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ON THE HYPERSURFACES OF HERMITIAN SYMMETRIC SPACES OF COMPACT TYPE II

YOSHIO KIMURA

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1. Introduction.

Let M be an irreducible Hermitian symmetric space of compact type and let L be a holomorphic line bundle over M. Denote by $\Omega^p(L)$ the sheaf of germs of L-valued holomorphic p-forms on M. In the previous paper [1] we have studied the cohomology groups $H^q(M, \Omega^p(L))$ of M if M is of type *BDI*, *EIII* or *EVII*. This note is the continuation of [1], and we retain the notations introduced in [1]. In this note we study the cohomology groups $H^q(M, \Omega^p(L))$ of M of type *AIII*, *CI* or *EIII* and show the following theorem.

Theorem. *Let M be an irreducible Hermitian symmetric space of compact type but not a complex projective space nor a complex quadric of even dimension. Let V be a hypersurface of M whose degree $\geqq 2$. Then*

$$H^0(V, \Theta) = (0)$$

where Θ is the sheaf of germs of holomorphic vector fields on V.

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2. Proof of the Theorem.

Theorem 8 and Lemma 3 in the previous paper [1] is incorrect. The followings are true.

Theorem 8. *Let M be an irreducible Hermitian symmetric space of type EIII, EVII or a complex quadric of odd dimension (resp. a complex quadric of even dimension $\geqq 4$), and let V be a hypersurface of M whose degree is d. Then*

$$H^0(V, \Theta) = (0) \qquad \text{if } d \geqq 2 \text{ (resp. } d \geqq 3)$$

Lemma 3. *Let M be an n-dimensional irreducible Hermitian symmetric space of compact type EIII, EVII or a complex quadric of odd dimension (resp. a* *complex quadric of even dimension* $\geqq 4$). *Then*

$$H^q(M, \Omega^p(E_{-k\omega_j})) = (0)\,, \quad H^{q+1}(M, \Omega^p(E_{-(k-d)\omega_j})) = (0)$$

for $p+q=n+1$, $k=pd-\lambda$ *if* $2\leqq p\leqq n-1$ *and* $d\geqq 2$ (*resp.* $d\geqq 3$).

In the proof of Theorem 8 in [1], we have to replace n by $n-1$ since $\dim V=n-1$. Thus we need the above Lemma 3, which is verified by the computations in [1].

From the above theorem we may assume that M is of type *AIII*, *CI* or *DIII* but not a complex projective space nor a complex quadric. If we prove the following proposition for such a space M we get the Theorem the same way as in the proof of Theorem 8 in [1].

Proposition 1. If $d\geqq 2$

$$H^q(M, \Omega^p(E_{-k\omega_j})) = (0)\,, \quad H^{q+1}(M, \Omega^p(E_{-(k-d)\omega_j})) = (0)\,,$$

for $p+q\geqq n+1$, $k=pd-\lambda$.

By Theorems 1 and 2 in [1], we get Proposition 1 if we prove the following inequalities:

$$\#\{\beta\in\Delta(\mathfrak{n}^+);\, (\sigma\delta+(dn(\sigma)-\lambda)\omega_j,\, \beta)<0\}<n+1-n(\sigma)\,,$$
$$\#\{\beta\in\Delta(\mathfrak{n}^+);\, (\sigma\delta+(dn(\sigma)-d-\lambda)\omega_j,\, \beta)<0\}<n+2-n(\sigma)\,,$$

for $\sigma\in W^1$ and $d\geqq 2$.

Since $(\omega_j, \beta)>0$ for $\beta\in\Delta(\mathfrak{n}^+)$, we only have to prove the inequalities in the case of $d=2$. Recall that $\#\Delta(\mathfrak{n}^+)=n$. We can restate the inequalities, in the case of $d=2$, as follows:

Proposition 2. For $\sigma\in W^1$

$$\#\{\beta\in\Delta(\mathfrak{n}^+);\, (\sigma\delta,\, \beta)\geqq((\lambda-2n(\sigma))\omega_j,\, \beta)\}>n(\sigma)-1\,,$$
$$\#\{\beta\in\Delta(\mathfrak{n}^+);\, (\sigma\delta,\, \beta)\geqq((\lambda+2-2n(\sigma))\omega_j,\, \beta)\}>n(\sigma)-2\,.$$

In the following we shall prove Proposition 2 in each case.

2.1. The case that M is of type AIII but not a complex projective space, that is $M=SU(l+1)/S(U(j)\times U(l+1-j))$, $l\geqq 3$ and $2\leqq j\leqq l-1$. We immediately see that $n=j(l+1-j)$ and $\lambda=l+1$. The Dynkin diagram of Π is as follows:

$$\underset{\alpha_1}{\circ}\text{———}\underset{\alpha_2}{\circ}\text{—}\cdots\text{—}\underset{\alpha_j}{\circ}\text{—}\cdots\text{—}\underset{\alpha_{l-1}}{\circ}\text{———}\underset{\alpha_l}{\circ}$$

Let $\{\varepsilon_i;\, 1\leqq i\leqq l+1\}$ be a usual basis of R^{l+1}. Then we have:

$$\begin{aligned}
&\mathfrak{h}_0 = \{\sum_{i=1}^{l+1} a_i\varepsilon_i \in R^{l+1};\ \sum_{i=1}^{l+1} a_i = 0\}\ ,\\
&\Delta = \{\varepsilon_i - \varepsilon_k;\ 1\leqq i,\ k\leqq l+1,\ i\neq k\}\ ,\\
&\Pi = \{\alpha_1 = \varepsilon_1-\varepsilon_2,\ \alpha_2 = \varepsilon_2-\varepsilon_3,\ \cdots,\ \alpha_l = \varepsilon_l-\varepsilon_{l+1}\}\ ,\\
&\Delta(\mathfrak{n}^+) = \{\varepsilon_i-\varepsilon_k;\ 1\leqq i\leqq j<k\leqq +1\}\ ,\\
&2\delta = l\varepsilon_1+(l-2)\varepsilon_2+(l-4)\varepsilon_3+\cdots-(l+2)\varepsilon_l-l\varepsilon_{l+1}\,,\\
&\omega_j = \varepsilon_1+\cdots+\varepsilon_j-\frac{j}{l+1}\sum_{i=1}^{l+1}\varepsilon_i\,.
\end{aligned}$$

An element $\sigma\in W$ acts on R^{l+1} by $\sigma\varepsilon_i=\varepsilon_{\sigma(i)}$ for $1\leqq i\leqq l+1$, where σ in the index is a permutation of $\{1, 2, \cdots, l+1\}$. We represent σ by

$$\begin{pmatrix} 1 & 2 & \cdots & l+1 \\ \sigma(1) & \sigma(2) & \cdots & \sigma(l+1)\end{pmatrix}.$$

Then

$$W^1 = \left\{\sigma\in W;\ \sigma^{-1} = \begin{pmatrix} 1 & \cdots & l+1 \\ \sigma^{-1}(1) & \cdots & \sigma^{-1}(l+1)\end{pmatrix},\ \begin{matrix}\sigma^{-1}(1)<\cdots<\sigma^{-1}(j) \\ \sigma^{-1}(j+1)<\cdots<\sigma^{-1}(l+1)\end{matrix}\right\}.$$

The index $n(\sigma)$ of $\sigma\in W^1$ is given by

$$n(\sigma) = \sum_{i=1}^{j}(\sigma^{-1}(i)-i)$$

(Takeuchi [2]). We see easily that

$$\begin{aligned}
&(\omega_j,\ \beta) = 1 \qquad \text{for any } \beta\in\Delta(\mathfrak{n}^+)\,,\\
&(\sigma\delta,\ \varepsilon_i-\varepsilon_k) = \sigma^{-1}(k)-\sigma^{-1}(i) \qquad \text{for } 1\leqq i,\ k\leqq l+1\,.
\end{aligned}$$

Therefore we have to prove that the following two inequalities are true for any $\sigma\in W^1$

(1.1) $\#\{(i,\ k);\ 1\leqq i\leqq j<k\leqq l+1,\ \sigma^{-1}(k)-\sigma^{-1}(i)\geqq l+1-2n(\sigma)\}>n(\sigma)-1\,,$

(1.2) $\#\{(i,\ k);\ 1\leqq i\leqq j<k<l+1,\ \sigma^{-1}(k)-\sigma^{-1}(i)\geqq l+3-2n(\sigma)\}>n(\sigma)-2\,.$

First we prove the inequality (1.1).

Lemma 1.1. *Let* $\sigma\in W^1$. *If* $n(\sigma)\geqq l+1$, *the inequality* (1.1) *is true.*

Proof. Since $n(\sigma)\geqq l+1$, $l+1-2n(\sigma)\leqq -(l+1)$. There exist no pair $(i,\ k)$, $i\neq k$, which satisfies

$$\sigma^{-1}(k)-\sigma^{-1}(i)<-(l+1)\,.$$

Therefore

$$\#\{(i,\ k);\ 1\leqq i\leqq j<k\leqq l+1,\ \sigma^{-1}(k)-\sigma^{-1}(i)\geqq l+1-2n(\sigma)\} = n\,.$$

From the definition of the index $n(\sigma)\leqq n$, it follows that $n>n(\sigma)-1$. Q.E.D.

Lemma 1.2. *Let* $\sigma\in W^1$. *Assume that* $\sigma(1)\neq 1$ *and* $\sigma(l+1)\neq l+1$. *Then* $n(\sigma)\geqq l$.

Proof. By the assumption $\sigma^{-1}(j)=l+1$ and $\sigma^{-1}(i)-i\geqq 1$, $1\leqq i\leqq j$. Therefore

$$\begin{aligned} n(\sigma) &= \sum_{i=1}^{j}(\sigma^{-1}(i)-i) \\ &= \sigma^{-1}(j)-j+\sum_{i=1}^{j-1}(\sigma^{-1}(i)-i) \\ &\geqq (l+1-j)+(j-1) \\ &= l\,. \end{aligned}$$

Q.E.D.

Lemma 1.3. *Let* $\sigma\in W^1$. *Assume that* $\sigma(1)\neq 1$ *and* $\sigma(l+1)\neq l+1$. *Then the inequality* (1.1) *is true.*

Proof. By Lemmas 1.1 and 1.2 we may assume that $n(\sigma)=l$. Then such an element σ is unique and given by

$$\sigma^{-1}=\begin{pmatrix} 1 \cdots j-1 & j & j+1 & j+2 \cdots l+1 \\ 2 \cdots \ j & l+1 & 1 & j+1 \cdots \ l \end{pmatrix}.$$

The pair (i, k), $1\leqq i\leqq j<k\leqq l+1$, which satisfies

$$\sigma^{-1}(k)-\sigma^{-1}(i)<l+1-2n(\sigma) = 1-l$$

is $(j, j+1)$. Hence

$$\#\{(i, k);\ 1\leqq i\leqq j<k\leqq l+1,\ \sigma^{-1}(k)-\sigma^{-1}(i)\geqq 1-l\} = n-1>n(\sigma)-1\,.$$

Q.E.D.

Lemma 1.4. *If* $j=2$, *the inequality* (1.1) *is true for any* $\sigma\in W^1$.

Proof. From the definition of $n(\sigma)$

$$n(\sigma) = \sigma^{-1}(1)+\sigma^{-1}(2)-3\,. \tag{1.3}$$

If $n(\sigma)=0$, the inequality (1.1) is clearly true. Let $n(\sigma)=1$. Then $\sigma^{-1}(1)=1$, $\sigma^{-1}(2)=3$ and

$$\sigma^{-1}(l+1)-\sigma^{-1}(1) = l>l+1-2n(\sigma)\,.$$

It follows that the inequality (1.1) is true. Let $n(\sigma)=2$. It is easy to see that the inequality (1.1) is true.

By Lemma 1.1 we have already seen that if $n(\sigma)\geqq l+1$ the inequality is true. Hence we only have to show that (1.1) is true under the following condition:

(1.4) $$5<\sigma^{-1}(1)+\sigma^{-1}(2)<l+4\,.$$

By (1.3)

$$l+1-2n(\sigma)=l+7-2(\sigma^{-1}(1)+\sigma^{-1}(2))\,.$$

Since $\sigma^{-1}(k)\geqq k-2$ for $2<k\leqq l+1$,

$$\begin{aligned}&\#\{k;\ 2<k\leqq l+1,\ \sigma^{-1}(k)-\sigma^{-1}(1)\geqq l+7-2(\sigma^{-1}(1)+\sigma^{-1}(2))\}\\ &\geqq\min\{\sigma^{-1}(1)+2\sigma^{-1}(2)-7,\ l-1\}\,.\end{aligned}$$

Similarly

$$\begin{aligned}&\#\{k;\ 2<k\leqq l+1,\ \sigma^{-1}(k)-\sigma^{-1}(2)\geqq l+7-2(\sigma^{-1}(1)+\sigma^{-1}(2))\}\\ &\geqq\min\{2\sigma^{-1}(1)+\sigma^{-1}(2)-7,\ l-1\}\,.\end{aligned}$$

Therefore

$$\begin{aligned}&\#\{(i,k);\ 1\leqq i\leqq 2<k\leqq l+1,\ \sigma^{-1}(k)-\sigma^{-1}(i)\geqq l+1-2n(\sigma)\}\\ &\geqq\min\{3(\sigma^{-1}(1)+\sigma^{-1}(2))-14,\ l+2\sigma^{-1}(1)+\sigma^{-1}(2)-8,\ 2l-2\}\,.\end{aligned}$$

It is easy to see that $3(\sigma^{-1}(1)+\sigma^{-1}(2))-14$, $l+2\sigma^{-1}(1)+\sigma^{-1}(2)-8$ and $2l-2$ are both larger than $n(\sigma)-1=\sigma^{-1}(1)+\sigma^{-1}(2)-4$ under the condition (1.4). Q.E.D.

We get the following lemma in the similar way as above.

Lemma 1.5. *If $j=l-1$, the inequality* (1.1) *is true for any $\sigma\in W^1$.*

We shall prove that the inequality (1.1) is true for any $\sigma\in W^1$ by using induction on l. If $l=3$ so that $j=2$, it follows, by Lemma 1.4, our assertion is true.

Let $l=l_0\geqq 4$. We can assume that $3\leqq j=j_0\leqq l_0-2$ and either $\sigma(1)=1$ or $\sigma(l_0+1)=l_0+1$ by Lemmas 1.3, 1.4 and 1.5.

Case 1: $\sigma(1)=1$. Define the element τ of W^1, which is considered as an element of W^1 for $l=l_0-1$ and $j=j_0-1$, by

$$\tau^{-1}=\begin{pmatrix}1 & 2 & \cdots & l_0\\ \sigma^{-1}(2)-1 & \sigma^{-1}(3)-1 & \cdots & \sigma^{-1}(l_0+1)-1\end{pmatrix}.$$

We immediately see that $2\leqq j\leqq l-2$ and $n(\tau)=n(\sigma)$. By the assumption of the induction,

$$\#\{(i,k);\ 1\leqq i\leqq j_0-1<k\leqq l_0,\ \tau^{-1}(k)-\tau^{-1}(i)\geqq l_0-2n(\tau)\}>n(\tau)-1\,.$$

Hence

(1.5) $\#\{(i, k);\ 2\leqq i\leqq j_0<k\leqq l_0+1,\ \sigma^{-1}(k)-\sigma^{-1}(i)\geqq l_0-2n(\sigma)\}>n(\sigma)-1$.

For any k, $j_0\leqq k\leqq l_0+1$, if there exists i, $2\leqq i\leqq j_0$, which satisfies the following:

$$\sigma^{-1}(k)-\sigma^{-1}(i)=l_0-2n(\sigma)\,,$$

such an integer i is unique and

$$\sigma^{-1}(k)-\sigma^{-1}(1)\geqq l_0+1-2n(\sigma)\,.$$

Hence (1.5) leads to (1.1).

Case 2: $\sigma(l_0+1)=l_0+1$. Define the element $\tau\in W^1$, which is considered as an element of W^1 for $l=l_0-1$ and $j=j_0$, by

$$\tau^{-1}=\begin{pmatrix} 1 & \cdots & l_0 \\ \sigma^{-1}(1) & \cdots & \sigma^{-1}(l_0)\end{pmatrix}.$$

Then $3\leqq j\leqq l-1$ and $n(\tau)=n(\sigma)$. By the assumption of the induction,

(1.6) $\#\{(i, k);\ 1\leqq i\leqq j_0<k\leqq l_0,\ \sigma^{-1}(k)-\sigma^{-1}(i)\geqq l_0-2n(\sigma)\}>n(\sigma)-1$.

For any i, $1\leqq i\leqq j_0$, if there exists k, $j_0<k\leqq l_0$, which satisfies the following:

$$\sigma^{-1}(k)-\sigma^{-1}(i)=l_0-2n(\sigma)\,,$$

such an integer k is unique and

$$\sigma^{-1}(l_0+1)-\sigma^{-1}(i)\geqq l_0+1-2n(\sigma)\,.$$

Hence (1.5) leads to (1.1).

Thus we have proved that the inequality (1.1) is true for any $\sigma\in W^1$.

In the following we shall prove that the inequality (1.2) is true for any $\sigma\in W^1$.

Lemma 1.6. *Let $\sigma\in W^1$. If $n(\sigma)\geqq l+1$, the inequality* (1.2) *is true.*

Proof. Since $n(\sigma)\geqq l+1$

$$l+3-2n(\sigma)\leqq 1-l\,.$$

If there exists a pair (i, k), $i\neq k$, which satisfies

$$\sigma^{-1}(k)-\sigma^{-1}(i)<1-l\,,$$

such a pair is unique. Therefore

$$\#\{(i, k);\ 1\leqq i\leqq j\leqq<k\leqq l+1,\ \sigma^{-1}(k)-\sigma^{-1}(i)\geqq l+1-2n(\sigma\}\geqq)n-1>n(\sigma)-2\,.$$

Q.E.D.

Lemma 1.7. *Let $\sigma \in W^1$. Assume that $\sigma(1) \neq 1$ and $\sigma(l+1) \neq l+1$. Then the inequality* (1.2) *is true.*

Proof. By Lemma 1.2 and 1.6 we may assume that $n(\sigma)=l$. Such an element σ is unique and represented by

$$\sigma^{-1}=\begin{pmatrix} 1 \cdots j-1 & j & j+1 & j+2 \cdots l+1 \\ 2 \cdots\ \ j & l+1 & 1 & j+1 \cdots\ \ l \end{pmatrix}.$$

The number of the pairs (i, k), $l \leqq i \leqq j < k \leqq l+1$, which satisfies

$$\sigma^{-1}(k)-\sigma^{-1}(i) < l+3-2n(\sigma) = 3-l$$

is at most 2. Therefore

$$\#\{(i, k);\ 1 \leqq i \leqq j < k \leqq l+1,\ \sigma^{-1}(k)-\sigma^{-1}(i) \geqq l+3-2n(\sigma)\} \geqq n-2\,.$$

Since $n(\sigma)=l$, $n(\sigma)<n$. It follows that (1.2) is ture. Q.E.D.

Lemma 1.8. *If $j=2$, the inequality* (1.2) *is true for any $\sigma \in W^1$.*

Proof. It is easy to see that (1.2) is true if $n(\sigma) \leqq 3$. By Lemma 1.6 and (1.3), we only have to show that (1.2) is true under the condition:

$$6 < \sigma^{-1}(1)+\sigma^{-1}(2) < l+4\,. \tag{1.7}$$

We get the following inequality in the same way as in the proof of Lemma 1.4.

$$\begin{aligned}&\#\{(i, k);\ 1 \leqq i \leqq 2 < k \leqq l+1,\ \sigma^{-1}(k)-\sigma^{-1}(i) \leqq l+3-2n(\sigma)\} \\ &\geqq \min\{3(\sigma^{-1}(1)+\sigma^{-1}(2))-18,\ l+2\sigma^{-1}(1)+\sigma^{-1}(2)-10,\ 2l-2\}\,.\end{aligned}$$

It is easy to see that $3(\sigma^{-1}(1)+\sigma^{-1}(2))-18$, $l+2\sigma^{-1}(1)+\sigma^{-1}(2)-10$ and $2l-2$ are both larger than $n(\sigma)-2=\sigma^{-1}(1)+\sigma^{-1}(2)-5$ under the condition (1.7). Q.E.D.

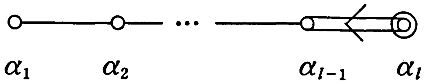
We get the following lemma in the similar way as above.

Lemma 1.9. *If $j=l-1$, the inequality* (1.2) *is true for any $\sigma \in W^1$.*

From Lemmas 1.7, 1.8 and 1.9, we can prove that the inequality (1.2) is true for any $\sigma \in W^1$ in the same way as in the proof of the inequality (1.1).

2.2. The case that M is of type CI, that is $M=Sp(l)/U(l)$. If $l=1$, $M=P_1(C)$. If $l=2$, M is a complex quadroic of dimension 3. Hence we assume that $l \geqq 3$.

In this case $n=\frac{1}{2}l(l+1)$ and $\lambda=l+1$. The Dynkin diagram of Π is as follows:



where α_l◎ shows $\alpha_j=\alpha_l$. Let $\{\varepsilon_i;\ 1\leqq i\leqq l\}$ be the basis of $\mathfrak{h}_0$ which satisfies $(\varepsilon_i, \varepsilon_j)=\delta_{ij}$. Then we have:

$$\begin{aligned}
&\Delta = \{\pm 2\varepsilon_i;\ 1\leqq i\leqq l,\ \pm\varepsilon_i\pm\varepsilon_j;\ 1\leqq i<j\leqq l\}\ ,\\
&\Pi = \{\alpha_1 = \varepsilon_1-\varepsilon_2, \cdots, \alpha_{l-1} = \varepsilon_{l-1}-\varepsilon_l,\ \alpha_l = 2\varepsilon_l\}\ ,\\
&\Delta(\mathfrak{n}^+) = \{2\varepsilon_i;\ 1\leqq i\leqq l,\ \varepsilon_i+\varepsilon_j;\ 1\leqq i<j\leqq l\}\ ,\\
&\delta = l\varepsilon_1+(l-1)\varepsilon_2+\cdots+\varepsilon_l\, ,\\
&\omega_l = \varepsilon_1+\cdots+\varepsilon_l\, .
\end{aligned}$$

An element $\sigma\in W$ acts on $\mathfrak{h}_0$ by $\sigma\varepsilon_i=\pm\varepsilon_{\bar\sigma(i)}$ for $1\leqq i\leqq l$, where $\bar\sigma$ is a permutation of $\{1, 2, \cdots, l\}$. We denote the element $\sigma\in W$ by the symbol

$$\begin{pmatrix} 1 & 2 & \cdots & l \\ \pm\bar\sigma(1) & \pm\bar\sigma(2) & \cdots & \pm\bar\sigma(l) \end{pmatrix}$$

Then

$$W^1 = \left\{\sigma\in W;\ \bar\sigma^{-1} = \begin{pmatrix} 1 & \cdots & r & r+1 & \cdots & l \\ \bar\sigma^{-1}(1) & \cdots & \bar\sigma^{-1}(r) & -\bar\sigma^{-1}(r+1) & \cdots & -\bar\sigma^{-1}(l) \end{pmatrix}\right.$$
$$\left.\text{for } 0\leqq r\leqq l,\ \bar\sigma^{-1}(1)<\cdots<\bar\sigma^{-1}(r),\ \bar\sigma^{-1}(r+1)>\cdots>\bar\sigma^{-1}(l)\right\}.$$

The index $n(\sigma)$ of $\sigma\in W^1$ is given by

$$n(\sigma) = \sum_{i=1}^{r}(\bar\sigma^{-1}(i)-i)+{}_{l+1-r}C_2$$

(Takeuchi [2]). We see easily that

$$\begin{aligned}
&(\omega_l, \beta) = 2 \qquad \text{for any } \beta\in\Delta(\mathfrak{n}^+)\, ,\\
&(\sigma\delta, \varepsilon_i) = \begin{cases} (l+1-\bar\sigma^{-1}(i)) & \text{if } 1\leqq i\leqq r \\ -(l+1-\bar\sigma^{-1}(i)) & \text{if } r<i\leqq l\, . \end{cases}
\end{aligned}$$

Therefore we have to prove that the following inequalities are true for any $\sigma\in W^1$

(2.1) $\quad \#\{\beta\in\Delta(\mathfrak{n}^+);\ (\sigma\delta, \beta)\geqq 2(l+1)-4n(\sigma)\}>n(\sigma)-1\ ,$

(2.2) $\quad \#\{\beta\in\Delta(\mathfrak{n}^+);\ (\sigma\delta, \beta)\geqq 2(l+3)-4n(\sigma)\}>n(\sigma)-2\, .$

Since $(\sigma\delta, \beta)\geqq -2l$, $\beta\in\Delta(\mathfrak{n}^+)$, we immedietely see that if $n(\sigma)\geqq l+1$ (resp. $l+2$), the inequality (2.1) (resp. (2.2)) is true for any $\sigma\in W^1$.

Lemma 2.1. *Let* $\sigma\in W^1$. *If* $n(\sigma)\geqq l$, *the inequality* (2.1) *is true.*

Proof. From the above notice we can assume that $n(\sigma)=l$. In this case

$$2(l+1)-4n(\sigma) = 2-2l\,.$$

It is easy to see that

$$\#\{\beta\in\Delta(\mathfrak{n}^+);\ (\sigma\delta,\ \beta)<2-2l\}\leqq 2\,.$$

Hence

$$\#\{\beta\in\Delta(\mathfrak{n}^+);\ (\sigma\delta,\ \beta)\geqq 2-2l\}\geqq {}_{l+1}C_2-2>l-1 = n(\sigma)-1\,.$$

Q.E.D.

Lemma 2.2. *Let $\sigma\in W^1$. If $n(\sigma)\geqq l$, the inequality* (2.2) *is true.*

Proof. If $n(\sigma)\geqq l+1$, the inequality is ture in the same way as above. Therefore we may assume that $n(\sigma)=l$.

Case 1: $l=3$. If $r=0$, $n(\sigma)=6\neq 3$. Hence $r>0$, and σ is one of the following elements:

$$\begin{pmatrix}1 & 2 & 3\\ 1 & -3 & -2\end{pmatrix},\quad \begin{pmatrix}1 & 2 & 3\\ 2 & 3 & -1\end{pmatrix}.$$

In each case (2.2) is true.

Case 2: $l=4$. If $r\leqq 1$, $n(\sigma)\geqq 6>4$. Hence $r\geqq 2$. It follows that $(\sigma\delta,\ 2\varepsilon_1)$, $(\sigma\delta,\ \varepsilon_1+\varepsilon_2)$ and $(\sigma\delta,\ 2\varepsilon_2)$ are larger than $2(l+3)-4n(\sigma)=-2$. On the other hand $n(\sigma)-2=2$. Therefore (2.2) is true.

Case 3: $l\geqq 5$. If $\beta\in\Delta(\mathfrak{n}^+)$ satisfies

$$(\sigma\delta,\ \beta)<2(l+3)-4n(\sigma) = 6-2l\,,$$

β is one of the following 12 elements:

$$\begin{aligned}&2\varepsilon_l,\ \varepsilon_l+\varepsilon_{l-1},\ \varepsilon_l+\varepsilon_{l-2},\ \varepsilon_l+\varepsilon_{l-3},\ \varepsilon_l+\varepsilon_{l-4},\ \varepsilon_l+\varepsilon_{l-5},\\ &2\varepsilon_{l-1},\ \varepsilon_{l-1}+\varepsilon_{l-2},\ \varepsilon_{l-1}+\varepsilon_{l-3},\ \varepsilon_{l-1}+\varepsilon_{l-4},\ 2\varepsilon_{l-2},\ \varepsilon_{l-2}+\varepsilon_{l-3}\,.\end{aligned}$$

On the other hand

$$\begin{aligned}&{}_{l+1}C_2-12-(l-2)\\ &=\tfrac{1}{2}\{(l(l+1)-20-2l\}\\ &=\tfrac{1}{2}(l^2+l-20)\\ &=\tfrac{1}{2}(l+4)(l-5)\geqq 0\,.\end{aligned}$$

The equality holds only in the case $l=5$. But if $l=5$, $\varepsilon_l+\varepsilon_{l-5}\notin\Delta(\mathfrak{n}^+)$. Therefore the inequality is ture. Q.E.D.

Lemma 2.3. *Let $\sigma\in W$. If $\sigma(1)\neq 1$, $n(\sigma)\geqq l$.*

Proof. By the assumption,

$$\sum_{i=1}^{r}(\sigma(i)-i)\geqq r\,.$$

Hence

$$\begin{aligned}&n(\sigma)-l\\ &\geqq r+{}_{l+1-r}C_2-l\\ &=\tfrac{1}{2}(l-r-1)(l-r)\geqq 0\,.\end{aligned}$$

Q.E.D.

We shall prove that the inequality (2.1) is true for any $\sigma\in W^1$ by using induction on l. Let $l=3$. If $n(\sigma)\geqq 3$, the inequality is true by Lemma 2.1. If $n(\sigma)=0$, the inequality is also true for $n(\sigma)-1<0$. If $n(\sigma)=1$ (resp. 2),

$$\sigma=\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & -3\end{pmatrix}\left(\text{resp.}\begin{pmatrix}1 & 2 & 3\\ 1 & 3 & -2\end{pmatrix}\right),$$

and (2.1) is true.

Let $l=l_0>3$. By Lemmas 2.1 and 2.3, we may assume that $\sigma(1)=1$. Define the element $\tau\in W^1$, which is considered as an element of W^1 for $l=l_0-1$, by

$$\tau^{-1}=\begin{pmatrix}1 & \cdots & r-1 & r & \cdots & l_0-1\\ \sigma^{-1}(2)-1 & \cdots & \sigma^{-1}(r)-1 & -(\sigma^{-1}(r+1)-1) & \cdots & -\sigma^{-1}(l_0)-1)\end{pmatrix}.$$

We easily see that $n(\tau)=n(\sigma)$. By the assumption of the induction,

$$\#\{\varepsilon_i+\varepsilon_j;\ 1\leqq i,j\leqq l_0-1,\ (\tau\delta',\ \varepsilon_i+\varepsilon_j)\geqq 2l_0-4n(\tau)\}>n(\tau)-1\,,$$

where $\delta'=(l_0-1)\varepsilon_1+(l_0+2)\varepsilon_2+\cdots+\varepsilon_{l_0-1}$. It follows, by the fact that $(\tau\delta',\ \varepsilon_{i-1})=(\sigma\delta,\ \varepsilon_i)$ for $2\leqq i\leqq l_0$, that

$$\#\{\varepsilon_i+\varepsilon_j;\ 2\leqq i,j\leqq l_0,\ (\sigma\delta,\ \varepsilon_i+\varepsilon_j)\geqq 2l_0-4n(\sigma)\}>n(\sigma)-1\,. \tag{2.3}$$

Lemma 2.4. *Let*

$$\begin{aligned}s=\#\{\varepsilon_i;\ 2\leqq i\leqq l_0,\ \exists\varepsilon_j,\ 2\leqq j\leqq l_0,\ j\neq i,\ \textit{such that}\\ (\sigma\delta,\ \varepsilon_i+\varepsilon_j)=2l_0-4n(\sigma)\ \text{or}\ 2l_0+1-4n(\sigma)\}\,.\end{aligned}$$

Then

$$\begin{aligned}\#\{\varepsilon_i+\varepsilon_j;\ 2\leqq i<j\leqq l_0,\ (\sigma\delta,\ \varepsilon_i+\varepsilon_j)=2l_0-4n(\sigma)\ \textit{or}\\ 2l_0+1-4n(\sigma)\}\leqq s-1\,.\end{aligned}$$

Proof. Let ε_i, $2\leqq i\leqq l_0$, satisfy the condition that there exists ε_j, $2\leqq j\leqq l_0$, $j\neq i$, such that $(\sigma\delta,\ \varepsilon_i+\varepsilon_j)=2l_0-n(\sigma)$ or $2l_0+1-n(\sigma)$. For the element ε_i

$$\begin{aligned}\#\{\varepsilon_i+\varepsilon_j;\ 2\leqq j\leqq l_0,\ j\neq i,\ (\sigma\delta,\ \varepsilon_i+\varepsilon_j)=2l_0-4n(\sigma)\\ \text{or}\ 2l_0+1-4n(\sigma)\}\leqq 2\,.\end{aligned} \tag{2.4}$$

In this way we find at most $2s$ ordered pairs (i, j), $2\leqq i,\ j\leqq l_0$, $j\neq i$, which satisfies $(\sigma\delta,\ \varepsilon_i+\varepsilon_j)=2l_0-4n(\sigma)$ or $2l_0+1-4n(\sigma)$. On the other hand the distinct pairs (i, j) and (j, i) induce the same element $\varepsilon_i+\varepsilon_j$. Therefore

$$\text{(2.5)}\quad \#\{\varepsilon_i+\varepsilon_j;\ 2\leqq i<j\leqq l_0,\ (\sigma\delta,\ \varepsilon_i+\varepsilon_j)=2l-4n(\sigma) \text{ or } 2l+1-4n(\sigma)\}\leqq s\,,$$

and the equality holds if and only if the equality in (2.4) holds for any ε_i, $2\leqq i\leqq l_0$, such that $(\sigma\delta,\ \varepsilon_i+\varepsilon_j)=2l-n(\sigma)$ or $2l+1-n(\sigma)$.

Define the integer i_0 (resp. i_m) by

$$\begin{aligned}&\min(\text{resp. }\max)\{i;\ 2\leqq i\leqq l_0,\ \exists j,\ 2\leqq j\leqq l_0,\ j\neq i \text{ such that}\\ &(\sigma\delta,\ \varepsilon_i+\varepsilon_j)=l-2n(\sigma) \text{ or } l+1-2n(\sigma)\}\,.\end{aligned}$$

If the equalities in (2.4) for ε_{i_0} and ε_{i_m} hold, there exist the integers i and j such that

$$\begin{aligned}&(\sigma\delta,\ \varepsilon_{i_0}+\varepsilon_j)=l-2n(\sigma) \text{ or } l+1-2n(\sigma)\,,\\ &(\sigma\delta,\ \varepsilon_i+\varepsilon_{i_m})=l-2n(\sigma) \text{ or } l+1-2n(\sigma)\,,\\ &i_0<i \text{ and } j<i_m\,.\end{aligned}$$

Hence

$$(\sigma\delta,\ \varepsilon_i+\varepsilon_{i_m})\leqq(\sigma\delta,\ \varepsilon_{i_0}+\varepsilon_j)-2\,.$$

This is impossible, and therefore, the equality in (2.5) does not hold. Q.E.D.

Let ε_i, $2\leqq i\leqq l_0$, satisfy that there exists ε_j, $2\leqq j\leqq l_0$, $j\neq i$, such that

$$(\sigma\delta,\ \varepsilon_i+\varepsilon_j)=2l-4n(\sigma) \text{ or } 2l+1-4n(\sigma)\,.$$

For this element ε_i,

$$(\sigma\delta,\ \varepsilon_i+\varepsilon_1)\geqq 2l+2-4n(\sigma)\,,$$

in all but the following case:

$$(\sigma\delta,\ \varepsilon_i+\varepsilon_2)=2l-4n(\sigma)\,.$$

Therefore, by Lemma 2.4,

$$\begin{aligned}&\#\{\varepsilon_i+\varepsilon_j;\ 1\leqq i<j\leqq l_0,\ (\sigma\delta,\ \varepsilon_i+\varepsilon_j)\geqq 2l_0+2-4n(\sigma)\}\\ \geqq&\#\{\varepsilon_i+\varepsilon_j;\ 2\leqq i<j\leqq l_0,\ (\sigma\delta,\ \varepsilon_i+\varepsilon_j)\geqq 2l_0-4n(\sigma)\}\,.\end{aligned}$$

There exist at most one element ε_i, $2\leqq i\leqq l_0$, such that

$$(\sigma\delta,\ 2\varepsilon_i)=2l-4n(\sigma) \text{ or } 2l+1-4n(\sigma)\,.$$

If such ε_i exists,

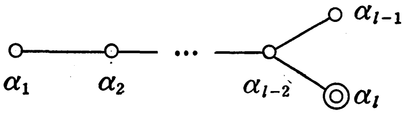
$$(\sigma\delta,\ 2\varepsilon_1)\geqq 2l+2-4n(\sigma)\,.$$

Therefore the inequality (2.1) is true.

Thus we have proved that the inequality (2.1) is true for any $\sigma\in W^1$.

From Lemmas 2.2 and 2.3, we can prove that the inequality (2.2) is true for any $\sigma\in W^1$ in the same way as above.

2.3. The case that M is of type DIII, that is $M=SO(2l)/U(l)$. If $l=3$, $M=P_3(C)$. If $l\geqq 4$, M is a complex quadric of dimension 6. Hence we assume that $l\geqq 5$. In this case $n=\frac{1}{2}l(l-1)$ and $\lambda=2l-2$. The Dinkin diagram of Π is as follows:



where α_l ◎ shows $\alpha_j=\alpha_l$. Let $\{\varepsilon_i;\ 1\leqq i\leqq l\}$ be the basis of $\mathfrak{h}_0$ which satisfies $(\varepsilon_i, \varepsilon_j)=\delta_{ij}$. Then we have:

$$\begin{aligned}
\Delta &= \{\pm\varepsilon_i\pm\varepsilon_j;\ 1\leqq i<j\leqq l\}\ ,\\
\Pi &= \{\alpha_1=\varepsilon_1-\varepsilon_2, \cdots, \alpha_{l-1}=\varepsilon_{l-1}-\varepsilon_l,\ \alpha_l=\varepsilon_{l-1}+\varepsilon_l\}\ ,\\
\Delta(\mathfrak{n}^+) &= \{\varepsilon_i+\varepsilon_j;\ 1\leqq i<j\leqq l\}\ ,\\
\delta &= (l-1)\varepsilon_1+(l-2)\varepsilon_2+\cdots+\varepsilon_{l-1}\,,\\
\omega &= \tfrac{1}{2}(\varepsilon_1+\cdots+\varepsilon_l)\,.
\end{aligned}$$

An element $\sigma\in W$ acts on $\mathfrak{h}_0$ by $\sigma\varepsilon_i=\pm\varepsilon_{\bar\sigma(i)}$ for $1\leqq i\leqq l$, where $\bar\sigma$ is a permutation of $\{1, 2, \cdots, l\}$. We denote the element $\sigma\in W$ by the symbol

$$\begin{pmatrix} 1 & 2 & \cdots & l \\ \pm\bar\sigma(1) & \pm\bar\sigma(2) & \cdots & \pm\bar\sigma(l)\end{pmatrix}.$$

Then

$$W^1=\left\{\sigma\in W;\ \sigma^{-1}=\begin{pmatrix} 1 & \cdots & r & r+1 & \cdots & l \\ \bar\sigma^{-1}(1) & \cdots & \bar\sigma^{-1}(r) & -\bar\sigma^{-1}(r+1) & \cdots & -\bar\sigma^{-1}(l)\end{pmatrix},\right.$$

$$\left. l-r \text{ is even},\ \bar\sigma^{-1}(1)<\cdots<\bar\sigma^{-1}(r),\ \bar\sigma^{-1}(r+1)>\cdots>\bar\sigma^{-1}(l)\right\}.$$

The index $n(\sigma)$ of $\sigma\in W^1$ is given by

$$n(\sigma)=\sum_{i=1}^{r}(\bar\sigma^{-1}(i)-i)+{}_{l-r}C_2$$

(Takeuchi [2]). We see easily that

$$\begin{aligned}
(\omega_l, \beta) &= 1 \qquad \text{for any } \beta\in\Delta(\mathfrak{n}^+)\,,\\
(\sigma\delta, \varepsilon_i) &= \begin{cases} l-\sigma^{-1}(i) & \text{if } 1\leqq i\leqq r \\ -(l-\sigma^{-1}(i)) & \text{if } r<i\leqq l\,. \end{cases}
\end{aligned}$$

Therefore we have to prove that the following inequalities are true for any $\sigma\in W^1$.

$$\sharp\{\beta\in\Delta(\mathfrak{n}^+);\ (\sigma\delta,\ \beta)\geqq 2l-2-2n(\sigma)\}>n(\sigma)-1\,, \tag{3.1}$$

$$\sharp\{\beta\in\Delta(\mathfrak{n}^+);\ (\sigma\delta,\ \beta)\geqq 2l-2n(\sigma)\}>n(\sigma)-2\,. \tag{3.2}$$

Lemma 3.1. *Let $\sigma\in W^1$. If $n(\sigma)\geqq 2l-3$, the inequality* (3.1) *is true.*

Proof. By the assumption $2l-2-2n(\sigma)\leqq 4-2l$. Let β be an element of $\Delta(\mathfrak{n}^+)$ which satisfies that

$$(\sigma\delta,\ \beta)<4-2l\,,$$

then $\beta=\varepsilon_{l-1}+\varepsilon_l$. Therefore

$$\sharp\{\beta\in\Delta(\mathfrak{n}^+);\ (\sigma\delta,\ \beta)\geqq 2l-2-2n(\sigma)\}\geqq n-1\,.$$

If the equality holds, $n(\sigma)=2l-3$ and $n-n(\sigma)=\frac{1}{2}(l-2)\,(l-3)>0$. Q.E.D.

Lemma 3.2. *Let $\sigma\in W^1$. If $n(\sigma)\geqq 2l-3$, the inequality* (3.2) *is true.*

Proof. If $n(\sigma)\geqq -2l-3$, the inequality is true in the same way as above. Therefore we assume that $n(\sigma)=2l-3$. The number of the elements $\beta\in\Delta(\mathfrak{n}^+)$ such that

$$(\sigma\delta,\ \beta)<2l-2n(\sigma)=6-2l$$

is at most 4. Since $l\geqq 5$,

$$(n-4)-(n(\sigma)-2)=\frac{1}{2}l(l-1)-4-2l+5=\frac{1}{2}l(l-5)+1>0\,.$$

Q.E.D.

Lemma 3.3. *If $\bar{\sigma}^{-1}(1)\geqq 3$, then $n(\sigma)\geqq 2l-3$.*

Proof. By the assumption

$$\sum_{i=1}^{r}(\bar{\sigma}^{-1}(i)-i)\geqq 2r\,.$$

It follows that

$$\begin{aligned}&n(\sigma)-(2l-3)\\ &\geqq 2r+{}_{l-r}C_2-(2l-3)\\ &=\frac{1}{2}(l-r-2)(l-r-3)\geqq 0\,.\end{aligned}$$

Q.E.D.

We prove that the inequality (3.1) is true for all $\sigma\in W^1$ by using induction on l. If $l=5$, we easily see that the inequality is true.

Let $l=l_0>5$. By Lemmas 3.1 and 3.3, we can assume that $\bar{\sigma}^{-1}(1)=1$ or 2.

Case 1: $\bar{\sigma}^{-1}(1)=1$. Define the element $\tau\in W^1$, which is considered as an element of W^1 for $l=l_0-1$, by

$$\tau^{-1}=\begin{pmatrix} 1 & \cdots & r-1 & r & \cdots & l_0-1 \\ \bar{\sigma}^{-1}(2)-1 & \cdots & \bar{\sigma}^{-1}(r)-1 & -(\bar{\sigma}^{-1}(r+1)-1) & \cdots & \bar{\sigma}^{-1}((l_0)-1) \end{pmatrix}.$$

Then $n(\tau)=n(\sigma)$. By the assumption of the induction,

$$\#\{\varepsilon_i+\varepsilon_j;\ 2\leqq i<j\leqq l_0,\ (\sigma\delta,\ \varepsilon_i+\varepsilon_j)\geqq 2l_0-4-2n(\sigma)\}>n(\sigma)-1\,.$$

Let

$$s=\#\{\varepsilon_i;\ 2\leqq i\leqq l_0,\ \exists\varepsilon_j,\ 2\leqq j\neq i\leqq l_0,\ \text{such that} \\ (\sigma\delta,\ \varepsilon_i+\varepsilon_j)=2l_0-4-2n(\sigma)\ \text{or}\ 2l_0-3-2n(\sigma)\}\,.$$

Then, in the same way as in Lemma 2.4, we see that

$$\#\{\varepsilon_i+\varepsilon_j;\ 2\leqq i<j\leqq l_0,\ (\sigma\delta,\ \varepsilon_i+\varepsilon_j)=2l_0-4-2n(\sigma)\ \text{or}\ 2l_0-3-2n(\sigma)\}\leqq s-1\,.$$

Let ε_i satisfy that there exists ε_j, $2\leqq j\leqq l_0$, $j\neq i$, such that

$$(\sigma\delta,\ \varepsilon_i+\varepsilon_j)=2l_0-4-2n(\sigma)\ \text{or}\ 2l_0-3-2n(\sigma)\,.$$

Then

$$(\sigma\delta,\ \varepsilon_i+\varepsilon_1)\geqq 2l_0-2-2n(\sigma)$$

in all but the following case:

$$(\sigma\delta,\ \varepsilon_i+\varepsilon_2)=2l_0-4-2n(\tau)\ \text{and}\ \bar{\sigma}^{-1}(2)=2\,.$$

Therefore the inequality is true.

Case 2: $\bar{\sigma}^{-1}(1)=2$. By the definition of W^1

$$\sigma^{-1}=\begin{pmatrix} 1 & 2 & \cdots & r & r+1 & \cdots & l_0 \\ 2 & \bar{\sigma}^{-1}(2) & \cdots & \bar{\sigma}^{-1}(r) & -\bar{\sigma}^{-1}(r+1) & \cdots & -1 \end{pmatrix}.$$

Define the element $\sigma'\in W^1$ by

$$(\sigma')^{-1}=\begin{pmatrix} 1 & 2 & \cdots & r & r+1 & \cdots & l_0-1 & l_0 \\ 1 & \bar{\sigma}^{-1}(2) & \cdots & \bar{\sigma}^{-1}(r) & -\bar{\sigma}^{-1}(r+1) & \cdots & -\bar{\sigma}^{-1}(l_0-1) & -2 \end{pmatrix}.$$

Then $n(\sigma')=n(\sigma)-1$. Define another element $\tau\in W^1$, which is considered for $l=l_0-1$, by

$$\tau^{-1}=\begin{pmatrix} 1 & \cdots & r & r+1 & \cdots & l_0-1 \\ \bar{\sigma}^{-1}(2)-1 & \cdots & \bar{\sigma}^{-1}(r)-1 & -(\bar{\sigma}^{-1}(r+1)-1) & \cdots & -1 \end{pmatrix}.$$

Then $n(\tau)=n(\sigma')$.

Assume that the inequality (3.2) is true for τ. If we notice that $(\overline{\sigma'})^{-1}(2)>2$, we get the following inequality in the same way as in case 1.

$$\#\{\beta\in\Delta(\mathfrak{n}^+);\ (\sigma'\delta,\ \beta)\geqq 2l_0-2-2n(\sigma')\}>n(\sigma')\,.$$

Clearly

$$(\sigma\delta,\ \beta)\geqq(\sigma'\delta,\ \beta)-2 \qquad \text{for any } \beta\in\Delta(\mathfrak{n}^+)\,.$$

Hence if $\beta\in\Delta(\mathfrak{n}^+)$ satisfies that

$$(\sigma'\delta,\ \beta)\geqq 2l_0-2-2n(\sigma')\,,$$

then

$$(\sigma\delta,\ \beta)\geqq 2l_0-2-2n(\sigma)\,.$$

Therefore

$$\#\{\beta\in\Delta(\mathfrak{n}^+);\ (\sigma\delta,\ \beta)\geqq 2l_0-2-2n(\sigma)\}>n(\sigma)-1\,.$$

Thus we have proved that the inequality (3.1) is true for any $\sigma\in W^1$. We can prove that the inequality (3.2) is true in the same way as above.

References

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Kobe University of Commerce
Tarumi, Kobe 655, Japan