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LEFT NOETHERIAN MULTIPLICATION RINGS

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Recently in [9], we defined a multiplication ring, shortly an M-ring, as a ring such that for any two ideals $\mathfrak{a}$, $\mathfrak{b}$ with $\mathfrak{a}<\mathfrak{b}$, there exists $\mathfrak{c}$, $\mathfrak{c}'$ such that $\mathfrak{a}=\mathfrak{b}\mathfrak{c}=\mathfrak{c}'\mathfrak{b}$, where "$<$" means a proper inclusion. An M-ring R is called to be *non-idempotent* if $R>R^2$, and now we call an M-ring R with $R=R^2$ *idempotent*. In [9] we have proved that the unique maximal idempotent ideal $\mathfrak{d}$ of a non-idempotent M-ring can be obtained as an intersection of some sequence of ideals $\{\mathfrak{d}_\alpha\}_\Lambda$ ([9], Theorem 5): $\mathfrak{d}=\bigcap_{\alpha\in\Lambda}\mathfrak{d}_\alpha$. In this note we shall prove that any left Noetherian M-ring is a so-called "general ZPI-ring" in a commutative case ([2]), i.e. each ideal can be written as a product of prime ideals, and as a consequence the multiplication of ideals is commutative. §1 is preliminaries of our study, and in §2 we consider left Noetherian non-idempotent M-rings, and in §3 we study an idempotent case, and the last chapter is the summary of our study. In the following arguments we do not assume the existence of the identity.

1. Preliminaries

In general, we have the following:

Theorem 1. *Let R be an M-ring, and let $\mathfrak{a}$, $\mathfrak{b}$ be two idelas of R with $\mathfrak{a}<\mathfrak{b}$. Then there exists the unique maximal element in the set of ideals $\mathfrak{c}$ with $\mathfrak{a}=\mathfrak{b}\mathfrak{c}$ and also in the set of ideals $\mathfrak{c}'$ of R with $\mathfrak{a}=\mathfrak{c}'\mathfrak{b}$.*

Proof. Let M denote the set $\{\mathfrak{c}$: ideals of R with $\mathfrak{a}=\mathfrak{b}\mathfrak{c}\}$. By Zorn's Lemma, there exist maximal elements in M. Let $\mathfrak{c}_1$, $\mathfrak{c}_2$ be two maximal elements in M. In this case $\mathfrak{a}=\mathfrak{b}\mathfrak{c}_1$ and $\mathfrak{a}=\mathfrak{b}\mathfrak{c}_2$. So $\mathfrak{a}=(\mathfrak{a}, \mathfrak{a})=(\mathfrak{b}\mathfrak{c}_1, \mathfrak{b}\mathfrak{c}_2)=\mathfrak{b}(\mathfrak{c}_1, \mathfrak{c}_2)$. Obviously $(\mathfrak{c}_1, \mathfrak{c}_2)\supseteq \mathfrak{c}_1, \mathfrak{c}_2$. Since both $\mathfrak{c}_1$ and $\mathfrak{c}_2$ are maximal in M, we have $\mathfrak{c}_1=(\mathfrak{c}_1, \mathfrak{c}_2)=\mathfrak{c}_2$.

DEFINITION. We shall call a prime ideal $\mathfrak{p}$ of R a minimal prime divisor of an ideal $\mathfrak{a}$ if $\mathfrak{a}\subseteq\mathfrak{p}$ and there is no prime ideal $\mathfrak{p}'$ with $\mathfrak{a}\subseteq\mathfrak{p}'<\mathfrak{p}$. (c.f. [6]).

Lemma 1. *Let R be an M-ring and $\mathfrak{a}$, $\mathfrak{b}$ two ideals of R with $\mathfrak{a}<\mathfrak{b}$. If $\mathfrak{b}$ is an idempotent ideal, then* $\mathfrak{a}=\mathfrak{a}\mathfrak{b}=\mathfrak{b}\mathfrak{a}$.

Proof. $\mathfrak{a}=\mathfrak{b}\mathfrak{c}=\mathfrak{c}'\mathfrak{b}$ for some ideals $\mathfrak{c}$, $\mathfrak{c}'$ of R. Therefore $\mathfrak{b}\mathfrak{a}=\mathfrak{b}\cdot\mathfrak{b}\mathfrak{c}=\mathfrak{b}^2\mathfrak{c}=\mathfrak{b}\mathfrak{c}=\mathfrak{a}$. Similarly $\mathfrak{a}\mathfrak{b}=\mathfrak{a}$.

Lemma 2. *Let R be a left Noetherian ring, $\mathfrak{a}$ an ideal of R and $\mathfrak{q}_1, \cdots, \mathfrak{q}_k$ be the set of minimal prime divisors of $\mathfrak{a}$. If $\mathfrak{a}=\mathfrak{q}_1\mathfrak{q}_2\cdots\mathfrak{q}_k\mathfrak{a}$, then, for some postitve integer ρ, $(\mathfrak{q}_1\cdots\mathfrak{q}_k)^\rho\subseteq\mathfrak{a}$ and $\mathfrak{a}$ is an idempotent ideal.*

Proof. We set $\bar{R}=R/\mathfrak{a}$. Let $\bar{N}_3$ denote the nil-radical of $\bar{R}$ and $\bar{N}_1$ the intersection of minimal prime ideals of $\bar{R}$, i.e. $\bar{N}_1=\bar{\mathfrak{q}}_1\cap\cdots\cap\bar{\mathfrak{q}}_k$. Then $\bar{N}_1(=\bar{N}_3)$ is nilpotent, i.e. $\bar{N}_1^\rho=\{\bar{0}\}$ for some positive integer ρ. Therefore $\overline{(\mathfrak{q}_1\cdots\mathfrak{q}_k)^\rho}=(\bar{\mathfrak{q}}_1\cdots\bar{\mathfrak{q}}_k)^\rho\subseteq(\bar{\mathfrak{q}}_1\cap\cdots\cap\bar{\mathfrak{q}}_k)^\rho=\bar{N}_1^\rho=\{\bar{0}\}$, i.e. $(\mathfrak{q}_1\cdots\mathfrak{q}_k)^\rho\subseteq\mathfrak{a}$. Hence $\mathfrak{a}=(\mathfrak{q}_1\cdots\mathfrak{b}_k)^\rho\mathfrak{a}\subseteq\mathfrak{a}\mathfrak{a}=\mathfrak{a}^2$, i.e. $\mathfrak{a}^2\supseteq\mathfrak{a}$ so $\mathfrak{a}=\mathfrak{a}^2$.

2. Left Noetherian non-idempotent M-rings

Proposition 3. *Let R be a left Notherian non-idempotent M-ring. Assume $N<\mathfrak{d}$, where N denote the Jacobson radical of R and $\mathfrak{d}$ denote the unique maximal idempotent ideal of R. Then each ideal of R, properly contained in $\mathfrak{d}$, can be written as a product of prime ideals of R which are properly contained in $\mathfrak{d}$.*

Proof. Let $\mathfrak{a}$ be an ideal of R with $\mathfrak{a}<\mathfrak{d}$. Then, by Proposition 6 of [10], there exists a prime ideal $\mathfrak{p}_1$ with $\mathfrak{a}\subseteq\mathfrak{p}_1<\mathfrak{d}$. By the results of McCoy ([6], pp. 829), there exists a minimal prime divisor $\mathfrak{p}_1^0$ of $\mathfrak{a}$ with $\mathfrak{a}\subseteq\mathfrak{p}_1^0\subseteq\mathfrak{p}_1<\mathfrak{d}$ and so the set of minimal prime divisors of $\mathfrak{a}$ is not empty, and it is a finite set ([4]). Let $\mathscr{P}_\mathfrak{a}$ denote the set of minimal prime divisors of an ideal $\mathfrak{a}$. If there exists some $\mathfrak{p}_1^0$ in $\mathscr{P}_\mathfrak{a}$ with $\mathfrak{a}=\mathfrak{p}_1^0$, then there is nothing to prove. If, for each $\mathfrak{p}_1^0\in\mathscr{P}_\mathfrak{a}$, $\mathfrak{a}<\mathfrak{p}_1^0$, then $\mathfrak{a}=\mathfrak{p}_1^0\mathfrak{a}_1$ for some ideal $\mathfrak{a}_1$, and $\mathfrak{a}\subseteq\mathfrak{a}_1$. Now by Theorem 1 we can choose the unique maximal ideal $\mathfrak{a}_1$ in the set $\{\mathfrak{a}_1$: ideals of R with $\mathfrak{a}=\mathfrak{p}_1^0\mathfrak{a}_1\}$, of course $\mathfrak{a}_1$ is uniquely determined by $\mathfrak{p}_1^0$. Now we consider the ordered pairs $(\mathfrak{p}_1^0, \mathfrak{a}_1)$. Let $\{(\mathfrak{p}_1^0, \mathfrak{a}_1)\}_\mathfrak{a}$ denote the set of such pairs. If there exists some $(\mathfrak{p}_1^0, \mathfrak{a}_1)\in\{(\mathfrak{p}_1^0, \mathfrak{a}_1)\}_\mathfrak{a}$ with $\mathfrak{a}<\mathfrak{a}_1$, then we choose the pair $(\mathfrak{p}_1^0, \mathfrak{a}_1)$; by Theorem 5 of [9] we have $\mathfrak{a}_1\subseteq\mathfrak{d}$ or $\mathfrak{a}_1=\mathfrak{d}_\alpha^\rho$ for some ordinal α and some positive integer ρ. If $\mathfrak{a}_1=\mathfrak{d}$ or $\mathfrak{a}_1=\mathfrak{d}_\alpha^\rho$, then by Lemma 1 of [10] $\mathfrak{a}=\mathfrak{p}_1^0$, which is a contradiction, so $\mathfrak{a}_1<\mathfrak{d}$. Thus $\mathfrak{a}<\mathfrak{a}_1<\mathfrak{d}$. By the same arguments, there exists a minimal prime divisor $\mathfrak{p}_2^0$ of $\mathfrak{a}_1$ with $\mathfrak{a}_1\subseteq\mathfrak{p}_2^0<\mathfrak{d}$. Let $\mathscr{P}_{\mathfrak{a}_1}$ denote the set of minimal prime divisors of $\mathfrak{a}_1$. If there exists some $\mathfrak{p}_2^0$ in $\mathscr{P}_{\mathfrak{a}_1}$ with $\mathfrak{a}_1=\mathfrak{p}_2^0$, then there is nothing to prove, because we have $\mathfrak{a}=\mathfrak{p}_1^0\mathfrak{p}_2^0$. If, for each $\mathfrak{p}_2^0\in\mathscr{P}_{\mathfrak{a}_1}$, $\mathfrak{a}_1<\mathfrak{p}_2^0$, then $\mathfrak{a}_1=\mathfrak{p}_2^0\mathfrak{a}_2$ for some ideal $\mathfrak{a}_2$, and $\mathfrak{a}<\mathfrak{a}_1\subseteq\mathfrak{a}_2$. By Theorem 1 we can choose the unique maximal ideal $\mathfrak{a}_2$ in the set $\{\mathfrak{a}_2$: ideals of R with $\mathfrak{a}_1=\mathfrak{p}_2^0\mathfrak{a}_2\}$. By the same arguments as above, we consider the set of pairs $\{(\mathfrak{p}_2^0, \mathfrak{a}_2)\}_{\mathfrak{a}_1}$. If there exists some $(\mathfrak{p}_2^0, \mathfrak{a}_2)$ in $\{(\mathfrak{p}_2^0, \mathfrak{a}_2)\}_{\mathfrak{a}_1}$ with $\mathfrak{a}_1<\mathfrak{a}_2$, then we choose the pair $(\mathfrak{p}_2^0, \mathfrak{a}_2)$, so we have $\mathfrak{a}=\mathfrak{p}_1^0\mathfrak{p}_2^0\mathfrak{a}_2$ and $\mathfrak{a}<\mathfrak{a}_1<\mathfrak{a}_2<\mathfrak{d}$. Repeating the same arguments, since R is left Noetherian, we must arrive at the following situation: $\mathfrak{a}_{m-1}<\mathfrak{p}_m^0<$

$\mathfrak{d}$, $\mathfrak{a}_{m-1}=\mathfrak{p}_m^0\mathfrak{a}_m$, where $\mathfrak{p}_m^0$ is a minimal prime divisor of $\mathfrak{a}_{m-1}$, $\mathfrak{a}_m$ is the unique maximal ideal in the set $\{\mathfrak{a}_m$: ideals of R with $\mathfrak{a}_{m-1}=\mathfrak{p}_m^0\mathfrak{a}_m\}$ and there is no pair in $\{(\mathfrak{p}_m^0, \mathfrak{a}_m)\}_{\mathfrak{a}_{m-1}}$ with $\mathfrak{a}_{m-1}<\mathfrak{a}_m$, i.e. for every pair $(\mathfrak{p}_m^0, \mathfrak{a}_m)$, $\mathfrak{a}_{m-1}=\mathfrak{a}_m$. Let $\{\mathfrak{p}_{m1}^0, \mathfrak{p}_{m2}^0, \cdots, \mathfrak{p}_{mk}^0\}$ denote the set of minimal prime divisors of $\mathfrak{a}_{m-1}$. Then we have $\mathfrak{a}_m=\mathfrak{p}_{m1}^0\mathfrak{a}_m=\mathfrak{p}_{m2}^0\mathfrak{a}_m=\cdots=\mathfrak{p}_{mk}^0\mathfrak{a}_m$ and so $\mathfrak{a}_m=\mathfrak{p}_{m1}^0\mathfrak{p}_{m2}^0\cdots\mathfrak{p}_{mk}^0\mathfrak{a}_m$. By Lemma 2, $(\mathfrak{p}_{m1}^0\mathfrak{p}_{m2}^0\cdots\mathfrak{p}_{mk}^0)^\rho\subseteq\mathfrak{a}_m$ for some positive integer ρ and $\mathfrak{a}_m^2=\mathfrak{a}_m$. If $(\mathfrak{p}_{m1}^0\mathfrak{p}_{m2}^0\cdots\mathfrak{p}_{mk}^0)^\rho=\mathfrak{a}_m$, there remains nothing to prove. If $(\mathfrak{p}_{m1}^0\mathfrak{p}_{m2}^0\cdots\mathfrak{p}_{mk}^0)^\rho<\mathfrak{a}_m$, then by Lemma 1 $(\mathfrak{p}_{m1}^0\mathfrak{p}_{m2}^0\cdots\mathfrak{p}_{mk}^0)^\rho\mathfrak{a}_m=(\mathfrak{p}_{m1}^0\mathfrak{p}_{m2}^0\cdots\mathfrak{p}_{mk}^0)^\rho$. Since $\mathfrak{a}_m=(\mathfrak{p}_{m1}^0\mathfrak{p}_{m2}^0\cdots\mathfrak{p}_{mk}^0)^\rho\mathfrak{a}_m$, we have $\mathfrak{a}_m=(\mathfrak{p}_{m1}^0\mathfrak{p}_{m2}^0\cdots\mathfrak{p}_{mk}^0)^\rho$, which is a contradiction. Thus $\mathfrak{a}$ is a product of prime ideals.

Lemma 4. *Let R, N and $\mathfrak{d}$ be as above. Assume $N<\mathfrak{d}$. Then every ideal $\mathfrak{a}$ of R with $\mathfrak{a}<\mathfrak{d}$, can be written as a product of minimal prime divisors of $\mathfrak{a}$, i.e. $\mathfrak{a}=\mathfrak{p}_1^{e_1}\mathfrak{p}_1^{e_2}\cdots\mathfrak{p}_k^{e_k}$, where $\{\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_k\}$ is the set of minimal prime divisors of $\mathfrak{a}$, such that $\mathfrak{p}_i<\mathfrak{d}$, $i=1, 2, \cdots, k$.*

Proof. By Proposition 3, $\mathfrak{a}$ can be written as a product of prime ideals of R, and each minimal prime divisors of $\mathfrak{a}$ appears as a factor of $\mathfrak{a}$. On the other hand each prime ideal containing $\mathfrak{a}$ contains some of minimal prime divisors of $\mathfrak{a}$ ([6]). Now our statement follows by Proposition 1 of [9].

Proposition 5. *Let R be a left Noetherian non-idempotent M-ring, then every ideal $\mathfrak{a}$ can be written as a product of minimal prime divisors of $\mathfrak{a}$: $\mathfrak{a}=\mathfrak{p}_1^{e_1}\mathfrak{p}_2^{e_2}\cdots\mathfrak{p}_k^{e_k}$, $e_i>0$ for $i=1, 2, \cdots, k$ and $\{\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_k\}$ is the set of minimal prime divisors of $\mathfrak{a}$.*

Proof. First we assume $N<\mathfrak{d}$. By Lemma 4, we consider only the case that $\mathfrak{a}=\mathfrak{d}$ or $\mathfrak{a}=\mathfrak{d}_\alpha^i$ for some ordinal α and some positive integer i. By Proposition 8 of [9], we have for the first time $\mathfrak{d}_\lambda^j=\mathfrak{d}_\lambda^{j+1}=\cdots$ for some ordinal λ and some positive integer j, then $\mathfrak{d}=\mathfrak{d}_\lambda^j$. In the case $j=1$, if $\mathfrak{a}=\mathfrak{d}_\alpha^i$, $\alpha<\lambda$, $i>0$, then $\mathfrak{d}_\alpha$ is a minimal prime divisor of $\mathfrak{a}$ by Theorem 4 of [9], and if $\mathfrak{a}=\mathfrak{d}$, $\mathfrak{a}$ itself is a prime ideal. In the case $j>1$, if $\mathfrak{a}=\mathfrak{d}_\alpha^i$, $\alpha<\lambda+1$, $i>0$ (including the case $\mathfrak{a}=\mathfrak{d}$), by Theorem 4 of [9] $\mathfrak{d}_\alpha$ is a minimal prime divisor of $\mathfrak{a}$. Finally we assume that $N\nless\mathfrak{d}$. Then $N=\mathfrak{d}$ or $N=\mathfrak{d}_\alpha^i$ for some ordinal α and some positive integer i and so $\mathfrak{d}=\{0\}$ by Lemma 1 of [10] and Nakayama's Lemma. The statements are also valid in this case.

3. Left Noetherian idempotent M-ring

DEFINITION. An M-ring R is called to be *idempotent* if $R=R^2$.

By Lemma 1 we have

Lemma 6. *Let R be an idempotent M-ring, then for each ideal $\mathfrak{a}$ of R $\mathfrak{a}=R\mathfrak{a}=\mathfrak{a}R$.*

Lemma 7. *Let R be a ring such that $R=R^2$, then each maximal ideal is a prime ideal of R.*

Now by the quite similar arguments used in proving Proposition 3, we can prove the following:

Proposition 8. *Let R be a left Noetherian idempotent M-ring. Then each ideal of R, properly contained in R, can be written as a product of prime ideals of R.*

Also by the similar consideration used in the proof of Lemma 4, we have

Poposition 9. *Let R be a left Noetherian idempotent M-ring. Then every ideal $\mathfrak{a}$ of R can be written as a product of minimal prime divisors of $\mathfrak{a}$, i.e. $\mathfrak{a}=\mathfrak{p}_1^{e_1}\mathfrak{p}_2^{e_2}\cdots\mathfrak{p}_k^{e_k}$ $e_i>0$ for $i=1, 2, \cdots, k$ and $\{\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_k\}$ is the set of minimal prime divisors of $\mathfrak{a}$.*

4. Left Noetherian *M*-rings

We summarize the preceding results and we have

Theorem 2. *Let R be a left Noetherian M-ring, then each ideal can be written as a product of prime ideals of R, i.e. R is a "general ZPI-ring."*

Theorem 3. *Let R be a left Noetherian M-ring, then every ideal $\mathfrak{a}$ of R can be written as a product of minimal prime divisors of $\mathfrak{a}$, i.e. $\mathfrak{a}=\mathfrak{p}_1^{e_1}\mathfrak{p}_2^{e_2}\cdots\mathfrak{p}_k^{e_k}$ $e_i>0$ for $i=1, 2, \cdots, k$, where $\{\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_k\}$ is the set of minimal prime divisors of $\mathfrak{a}$.*

From the above results and Proposition 1 of [9] we have

Theorem 4. *Let R be a left Noetherian M-ring, then the multiplication of ideals is commutative.*

References

[1] K. Asano: Theory of rings and ideals, Kyoritsu-Shuppan, 1949 (in Japanese).

[2] M.D. Larsen and P.J. McCarthy: Multiplicative theory of ideals, Academic Press, New York and London, 1971.

[3] T. Nakayama and G. Azumaya: Algebra, vol. 2, Iwanami, 1954 (in Japanese).

[4] A.W. Goldie: *The structure of Notehrian rings,* Lecture Note in Mathenatics, 246, Springer-Verlag, 1972, 242–320.

[5] ———: *Semi-prime rings with maximum condition,* Proc. London Math. Soc. (3) **10** (1960), 201–220.

[6] N.H. McCoy: *Prime ideals in general rings*, Amer. J. Math. **71** (1949), 823–833.

[7] S. Mori: *Uber Idealtheorie der Multiplikationsringe*, J. Sci. Hiroshima Univ. Ser. A **19** (1956), 429–437.

[8] ———: *Struktur der Multiplikationsringe*, ibid. **16** (1952), 1–11.

[9] T. Ukegawa: *Some properties of non-commutative multiplication rings*, Proc. Japan Acad. **54 A** (1978), 279–284.

[10] ———: *On the unique maximal idempotent ideals of non-idempotent multiplication rings*, ibid. **55 A** (1979), 132–135.

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