let $E_i = \sigma_i^{-1}(P_{i-1})$ for $1 \leqq i \leqq N$ and let $E(\nu, j) = E_i$ if $i = q_0 + \cdots + q_{\nu-1} + j$ with $1 \leqq \nu \leqq \alpha$ and $1 \leqq j \leqq q_\nu$, where we set $q_0 := 0$; $P_i$ is the intersection point of the proper transform of $E(\nu-1, q_{\nu-1})$ on $V_i$ and the curve $E(\nu, j)$ if $i = q_0 + \cdots + q_{\nu-1} + j$ with $1 \leqq \nu \leqq \alpha$ and $1 \leqq j \leqq q_\nu$ ($1 \leqq j < q_\alpha$ if $\nu = \alpha$), where we set $E(0, q_0) = l_0$.

The composition $\sigma = \sigma_1 \cdot \cdots \cdot \sigma_N$ is called *the Euclidean transformation with re-*

*spect to a datum* $(P_0, l_0, d_0, d_1)$, or the Euclidean transformation, for short, if the datum $(P_0, l_0, d_0, d_1)$ is clear by the context. The related definitions are given in Miyanishi [5; p. 92 and p. 214]. In particular, the weighted dual graph of $\sigma^{-1}(l_0)$ is the same as given in [*loc. cit.*; p. 95]. If $C$ is an irreducible curve on $V_0$ such that $P_0$ is a one-place point of $C$ with $d_0=i(C, l_0; P_0)$ and $d_1=\mathrm{mult}_{P_0}C$ (=the multiplicity of $C$ at $P_0$), then the proper transform $C_i:=(\sigma_1\cdots\sigma_{i-1})'(C)$ passes through $P_i$ so that $d_\nu=(C_i\cdot E(\nu, j))$ and the intersection multiplicity of $C_i$ with the proper transform of $E(\nu-1, q_{\nu-1})$ on $V_i$ is $d_{\nu-1}-jd_\nu$, where $1\leqq i\leqq N$ and $i=q_0+\cdots+q_{\nu-1}+j$ with $1\leqq\nu\leqq\alpha$ and $1\leqq j\leqq q_\nu$; the smaller one of $d_\nu$ and $d_{\nu-1}-jd_\nu$ is the multiplicity of $C_i$ at $P_i$. We note that $\sigma'(C)$ meets only the last exceptional curve $E(\alpha, q_\alpha)$ and does not meet the other exceptional curves in the process $\sigma$.

(2) Let $V_0$, $P_0$ and $l_0$ be as above. Let $r$ be a positive integer. *An equi-multiplicity transformation of length r with center* $P_0$, or an *EM*-transformation, for short, is the composition $\sigma=\sigma_1\cdot\cdots\cdot\sigma_r$ of quadratic transformations defined as follows:

For $1\leqq i\leqq r$, $\sigma_i$ is the quadratic transformation with center $P_{i-1}$ and $P_i$ (for $i<r$) is a point on $\sigma_i^{-1}(P_{i-1})$ other than the point $\sigma_i'(\sigma_{i-1}^{-1}(P_{i-2}))\cap\sigma_i^{-1}(P_{i-1})$ $(\sigma_1'(l_0)\cap\sigma_1^{-1}(P_0)$ if $i=1)$. A related notion is the $(e,i)$-transformation defined in [*loc. cit.*; p. 100]. If $C$ is an irreducible curve on $V_0$ such that $P_0$ is a one-place point with $d_0:=i(C, l_0; P_0)$ equal to $d_1:=\mathrm{mult}_{P_0}C$, $P_1:=\sigma_1'(C)\cap\sigma_1^{-1}(P_0)$ differs from $\sigma_1'(l_0)\cap\sigma_1^{-1}(P_0)$. If $d_1^{(1)}:=\mathrm{mult}_{P_1}\sigma_1'(C)$ equals $d_1=(\sigma_1'(C)\cdot\sigma_1^{-1}(P_0))$ then $P_2:=(\sigma_1\sigma_2)'(C)\cap\sigma_2^{-1}(P_1)$ differs from $\sigma_2'(\sigma_1^{-1}(P_0))\cap\sigma_2^{-1}(P_1)$. Thus, this step can be repeated as long as the intersection multiplicity of the proper transform of $C$ with the last exceptional curve equals the multiplicity of the proper transform of $C$ at the intersection point, and the composition of performed quadratic transformations is an equi-multiplicity transformation with center $P_0$.

## 1. The pencil $\Lambda$

Let $D$ be a plane curve with $CLKD$ $-\infty$ and let $X:=\boldsymbol{P}_k^2-D$. Then $X$ is a nonsingular rational affine surface with logarithmic Kodaira dimension $\bar{\kappa}(X)=-\infty$. By virtue of the analogue of Enriques' characterization theorem of ruled surfaces (cf. Miyanishi-Sugie [7] and Fujita [1]), $X$ contains a cylinder-like open set $U\cong U_0\times\boldsymbol{A}^1$, where $U_0$ is a nonsingular rational affine curve. Then there exists an irreducible linear pencil $\Lambda$ on $\boldsymbol{P}_k^2$ such that, for a general member $C$ of $\Lambda$, $C\cap X$ is a general fiber of the canonical projection $p_1: U\to U_0$. Hence $C\cap X$ is isomorphic to the affine line $\boldsymbol{A}_k^1$. Let $d$ be the degree of $C$. Since $d>0$, $\Lambda$ has a unique base point $P_0$, which is a one-place point for a general member $C$ of $\Lambda$. Let $D_1$ be an irreducible component of the curve $D$. Then $D_1$ meets only at $P_0$ with general members of $\Lambda$. Hence $D_1$ is an irreducible component of a member of $\Lambda$. Therefore, we know that $D$ is a union of irreducible components of members of $\Lambda$. If $d=1$ the pencil $\Lambda$ consists of lines through $P_0$, and we have nothing more to show in the theorem; take the identity automorphism of $\boldsymbol{P}_k^2$ as the Cremona transformation $\varphi$. Thus, we shall assume, henceforth, that $d>1$.

Let $C$ be a general member of $\Lambda$ and let $l_0$ be the tangent line of $C$ at $P_0$, which is the line of maximal contact with $C$ at $P_0$. Then $l_0$ is the tangent line for almost all members of $\Lambda$. Indeed, if $d=2$ the pencil $\Lambda$ is spanned by a nonsingular conic $C$ through $P_0$ and $2l_0$, where $l_0$ is the tangent line of $C$ at $P_0$. Hence every irreducible member of $\Lambda$ has $l_0$ as its tangent line at $P_0$. If $d>2$ then $P_0$ is a singular point for almost all members of $\Lambda$. If the assertion does not hold, after the quadratic transformation $\sigma\colon V'\to V_0:=\boldsymbol{P}_k^2$, the exceptional curve $E':=\sigma^{-1}(P_0)$ is a quasi-section of the pencil $\sigma'\Lambda$ (=the proper transform of $\Lambda$ by $\sigma$), (cf. [5; p. 190]). Since $P_0$ is a one-place point for almost all members $C$ of $\Lambda$ and $(E'\cdot\sigma'(C))>1$, this is a contradiction.

We shall prove the following:

**Lemma 1.** *Let the notations and the assumptions be as above. Then the following assertions hold true:*

(1) *Every member $M$ of $\Lambda$ is irreducible and $M_{\text{red}}-\{P_0\}$ is isomorphic to the affine line $\boldsymbol{A}_k^1$.*

(2) *$\Lambda$ has at most two multiple members.*

(3) *If $\Lambda$ has only one multiple member then it is $dl_0$.*

Proof. Let $C_0$ be a general member of $\Lambda$. Then $C_0$ is a curve with $CLKD\ -\infty$. In order to prove the above assertions, we may replace $D$ by $C_0$ and assume that $D$ consists of only one general member of $\Lambda$. Then $\text{Pic}(X)$ is a cyclic group of order $d$, where $X=\boldsymbol{P}_k^2-D$. Then we shall show that the projection $p_1\colon U\to U_0$, $U$ being the cylinderlike open set of $X$ which defines the pencil, extends to a surjective morphism $\pi\colon X\to T\cong\boldsymbol{A}_k^1$, whose general fibers are isomorphic to $\boldsymbol{A}_k^1$. Indeed, since $P_0$ is the single base point of $\Lambda$, the rational map $\Phi_\Lambda\colon\boldsymbol{P}_k^2\to\boldsymbol{P}_k^1$ defined by $\Lambda$ is regular outside $P_0$. Hence the restriction $\pi:=\Phi_\Lambda|_X$ is a morphism. Since $D$ is a member of $\Lambda$, the image $T:=\pi(X)$ is isomorphic to the affine line. By construction, general fibers of $\pi$ are isomorphic to $\boldsymbol{A}_k^1$.

By virtue of Miyanishi [6; Lemma 1.1], we have

$$\text{rank Pic}\,(X)\underset{\boldsymbol{Z}}{\otimes}\boldsymbol{Q}=\sum_{P\in T}(\mu_P-1)\,,$$

where $\mu_P$ is the number of irreducible components of the fiber $\pi^{-1}(P)$ and where $P$ ranges over all points of $T$. Since $\text{Pic}(X)$ is a cyclic group of order $d$, we know that every fiber of $\pi$ is irreducible; we know also by the same result (*loc.*

*cit.*) that every fiber of $\pi$ is isomorphic to $\boldsymbol{A}_k^1$. This implies that every member of $\Lambda$ is irreducible and nonsingular outside the point $P_0$. This proves the assertion (1). We can strengthen, in effect, the cited result as follows: Let $P_1, \cdots, P_s$ exhaust the points of $T$ such that $\pi^*(P_i)=a_i\pi^{-1}(P_i)$ with $a_i \geqq 2$ for $1 \leqq i \leqq s$; then $\mathrm{Pic}(X)$ is a finite abelian group with generators $\xi_1, \cdots, \xi_s$ and relations $a_1\xi_1=\cdots=a_s\xi_s=0$. Hence $a_1, \cdots, a_s$ are pairwise coprime, and $d=a_1\cdots a_s$.

If $s=1$, we know thence that $a_1=d$. Let $F_1$ be the closure in $\boldsymbol{P}_k^2$ of the irreducible curve $\pi^{-1}(P_1)$. Then $dF_1$ is linearly equivalent to $C$. Hence $F_1$ is a line. Since $F_1 \cap C=\{P_0\}$, we know that $F_1=l_0$. Hence $dl_0$ is a member of $\Lambda$. This proves the assertion (3).

Let $\tilde{\sigma}: \tilde{V} \rightarrow V_0 := \boldsymbol{P}_k^2$ be the shortest succession of quadratic transformations with centers at base points (including infinitely near base points) of $\Lambda$ such that the proper transform $\tilde{\Lambda}$ of $\Lambda$ by $\tilde{\sigma}$ has no base points. Let $\tilde{E}$ be the exceptional curve obtained by the last quadratic transformation in the process $\tilde{\sigma}$. Then $\tilde{E}$ is a cross-section of $\tilde{\Lambda}$, i.e., a general member of $\tilde{\Lambda}$ meets $\tilde{E}$ transversally in a single point. Let $F_i$ be the closure in $\boldsymbol{P}_k^2$ of $\pi^{-1}(P_i)$, let $\tilde{F}_i$ be the proper transform of $F_i$ by $\tilde{\sigma}$ and let $\Gamma_i$ be the member of $\tilde{\Lambda}$ for which $\tilde{F}_i$ is an irreducible component, where $1 \leqq i \leqq s$. Since $a_i \geqq 2$, $\Gamma_i$ is a reducible member. Namely, $\Gamma_i$ contains other irreducible components by means of which the component $\tilde{F}_i$ is connected to the cross-section $\tilde{E}$. Those irreducible components of $\Gamma_i$'s other than $\tilde{F}_i$'s for $1 \leqq i \leqq s$ are exceptional curves obtained by the quadratic transformations in the process $\tilde{\sigma}$. The composition $\tilde{\sigma}$ of quadratic transformations is, in effect, a composition of Euclidean transformations and *EM*-transformations, which are uniquely determined by a general member $C$ of $\Lambda$ as explained in the Introduction. This is due to the fact that $P_0$ is a one-place point for a general member $C$ of $\Lambda$. By looking at the configurations of exceptional curves obtained by the Euclidean transformations or the *EM*-transformations (cf. [5; p. 95]), we know that the totality of exceptional curves (except $\tilde{E}$) obtained in the process $\tilde{\sigma}$ is divided into at most two connected components, and that each of (irreducible) exceptional curves (except $\tilde{E}$) is contained in a member of $\tilde{\Lambda}$. This observation implies that $\tilde{\Lambda}$ contains at most two reducible fibers. Therefore $\Lambda$ has at most two multiple members. This proves the assertion (2). Q.E.D.

We have also the following:

**Lemma 2.** *Let the notations and the assumptions be as above. Then the following conditions are equivalent to each other*:

(1) *For a general member $C$ of $\Lambda$, we have*

$$(C \cdot l_0) = i(C, l_0; P_0) = d\,.$$

(2) $dl_0 \in \Lambda$, i.e., *$D$ is of the first kind.*
(3) *$\Lambda$ has only one multiple member.*

Proof. (1)$\Rightarrow$(2): By assumption, $C-\{P_0\}$ is contained in the affine plane $X_0 := \boldsymbol{P}_k^2 - l_0$. Since $C-\{P_0\}$ is isomorphic to the affine line, we may choose coordinates $x, y$ on $X_0$ so that the curve $C-\{P_0\}$ is defined by $x=0$ (cf. Abhyankar-Moh's Embedding theorem [5; p. 90]). Let $C'$ be another general member of $\Lambda$. Then $C'-\{P_0\}$ is defined by $f(x,y)=0$, where $f(x,y) \in k[x,y]$. Since $C \cap C' = \{P_0\}$, we have $f(0,y) \neq 0$, i.e., $f(x,y)=c+xg(x,y)$ with $c \in k^* := k-(0)$ and $g(x,y) \in k[x,y]$. Since $k[x,y]/(f(x,y))$ is a polynomial ring in one variable over $k$, there exists an element $c' \in k^*$ such that $x-c'$ is divisible by $f(x,y)$. Hence we may assume that $f(x,y)=x-c'$. Since $\Lambda$ is spanned by $C$ and $C'$, almost all members of $\Lambda$ restricted on $X$ are defined by equations of the form $x=c$ with $c \in k^*$. This implies that $\Lambda$ is spanned by $C$ and $dl_0$, and that every member of $\Lambda$ except $dl_0$ is irreducible and reduced.
(2)$\Rightarrow$(1): Since $P_0$ is the unique base point of $\Lambda$, we have

$$i(C, l_0; P_0) = (C \cdot l_0) = d\,.$$

(2)$\Rightarrow$(3): We have shown, in the course of the proof (1)$\Rightarrow$(2), that $dl_0$ is the unique multiple member of $\Lambda$. The implication (3)$\Rightarrow$(2) is proved in Lemma 1. Q.E.D.

Assume that $\Lambda$ has two multiple members $aF$ and $bG$. Assuming, as in the proof of Lemma 1, that $D$ consists of only one general member of $\Lambda$, Pic($X$) (with $X := \boldsymbol{P}_k^2 - D$) is a cyclic group of order $d$ generated by $[F]$ and $[G]$ with relations $a[F]=b[G]=0$, where $[F]$ and $[G]$ are the divisor classes represented by $F$ and $G$, respectively. Then it is clear that $b=\deg F$, $a=\deg G$, $d=ab$ and $G.C.D.(a,b)=1$. Therefore, we proved the assertion (1) of the theorem.

## 2. The Cremona transformation $\varphi$

We retain the notations in the previous section. Let $\tilde{\sigma}: \tilde{V} \to V_0 := \boldsymbol{P}_k^2$ be the shortest succession of quadratic transformations with centers at base points (including infinitely near base points) of $\Lambda$ such that the proper transform $\tilde{\Lambda}$ of $\Lambda$ by $\tilde{\sigma}$ has no base points. As already remarked, $\tilde{\sigma}$ is the composition of Euclidean transformations and $EM$-transformations. Since the process $\tilde{\sigma}$ is uniquely determined, the Euclidean trasnformations and the $EM$-transformations which constitute $\tilde{\sigma}$ are uniquely determined. Let $C$ be a general member of $\Lambda$, let $d_0 = i(C, l_0; P_0)$ and let $d_1 = \mathrm{mult}_{P_0} C$ (=the multiplicity of $C$ at $P_0$). Then we have $d_1 < d_0 \leqq d$. Hence the process $\tilde{\sigma}$ starts with the Euclidean transformation $\sigma_1: V_1 \to V_0$ with respect to a datum $(P_0, l_0, d_0, d_1)$. Write

$$\tilde{\sigma} = \sigma_1 \tau_1 \sigma_2 \tau_2 \cdots \sigma_n \tau_n\,,$$

where $\sigma_i$'s ($1\leqq i\leqq n$) are Euclidean transformations and $\tau_j$'s ($1\leqq j\leqq n$) are the $EM$-transformations; if the $EM$-transformation, say $\tau_j$, is not necessary in the process $\tilde{\sigma}$, we understand $\tau_j=id.$.

Let $\rho_m=\sigma_1\tau_1\cdots\sigma_m$ with $m<n$. Then the totality of (irreducible) exceptional curves obtained in the process $\rho_m$ is contained in one and the same member of the pencil $\rho'_m\Lambda$. Indeed, the pencil $\rho'_m\Lambda$ has the unique base point $P_m$ on the last exceptional curve $E_m$ which does not lie on any other exceptional curves appeared in the process $\rho_m$. Since each exceptional curve, which is an irreducible component of $\rho_m^{-1}(P_0)$, is contained in a member of $\rho'_m\Lambda$ and since $\sigma_m^{-1}(P_0)$, is connected, $E_m$ and all other exceptional curves are contained in one and the same member of $\rho'_m\Lambda$. A similar assertion holds true for $\rho_m\tau_m$. Since the last exceptional curve $\tilde{E}$ obtained in the process $\tilde{\sigma}$ is a cross-section of the pencil $\tilde{\Lambda}=\tilde{\sigma}'\Lambda$, we know easily that $\tau_n\neq id.$ if $D$ is of the first kind and $\tau_n=id.$ if $D$ is of the second kind. Moreover, if $D$ is of the second kind, one of two multiple members $aF$ and $bG$ of $\Lambda$, say $aF$, has the corresponding irreducible member $a\psi'(F)$ in the pencil $\psi'\Lambda$, where $\psi=\rho_m$ or $\rho_m\tau_m$ with $m<n$; $\psi'(G)$ then belongs to the member of $\psi'\Lambda$ containing all the exceptional curves; in the final step, $\tilde{\sigma}'(F)$ becomes an irreducible component of a reducible member of $\tilde{\sigma}'\Lambda$, whose irreducible components (except $\tilde{\sigma}'F$) are exceptional curves arising from the Euclidean transformation $\sigma_n$ (cf. the configuration in [5; p. 95]).

Since $\tilde{\Lambda}=\tilde{\sigma}'\Lambda$ has no base points, $\tilde{\Lambda}$ defines a surjective morphism $p:=\Phi_{\tilde{\Lambda}}$: $\tilde{V}\to\boldsymbol{P}_k^1$ such that, if $\boldsymbol{P}_k^2-\{P_0\}$ is naturally identified with an open set of $\tilde{V}$, $p$ coincides with the rational map $\Phi_\Lambda$ on $\boldsymbol{P}_k^2-\{P_0\}$, and that the general fibers of $p$ are non-singular rational curves. The curve $\tilde{E}$ is a cross-section of $p$. We shall construct a $\boldsymbol{P}^1$-bundle from $\tilde{V}$ by contracting, one by one, all possible exceptional curves of the first kind contained in fibers of $p$. This contraction process is possible by virtue of the following:

**Lemma 3** [5; Lemma 2.2, p. 115]. *Let $f: V\to B$ be a surjective morphism from a nonsingular projective surface $V$ onto a nonsingular complete curve $B$ such that almost all fibers are isomorphic to $\boldsymbol{P}_k^1$. Let $F=n_1C_1+\cdots+n_rC_r$ be a singular fiber of $f$, where $C_i$ is an irreducible curve, $C_i\neq C_j$ if $i\neq j$, and $n_i>0$. Then we have:*

(1) *G.C.D.$(n_1,\cdots,n_r)=1$ and* $\mathrm{Supp}(F):=\bigcup_{i=1}^{r}C_i$ *is connected.*

(2) *For $1\leqq i\leqq r$, $C_i$ is isomorphic to $\boldsymbol{P}_k^1$ and $(C_i^2)<0$.*

(3) *For $i\neq j$, $(C_i\cdot C_j)=0$ or 1.*

(4) *For three distinct indices $i$, $j$ and $l$, $C_i\cap C_j\cap C_l=\phi$.*

(5) *One of $C_i$'s, say $C_1$, is an exceptional curve of the first kind. If $\tau$: $V\to W$ is the contraction of $C_1$, then $f$ factors as $f: V\xrightarrow{\tau}W\xrightarrow{g}B$, where $g: W\to B$ is a fibration by $\boldsymbol{P}_k^1$.*

(6) *If one of $n_i$'s, say $n_1$, equals 1 then there is an exceptional curve of the first*

*kind among $C_i$'s with $2 \leqq i \leqq r$.*

Let $\theta_1$: $\tilde{V} \to W$ be the contraction of all possible exceptional curves of the first kind contained in the reducible fibers of $p$. Then $p$ factors as $p: \tilde{V} \xrightarrow{\theta_1} W \xrightarrow{q} \boldsymbol{P}_k^1$ (cf. Lemma 3, (5)) and $W$ is a $\boldsymbol{P}^1$-bundle over $\boldsymbol{P}_k^1$. If $\Delta$ is a reducible fiber of $p$, the irreducible component $\Delta_1$ of $\Delta$ meeting the cross-section $\tilde{E}$ has multiplicity 1. Hence we may assume that $\Delta_1$ is not contracted in the process $\theta_1$, i.e., $\theta_1(\Delta_1)$ is the fiber of $q$ lying over the point $p(\Delta)$, (cf. Lemma 3, (6)); we assume that this assumption is valid for all reducible fibers of $p$. Then $\theta_1(\tilde{E})$ is a cross-section of $q$ with $(\theta_1(\tilde{E})^2) = -1$. This implies that $W$ is the Hirzebruch surface of degree 1, i.e., $W = \mathrm{Proj}(\boldsymbol{O}_{P^1} \oplus \boldsymbol{O}_{P^1}(1))$. Then, by contracting $\theta_1(\tilde{E})$, we obtain the projective plane $\boldsymbol{P}_k^2$ and all fibers of $q$ become lines through the point $Q_0 := \theta_2(\theta_1(\tilde{E}))$, where $\theta_2$: $W \to \boldsymbol{P}_k^2$ is the contraction of $\theta_1(\tilde{E})$. Let $\theta = \theta_2 \cdot \theta_1$ and let $\varphi := \theta \cdot \tilde{\sigma}^{-1}: \boldsymbol{P}_k^2 \to \tilde{V} \to \boldsymbol{P}_k^2$. Then $\varphi$ is the required Cremona transformation. By construction of $\varphi$, it is now straightforward to see that the assertion (2) of the theorem is verified by $\varphi$.

The converse of the theorem follows from the analogue of Enriques' characterization theorem of ruled surfaces (cf. [7] and [11]).

## 3. Further properties of $\Lambda$

In this section, we consider the case where $D$ is of the second kind. We retain the notations and the assumptions in the previous section. Let $C$ be a general member of $\Lambda$, and let $d_0$ and $d_1$ be the same as defined as before. The pencil $\Lambda$ has two multiple members $aF$ and $bG$; we assume that $a\psi'(F)$ is an irreducible member of $\psi'\Lambda$ for $\psi = \rho_m$ or $\rho_m \tau_m$ with $m < n$. Let $b_0 = i(F, l_0; P_0)$ and let $b_1 = \mathrm{mult}_{P_0} F$; then $b_1 < b_0 \leqq b = \deg F$. Define positive integers $d_2, \cdots, d_\alpha$ and $q_1, \cdots, q_\alpha$ by the Euclidean algorithm with respect to $d_0$ and $d_1$ (cf. Introduction). Similarly, define positive integers $b_2, \cdots, b_\beta$ and $p_1, \cdots, p_\beta$ by the Euclidean algorithm with respect to $b_0$ and $b_1$.

Let $\sigma_1$: $V_1 \to V_0 := \boldsymbol{P}_k^2$ be the first Euclidean transformation with respect to a datum $(P_0, l_0, d_0, d_1)$. Then, since

$$(\sigma_1'(C)^2) = (C^2) - \sum_{i=1}^{\alpha} q_i d_i^2 = d^2 - d_0 d_1 > 0\,,$$

the pencil $\sigma_1'\Lambda$ has still a base point $P_1'$ on the last exceptional curve $E_1'$ obtained in the process $\sigma_1$. This implies that $\alpha = \beta$, $p_i = q_i$ for $1 \leqq i \leqq \alpha$ and $d_\alpha = (\sigma_1'(C) \cdot E_1') = (a\sigma_1'(F) \cdot E_1') = ab_\beta$. Therefore we have $d_i = ab_i$ for $0 \leqq i \leqq \alpha$.

For $1 \leqq m < n$, let $\eta_m = \rho_m \tau_m$, let $E_m$ be the last exceptional curve obtained in the process $\eta_m$, and let $P_m$ be the base point of the pencil $\eta_m'\Lambda$, i.e., $P_m = \eta_m'(C) \cap E_m$. Let $d_0^{(m)} = i(\eta_m'C, E_m; P_m) = (\eta_m'C \cdot E_m)$, $d_1^{(m)} = \mathrm{mult}_{P_m} \eta_m'C$, $b_0^{(m)} = i(\eta_m'F, E_m; P_m) = (\eta_m'F \cdot E_m)$ and $b_1^{(m)} = \mathrm{mult}_{P_m} \eta_m'F$. Set $d_0^{(0)} = d_0$, $d_1^{(0)} = d_1$, $b_0^{(0)} = b_0$ and $b_1^{(0)} = b_1$.

Then it is easy to obtain the following relations:

(1) $d_0^{(m)}=G.C.D.(d_0^{(m-1)}, d_1^{(m-1)})$ *and*
$b_0^{(m)}=G.C.D.(b_0^{(m-1)}, b_1^{(m-1)})$ *for* $1\leqq m<n$;

(2) $G.C.D.(d_0^{(n-1)}, d_1^{(n-1)})=G.C.D.(b_0^{(n-1)}, b_1^{(n-1)})=1$;

(3) $d_0^{(m)}=ab_0^{(m)}$ *and* $d_1^{(m)}=ab_1^{(m)}$ *for* $0\leqq m<n-1$.

Note that $\tilde{\Lambda}=\sigma'\Lambda$ has no base points and that $\sigma'F$ is the unique exceptional curve of the first kind in the reducible member of $\tilde{\Lambda}$ of which $\sigma'F$ is an irreducible component. Hence we obtain:

(4) $((\eta'_{n-1}C)^2)=d_0^{(n-1)}d_1^{(n-1)}$ *and* $((\eta'_{n-1}F)^2)=b_0^{(n-1)}b_1^{(n-1)}-1$.

Since $\eta'_{n-1}C$ is linearly equivalent to $a\eta'_{n-1}F$ and since $d_0^{(n-1)}=ab_0^{(n-1)}$, we obtain by a straightforward computation with the relation (4) taken into account:

(5) $d_0^{(n-1)}=a^2$, $d_1^{(n-1)}=ab_1^{(n-1)}-1$ *and* $b_0^{(n-1)}=a$.

Let $r_m$ be the length of the $EM$-transformation $\tau_m$ for $1\leqq m<n$. Then we have:

(6) $d^2=\sum_{m=0}^{n-1}d_0^{(m)}d_1^{(m)}+\sum_{m=1}^{n-1}(d_0^{(m)})^2r_m$, *and* $b^2=\sum_{m=0}^{n-1}b_0^{(m)}b_1^{(m)}+\sum_{m=1}^{n-1}(b_0^{(m)})^2r_m-1$.

Since $p_a(\sigma'C)=p_a(\sigma'F)=0$, we obtain:

(7) $3d=d_0+\sum_{m=0}^{n-1}d_1^{(m)}+\sum_{m=1}^{n-1}d_0^{(m)}r_m+1$, *and* $3b=b_0+\sum_{m=0}^{n-1}b_1^{(m)}+\sum_{m=1}^{n-1}b_0^{(m)}r_m$.

Since $d_0<d=ab$ and $a^2|d_0$ by (5), we have:

(8) $a<b$.

Since $b_0^{(n-1)}=a$ by (5) and $b_0^{(n-1)}|b_0^{(m)}$ for $0\leqq m<n$, we obtain from (6):

(9) $a|(b^2+1)$.

On the other hand, by viıtue of Noether's inequality (cf. Hudson [2; p. 9]), the sum of three highest (or three equally highest) multiplicities of the singular points (including infinitely near singular points) of $C$ centered at $P_0$ is larger than $d$. Hence we have:

(10) *If* $\alpha\geqq 2$ *then* $q_1=1$; *if* $\alpha=1$ *then* $q_1\leqq 2$. *Hence we have* $d<3d_1$ *and* $b<3b_1$.

Finally, we consider Yoshihara's quintic rational curve $F$ which is defined by the following equation:

$$F\colon (YZ-X^2)(YZ^2-X^2Z-2XY^2)+Y^5=0.$$

Then $F$ has only one singular point $P_0:=(X=0, Y=0, Z=1)$, which is a cuspidal point (i.e., a one-place point) of multiplicity 2. The tangent line of $F$ at $P_0$ is given by

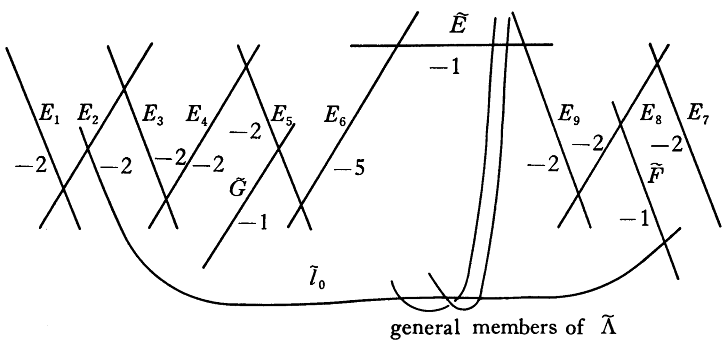
$$l_0\colon \boldsymbol{Y}=0\,.$$

New consider the conic $G$ defined by

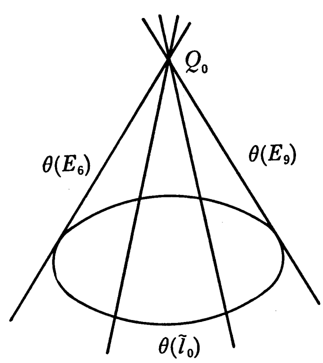
$$G\colon YZ-X^2=0\,.$$

Let $\Lambda$ be the linear pencil on $\boldsymbol{P}_k^2$ spanned by $2F$ and $5G$. Then, with the foregoing notations, we have:

$d=10$, $a=2$, $b=5$, $d_0=8$, $d_1=4$, $b_0=4$, $b_1=2$. Then $\sigma^{-1}(l_0\cup F\cup G)$ has the following configuration on $\tilde{V}$:




where $\tilde{l}_0:=\sigma'(l_0)$, $\tilde{F}:=\sigma'(F)$ and $\tilde{G}:=\sigma'(G)$. Let $\theta$ be the contraction of $\tilde{G}$, $E_5$, $E_4$, $E_3$, $E_2$, $E_1$, $\tilde{F}$, $E_8$, $E_7$ and $\tilde{E}$ in this order. Then the resulting configuration on $\boldsymbol{P}_k^2$ is given by:




where $\theta(\tilde{l}_0)$ is a nonsingular conic. Moreover, $\varphi=\theta\cdot\sigma^{-1}$ is a Cremona transformation of degree 10.

## References

[1] T. Fujita: *On Zariski Problem,* Proc. Japan Acad. **55** (1979), 106–110.

[2] H.P. Hudson: Cremona transformations in plane and space, Cambridge Univ. Press, 1927.

[3] S. Iitaka: *On logarithmic Kodaira dimension of algebraic varieties,* Complex analysis and algebraic geometry, 1977, Iwanami, Tokyo.

[4] S. Iitaka: *Geometry on complements of lines in* $\boldsymbol{P}^2$, Tokyo J. Math. **1** (1978), 1–19.

[5] M. Miyanishi: Lectures on curves on rational and unirational surfaces, Tata Institute of Fundamental Research, 1978, Springer, Berlin-Heidelberg-New York.

[6] M. Miyanishi: *Regular subrings of a polynomial ring,* Osaka J. Math. **17** (1980), 329–338.

[7] M. Miyanishi and T. Sugie: *Affine surfaces containing cylinderlike open sets,* J. Math. Kyoto Univ., **20** (1980), 11–42.

[8] H. Yoshihara: *A problem on plane rational curves,* Sugaku **31** (1979), 256–261, (in Japanese).

Masayoshi Miyanishi
Department of Mathematics
Osaka University
Toyonaka, Osaka 560
Japan

Tohru Sugie
Department of Mathematics
Kyoto University
Kyoto 606
Japan