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ON HOLOMORPHIC SECTIONS WITH SLOW GROWTH OF HERMITIAN LINE BUNDLES ON CERTAIN KÄHLER MANIFOLDS WITH A POLE

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1. Introduction. We call (M, o) a *Riemannian manifold with a pole* iff M is a Riemannian manifold and $\exp_o$: $M_o \to M$ is a global diffeomorphism. We write $r(x)$ for the distance function from o. Suppose now our (M, o) satisfies the following condition:

(1–1) There exist C^∞ functions k, K: $[0, \infty) \to [0, \infty)$ such that
 (i) $-k(r(x)) \leqq$ all the radial curvature at $x \leqq K(r(x))$,
 (ii) $\int_0^\infty tk(t)dt < \infty$,
 (iii) $\int_0^\infty tK(t)dt \leqq 1$.

In (i) above, a *radial curvature* at an $x \in M$ denotes the sectional curvature of a 2-dimensional plane in M_x which is tangent to the unique geodesic joining the pole o of M to x (if $x=o$, then simply define a radial curvature to be a sectional curvature at o). R. Greene and H. Wu have studied general properties of Riemannian manifolds with a pole in [1]. Among other things, they have shown that Riemannian manifolds with a pole satisfying condition (1–1) give r se to a very interesting class of Riemannian manifolds. Making use of their results, we shall prove the following theorem.

Theorem 1. *Let (M, o) be an m-dimensional Kähler manifold with a pole satisfying condition* (1–1) *above* $(m \geqq 2)$. *Let $L \to M$ be a holomorphic line bundle over M with a hermitian fibre metric h. Suppose the Chern form $\omega = -(i/2\pi)\partial\bar{\partial} \log h$ of the hermitian line bundle $\{L, h\}$ satisfies one of the following conditions*:

(1–2) $$\omega \text{ is non positive,}$$

(1–3) $$||\omega(x)|| \leqq v(r(x)) \qquad (x \in M),$$

where $v(t)$ is a nonnegative function on $[0, \infty)$ *which satisfies*

(1–4) $$\int_0^\infty tv(t)dt < \infty .$$

Then there exists a positive number v_0 *such that if* s *is a non-zero holomorphic section of* L *over* M *which satisfies*

$$\|s(x)\| \leqq C(1+r(x))^{v} \tag{1-5}$$

on M *for some constant* $C>0$ *and some* $v<v_0$, *then* s *is nowhere zero on* M.

When (M, o) has negative curvatures everywhere, above Theorem 1 has been proved by Greene and *Wu* (cf. Step III in the proof of Theorem J in [1]). Before them, Siu and Yau have proved above Theorem 1 when M is negatively curved and $k(t)=At^{-2-\varepsilon}$ $(\varepsilon>0)$ (cf. Proposition 2-4 in [5]). Our proof of Theorem 1 will be given by generalizing the agruments in the proofs of Step III in [1] and Proposition 2–4 in [5] cited above.

It has been conjectured that an m-dimensional Kähler manifold (M, o) with a pole satisfying condition (1–1) should be biholomorphic to C^m. In fact this is true in the case where (M, o) is negatively curved and $k(t)=At^{-2-\varepsilon}$ ([5]). More generally Greene and *Wu* have verified this conjecture in the case where (M, o) is negatively curved and $k(t)$ is nondecreasing on $[\theta, \infty)$ for some $\theta>0$ (cf. Theorem J in [1]). In the proofs of these results, one of the crucial steps was to prove above Theorem 1 in case (M, o) is negatively curved (Step III, [1], p. 188). Therefore our Theorem 1 will be of some use to study the conjecture mentioned above. In fact an application of our Theorem 1 to the case where (M, o) is positively curved will be published elsewhere.

2. Preliminaries. Let (M, o) be an m-dimensional Kähler manifold with a pole which satisfies condition (1–1). We recall several facts from Theorem C in [1] and Theorem in [4] as follows.

Fact 2–1. *Define* C^∞ *functions* $p(t)$ *and* $q(t)$ *by*

$$p''-kp=0,\ p(0)=0 \text{ and } p'(0)=1\,, \tag{2-1}$$

$$q''+Kq=0,\ q(0)=0 \text{ and } q'(0)=1\,. \tag{2-1}$$

Then the following inequalities hold on $[0, \infty)$:

$$1\leqq p'(t)\leqq \eta \text{ and } t\leqq p(t)\leqq \eta t\,, \tag{2-3}$$

$$1\geqq q'(t)\geqq \mu \text{ and } t\geqq q(t)\geqq \mu t\,, \tag{2-4}$$

where the constants η *and* μ *are positive and satisfy*

$$1\leqq \eta\leqq \exp\left\{\int_0^\infty tk(t)dt\right\}, \tag{2-5}$$

$$1\geqq \mu\geqq 1-\int_0^\infty tK(t)dt\,. \tag{2-6}$$

Fact 2–2. *Let $p(t)$ and $q(t)$ be as in* Fact 2–1. *Set $\eta^*(t)=tp'(t)/p(t)$ and $\mu^*(t)=tq'(t)/q(t)$. Then for any $t\geqq 0$, we have*

$$1\leqq\eta^*(t)\leqq\eta\,, \tag{2–7}$$

$$1\geqq\mu^*(t)\geqq\mu\,. \tag{2–8}$$

If D^2r (resp. D^2r^2) denotes the Hessian of the function r (resp. the function r^2), then the following inequalities hold on $M-\{o\}$:

$$\frac{\mu^*(r)}{r}(g-dr\otimes dr)\leqq D^2r\leqq\frac{\eta^*(r)}{r}(g-dr\otimes dr)\,, \tag{2–9}$$

$$2\mu g\leqq D^2r^2\leqq 2\eta g\,, \tag{2–10}$$

where $g=2\sum g_{j\bar{k}}dz^jd\bar{z}^k$ is the Kähler metric of (M, o).

As usual the associated Kähler form Ω of the Kähler metric $g=2\sum g_{j\bar{k}}dz^j\cdot d\bar{z}^k$ is defined by $\Omega=i\sum g_{j\bar{k}}dz^j\wedge d\bar{z}^k$.

Lemma 2–1. *The following inequalities hold in $M-\{o\}$:*

$$\frac{\eta^*(r)}{r}(\Omega-i\partial r\wedge\bar{\partial}r)\geqq i\partial\bar{\partial}r\geqq\frac{\mu^*(r)}{r}(\Omega-i\partial r\wedge\bar{\partial}r)\,, \tag{2–11}$$

$$2\eta\Omega\geqq i\partial\bar{\partial}r^2\geqq 2\mu\Omega\,, \tag{2–12}$$

$$\Omega\geqq 2i\partial r\wedge\bar{\partial}r\,. \tag{2–13}$$

Proof. Let J be the natural almost complex structure on M. Define J-invariant symmetric covariant two tensors h_1 and h_2 by

$$h_1(X, Y) = (dr\otimes dr)(X, Y)+(dr\otimes dr)(JX, JY)\,,$$
$$h_2(X, Y) = D^2r(X, Y)+D^2r(JX, JY)\,.$$

Then $h_1(JX, Y)=2i\partial r\wedge\bar{\partial}r(X, Y)$. Since $h_1(X, X)\leqq g(X, X)$, we have (2–13). On the other hand, since g is a Kähler metric, we have $h_2(x, x)=(2i\partial\bar{\partial}r)(X, JX)$. Then (2–9) and (2–13) imply (2–11). Finally (2–10) implies (2–12). q.e.d.

We need a few more facts from [1]. A Riemannian manifold with a pole (N, e) is called a *model* iff the linear isotropy group of isometries at the pole e is the full orthogonal group. Then for a point $x\in N$, all the radial curvature at x are the same. Hence there exists a C^∞ function K_N: $[0, \infty)\to\boldsymbol{R}$ such that for a point x, any radial curvature at x is equal to $k_N(r_N(x))$, where $r_N: N\to[0, \infty)$ is the distance function from e. The k_N is called the *radial curvature function* of the model (N, e). Moreover the metric g_N of N relative to geodesic polar coordinate centered at e assumes the form

$$g_N = dt^2+f(t)^2d\theta^2\,, \tag{2–14}$$

where f is a C^∞ function on $[0, \infty)$ which satisfies $f>0$ on $(0, \infty)$ and

(2–15) $$f''+k_N f=0 \text{ with } f(0)=0 \text{ and } f'(0)=1\,.$$

Conversely for any C^∞ function $f(t)$ on $[0, \infty)$ satisfying $f>0$ on $(0, \infty)$, $f(0)=0$ and $f'(0)=1$, there exists uniquely (up to isometry) a model (N, e) such that (2–14) holds. Then the radial curvature function k_N is equal to $-f''/f$. Therefore by (2–3) we have the following fact (cf. p. 60 of [1]).

Fact 2–3. *Consider the function $p(t)$ defined in Fact 2–1. Then there exists a 2 m dimensional model (N, e) whose metric relative to geodesic polar coordinates centered at e is given by*

$$g_N = dt^2 + p(t)^2 d\theta^2\,,$$

and the radial curvature function k_N is exactly $-k$.

Now by Proposition 2.15 (Laplacian Comparison Theorem) of [1], we have the following fact.

Fact 2–4. *Let (M, o) be a Kähler manifold with a pole satisfying condition* (1–1) *and (N, r) the model constructed in Fact 2–3. Let $f(t)$ be a nondecreasing C^∞ function on $(0, \infty)$. Then for every $x\in M-\{o\}$ and $y\in N-\{e\}$ such that $r(x)=r_N(y)$, we have*

$$\Delta f(r)\,(x) \leqq \Delta f(r_N)\,(y)\,.$$

Lemma 2–2. *Let (M, o) be a Kähler manifold with a pole satisfying condition* (1–1). *Let $p(x)$ be the C^∞ function defined in Fact 2–1. For a positive number $R>0$, define a C^∞ function f_R on $(0, \infty)$ by*

$$f_R(t) = \int_t^R \frac{ds}{p(s)^{2m-1}}\,.$$

Then we have $\Delta f_R(r) \geqq 0$ on $M-\{o\}$.

Proof. Let (N, e) be the model constructed in Fact 2–3. Let $\{x^1, \cdots, x^{2m}\}$ be the geodesic polar coordinate system of N centered at e such that $x^1 = r_N$. Then on $N-\{e\}$ we see

$$\begin{aligned}\Delta f_R(r_N) &= \frac{1}{\sqrt{G}} \sum_{A,B} \frac{\partial}{\partial x^A}\left(\sqrt{G}\, g_N^{AB} \frac{\partial}{\partial x^B} f_R(r_N)\right)\\ &= (p(t)^{2(2m-1)})^{-1/2} \frac{d}{dt}\left\{(p(t)^{2(2m-1)})^{1/2} \frac{d}{dt} f_R(t)\right\}\\ &= 0\,.\end{aligned}$$

Since $-f_R(t)$ is atnondecreasing C^∞ function on $(0, \infty)$, Fact 2–4 implies $\Delta(-f_R(r))$

$(\leqq \Delta(-f_R(r_N))=0$ on $M-\{o\}$. q.e.d.

Lemma 2–3. *Let* (M, o) *be a Kähler manifold with a pole satisfying condition* (1–1). *Let* $q(t)$ *be the* C^∞ *function defined in Fact* 2–1. *Set* $F(t)=\exp\left(2\int_1^t \frac{dt}{q}\right)$. *Then we have the following*:

(i) $F(r)$ *is an* C^∞ *function on* M,

(ii) $F(r)/q(r)^2$ *is a positive monotone increasing* C^∞ *function on* $[0, \infty)$

(iii) $\frac{2\mu F(r)}{q(r)^2}\Omega \leqq i\partial\bar{\partial}F(r) \leqq \frac{2(1+\eta)F}{q(r)^2}$,

(iv) $i\partial\bar{\partial}\log F(r)\geqq 0$ *on* $M-\{o\}$.

Proof. As we obtain Fact 2–3, there exists a Riemannian metric g on R^2 which can be written as

$$g = dt^2+q(t)^2 d\theta^2$$

on $R^2-\{o\}$, where (t, θ) is the usual polar coordinates. We put a complex structure on R^2 so that g becomes a Kähler metric. Define a map I: $R^2-\{o\}\to R^2-\{o\}$ by

$$I(t, \theta) = \left(\exp\int_1^t \frac{dt}{q}, \theta\right),$$

where (t, θ) is the polar coordinates on $R^2-\{o\}$. Then I is a diffeomorphism. Since we have

$$I^*(dt^2+t^2d\theta^2) = \left\{\frac{\exp\left(\int_1^t \frac{dt}{q}\right)}{q}\right\}^2 (dt^2+q^2d\theta^2)\,, \tag{2–16}$$

I is a conformal map. Hence, if we consider I to be a C-valued function, I is a holomorphic function on $R^2-\{o\}$. Since I is bounded on a neighbourhood of o, I can be extended holomorphically to o and we have $I(o)=o$. Set $\tilde{F}(s)=|I((s, o))|^2$ for any $s\in R$. Then $\tilde{F}$ is an even C^∞ function on R and $F(t)=\tilde{F}(t)$ for any $t>0$. Since r^2 is a C^∞ function on M, we know that $F(r)$ is a C^∞ function on M. Since I is holomorphic at o, I is a biholomorphic map. Consequently, (2–16) implies that $F(t)/q(t)^2$ is an even positive C^∞ function. Hence $F(r)/q(r)^2$ a positive C^∞ function on M. On the other hand, we have

$$\left(\frac{F(t)}{q(t)^2}\right)' = \frac{2F(t)}{q(t)^3}(1-q'(t))\,.$$

By (2–4), we see $F(t)/q(t)^2$ is an increasing function. Since we have

$$i\partial\bar{\partial}F(r) = F'(r)i\partial\bar{\partial}r+F''(r)i\partial r\wedge\bar{\partial}r$$

$$= \frac{2F(r)}{q(r)} i\partial\bar{\partial}r + \frac{2F(r)}{q(r)^2}(2-q'(r))i\partial r\wedge\bar{\partial}r\,,$$

by (2·–11) and (2–4), we see

$$\begin{aligned} i\partial\bar{\partial}F(r) &\geqq \frac{2F(r)}{q(r)}\frac{\mu^*(r)}{r}(\Omega - i\partial r\wedge\bar{\partial}r) + \frac{2F(r)}{q(r)^2}(2-q'(r))i\partial r\wedge\bar{\partial}r \\ &= \frac{2F(r)}{q(r)}\{q'\Omega + 2(1-q')i\partial r\wedge\bar{\partial}r\} \geqq \frac{2\mu F(r)}{q(r).^2}\Omega\,. \end{aligned}$$

On the other hand, by (2.11) and (2.3), we see

$$\begin{aligned} i\partial\bar{\partial}F(r) &\leqq \frac{2F(r)}{q(r)}\frac{\eta^*(r)}{r}(\Omega - i\partial r\wedge\bar{\partial}r) + \frac{2F(r)}{q(r)^2}(2-q'(r))i\partial r\wedge\bar{\partial}r \\ &= \frac{2F(r)}{q(r)}\frac{p'(r)}{p(r)}(\Omega - i\partial r\wedge\bar{\partial}r) + \frac{2F(r)}{q(r)^2}(2-q'(r))i\partial r\wedge\bar{\partial}r \\ &\leqq \frac{2F(r)}{q(r)}\frac{p'(r)}{p(r)}\Omega + \frac{2F(r)}{q(r)^2}\cdot 2i\partial r\wedge\bar{\partial}r \\ &\leqq \frac{2\eta F(r)}{q(r)^2}\Omega + \frac{2F(r)}{q(r)^2}\Omega\,. \end{aligned}$$

Finally by (2–11), we have

$$\begin{aligned} i\partial\bar{\partial}\log F(r) &= \frac{2}{q(r)}i\partial\bar{\partial}r - \frac{2q'(r)}{q(r)^2}i\partial r\wedge\bar{\partial}r \\ &\geqq \frac{2}{q(r)}\frac{\mu^*(r)}{r}(\Omega - i\partial r\wedge\bar{\partial}r) - \frac{2q'(r)}{q(r)^2}i\partial r\wedge\bar{\partial}r \\ &= \frac{2q'(r)}{q(r)^2}\Omega - \frac{4q'(r)}{q(r)^2}i\partial r\wedge\bar{\partial}r \\ &\geqq 0\,. \end{aligned}$$

q.e.d.

3. A volume estimate for analytic subsets. Let V be a closed analytic subset in M of pure dimension n. For a positive number t, we set $B(t)=\{x\in M,\ r(x)<t\}$ and $\partial B(t)=\{x\in M,\ r(x)=t\}$. Then $\overline{B(t)}$ is compact and $\partial B(t)$ is a hypersurface in M. We write Vol $(V\cap B(t))$ for the volume of $V\cap B(t)$. Then we have

$$\mathrm{Vol}\,(V\cap B(t)) = \frac{1}{2^n n!}\int_{V\cap B(t)}\Omega^n\,.$$

As usual we set $d^c = i(\bar{\partial}-\partial)/2$ so that $dd^c = i\partial\bar{\partial}$. In this section we shall prove

Proposition 3–1. *Let (M, o) be as in Theorem 1 in section 1. Then there exists a positive constant A depending only on (M, o) such that for any closed an-*

alytic subset V in M of pure dimension n, we have Vol $(V\cap B(t))\geqq Al(V)t^{2n}$ for $t\geqq 0$, where $l(V)$ is the multiplicity of V at o.

Proof. Set $B(t,s)=B(s)-\overline{B(t)}$ for $0<t<s$. Using Stokes Theorem for analytic subsets (cf. [3] or Theorem 1.28 in [2]), for $0<t<s$, we have

$$\begin{aligned}
&\frac{1}{F(s)^n}\int_{V\cap B(s)}(dd^cF(r))^n-\frac{1}{F(t)^n}\int_{V\cap B(t)}(dd^cF(r))^n\\
&=\frac{1}{F(s)^n}\int_{V\cap\partial B(s)}d^cF(r)\wedge(dd^cF(r))^{n-1}\\
&\quad-\frac{1}{F(t)^n}\int_{V\cap\partial B(t)}d^cF(r)\wedge(dd^cF(r))^{n-1}\\
&=\int_{V\cap\partial B(t)}\frac{d^cF(r)}{F(s)}\wedge\left(\frac{dd^cF(r)}{F(s)}-\frac{dF(r)\wedge d^cF(r)}{F(s)^2}\right)^{n-1}\\
&\quad-\int_{V\cap\partial B(t)}\frac{d^cF(r)}{F(t)}\wedge\left(\frac{dd^cF(r)}{F(t)}-\frac{dF(r)\wedge d^cF(r)}{F(t)^2}\right)^{n-1}
\end{aligned}$$

($dF(r)$ being zero on $B(t)$ and $B(s)$)

$$\begin{aligned}
&=\int_{V\cap\partial B(t)}d^c\log F(r)\wedge(dd^c\log F(r))^{n-1}\\
&\quad-\int_{V\cap\partial B(t)}d^c\log F(r)\wedge(dd^c\log F(r))^{n-1}\\
&=\int_{V\cap B(s)}(dd^c\log F(r))^n-\int_{V\cap B(t)}(dd^c\log F(r))^n\\
&=\int_{V\cap B(t,s)}(dd^c\log F(r))^n\\
&\geqq 0
\end{aligned}$$

(cf. (ii) of Lemma 2–3). Therefore we know

$$\frac{1}{F(t)^n}\int_{V\cap B(t)}(dd^cF(r))^n$$

is a non-negative increasing function for $t>0$. In particular there exists

$$\lim_{t\downarrow 0}\frac{1}{F(t)^n}\int_{V\cap B(t)}(dd^cF(r))^n$$

which is denoted by $n^*(V, o)$. Now by (iii) and (ii) of Lemma 2–3, we have

$$\begin{aligned}
\text{(3–1)}\qquad \frac{\mathrm{Vol}\,(V\cap B(t))}{t^{2n}}&=\frac{1}{2^n n!t^{2n}}\int_{V\cap B(t)}\Omega^n\\
&\geqq\frac{1}{2^{2n}n!(1+\eta)^n t^{2n}}\cdot\int_{V\cap B(t)}\left(\frac{q(r)^2}{F(r)}\right)^n(dd^cF(r))^n
\end{aligned}$$

$$\geqq \frac{1}{2^{2n}n!(1+\eta)^n t^{2n}} \cdot \frac{q(t)^{2n}}{F(t)^n} \int_{V\cap B(t)} (dd^c F(r))^n$$

$$\geqq \frac{\mu^{2n}}{2^{2n}n!(1+\eta)^n} \frac{1}{F(t)^n} \int_{V\cap B(t)} (dd^c F(r))^n .$$

Hence we have

$$\text{(3–2)} \qquad \mathrm{Vol}\,(V\cap B(t)) \geqq \frac{\mu^{2n}}{2^{2n}n!(1+\eta)^n} n^*(V, o) t^{2n}$$

for $t\geqq 0$. By (2–4), we see

$$\text{(3–3)} \qquad \frac{1}{t^2} \leqq \frac{1}{F(t)} \leqq \frac{1}{t^{2\mu}}$$

for $0<t\leqq 1$. By (ii) and (iii) of Lemma 2–3, there exists a positive constant B_1 such that

$$\text{(3–4)} \qquad i\partial\bar{\partial}F(r) \geqq B_1\Omega$$

on $B(1)$. Then (3–3) and (3–4) imply

$$\text{(3–5)} \qquad \frac{1}{F(t)^n}\int_{V\subset B(t)} (dd^c F(r))^n \geqq \frac{B_1{}^n}{t^{2n}} \int_{V\cap B(t)} \Omega^n$$

for $0<t\leqq 1$. Now take sufficiently small $\varepsilon>0$ so that $B(\varepsilon)$ is a holomorphic local coordinate neighbourhood with a holomorphic local coordinate system $\{z^1, \cdots, z^m\}$ ($z^i(o)=0$, $1\leqq i\leqq m$). Let $g_0=\sum dz^j d\bar{z}^j$ be the usual flat Kähler metric on $B(\varepsilon)$. Then $\alpha_0=\{i\partial\bar{\partial}(\sum|z^j|^2)\}/2$ is the associated Kähler form of g_0. By using $i\partial\bar{\partial}\log(\sum|z^j|^2)\geqq 0$ on $B(\varepsilon)-\{o\}$, the same argument to have obtained $n^*(V, o)$ implies that if we set

$$n(t, o) = \frac{1}{t^{2n}}\int_{V\cap\{\sum|z^j|^2<t^2\}} \alpha_0{}^n ,$$

then $n(t, o)$ is an increasing function of t and $\lim_{t\to 0} n(t, o)=B_2 l(V)$ where B_2 is a universal constant (cf. Corollary 1.29 in [2]). Since there exists a positive constant B_3 such that

$$\frac{1}{B_3}\alpha_0 \leqq \Omega \leqq B_3\alpha_0$$

on $B\left(\frac{\varepsilon}{2}\right)$, we know there exists a positive constant B_4 such that

$$\text{(3–6)} \qquad \frac{1}{t^{2n}}\int_{V\cap B(t)} \Omega^n \geqq B_4 l(V)$$

for sufficiently small t. By (3–5) and (3–6) we have

$$n^*(V, o) \geqq B_1^n B_4 l(V)\,. \tag{3–7}$$

Then (3–7) and (3–2) imply Proposition 3–1. q.e.d.

Corollary. *For a positive number R, there exists a positive constant $B^*(R)$ such that for $t \geqq R$ we have*

$$\mathrm{Vol}\,(V \cap B(t)) \geqq B^*(R) V^*(R) t^{2n}\,,$$

where $V^*(R) = \int_{V \cap B(R)} (dd^c F(r))^n$.

Proof. By (3–1) we see

$$\frac{\mathrm{Vol}\,(V \cap B(t)}{t^{2n}} \geqq \frac{2^{2n} n!\,(1+\eta)^n}{2n} \frac{1}{F(R)^n} V^*(R)\,. \qquad \text{q.e.d.}$$

4. Proof of Theorem 1. We keep the notation of the previous sections. Let $p(t)$ be the function defined in Fact 2–1. For any positive number R, define a C^∞ function F_R on $M-\{o\}$ by $F_R(x) = f_R(r(x))$ where f_R is the function defined in Lemma 2–2.

Let s by any nonzero holomorphic section of L such that

$$\{x \in M;\ s(x) = 0\} \text{ is not empty,} \tag{4–1}$$

$$\|s(x)\| \leqq C(1+r(x))^\nu \tag{4–2}$$

for some positive constants C and ν. Let V be the divisor defined by the zeros of s. For $R>1$, we set $B(R, 1) = B(R) - B(1)$. Then Green's formula implies

$$\begin{aligned}
&\int_{B(R,1)} \Delta F_R \cdot \log \|s\|^2 - \int_{B(R,1)} F_R \cdot \Delta \log \|s\|^2 \\
&= \int_{\partial B(R)} \log \|s\|^2 * dF_R - \int_{\partial B(1)} \log \|s\|^2 * dF_R \\
&- \int_{\partial B(r)} F_R * d \log \|s\|^2 + \int_{\partial B(1)} F_R * d \log \|s\|^2\,.
\end{aligned}$$

By (2–3) we have

$$\frac{1}{\eta^{2m-1}} \int_t^R \frac{1}{t^{2m-1}} \leqq f_R(t) \leqq \int_t^R \frac{dt}{t^{2m-1}}\,.$$

Hence we obtain

$$\frac{1}{(2m-1)\eta^{2m-1}} \left(\frac{1}{t^{2m-2}} - \frac{1}{R^{2m-2}} \right) \leqq f_R(t) \leqq \frac{1}{2m-2} \left(\frac{1}{t^{2m-2}} - \frac{1}{R^{2m-2}} \right). \tag{4–1}$$

Therefore we have an estimate:

$$\begin{aligned}&\left|-\int_{\partial B(1)}\log\|s\|^2*dF_R+\int_{\partial B(1)}F_R*d\log\|s\|^2\right|\\&=\left|-f_R'(1)\int_{\partial B(1)}\log\|s\|^2*dr+f_R(1)\int_{\partial B(1)}*d\log\|s\|^2\right|\\&\leqq\frac{1}{p(1)^{2m-1}}\left|\int_{\partial B(1)}\log\|s\|^2*dr\right|+\frac{1}{2m-2}\left(1-\frac{1}{R^{2m-2}}\right)\left|\int_{\partial B(1)}*d\log\|s\|^2\right|\\&=0(1),\end{aligned}$$

where $0(1)$ stands for a bounded term as $R\to\infty$. Since

$F_R\equiv 0$ on $\partial B(R)$, we have

$$\text{(4–2)}\qquad \begin{aligned}&\int_{B(R,1)}\Delta F_R\cdot\log\|s\|^2-\int_{B(R,1)}F_R\cdot\Delta\log\|s\|^2\\&=\int_{\partial B(R)}\log\|s\|^2 f_R'(R)*dr+O(1).\end{aligned}$$

On the other hand, Poincaré-Lelong's formula implies

$$\text{(4–3)}\qquad \frac{2^m m!}{2\pi}\int_{B(R,1)}F_R\Delta\log\|s\|^2=\int_{B(R\ 1)\cap V}F_R\Omega^{m-1}-\int_{B(R,1)}F_R\omega\wedge\Omega^{m-1}$$

(cf. [3] or Theorem 1.11 in [2]). Since $\Delta F_R\geqq 0$ by Lemma 2–2, we see by (4–2)

$$\int_{B(R,1)}\Delta F_R\cdot\log\|s\|^2\leqq\int_{B(R,1)}\Delta F_R\cdot\log\{C(1+r)^\nu\}.$$

Now by Green's formula, we have

$$\begin{aligned}&\int_{B(R,1)}\Delta F_R\cdot\log\{C(1+r)^\nu\}\\&=\int_{B(R,1)}F_R\cdot\Delta\log\{C(1+r)^\nu\}+\int_{\partial B(R)}\log\{C(1+r)^\nu\}*dF_R\\&\quad-\int_{\partial B(1)}\log\{C(1+r)^\nu\}*dF_R-\int_{\partial B(R)}F_R*d\log\{C(1+r)^\nu\}\\&\quad+\int_{\partial B(1)}F_R*d\log\{C(1+r)^\nu\}\\&=\int_{B(R,1)}F_R\cdot\Delta\log C(1+r)^\nu\}+\int_{\partial B(R)}\log\{C(1+r)^\nu\}*dF_R+O(1).\end{aligned}$$

Therefore we have

$$\text{(4–4)}\qquad \int_{B(R,1)}\Delta F_R\cdot\log\|s\|^2\leqq\int_{B(R,1)}F_R\cdot\Delta\log\{C(1+r)^\nu\}$$

$$+\int_{\partial B(R)}\log\{C(1+r)^\nu\}\, f_R'(R)*dr+O(1)\,.$$

From (4–4), (4–2) and (4–3), we obtain

$$\begin{aligned}&\int_{B(R,1)}F_R\cdot\Delta\log\{C(1+r)^\nu\}+\int_{\partial B(R)}\log\{C(1+r)^\nu\}\, f_R'(R)*dr+O(1)\\&\geqq\int_{B(R,1)}\Delta F_R\cdot\log\|s\|^2\\&=\int_{B(R,1)}F_R\cdot\Delta\log\|s\|^2+\int_{\partial B(R)}\log\|s\|^2 f_R'(R)*dr+O(1)\\&=\frac{2\pi}{2^m m!}\int_{B(R,1)\cap V}F_R\Omega^{m-1}-\frac{2\pi}{2^m m!}\int_{B(R,1)}F_R\omega\wedge\Omega^{m-1}\\&\quad+\int_{\partial B(R)}\log\|s\|^2 f_R'(R)*dr+O(1)\,.\end{aligned}$$

Since $f_R'(R)<0$, we see

$$\int_{\partial B(R)}\log\{C(1+r)^\nu\}\, f_R'(R)*dr\leqq\int_{\partial B(R)}\log\|s\|^2 f_R'(R)*dr\,.$$

Therefore we obtain

$$\begin{aligned}\text{(4–5)}\qquad&\frac{2^m m!}{2\pi}\int_{B(R,1)}F_R\cdot\Delta\log\{C(1+r)^\nu\}\\&\geqq\int_{B(R,1)\cap V}F_R\Omega^{m-1}-\int_{B(R,1)}F_R\omega\wedge\Omega^{m-1}+O(1)\,.\end{aligned}$$

Now by (2–9) and (2–7), we have

$$\text{(4–6)}\qquad \Delta r\leqq\frac{(2m-1)\eta}{r}\,.$$

Then (4–6) implies

$$\begin{aligned}\text{(4–7)}\qquad \Delta\log C(1+r)&=\frac{\Delta r}{1-r}-\frac{1}{(1+r)^2}\\&\leqq\frac{(2m-1)\eta}{(1+r)r}\,.\end{aligned}$$

Now (4–7) and (4–1) imply

$$\begin{aligned}\text{(4–8)}\qquad \int_{B(R,1)}F_R\Delta\log\{C(1+r)^\nu\}&\leqq\nu\int_{B(R,1)}\frac{1}{(2m-2)r^{2m-2}}\cdot\frac{(2m-1)\eta}{(1+r)r}\\&\leqq\frac{(2m-1)S(2m-1)\eta^{2m}\nu}{2m-2}\log(1+R)\,,\end{aligned}$$

where $S(2m-1)$ denotes the Euclidian volume of $(2m-1)$ dimensional unit sphere. By the same way we have

$$\begin{aligned}\int_{B(R,1)\cap V} F_R\Omega^{m-1} &= \int_1^R dt\int_{\partial B(t)\cap V} F_R\iota\Big(\frac{\partial}{\partial r}\Big)\Omega^{m-1} \\ &= \int_1^R f_R(t)dt\int_{\partial B(t)\cap V}\iota\Big(\frac{\partial}{\partial r}\Big)\Omega^{m-1} \\ &= \int_1^R\Big[\frac{d}{dt}\Big\{f_R(t)\int_1^t dt\int_{\partial B(t)\cap V}\iota\Big(\frac{\partial}{\partial r}\Big)\Omega^{m-1}\Big\}\Big]dt \\ &\quad -\int_1^R\Big\{f_R'(t)\int_1^t dt\int_{\partial B(t)\cap V}\iota\Big(\frac{\partial}{\partial r}\Big)\Omega^{m-1}\Big\}dt \\ &= 2^{m-1}(m-1)!\,[f_R(t)\operatorname{Vol}(B(t)\cap V)]_1^R \\ &\quad +2^{m-1}(m-1)!\int_1^R\frac{\operatorname{Vol}(B(t,1)\cap V)}{p(t)^{2m-1}}\,.\end{aligned}$$

Then by (2–3) and $f_R(R)=0$, we have

$$\text{(4–9)}\qquad \int_{B(R,1)\cap V} F_R\Omega^{m-1} \geqq \frac{2^{m-1}(m-1)!}{\eta^{2m-1}}\int_1^R\frac{\operatorname{Vol}(B(t)\cap V)}{t^{2m-1}}dt+O(1)\,.$$

From (4–5), (4–8) and (4–9) we obtain

$$\text{(4–10)}\qquad \begin{aligned}E_1\eta\nu\log(1+R) &\geqq \frac{2^{m-1}(m-1)!}{\eta^{2m-1}}\int_1^R\frac{\operatorname{Vol}(B(t)\cap V)}{t^{2m-1}}dt \\ &\quad -\int_{B(R,1)} F_R\omega\wedge\Omega^{m-1}+O(1)\,,\end{aligned}$$

where E_1 is a positive constant depending only on m.

If the Chern form ω satisfies (1–3), we have

$$\text{(4–11)}\qquad -\int_{B(R,1)} F_R\omega\wedge\Omega^{m-1}\geqq 0\,.$$

If ω satisfies (1–4), we see by (4–1)

$$\text{(4–12)}\qquad \begin{aligned}\Big|\int_{B(R,1)} F_R\omega\wedge\Omega^{m-1}\Big| &\leqq D\int_{B(R,1)} F_R\|\omega\|\Omega^m \\ &\leqq D'S(2m-1)\int_1^R f_R(t)v(t)t^{2m-1}dt \\ &\leqq D''\int_1^R tv(t)dt = O(1)\,,\end{aligned}$$

where D, D', D'' are constants independent of s. Therefore by (4–10), (4–11) and (3–12), we have

(4–13) $$E_1\eta\nu\log(1+R) \geqq \frac{2^{m-1}(m-1)!}{\eta^{2m-1}}\int_1^R \frac{\mathrm{Vol}(B(t)\cap V)}{t^{2m-1}}dt+O(1)\,.$$

Therefore from Proposition 3–1, we obtain

$$\begin{aligned}E_1\nu\log(1+R) &\geqq \frac{2^{m-1}(m-1)!}{\eta^{2m}}\int_1^R \frac{Al(V)t^{2m-2}}{t^{2m-1}}dt+O(1)\\ &\geqq E_2 l(V)\log R+O(1)\,,\end{aligned}$$

where E_2 is a positive constant depending only on (M, o). Hence, taking the limit, we have

(4–14) $$\nu \geqq \frac{E_2}{E_1}l(V)\,,$$

where E_1 and E_2 are positive constants depending only on (M, o).

Lemma 4–1. *Let* (M, o) *and* $\{L, h\}$ *be as in* Theorem 1. *For a positive number* ν, *denote by* $\Gamma(M, L; \nu)$ *the complex vector space of holomorphic sections* s *over* M *which satisfy*

(4–15) $$||s(x)|| \leqq C(1+r(x))^{\nu}$$

for some positive C. *Then there exists a positive number* ν^* *depending only on* (M, o) *such that the dimension of* $\Gamma(M, L; \nu^*)$ *is at most one.*

Proof. Take $E_2/2E_1$ as ν^*, where E_1, E_2 are as in (4–14). Take any holomorphic section s in $\Gamma(M, L; \nu^*)$. Then by (4–15), we see $l(V)=0$, *i.e.*, $s(o)\neq 0$. Suppose there were two elements s_1 and s_2 in $\Gamma(M, L; \nu^*)$ which are linearly independent. Since $s_1(o)\neq 0$ and $s_2(o)\neq 0$, there would exist a number a such that $(as_1+s_2)\,(o)=0$. Then as_1+s_2 should be zero. This is a contradiction.

Proof of Theorem 1. Let ν^* be as in Lemma 4–1. It is enough to check the case when $\Gamma(M, L; \nu^*)$ contains an element s_0 such that $\{x\in M; s_0(x)=0\}$ is non empty. Fix a sufficiently large number R_0 so that

(4–16) $$B(R)\cap\{x\in M;\, s_0(x)=0\}\neq\emptyset\,.$$

Then by (4–13) and Corollary to Proposition 3–1, for $R>R_0$ we have

$$\begin{aligned}E_1\nu\log(1+R) &\geqq \frac{2^{m-1}(m-1)!}{\eta^{2m}}\int_{R_0}^R \frac{\mathrm{Vol}(B(t)\cap V)}{t^{2m-1}}dt+O(1)\\ &\geqq \frac{2^{m-1}(m-1)!\,B^*(R_0)V^*(R_0)}{\eta^{2m}}\log R+O(1)\,.\end{aligned}$$

Hence, taking the limit, we see

$$\nu \geqq \frac{E_3}{E_1} B^*(R_0) V^*(R_0) , \tag{4–17}$$

where E_1, E_3 are positive constants depending only on (M, o), and $B^*(R_0)$ is positive. By (4–16), we see $V^*(R_0)$ is positive. Now set

$$\nu_0 = \min \left\{ \nu^*, \frac{E_3 B^*(R_0) V^*(R_0)}{2E_1} \right\} .$$

Then we have $\Gamma(M, L; \nu_0)=0$. In fact take any element s in $\Gamma(M, L; \nu_0)$. Then by Lemma 4–1, there exists a real number a such that $s=as_0$. Suppose $a \neq 0$. Then since $\{x \in M; s(x)=0\} = \{x \in M; s_0(x)=0\}$, (4–17) implies $\nu_0 \geqq E_3 B^*(R_0) V^*(R_0)/E_1$. This is a constradiction. q.e.d.

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