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Osaka J. Math.
17 (1980), 411–414

A REMARK ON THE CARTAN MATRIX OF A CERTAIN p-BLOCK

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(Received June 14, 1979)

1. Introduction

Let G be a finite group with order divisible by a fixed prime p. In this paper a 'block' means a 'p-block'. If B is a block of G with defect group D, we denote by C_B the Cartan matrix of B. Then it holds generally that det $C_B \geq |D|$. So it is interesting to consider when the equality sign holds in the above.

If D is cyclic, we can deduce from Dade's theorem [6] that det $C_B = |D|$. If D is a dihedral 2-group, Brauer [5], (4G) showed that det $C_B = |D|$. Also, Olsson ([9], Proposition 3.2) investigated the elementary divisors of C_B of B with quaternion or semi-dihedral defect group D.

The purpose of this paper is to prove the following

Theorem. *Let B be a block of G with defect group D and C_B the Cartan matrix of B. Suppose that the centralizer in G of any element of order p of D is p-nilpotent. Then det $C_B = |D|$, so one diviscr of C_B is $|D|$ and all other elementary divisors are* 1.

The set of elementary divisors of C_B coincides with the set of the order of defect groups of p-regular (conjugate) classes of G associated with B. (For selection of sets of conjugate classes for the blocks, see Brauer [1], [2], [4], Osima [11], and Iizuka [8].) Also the greatest elementary divisor of C_B is equal to $|D|$ and all other elementary divisors are less than $|D|$. Therefore det $C_B = |D|$ implies that $|D|$ is only one elementary divisor of C_B except 1's.

Let $^{\#}Bl_d(G)$ denote the number of blocks of G with defect d, $^{\#}Cl'_d(G)$ the number of p-regular classes of G with defect d, and p^a the order of a Sylow p-subgroup of G. The following is an immediate consequence of the theorem.

Corollary. *Suppose that the centralizer in G of any element of order p of G is p-nilpotent. Then*

$$^{\#}Bl_d(G) = {}^{\#}Cl'_d(G) \qquad \textit{for any positive integer} \quad d \leq a\,.$$

Acknowledgement. The present version of the theorem is due to Professor T. Okuyama. The author originally proved it under an additional hypothesis. The author wishes to thank Professor T. Okuyama for pointing out that the hypothesis is not necessary. Also the author thanks to Professor K. Iizuka and Professor Y. Tsushima for their comments which improved the presentation of the proof of the theorem.

NOTATION. Let p be a prime number, G a finite group of order $p^a g'$ with $(p, g')=1$, and $\mathfrak{p}$ a fixed prime divisor of p in the ring $\boldsymbol{Z}[\varepsilon]$, where ε is a primitive $|G|$-th root of 1. Then we denote by F the residue class field $\boldsymbol{Z}[\varepsilon]/\mathfrak{p}$, by FG the group algebra of G over F, and by $Z(FG)$ the center of FG. If B is a block of G, we denote by C_B the Cartan matrix of B and if Q is a p-subgroup of G, we denote by $m_B(Q)$ the number of p-regular classes of G associated with B which have Q as a defect group. We denote by $^{\sharp}Bl_d(G)$ and $^{\sharp}Cl'_d(G)$ the number of blocks of G with defect d and the number of p-regular classes of G with defect d respectively. For brevity we write $C(X)$ and $N(X)$ instead of $C_G(X)$ and $N_G(X)$ for a subset X of G respectively. If K is a conjugate class of G, we denote by $\hat{K}$ the class sum of K in the group algebra FG.

2. Proof of Theorem

We first state the following lemma of Brauer [4], (4C), which is useful in the proof of our theorem.

Lemma (Brauer). *Let B be a block of G with defect group D. For any subgroup Q of D*

$$m_B(Q) = \sum_b m_b(Q)$$

where b ranges over the blocks of $N(Q)$ with $b^G=B$.

Proof of the theorem. Suppose that the theorem is false and that G provides a counter example with minimal $|G|$. Let Q be a proper subgroup of D such that $m_B(Q)\neq 0$. By the above lemma, there exists a block b of $N(Q)$ with $b^G=B$ such that $m_b(Q)\neq 0$. Let X be a defect group of b. By Brauer's first main theorem, we may assume that $Q\subsetneqq X\subset D$. Therefore it follows from the minimal nature of G, that $G=N(Q)$.

Let E^* be the block idempotent of $Z(FG)$ corresponding to B. If $E^*=\sum_K a_K\hat{K}$, where K ranges over the p-regular classes of G and $a_K\in F$, $a_K\neq 0$ implies that a defect group of K contains Q. Therefore E^* is an idempotent of $Z(FQC(Q))$. Let $E^*=e^*+\cdots$ be the decomposition into block idempotents of $Z(FQC(Q))$ and b the block of $QC(Q)$ corresponding to e^*. Then $b^G=B$ and B is a unique block which covers b. Let T_b denote the inertia group of b, i.e. $T_b=\{g\in G\,|\,b^g=b\}$. Then there exists a unique block B' of T_b with defect group D which covers b, and furthermore B and B' have the same structure (see Fong [7]). Therefore it follows from the minimal nature of G, that $G=T_b$. Then $E^*=e^*$.

Since $m_B(Q)\neq 0$ there exists a p-regular class K of G with defect group Q such that $\hat{K}E^*\neq 0$. Since $K\subset QC(Q)$ we may write $K=L^G$, where L is a p-regular class of $QC(Q)$ and $L^G=\{x^g\,|\,x\in L, g\in G\}$. Also $m_B(D)=1$ since B has a defect group D, and this implies that there exists a p-regular class K' of G with defect group D such that $\hat{K}'E^*\neq 0$. Since $K'\subset QC(Q)$ we may write $K'=L'^G$, where L' is a p-regular class of $QC(Q)$. Then we have $\hat{L}E^*\neq 0$ and $\hat{L}'E^*\neq 0$. Since a defect group of K is different from that of K', $\hat{K}E^*$ and $\hat{K}'E^*$ are linearly independent. So this implies that $\hat{L}E^*$ and $\hat{L}'E^*$ are linearly independent. Indeed, if $D\cap QC(Q)\neq Q$, then defect groups of L and L' are different since L' has $D\cap QC(Q)$ as a defect group. So $\hat{L}E^*$ and $\hat{L}'E^*$ are linearly independent. If $D\cap QC(Q)=Q$, there exists a unique block B' of $DC(Q)$ with defect group D which covers b. Then $b^{DC(Q)}=B'$ and $B'^G=B$. So E^* is a block idempotent of $Z(DC(Q))$ corresponding to B'. Since K, $K'\subset DC(Q)$, there exist p-regular classes K_0 and K_0' of $DC(Q)$ contained by K and K' respectively such that $\hat{K}_0E^*\neq 0$ and $\hat{K}_0'E^*\neq 0$. Since defect groups of K_0 and K_0' is different, it follows from the minimal nature of G, that $G=DC(Q)$. Then $K'=L'$. If $a\hat{L}E^*+b\hat{K}'E^*=0$ for $a, b\in F$ and let $n=|K|/|L|$, then $a\hat{K}E^*+nb\hat{K}'E^*=0$. So this means $a=b=0$ because of the linear independency of $\hat{K}E^*$ and $\hat{K}'E^*$. So $\hat{L}E^*$ and $\hat{K}'E^*$ are linearly independent. Thus we have rank $C_b\geq 2$ (rank C_b is equal to the number of the irreducible Brauer characters in b).

On the other hand, if π is an element of order p of $Z(Q)$, $QC(Q)\subset C(\pi)$ and $C(\pi)$ is p-nilpotent. Therefore $QC(Q)$ is p-nilpotent. Hence every block of $QC(Q)$ has a unique irreducible Brauer character (see Osima [10]). Hence we have rank $C_b=1$, which is a contradiction.

REMARK. Under the situation of the theorem, the number $k(B)$ of irreducible ordinary characters in B is less than or equal to $|D|$. This follows from the following lemma, which is an immediate consequence of Brauer [3] II, (5D).

Lemma. *Let B be a block of G with defect group D, $s=(\pi, b)$ a major subsection associated with B and $1(b)$ the number of irreducible Brauer characters in b. If $1(b)=1$, then*

$$k(B)\leq |D|.$$

Indeed, under the assumption of the theorem, since $Z(D)\neq 1$, there exists a major subsection $s=(\pi, b)$ associated with B such that $1(b)=1$.

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