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CONFORMAL STRUCTURES ON THE REAL n-TORUS

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0. Introduction

As is well known, a doubly connected region B on the complex projective line $\boldsymbol{P}^1(\boldsymbol{C})$ can be mapped conformally onto the interior of a concentric annular region lying on $\boldsymbol{P}^1(\boldsymbol{C})$ with radii r_1, r_2 $(0\leq r_1<r_2\leq\infty)$. The quantity $\log(r_2/r_1)$, completely determined by B, is called *conformal modulus* of B. A pair of like regions B, B' is conformally equivalent if and only if their moduli are equal. Furthermore the conformal modulus is continuous to the effect that a proximity of B' to B in a certain sense (for example, with respect to Fréchet metric) implies the smallness of difference between their moduli.

Suppose given a real 2-torus $\boldsymbol{T}^2=S^1\times S^1$. The universal covering surface $\tilde{T}^2$ of $\boldsymbol{T}^2$, which is conformally equivalent to $\boldsymbol{C}$ gives rise to the group of its cover transformations $z\mapsto z+m_1\omega_1+m_2\omega_2$ with a pair of complex constants ω_1, ω_2 called canonical periods and with $m_1, m_2=0, \pm1, \pm2, \cdots$. One adopts the ratio $\tau=\omega_2/\omega_1$ as a *conformal modulus* for $\boldsymbol{T}^2$. But, because all the modular transformation $J(\tau)$ of τ corresponds to the same $\boldsymbol{T}^2$, τ is not uniquely determined by $\boldsymbol{T}^2$. That will be the naivest aspect of what we understand and should expect under the terminology *conformal modulus* (cf. e.g., Oikawa [1]).

The present note has been written from an attempt to attach to every real n-torus $(n\geq2)$ a unique conformal modulus which accords with the postulates above reviewed. We set the problem simply as an immediate and natural extension of the one in the classical case of $\boldsymbol{C}=\boldsymbol{R}^2$. Our process to approach the purpose rests on the extremal length method also familiar to most of complex analysts. We collect notations employed afterwards, explain the basic objects and remark an elementary algebraic fact which plays an important rôle in the subsequent analytic considerations in § 1. In §§ 2∼3 we study the foliation of vector fields generated by axes of infinitesimal ellipsoids which characterize the given diffeomorphism. §§ 4∼5 deals, after certain heuristic examinations, with the extremal quasiconformal mapping between rectangular parallelepipeds which turns out an n-dimensional version of Grötzsch's *möglichst konforme Abbildung* of a rectangle onto another rectangle. We state and prove the main theorem in §§ 5∼6 which asserts the existence and uniqueness of the extremal quasiconformal diffeomorphism between given tori. It leads to equivalent notions to Teichmüller distance as well as to coefficient of the Beltrami equation for Teichmüller mappings. The final section makes a brief sketch on the manner in which our conformal modulus behaves as the torus in question varies.

1. Preliminaries

The notations employed throughout this paper are as follows:

$\boldsymbol{Z}$: the set of all integers

$\boldsymbol{Z}^+$: the set of all positive integers

$\boldsymbol{R}$: the set of all real numbers

$\boldsymbol{R}^n$: the n-dimensional Euclidean space

δ_{ij}: the Kronecker's symbol $(i, j=1, 2, \cdots, n)$

$A=(a_{ij})_{i,j=1,\cdots,n}$: a square matrix with (i, j)-component a_{ij}

$\det A$: the determinant of a square matrix A

$\boldsymbol{e}_j$ $(j=1, 2, \cdots, n)$: the coordinate unit column vectors, i.e.,

$$\boldsymbol{e}_1=\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}, \cdots, \boldsymbol{e}_n=\begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix}.$$

The coordinates of a point $\boldsymbol{x}\in\boldsymbol{R}^n$ are denoted by $x^1, x^2, \cdots, x^n$. So $\boldsymbol{x}=x^1\boldsymbol{e}_1+\cdots+x^n\boldsymbol{e}_n$.

$E=(\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n)$: the (n, n)-unit matrix

Q: the unit cube spanned by $\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n$

$\|\cdot\|$: the Euclidean norm of a vector $\cdot$

$U(\boldsymbol{x}_0)$: an n-dimensional neighbourhood of a point $\boldsymbol{x}_0\in\boldsymbol{R}^n$

$\dot{U}(\boldsymbol{x}_0)$: an n-dimensional deleted neighbourhood of a point $\boldsymbol{x}_0\in\boldsymbol{R}^n$

G^c: the complement of a subset G of $\boldsymbol{R}^n$

cl G: the closure of a point set $G\subset\boldsymbol{R}^n$

Δ: an open subset of $\boldsymbol{R}^n$

$C^j[\Delta]$: the class of j-times continuously differentiable functions or mappings in Δ.

$|\cdot|$: the carrier of a submanifold $\cdot$ of $\boldsymbol{R}^n$, i.e., one employs the notation when one regards the submanifold $\cdot$ simply as a point subset of $\boldsymbol{R}^n$ discarding the local coordinations;

Γ: a family of locally rectifiable paths γ lying in a bounded subregion of $\boldsymbol{R}^n$

mes e: n-dimensional Lebesgue measure of a measurable subset e of $\boldsymbol{R}^n$, which is sometimes written as $m_n(e)$, too;

$\rho(\boldsymbol{x})$: a non-negative Borel function with compact support in $\boldsymbol{R}^n$

mod $\Gamma=\inf_\rho [A(\rho)/\{L(\rho)\}^n]$: the modulus of a path family Γ which is defined in terms of the two integrals $A(\rho)=\int \{\rho(\boldsymbol{x})\}^n dm_n(\boldsymbol{x})$ (with the integration domain $\boldsymbol{R}^n$) and $L(\rho)=\inf_{\gamma\in\Gamma}\int_\gamma \rho(\boldsymbol{x})||d\boldsymbol{x}||$.

Here let us recall one fundamental proposition in Linear Algebra:

Lemma 1 (cf. Satake [3], p. 179). *Let A be an arbitrary non-singular real square matrix and ${}^tA$ the transposed of A. Then*

(a) *all the eigen values of the symmetric matrix ${}^tAA$ are positive*;

(b) *there exists a pair of orthogonal matrices T_1, T_2 which makes T_2AT_1 diagonal*;

(c) *it is possible to choose T_1, T_2 so that* $\det(T_2AT_1)=\det A$.

Proof. We assume that A is of rank n. The symmetric matrix ${}^tAA$, known to have real eigen values $\beta_1, \cdots, \beta_n$, can be diagonalized by an orthogonal matrix T_1 whose i-th column is the i-th eigen vector $\boldsymbol{x}_i$ belonging to the eigen value β_i so that

$$(1)\qquad {}^t(AT_1)(AT_1)={}^tT_1({}^tAA)T_1=\begin{pmatrix}\beta_1 & 0\cdots 0\\ 0 & \beta_2 \quad 0\\ \vdots & \ddots\ \vdots\\ 0 & 0\cdots\ \beta_n\end{pmatrix}.$$

We have $0\le||A\boldsymbol{x}_i||^2=(\boldsymbol{x}_i, {}^tAA\boldsymbol{x}_i)=(\boldsymbol{x}_i, \beta_i\boldsymbol{x}_i)=\beta_i||\boldsymbol{x}_i||^2$, while $\det A\neq 0$ implies $\beta_i\neq 0$. This prove (a).

Since the columns $\boldsymbol{t}_i$ $(i=1, 2, \cdots, n)$ of AT_1 are orthogonal to each other by (1), we can compose an orthogonal matrix T_2 with the i-th row $\pm\boldsymbol{t}_i/\sqrt{\beta_i}$, so that

$$(2)\qquad T_2AT_1=\begin{pmatrix}\pm\sqrt{\beta_1} & 0 \cdots\cdots 0\\ 0 & \pm\sqrt{\beta_2}\cdots 0\\ \vdots & \vdots\\ 0 & 0\cdots\cdots\pm\sqrt{\beta_n}\end{pmatrix},$$

which proves (b).

(c) is trivial, because the double signs in (2) can be suitably chosen. q.e.d.

In denoting a linear transformation with its coefficient matrix A, B, etc., we use the same notation A, B, etc., because it will bring no confusion hereafter. Suppose given n real n-vectors $\boldsymbol{a}_i=\begin{pmatrix}a_{1i}\\ a_{2i}\\ \vdots\\ a_{ni}\end{pmatrix}$ $(i=1, 2, \cdots, n)$ which are linearly independent over $\boldsymbol{R}$. We denote by $\mathfrak{S}=\mathfrak{S}(\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n)$ the group of parallel displacements of $\boldsymbol{R}^n$ onto itself, each element of which affects the motion

$$\boldsymbol{R}^n\ni\boldsymbol{x}\mapsto\boldsymbol{x}+\sum_{i=1}^n m_i\boldsymbol{a}_i \qquad \text{for some}\quad m_i\in\boldsymbol{Z}.$$

Now we consider the quotient space $\boldsymbol{R}^n/\mathfrak{S}(\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n)$ endowed with the usual quotient topology and denote it by $\boldsymbol{T}^n=\boldsymbol{T}^n(\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n)$. This is just the *real n-torus* with which we are going to concern ourselves as the main subject in the present investigation. The vectors $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n$ are called *periods* of the torus. If every $\boldsymbol{a}_i$ coincides, in particular, with a coordinate vector $\boldsymbol{b}_i=b^i\boldsymbol{e}_i$ with some $b^i\in\boldsymbol{R}$ $(i=1, 2, \cdots, n)$, the period-parallellepiped of the torus $\boldsymbol{T}^n(\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n)$ will often be referred to as the cylindrical region $\boldsymbol{Z}^n(\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n)$. We denote by I_i the linear interval $[0, b^i)$ or $(b^i, 0]$ according as $b^i>0$ or $b^i<0$ $(i=1, 2, \cdots, n)$. The Cartesian product $I_1\times\cdots\times I_n$, with all the opposite faces identified, shall be named period-interval and be denoted by $\boldsymbol{I}^n(\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n)$. The subset

$$H_j^-=\{(x^1, \cdots, x^n): x^i\in I_i\ (i\neq j),\ x^j=0\} \qquad (j=1, 2, \cdots, n)$$

of $\boldsymbol{I}^n(\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n)$ shall be named the j-th lower base of $\boldsymbol{I}^n(\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n)$. The j-th upper base is similarly defined as

$$H_j^+=\{(x^1, \cdots, x^n): x^i\in I_i\ (i\neq j),\ x^j=b^j\} \qquad (j=1, 2, \cdots, n).$$

2. Vector fields and their orbits

Suppose given a continuously differentiable non-vanishing vector field $\boldsymbol{v}(\boldsymbol{x})$ in $U(\boldsymbol{x}_0)\subset\boldsymbol{R}^n$, namely a vector $\boldsymbol{v}(\boldsymbol{x})$ with the i-th component $l^i(\boldsymbol{x})\in C^1[U(\boldsymbol{x}_0)]$ $(i=1, 2, \cdots, n)$ assigned to every point $\boldsymbol{x}$ of $U(\boldsymbol{x}_0)$. Then we can have a system of ordinary differential equations

$$\frac{dx^1}{l^1(\boldsymbol{x})}=\frac{dx^2}{l^2(\boldsymbol{x})}=\cdots=\frac{dx^n}{l^n(\boldsymbol{x})} \tag{3}$$

in $U(\boldsymbol{x}_0)$: when some of the denominators vanishes, the corresponding numerator is assumed to vanish too as usual. The classical existence and uniqueness theorem assures that through every point $\boldsymbol{x}$ of $U(\boldsymbol{x}_0)$ there passes a unique solution arc $C(\boldsymbol{x})$ of (3), i.e., an n-tuple of continuously differentiable functions $\{x^i(t)\}_{i=1,\cdots,n}$ in some real parameter t satisfies (3) identically in $U(\boldsymbol{x}_0)$. $C(\boldsymbol{x})$ can be regarded as a smooth arc through the point $\boldsymbol{x}$ with the parametrization $x^i=x^i(t)$ $(i=1, 2, \cdots, n)$. Conversely, if a unique simple smooth arc passes through every point $\boldsymbol{x}\in U(\boldsymbol{x}_0)$, the tangent vector field $\{\boldsymbol{v}(\boldsymbol{x})\,|\,\boldsymbol{x}\in U(\boldsymbol{x}_0)\}$ is well defined, which gives rise to equations like (3). Thus we can identify any system of ordinary differential equations of type (3) with a family of smooth arcs $\{C(\boldsymbol{x})\}=\{C(\boldsymbol{x})\,|\,\boldsymbol{x}\in U(\boldsymbol{x}_0)\}$, which we shall call the family of *orbital arcs* (or briefly *orbits*) defined by (3).

DEFINITION 1. If p systems of differential equations $(p\in\boldsymbol{Z},\ 2\leq p\leq n)$

$$\frac{dx^1}{l_i^1(\boldsymbol{x})}=\frac{dx^2}{l_i^2(\boldsymbol{x})}=\cdots=\frac{dx^n}{l_i^n(\boldsymbol{x})} \qquad (i=1, 2, \cdots, p)$$

which define p vector fields, satisfy the condition

$$\sum_{k=1}^{n} l_i^k(\boldsymbol{x}) l_j^k(\boldsymbol{x}) = 0 \qquad (i \neq j;\ i, j=1, 2, \cdots, p)$$

in $U(\boldsymbol{x}_0)$, p smooth arcs intersect each other orthogonally at every point $\boldsymbol{x} \in U(\boldsymbol{x}_0)$. Then, denoting these families of arcs by $\{C_i(\boldsymbol{x})\}$ $(i=1, 2, \cdots, p)$, we say that *orthogonal p systems of orbital arcs* are defined in $U(\boldsymbol{x}_0)$.

Theorem 1. *Suppose that there are orthogonal 2 systems of orbits $\{C_i(\boldsymbol{x})\}$, $\{C_j(\boldsymbol{x})\}$ in a neighbourhood $U(\boldsymbol{x}_0)$ and that an arbitrary point $\boldsymbol{x}^*$ is prescribed in $U(\boldsymbol{x}_0)$. Then an ordered triple $\boldsymbol{x}^*$, $\{C_j\}$ and $\{C_i\}$ determines a smooth surface $S_{ij}(\boldsymbol{x}^*)$ uniquely to the effect that through every point of $S_{ij}(\boldsymbol{x}^*)$ there passes one and only one arc of $\{C_i(\boldsymbol{x}) \mid \boldsymbol{x} \in C_j(\boldsymbol{x}^*)\}$.*

Proof. Let $\boldsymbol{x}=\boldsymbol{x}(t)$ be a parametric equation for $C_j(\boldsymbol{x}^*)$ represented by the arc-length parameter t such that $\boldsymbol{x}(0)=\boldsymbol{x}^*$ and let

$$\boldsymbol{x} = \boldsymbol{x}(s, t) \tag{4}$$

the solution of the system of differential equations

$$\frac{dx^1}{l_j^1(\boldsymbol{x})} = \frac{dx^2}{l_j^2(\boldsymbol{x})} = \cdots = \frac{dx^n}{l_j^n(\boldsymbol{x})} \tag{5}$$

with initial condition $\boldsymbol{x}(0, t)=\boldsymbol{x}(t)$. Then (4) is seen to be a smooth function in the two variables s, t on account of the assumption imposed on the coefficients of the differential equation of type (3) and can be regarded as a parametric equation for some smooth surface immersed in $\boldsymbol{R}^n$. q.e.d.

REMARK 1. The subsurface $S_{ij}(\boldsymbol{x}^*)$ is equivalent to the solution of the quasilinear partial differential equation

$$\sum_{k=1}^{n-1} l_i^k(x^1, \cdots, x^{n-1}, x^n) \frac{\partial x^n}{\partial x^k} = l_i^n(x^1, \cdots, x^{n-1}, x^n) \tag{6}$$

with the initial curve $C_j(\boldsymbol{x}^*)$.

In order to attain a more general concept of orbital submanifolds which include points, orbital arcs and orbital surfaces (existence of which has just been proved in Theorem 1) as 0-dimensional, 1-dimensional and 2-dimensional special cases respectively, we try to transpose the locution in the following way:

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a 0-dimensional orbital sub nanifold $\boldsymbol{x}^*$ is the projection of a 1-dimensional orbital submanifold $C_i(\boldsymbol{x})$ along C_i;	an orbital arc $C_i(\boldsymbol{x})$ passes through a point $\boldsymbol{x}^*$.
a 1-dimensional orbital submanifold $C_j(\boldsymbol{x}^*)$ is the projection of a 2-dimensional orbital submanifold $S_{ij}(\boldsymbol{x}^*)$ along C_i;	an orbital surface $S_{ij}(\boldsymbol{x})$ contains an orbital arc $C_j(\boldsymbol{x}^*)$.

Thus the notion of projection map π along C_i has been set up so as to satisfy

$$\pi_{C_i}: C_i(\boldsymbol{x}) \mapsto \boldsymbol{x}^*, \quad \pi_{C_i}: S_{ij}(\boldsymbol{x}^*) \mapsto C_j(\boldsymbol{x}^*).$$

The pre-images in the last relationships may be regarded as product sets specifically constructed: $C_i(\boldsymbol{x}^*) = C_i(\boldsymbol{x}) \times \boldsymbol{x}^*$, $S_{ij}(\boldsymbol{x}^*) = C_i(\boldsymbol{x}) \times C_j(\boldsymbol{x}^*)$ $(\boldsymbol{x} \in C_j(\boldsymbol{x}^*))$. We arrive at a motivation: an operator $C_i = \pi_{C_i}^{-1}$ acts on a 0-dimensional (resp. 1-dimensional) orbital submanifold $\boldsymbol{x}^*$ (resp. $C_j(\boldsymbol{x}^*)$) to produce a 1-dimensional (resp. 2-dimensional) orbital submanifold $C_i(\boldsymbol{x}^*)$ (resp. $S_{ij}(\boldsymbol{x}^*)$), i.e.,

$$C_i: \begin{cases} \boldsymbol{x}^* \mapsto C_i(\boldsymbol{x}^*) \\ C_j(\boldsymbol{x}^*) \mapsto S_{ij}(\boldsymbol{x}^*). \end{cases}$$

In defining the operator C_i, no confusion may occur if we use the same symbol by identifying the operator with the relevant orbital arc C_i etc. as

$$C_i \times \boldsymbol{x} = C_i(\boldsymbol{x}), \quad C_i \times C_j(\boldsymbol{x}) = \begin{cases} S_{ij}(\boldsymbol{x}), & (i \neq j) \\ C_i(\boldsymbol{x}), & (i = j). \end{cases}$$

Superposition of those operators will be interpreted at least formally by

$$\begin{gathered} (C_i \times C_j)(\boldsymbol{x}) = C_i \times C_j(\boldsymbol{x}), \\ (C_i \times (C_j \times C_k))(\boldsymbol{x}) = C_i \times (C_j \times C_k)(\boldsymbol{x}), \\ ((C_i \times C_j) \times C_k)(\boldsymbol{x}) = (C_i \times C_j) \times C_k(\boldsymbol{x}) = C_i \times (C_j \times C_k)(\boldsymbol{x}) \end{gathered}$$

so that the multiplication may be associtaive. What can we say, however, about the existence or the uniqueness of $(C_i \times C_j \times C_k)(\boldsymbol{x}_0)$ at all ? The answer is in the affirmative: For any permutation $(i_1, i_2, \cdots, i_p)$ of p indices chosen out of $1, 2, \cdots, n$, the submanifold $(C_{i_1} \times C_{i_2} \times \cdots \times C_{i_p})(\boldsymbol{x}_0)$ exists and is unique. In fact, if $p=2$, we are done. Next we suppose that $(C_{i_2} \times \cdots \times C_{i_p})(\boldsymbol{x}_0)$ uniquely exists. Then if we adopt $(C_{i_2} \times \cdots \times C_{i_p})(\boldsymbol{x}_0)$ as initial submanifold, the partial differential equation (6) with $i = i_1$ has a unique solution according to the Cauchy's theorem. These observaiions permit us to pose the

DEFINITION 2. Given any orthogonal n systems of orbits $\{C_\nu\}_{\nu=1,\cdots,n}$, we set as follows:

$$M_i(\boldsymbol{x}) = C_i(\boldsymbol{x}), \quad M_{ij}(\boldsymbol{x}) = S_{ij}(\boldsymbol{x});$$

as for $p \geq 3$, let $i_1, i_2, \cdots, i_p$ be any permutation of p indices out of $1, 2, \cdots, n$. Then we set

$$M_{i_1 i_2 \cdots i_p}(\boldsymbol{x}) = C_{i_1} \times M_{i_2 \cdots i_p}(\boldsymbol{x}).$$

The commutativity holds too, namely

Theorem 2. *Suppose that there exist orthogonal* $p+1$ *systems of orbits* $C_j(x)$ $(j=1, 2, \cdots, p, p+1)$ *in* $U(x_0)$ $(2 \leq p \leq n-1)$. *Then for any permutation* $(i_1, \cdots, i_p)$ *of the multi-index* $(1, 2, \cdots, p)$ *we have*

$$(C_{i_1} \times C_{i_2} \times \cdots \times C_{i_p})(x) = (C_1 \times C_2 \times \cdots \times C_p)(x) \qquad (x \in U(x_0)).$$

Proof. Suppose first $p=2$. We lose no generality in assuming $i_1=1$, $i_2=2$. Let y_0 be an arbitrary point on the surface $S_{21}(x_0) \cap U(x_0)$. Since $S_{21}(x_0) \perp S_{13}(y_0)$, $S_{13}(y_0) \perp S_{23}(y_0)$ at y_0, we must have $S_{21}(x_0) \perp S_{23}(y_0)$ at y_0. Hence the intersection arc $\gamma = S_{21}(x_0) \cap S_{13}(y_0)$ is orthogonal to both $C_2(y_0)$ and $C_3(y_0)$ at y_0. It follows that the tangential direction at every point y on γ coincides with that to $C_1(y)$. So $|\gamma|$ coincides with $|C_1(y_0)|$. Since γ passes through the point $y_1 = S_{13}(y_0) \cap C_2(x_0)$, we see $|C_1(y_0)| = |C_1(y_1)|$. That is to say, every $C_1(x)$ $(x \in C_2(x_0))$ lies on $S_{21}(x_0)$. Interchanging i_1 with i_2, we conclude that $|S_{i_1 i_2}(x_0)| = |S_{i_2 i_1}(x_0)|$.

Next we shall show that $S_{12}(x_0) = S_{21}(x_0)$. Suppose that $S_{12}(x_0)$ is parametrized by $x = x(s, t)$ just as in the proof of Theorem 1. Take an arbitrary point x on $|S_{12}(x_0)| = |S_{21}(x_0)|$. The orbital arc $C_2(x)$, lying on $S_{12}(x_0)$, necessarily intersects the arc $C_1(x_0)$ through the point x_0. Let $y = y(\tilde{s})$ be the parametric representation of $C_1(x_0)$ with the arc length parameter $\tilde{s}$ such that $y(0) = x_0$. Solving the partial differential equation (6) with $i=2$ for the initial arc $C_1(x_0)$, we obtain a unique solution $x(\tilde{s}, \tilde{t})$ such that $x(\tilde{s}, 0) = y(\tilde{s})$. Since $x(\tilde{s}, \tilde{t}) = x(s, t)$, the smooth dependence of solutions for ordinary differential equations on their initial data yields the continuity of $\partial\tilde{s}/\partial s$, $\partial\tilde{s}/\partial t, \cdots, \partial t/\partial\tilde{s}$, $\partial t/\partial\tilde{t}$, namely the parameter change $\tilde{s} = \tilde{s}(s, t)$, $\tilde{t} = \tilde{t}(s, t)$ are seen to furnish us with different representations of a single smooth surface, which was to be proved.

For a unified description of higher dimensional orbital submanifolds $(C_{i_q} \times \cdots \times C_{i_p})(x_0)$ $(p=1, 2, \cdots, n;\ q<p)$ we denote the arc length of an orbital arc $C_{i_m}(x)$ $(m=1, 2, \cdots, p)$ generally by s^{i_m}, which was normalized above so that the parametric representation $x = x(s^{i_p})$ of $C_{i_p}(x_0)$ might satisfy $x(0) = x_0$. The parametric representation $x = x(s^{i_{p-1}}, s^{i_p})$ of $(C_{i_{p-1}} \times C_{i_p})(x_0)$ was likewise subject to the requirement that $x(0, s^{i_p})$ coincides with $C_{i_p}(x_0)$. As to the general $(p-q+1)$-dimensional orbital submanifold $(C_{i_q} \times \cdots \times C_{i_p})(x_0)$ also we shall commit ourselves to use the similar representation $x = x(s^{i_q}, \cdots, s^{i_p})$ with the initial condition that $x(0, s^{i_{q+1}}, \cdots, s^{i_p})$ coincides with the $(p-q)$-dimensional orbital submanifold $(C_{i_{q+1}} \times \cdots \times C_{i_p})(x_0)$.

Now we are ready to clarify the general commutativity property of those operators C_{i_m} $(m=1, 2, \cdots, p)$ announced in our proposition by means of the induction with respect to p. We have only to consider the case when $i_1 \neq 1$, since otherwise the conclusion follows trivially from the hypothesis of induction. Let $i_1 = q$. Then let us have the orbital submanifold $(C_{q+1} \times \cdots \times C_p)(x_0)$ in mind, on which we take an arbitrary point x_1 and fix it for a moment. Let the coordinates of $\boldsymbol{x}_1$ be denoted, in terms of our customary arc length, by $s^{q+1}=s_1^{q+1}, \cdots, s^p=s_1^p$. The orbital surface $(C_{q-1}\times C_q)(\boldsymbol{x}_1)$ (resp. $(C_q\times C_{q-1})(\boldsymbol{x}_1)$) can be represented by the equation $\boldsymbol{x}=\boldsymbol{x}(s^{q-1}, s^q, s_1^{q+1}, \cdots, s_1^p)$ (resp. $\boldsymbol{x}=\boldsymbol{x}(\tilde{s}^q, \tilde{s}^{q-1}, s_1^{q+1}, \cdots, s_1^p)$) and we have seen above that

$$\boldsymbol{x}(s^{q-1}, s^q, s_1^{q+1}, \cdots, s_1^p) = \boldsymbol{x}(\tilde{s}^q, \tilde{s}^{q-1}, s_1^{q+1}, \cdots, s_1^p)$$

and that $\tilde{s}^{q-1}, \tilde{s}^q$ are both smooth functions in s^{q-1}, s^q. When $\boldsymbol{x}_1$ varies, its coordinates $s^{q+1}, \cdots, s^p$ will deviate to some extent from their initial values $s_1^{q+1}, \cdots, s_1^p$. Then $\tilde{s}^{q-1}, \tilde{s}^q$ moves accordingly but it is in a smooth manner that the latters two depend upon the $p-q$ formers by the same reason as before. Hence

$$(7) \qquad \begin{aligned}&(C_1\times\cdots\times C_{q-1}\times C_q\times\cdots\times C_p)(\boldsymbol{x}_0)\\ &\quad=(C_1\times\cdots\times C_q\times C_{q-1}\times\cdots\times C_p)(\boldsymbol{x}_0).\end{aligned}$$

We repeat the same operation, until the factor C_q comes to the left end of the factors in the right hand product of (7). At every step of the procedure the identity (7) still remains to hold. We get finally

$$\begin{aligned}&(C_1\times\cdots\times C_{q-1}\times C_q\times C_{q+1}\times\cdots\times C_p)(\boldsymbol{x}_0)\\ &\quad=(C_q\times\cdots\times C_{q-1}\times C_{q+1}\times\cdots\times C_p)(\boldsymbol{x}_0).\end{aligned}$$

q.e.d.

DEFINITION 3. The two submanifolds $C_{i_1}\times\cdots\times C_{i_p}(\boldsymbol{x})$ and $C_{i_{p+1}}\times\cdots\times C_{i_n}(\boldsymbol{x})$ are said to be *orthogonal complements* each of the other.

Back to the situation at the beginning of this section, we assume given a single vector field $(P^1(\boldsymbol{x}), \cdots, P^n(\boldsymbol{x}))$ in an open subset Δ of $\boldsymbol{R}^n$, where the components $P^j(\boldsymbol{x})$ $(j=1, 2, \cdots, n)$ are all of $C^1[\Delta]$. Now supposing that there is an $(n-1)$-dimensional smooth submanifold M of $\boldsymbol{R}^n$ orthogonal to this vector field represented locally by a real-valued function $x^n=x^n(x^1, \cdots, x^{n-1})$ of the real variables $x^1, \cdots, x^{n-1}$, we shall seek a condition for M to satisfy.

Since the normal to M has the direction

$$\partial x^n/\partial x^1 : \partial x^n/\partial x^2 : \cdots : \partial x^n/\partial x^{n-1} : (-1),$$

we have

$$\frac{\partial x^n/\partial x^1}{P^1(\boldsymbol{x})} = \frac{\partial x^n/\partial x^2}{P^2(\boldsymbol{x})} = \cdots = \frac{\partial x^n/\partial x^{n-1}}{P^{n-1}(\boldsymbol{x})} = -\frac{1}{P^n(\boldsymbol{x})}.$$

A vector $\boldsymbol{x}_i=\begin{pmatrix} x_i^1 \\ x_i^2 \\ \vdots \\ x_i^n \end{pmatrix}$ $(i=1, 2, \cdots, n-1)$ becomes a solution of the quasilinear partial differential equation

$$(8)\qquad \frac{\partial x_i^n}{\partial x_i^j} = -\frac{P^j(\boldsymbol{x})}{P^n(\boldsymbol{x})}, \qquad (j=1, 2, \cdots, n-1),$$

if and only if $d\boldsymbol{x}_i$ satisfies the auxiliary differential equations

$$(9)\qquad \frac{dx_i^1}{0} = \frac{dx_i^2}{0} = \cdots = \frac{-dx_i^j}{P^n(\boldsymbol{x})/P^j(\boldsymbol{x})} = \cdots = \frac{dx_i^n}{1},$$

the numerators shall vanish everytime the corresponding denominators do. These vectors $\{d\boldsymbol{x}_i\}$ $(i=1, 2, \cdots, n-1)$ are all on the $(n-1)$-dimensional tangent space M requested orthogonal to the given vector ${}^t(P^1(\boldsymbol{x}),\cdots, P^n(\boldsymbol{x}))$. The system of equations (9) has the unique solution curve $C_i(\boldsymbol{x})$ through every point $\boldsymbol{x}\in\Delta$. $C_i(\boldsymbol{x})$ lies entirely in the linear subspace spanned by $\boldsymbol{e}_i$ and $\boldsymbol{e}_n$, hence $\{d\boldsymbol{x}_i\}_{i=1,\cdots,n-1}$ are linearly independent of each other. They can be orthogonalized (e.g., by Schmidt's method) into $d\boldsymbol{x}_i={}^t(Q_i^1(\boldsymbol{x}), \cdots, Q_i^n(\boldsymbol{x}))$ within the realm of M. The system of differential equations

$$\frac{dx^1}{Q_i^1(\boldsymbol{x})} = \frac{dx^2}{Q_i^2(\boldsymbol{x})} = \cdots = \frac{dx^n}{Q_i^n(\boldsymbol{x})}, \qquad i=1, 2, \cdots, n-1$$

is solved uniquely with a curve $\gamma_i(\boldsymbol{x})$ through arbitrarily given point $\boldsymbol{x}\in\Delta$. $\{\gamma_i(\boldsymbol{x})\}_{i=1,\cdots,n-1}$ constitute orthogonal $n-1$ systems of orbits which are orthogonal also to the given vector field. By virtue of Theorem 2 the desired $(n-1)$-dimensional submanifold M is thus constructible. We have proved

Theorem 3. *Let the orbits of a single system of differential equations of type* (3) *be well defined in an open subset Δ of $\boldsymbol{R}^n$. Then there exists a unique $(n-1)$-dimensional submanifold through an arbitrary point $\boldsymbol{x}_0\in\Delta$ as orthogonal complement to the prescribed* 1 *system of orbital arcs.*

3. Characteristic submanifolds

In this section F shall always denote a non-singular C^2-diffeomorphism $\boldsymbol{x}'=F(\boldsymbol{x})$ defined in a bounded open subset Δ of $\boldsymbol{R}^n$, which preserves the orientation and is nowhere conformal, unless otherwise stated explicitly and let $\mathcal{F}(\Delta)$ be the family of all such F.

Let $J(\boldsymbol{x})=J(\boldsymbol{x}; F)$ denote the Jacobian matrix $(\partial x^{i'}/\partial x^j)_{i,j=1,\cdots,n}$ of F at $\boldsymbol{x}\in\Delta$. Then the symmetric matrix ${}^tJ(\boldsymbol{x}; F)J(\boldsymbol{x}; F)$ has, by Lemma 1 (a), n positive eigen values $\lambda(\boldsymbol{x})=\lambda^i(\boldsymbol{x})$ $(i=1, 2, \cdots, n)$ which satisfy the characteristic equation

$$(10)\qquad \det({}^tJ(\boldsymbol{x}; F)J(\boldsymbol{x}; F)-\lambda(\boldsymbol{x})E) = 0\,.$$

Fixing a point $\boldsymbol{x}_0$ in Δ, we pose the

DEFINITION 4. The infinitesimal ellipsoid as counter-image by F^{-1} of the infinitesimal n-ball centred at $F(\boldsymbol{x}_0)$ is called *characteristic ellipsoid* of F at $\boldsymbol{x}_0$ and is denoted by $\mathfrak{E}(dx^1/\sqrt{\lambda^1(\boldsymbol{x}_0)}, \cdots, dx^n/\sqrt{\lambda^n(\boldsymbol{x}_0)})$: $\boldsymbol{x}_0$ is called to be a *point of semi-conformality* for F, if some of $\lambda^i(\boldsymbol{x}_0)$ $(i=1, 2, \cdots, n)$ coincide.

When $\boldsymbol{x}$ varies, the collection $\{\lambda^i(\boldsymbol{x})\}_{i=1,\cdots,n}$ will be expected to move nicely depending on $\boldsymbol{x}$, which we are going to show in what follows.

Consider a polynomial

$$g(t;\, \boldsymbol{x}) = a_0(\boldsymbol{x})+a_1(\boldsymbol{x})t+\cdots+a_{n-1}(\boldsymbol{x})t^{n-1}+a_n(\boldsymbol{x})t^n \qquad (a_n(\boldsymbol{x})\neq 0)$$

in a complex variable t and with coefficients in a vectorial parameter $\boldsymbol{x}$ ranging over the region $\Delta\subset \boldsymbol{R}^n$. For any $\boldsymbol{x}\in\Delta$, all the n roots of the algebraic equation $g(t;\, \boldsymbol{x})=0$ are finite, which we shall denote by $\alpha_i=\alpha_i(\boldsymbol{x})$ $(i=1, 2, \cdots, n)$.

Lemma 2. *Let $a_i(\boldsymbol{x})$ $(i=1, 2, \cdots, n)$ be continuous at $\boldsymbol{x}_0\in\Delta$. Then*

(a) *the continuity in $\boldsymbol{x}$ at $\boldsymbol{x}_0$ is inherited by $\alpha_i(\boldsymbol{x})$ $(i=1, 2, \cdots, n)$;*

(b) *the multiplicity of $\alpha_i(\boldsymbol{x})$ $(i=1, 2, \cdots, n)$ is an upper semi-continuous function in $\boldsymbol{x}$.*

Proof. It causes no loss of generality in assuming that $\alpha_1(\boldsymbol{x}_0)=\cdots=\alpha_p(\boldsymbol{x}_0)$ for some $p\in\boldsymbol{Z}^+$. There exists a neighbourhood $U\subset\boldsymbol{C}$ of $t=\alpha_1$ which contains no root of $g(t;\, \boldsymbol{x}_0)=0$ other than $\alpha_1(\boldsymbol{x}_0)$. As $\boldsymbol{x}\to\boldsymbol{x}_0$, so $a_i(\boldsymbol{x})\to a_i(\boldsymbol{x}_0)$ $(i=1, 2, \cdots, n)$. Hence $g(t;\, \boldsymbol{x})$ and $\partial g(t;\, \boldsymbol{x})/\partial t$ tends to $g(t;\, \boldsymbol{x}_0)$ and $\partial g(t;\, \boldsymbol{x}_0)/\partial t$ respectively on U where the convergence is uniform. Let $\varepsilon>0$ be an arbitrary small number and let C a closed contour in the ε-neighbourhood of $\alpha_1(\boldsymbol{x}_0)$. If $\boldsymbol{x}$ is sufficiently near $\boldsymbol{x}_0$ with reference to C, we see

$$-\frac{\sqrt{-1}}{2\pi}\int_C \frac{\partial g(t;\, \boldsymbol{x})/\partial t}{g(t;\, \boldsymbol{x})}dt = -\frac{\sqrt{-1}}{2\pi}\int_C \frac{\partial g(t;\, \boldsymbol{x}_0)/\partial t}{g(t;\, \boldsymbol{x}_0)}dt\,,$$

the contour integral in both sides being valued only at an integer. Therefore the left hand side must be equal to p, which proves (a).

The root $\alpha_1=\alpha_1(\boldsymbol{x})$ of the above $g(t;\, \boldsymbol{x})=0$ is of multiplicity p if and only if

$$\left(\frac{\partial}{\partial t}\right)^{\nu} g(t;\, \boldsymbol{x})\,|_{t=\alpha_1(x)} = 0 \qquad (\nu=1, 2, \cdots, p-1)\,,$$

$$\left(\frac{\partial}{\partial t}\right)^{p} g(t;\, \boldsymbol{x})\,|_{t=\alpha_1(x)} \neq 0\,.$$

If $\tilde{\boldsymbol{x}}$ is sufficiently near $\boldsymbol{x}$, $(\partial/\partial t)^p g(t;\, \boldsymbol{x})\,|_{t=\alpha_1(\tilde{x})}\neq 0$ by (a), which proves (b).

From Lemma 2 (b) follows at once

Theorem 4. *The set of $\boldsymbol{x}\in\boldsymbol{R}^n$ which renders an i-th eigen value $\lambda^i(\boldsymbol{x})$ of ${}^tJ(\boldsymbol{x};\, F)J(\boldsymbol{x};\, F)$ simple is, if non-void, open.*

On the other hand, one of the consequences of Lemma 2 (a) is

Lemma 3. *Let F belong to $\mathfrak{F}(\Delta)$ and let $\boldsymbol{x}_0$ be any point of Δ. If we denote by $\lambda^1(\boldsymbol{x}_0), \cdots, \lambda^n(\boldsymbol{x}_0)$ arbitrarily ordered eigen values of the matrix ${}^tJ(\boldsymbol{x}_0; F) \cdot J(\boldsymbol{x}_0; F)$, there exists, in a suitable neighbourhood $U(\boldsymbol{x}_0)$ a numbering $\lambda^i(\boldsymbol{x})$ $(i=1, 2, \cdots, n)$ for the set of eigen values of ${}^tJ(\boldsymbol{x}; F)J(\boldsymbol{x}; F)$,* such that we have $\lim \lambda^i(\boldsymbol{x})=\lambda^i(\boldsymbol{x}_0)$ $(i=1, 2, \cdots, n)$ *as $\boldsymbol{x}$ approaches $\boldsymbol{x}_0$ along any path in Δ.*

DEFINITION 5. $\{\lambda^i(\boldsymbol{x})\}_{i=1,\cdots,n}$ shall be named *prolongation* of $\{\lambda^i(\boldsymbol{x}_0)\}_{i=1,\cdots,n}$ in $U(\boldsymbol{x}_0)$.

For preciser informations than Lemma 2 about the eigen values of ${}^tJ(\boldsymbol{x}; F)J(\boldsymbol{x}; F)$ as well as the eigen vectors belonging to them we prepare

Lemma 4. *If the coefficients $a_i(\boldsymbol{x})$ $(i=1, 2, \cdots, n)$ of the polynomial $g(t; \boldsymbol{x})$ are of class $C^1[\Delta]$, so are the roots $\alpha_i(\boldsymbol{x})$ $(i=1, 2, \cdots, n)$.*

Proof. We first show that the smoothness of the coefficients $a_i(\boldsymbol{x})$ $(i=1, 2, \cdots, n)$ is inherited by a simple root. Taking an arbitrary point $\boldsymbol{x}_0 \in \Delta$, we suppose there is an index i such that $\alpha_i=\alpha_i(\boldsymbol{x}_0)$ is a simple root of $g(t; \boldsymbol{x}_0)=0$. Then we see that the partial derivative $\partial g(t; \boldsymbol{x})/\partial t$ does not vanish at the point $(\alpha_i, \boldsymbol{x}_0)$ and that $g(t; \boldsymbol{x})$, $\partial g(t; \boldsymbol{x})/\partial t$, $\partial g(t; \boldsymbol{x})/\partial x^j$ $(j=1, 2, \cdots, n)$ are all continuous in the variable $(t; \boldsymbol{x})$. Hence, by the implicit function theorem, $t=\alpha_i(\boldsymbol{x})$ is of class C^1 in some neighbourhood of $\boldsymbol{x}_0$.

Now we shall prove the proposition by induction with respect to the degree n of $g(t; \boldsymbol{x})$. If $n=1$, the assertion is trivial. So we assume that the proposition is valid for $n=k$.

1° The case in which the polynomial

$$g(t; \boldsymbol{x}_0) = a_0(\boldsymbol{x}_0)+a_1(\boldsymbol{x}_0)t+\cdots+a_k(\boldsymbol{x}_0)t^k+a_{k+1}(\boldsymbol{x}_0)t^{k+1}$$

has at least one simple root $\alpha_0=\alpha_0(\boldsymbol{x}_0)$. Since $\partial g(t; \boldsymbol{x})/\partial t \neq 0$ at $(\alpha_0, \boldsymbol{x}_0)$, $t=\alpha_0(\boldsymbol{x})$ is well defined in some neighbourhood $V(\boldsymbol{x}_0)$ and is of class $C^1[V(\boldsymbol{x}_0)]$. Put $g(t; \boldsymbol{x})=(t-\alpha_0(\boldsymbol{x}))g_1(t; \boldsymbol{x})$. Then the coefficients of $g_1(t; \boldsymbol{x})$ are of $C^1[V(\boldsymbol{x}_0)]$. Hence every zero of $g_1(t; \boldsymbol{x})$ is continuously differentiable in a neighbourhood of $\boldsymbol{x}_0$ by the hypothesis of induction. Thus all the zeros of $g(t; \boldsymbol{x})$ are continuously differentiable in the same neighbourhood of $\boldsymbol{x}_0$

2° The case in which $g(t; \boldsymbol{x}_0)$ has solely multiple zeros. Let β_j $(j=1, 2, \cdots, \nu)$ denote these zeros with multiplicities $m_j \geq 2$ $(j=1, 2, \cdots, \nu)$. Then $\partial g(t; \boldsymbol{x}_0)/\partial t$ is the polynomial of degree k with smooth coefficients, which remains to have all the $\beta_j(\boldsymbol{x}_0)$ $(j=1, 2, \cdots, \nu)$ as its zeros (but with the mutliplicities m_j-1 $(j=1, 2, \cdots, \nu)$). The hypothesis of the induction gives the C^1-smoothness of $\beta_j(\boldsymbol{x})$ $(j=1, 2, \cdots, \nu)$ at $\boldsymbol{x}_0$, which completes the proof.

Hence we obtain

Theorem 5. *The eigen values of* ${}^tJ(\boldsymbol{x}; F)J(\boldsymbol{x}; F)$ *for* $\mathcal{F} \in F(\Delta)$ *are of class* $C^1[\Delta]$.

The eigen space belonging to the i-th eigen value $\lambda^i(\boldsymbol{x})$ $(i=1, 2, \cdots, n)$ defined in (10) is spanned by one or more vectors $d\boldsymbol{x}=d\boldsymbol{x}_i$ $(i=1, 2, \cdots, n)$ which satisfies

$$({}^tJ(\boldsymbol{x}; F)J(\boldsymbol{x}; F)-\lambda^i(\boldsymbol{x})E)d\boldsymbol{x} = \mathbf{0}\,, \tag{11}$$

i.e., the semi-axis of the characteristic ellipsoid $\mathfrak{E}(dx^1/\sqrt{\lambda^1(\boldsymbol{x})}, \cdots, dx^n/\sqrt{\lambda^n(\boldsymbol{x})})$ of F at $\boldsymbol{x}$ corresponding to the length $1/\sqrt{\lambda^i(\boldsymbol{x})}$ lies on the eigen space defined by (11) (Lemma 1 (b)).

The totality of the eigen vectors and eigen spaces defined by (11) spans a tangent space at $\boldsymbol{x}$, which we call *tangent eigen space* of ${}^tJ(\boldsymbol{x}; F)J(\boldsymbol{x}; F)$ at $\boldsymbol{x}$. If $\lambda(\boldsymbol{x})=\lambda^i(\boldsymbol{x})$ is a root of (10) of multiplicity m, the corresponding tangent eigen space has dimension m there and vice versa.

EXAMPLE 1. The case $n=2$. $J(\boldsymbol{x}; F)=(\partial x^{i'}/\partial x^j)_{i,j=1,2}$. ${}^tJ(\boldsymbol{x}; F)J(\boldsymbol{x}; F)=(g_{ij})_{i,j=1,2}$, $g_{12}=g_{21}$. The roots of the characteristic equation

$$\det\begin{pmatrix} g_{11}-\lambda & g_{12} \\ g_{21} & g_{22}-\lambda \end{pmatrix} = 0$$

are $\lambda=(g_{11}+g_{22}\pm\sqrt{(g_{11}-g_{22})^2+4g_{12}^2})/2$, whose ratio λ^2/λ^1 just amounts to the square of dilatation-quotient. In search of eigen vectors $d\boldsymbol{x}_i$ corresponding to λ^i we set

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}\cdot\begin{pmatrix} dx_i^1 \\ dx_i^2 \end{pmatrix} = \lambda^i\begin{pmatrix} dx_i^1 \\ dx_i^2 \end{pmatrix}$$

or equivalently

$$\begin{cases} g_{11}dx_i^1+g_{12}dx_i^2 = \lambda^i dx_i^1 \\ g_{21}dx_i^1+g_{22}dx_i^2 = \lambda^i dx_i^2 \end{cases}, \quad i=1, 2.$$

Hence $\sum_{\mu,\nu=1}^{2} g_{\mu\nu}dx_i^\mu dx_i^\nu=\lambda^i((dx_i^1)^2+(dx_i^2)^2)$ $(i=1, 2)$ and we see that $d\boldsymbol{x}_i$ points to the direction of maximal or minimal stretching by F $(i=1, 2)$.

It may happen throughout Δ that some of the $\{\lambda^i(\boldsymbol{x})\}_{i=1,2,\cdots,n}$ assume the same value and some others not for a fixed $\boldsymbol{x}$. Let $\boldsymbol{x}_0$ be any point of Δ. Then there exists some neighbourhood $U(\boldsymbol{x}_0)$ satisfying the condition; for each index i the prolongation $\lambda^i(\boldsymbol{x})$ of $\lambda^i(\boldsymbol{x}_0)$ in $U(\boldsymbol{x}_0)$ has a multiplicity not greater than that of $\lambda^i(\boldsymbol{x}_0)$ $(i=1, 2, \cdots, n)$ (Lemma 2 (b), Theorem 4). Rearranging the order of the indices $1, 2, \cdots, n$ for a moment, we can find integers $p_1, p_2, \cdots, p_m$ such that $p_1+p_2+\cdots+p_m=n$ and that

$$\begin{aligned}
&\lambda^1(x_0) = \cdots = \lambda^{p_1}(x_0)\,, \\
&\lambda^{p_1+1}(x_0) = \cdots = \lambda^{p_1+p_2}(x_0)\,, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\lambda^{p_1+\cdots+p_{m-1}+1}(x_0) = \cdots = \lambda^{p_1+\cdots p_m}(x_0)\,.
\end{aligned}$$

Let ν denote any of these integers p_1, p_1+p_2, $\cdots$, $p_1+p_2+\cdots+p_{m-1}$ or $p_1+p_2+\cdots+p_m=n$. In $U(x_0)$ the differential equations (11) corresponding to $\lambda^\nu(x)$ may be written in terms of components in those matricial expressions as

$$(12)\qquad \begin{cases} P_1^{\nu 1}(x)dx^1+\cdots+P_1^{\nu n}(x)dx^n = 0 \\ P_2^{\nu 1}(x)dx^1+\cdots+P_2^{\nu n}(x)dx^n = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ P_n^{\nu 1}(x)dx^1+\cdots+P_n^{\nu n}(x)dx^n = 0\,, \end{cases}$$

where $P_i^{\nu j}(x)$ $(i, j=1, 2, \cdots, n)$ turn out to be of class $C^1[\Delta]$ (Theorem 5). The system (12) of differential equations includes only $n-p_\mu$ linearly independent relations, μ assuming some definite values of $1, 2, \cdots, m$ according as $\nu=p_1$, p_1+p_2, $\cdots$, or $p_1+p_2+\cdots+p_m=n$. Since the tangent vector $dx=\begin{pmatrix} dx^1 \\ dx^2 \\ \vdots \\ dx^n \end{pmatrix}$ satisfying (12) must be orthogonal to the vector fields $(P_i^{\nu 1}(x), \cdots, P_i^{\nu n}(x))$ $(i=1, 2, \cdots, n)$, those former vectors constitute, at every point $x \in U(x_0)$, the tangent eigen spaces of dimension p_μ $(\mu=1, 2, \cdots, m)$ which belongs to the eigen values $\lambda^\nu(x)$ as intersection of $n-p_\mu$ smooth submanifolds of dimension $n-1$ (Theorem 3). The observation so far motivates

DEFINITION 6. Let $(i_1, \cdots, i_l)$ denote any combination chosen out of $(1, 2, \cdots, m)$ $(l \leq m-1)$. The $(p_{i_1}+\cdots+p_{i_l})$-dimensional tangent eigen space just introduced is called *$(p_{i_1}+\cdots+p_{i_l})$-dimensional characteristic submanifold* of F in $U(x_0)$. Specifically, the characteristic submanifolds of dimension 1 and 2 will often be referred to as *characteristic arc* and *characteristic surface* respectively.

We have proved

Theorem 6. *Let an eigen value $\lambda(x)$ of the matrix ${}^tJ(x; F)J(x; F)$ be of multiplicity k at $x_0 \in \Delta$. If we write $\lambda(x_0)=\lambda^1(x_0)=\cdots=\lambda^k(x_0)$, a neighbourhood $U(x_0)$ exists such that*

(a) *there passes through every point $x \in U(x_0)$* a unique *k-dimensional characteristic submanifold $M_{12\cdots k}(x)$ belonging to the eigen value $\lambda(x)$;*

(b) *product spaces of the characteristic submanifolds belonging to different eigen values are characteristic submanifolds too;*

(c) *$M_{12\cdots k}(x)$ is orthogonal to every characteristic submanifolds of F belonging to the other eigen values than $\lambda(x)$.*

Corollary 1. *The i-th characteristic arc through a point* $\boldsymbol{x}$ *is determinate, if and only if the corresponding eigen value* $\lambda^i(\boldsymbol{x})$ *is a simple root of the characteristic equation* (10). *Characteristic arcs are open arcs.*

Corollary 2. *For a point* $\boldsymbol{x}^*$ *to be an end of the i-th characteristic arc it is necessary and sufficient that the i-th eigen value* $\lambda^i(\boldsymbol{x}^*)$ *is a multiple root of* (10) *and that any neighbourhood* $U(\boldsymbol{x}^*)$ *of* $\boldsymbol{x}^*$ *contains some point* $\boldsymbol{x}$ *at which the prolongation* $\lambda^i(\boldsymbol{x})$ *of* $\lambda^i(\boldsymbol{x}^*)$ *is a simple root of* (10).

Corollary 3. *The totality of points of semi-conformality of* $F\in\mathfrak{F}(\Delta)$ *constitutes a closed subset of the characteristic submanifold stated in Theorem* 6.

An i-th characteristic arc of $F\in\mathfrak{F}(\Delta)$ through a given point may be neither indefinitely prolongable nor definite everywhere as we have seen above. But we can do with some smooth arcs which take the place of those characteristics.

Suppose there is an index i such that $\lambda^i(\boldsymbol{x}^*)\neq\lambda^j(\boldsymbol{x}^*)$ holds for all $j=1, 2, \cdots, n$ except for $j=i$ at one given point $\boldsymbol{x}^*$. Then there passes the definite i-th characteristic arc $\gamma_i(\boldsymbol{x}^*)$ through $\boldsymbol{x}^*$. If $\gamma_i(\boldsymbol{x}^*)$ is prolongable unlimitedly in the region where F is defined as characteristics, we have nothing to worry. Let the closure of the open arc $\gamma_i(\boldsymbol{x}^*)$ contains at least one point of semi-conformality $\boldsymbol{x}^{**}$ such that $\lambda^i(\boldsymbol{x}^{**})$ ceases to be a simple root of the characteristic equation (10). The i-th characteristic arc is indeterminate at $\boldsymbol{x}^{**}$ which turns out an end point of $\gamma_i(\boldsymbol{x}^*)$. Corollary 3 to Theorem 6 says that $\boldsymbol{x}^{**}$ lies on a closed subregion Σ of a characteristic submanifold of dimension $m\geq 2$ (m being the multiplicity of $\lambda^i(\boldsymbol{x}^{**})$). Restarting at $\boldsymbol{x}^{**}$, one may proceed along any path $\tilde{\gamma}_i(\boldsymbol{x}^{**})$ within the realm of Σ. Hereupon $\gamma_i(\boldsymbol{x}^*)\cup\tilde{\gamma}_i(\boldsymbol{x}^{**})$ can be made smooth by suitable choice of $\tilde{\gamma}_i(\boldsymbol{x}^{**})$ itself. At the interior point of $\tilde{\gamma}_i(\boldsymbol{x}^{**})$, the i-th characteristic arc is in the least well defined in the proper sense. In the meantime one may encounter a point $\boldsymbol{x}^{***}$ such that $\lambda^i(\boldsymbol{x}^{***})$ is a simple root of (10). Then $\tilde{\gamma}_i(\boldsymbol{x}^{**})$ ends at $\boldsymbol{x}^{***}$. From this point on, the i-th characteristic arc $\gamma_i(\boldsymbol{x}^{***})$ becomes clear again, and so forth.

DEFINITION 7. $\gamma_i(\boldsymbol{x}^*)\cup\tilde{\gamma}_i(\boldsymbol{x}^{**})\cup\gamma_i(\boldsymbol{x}^{***})\cup\cdots$ shall be termed the *i-th characteristic arc of F in the wider sense* ($i=1, 2, \cdots, n$).

Theorem 7. *The i-th characteristic arc of* $F\in\mathfrak{F}(\Delta)$ *in the wider sense is prolongable unlimitedly throughout* Δ ($i=1, 2, \cdots, n$).

This section will be closed with

Theorem 8. *Let a diffeomorphism F of* $\boldsymbol{R}^n$ *onto itself belong to the family* $\mathfrak{F}(\boldsymbol{R}^n)$. *Consider an i-th characteristic arc* $\gamma_i(\boldsymbol{x}_0)$ *of F at one point* $\boldsymbol{x}_0\in\boldsymbol{R}^n$, *regardless of whether in the proper sense or not* ($i=1, 2, \cdots, n$). *Then the orthogonal complement of* $\gamma_i(\boldsymbol{x}_0)$ *through* $\boldsymbol{x}_0$ *is prolongable unlimitedly in* $\boldsymbol{R}^n$ *as* *an* $(n-1)$*-dimensional characteristic submanifold* (*but not uniquely in general*).

Proof. Fix an index i arbitrarily ($i=1, 2, \cdots, n$). The i-th eigen value $\lambda^i(\boldsymbol{x}_0)$ of ${}^tJ(\boldsymbol{x}_0; F)J(\boldsymbol{x}_0; F)$ shall be prolonged up to the whole space $\boldsymbol{R}^n$ so that $\lambda^i(\boldsymbol{x})$ may be a continuously differentiable function in the variable $\boldsymbol{x}\in\boldsymbol{R}^n$ (Theorem 5). Let us call $\boldsymbol{x}_0$ to be the *initial point* henceforth. For the $(n-1)$-dimensional i-th characteristic submanifold to become well defined in an open set $\Delta\subseteq\boldsymbol{R}^n$ it suffices that a 1 system of the i-th characteristic arcs (possibly in the wider sense) is assigned in Δ as normals to it (Theorem 3). Let O denote the open (possibly void) subset of $\boldsymbol{R}^n$ on which $\lambda^i(\boldsymbol{x})$ is simple (Theorem 4). O is decomposed into at most a countably infinite union of connected components G_n ($n=1, 2, \cdots$).

Let $\boldsymbol{x}^*$ be any point of $\boldsymbol{R}^n$. Denoting by $A_{i,\boldsymbol{x}^*,K}$ the affine stretching of $\boldsymbol{R}^n$ onto itself of magnitude K in the direction to the i-th characteristic arc $\gamma_i(\boldsymbol{x}^*)$ at $\boldsymbol{x}^*$, we set $\widetilde{F}_{i,\boldsymbol{x}^*,K}(\boldsymbol{x})=F\circ A_{i,\boldsymbol{x}^*,K}(\boldsymbol{x})$ in a sufficiently small neighbourhood of $\boldsymbol{x}^*$. The j-th semi-axis of the characteristic ellipsoid for $\widetilde{F}_{i,\boldsymbol{x}^*,K}$ shall simply be denoted by $\sqrt{\tilde{\lambda}^j(\boldsymbol{x})}$, which has been obtained by the smooth prolongation from $\sqrt{\tilde{\lambda}^j(\boldsymbol{x}^*)}$ ($j=1, 2, \cdots, n$). The characteristic arc corresponding to the eigen value $\tilde{\lambda}^i(\boldsymbol{x})$, provided it exists at all in the proper sense, is similarly denoted by $\tilde{\gamma}_i(\boldsymbol{x})$.

A. Letting $K\to 1$, we notice the followings:

(a) Let $\boldsymbol{v}(\boldsymbol{x})$ be any oriented segment starting at $\boldsymbol{x}$. Then the ray $\widetilde{F}_{i,\boldsymbol{x}^*,K}(\boldsymbol{v}(\boldsymbol{x}))$ returns to $\boldsymbol{v}(\boldsymbol{x})$ as $K\to 1$. Especially the tangent vector of the arc $\tilde{\gamma}_i(\boldsymbol{x})$ at $\boldsymbol{x}$ returns to that of $\gamma_i(\boldsymbol{x})$ at $\boldsymbol{x}$, if $\lambda^i(\boldsymbol{x})$ is simple.

(b) If, on the other hand, $\lambda^i(\boldsymbol{x})$ is multiple, the cluster set of the directional vectors of $\tilde{\gamma}_i(\boldsymbol{x})$ at $\boldsymbol{x}$ is contained in a characteristic submanifold of dimension ≥ 2; especially the directional vector of $\tilde{\gamma}_i(\boldsymbol{x}^*)$ at $\boldsymbol{x}^*$ tends to that of $\gamma_i(\boldsymbol{x}^*)$ at $\boldsymbol{x}^*$.

B. We shall show: wherever the initial point $\boldsymbol{x}_0$ may be located, there exists *locally* an $(n-1)$-dimensional characteristic submanifold $M_{1\cdots\check{i}\cdots n}(\boldsymbol{x}_0)$ looked for. The symbol $1\cdots\check{i}\cdots n$ means the absense of the index i.

(a) Suppose that $\boldsymbol{x}_0\in O$. We may assume without loss of generality that $\boldsymbol{x}_0\in G_1$. Since the i-th characteristic arc $\gamma_i(\boldsymbol{x})$ is well defined at every point $\boldsymbol{x}$ of G_1, there exists a unique $(n-1)$-dimensional characteristic submanifold $M_{1\cdots\check{i}\cdots n}(\boldsymbol{x}_0)$ in G_1 which complements the 1 system of characteristic arcs $\{\gamma_i(\boldsymbol{x})\}_{\boldsymbol{x}\in G_1}$ orthogonally and passes through $\boldsymbol{x}_0$ (Theorem 3).

(a_1) By the way we had better here to do with the proof of the fact that the submanifold $M_{1\cdots\check{i}\cdots n}(\boldsymbol{x}_0)$ is prolongable beyond the boundary ∂G_1 as the i-th orthogonal complement.

Consider a path C which starts at $\boldsymbol{x}_0$, lies entirely on $M_{1\cdots\check{i}\cdots n}(\boldsymbol{x}_0)$ and is characteristic in the wider sense for F. Since C is indefinitely prolongable

(Theorem 7), it necessarily intersects ∂G_1. Thus we have seen that the closure of $M_{1\cdots\check{i}\cdots n}(\boldsymbol{x}_0)$ intersects ∂G_1. Let $\boldsymbol{x}^*$ be an arbitrary point on the meet $\mathrm{cl}\, M_{1\cdots\check{i}\cdots n}(\boldsymbol{x}_0) \cap \partial G_1$ and let $\gamma_j(\boldsymbol{x}^*)$ $(j \neq i)$ any one of the $n-1$ charactristic arcs through $\boldsymbol{x}^*$ in the wider sense lying on $M_{1\cdots\check{i}\cdots n}(\boldsymbol{x}_0)$ with possible exception of the end point $\boldsymbol{x}^*$. There is a neighbourhood $U_1(\boldsymbol{x}^*)$ such that the tangential oscillation of $\gamma_j(\boldsymbol{x}^*)$ is less than $\pi/8n$ in $U_1(\boldsymbol{x}^*) \cap G_1^c$.

Now let us consider $\widetilde{F}_{i,\boldsymbol{x}^*,K}(\boldsymbol{x})$ in $U_1(\boldsymbol{x}^*)$. There is a small constant $c>0$ such that $\tilde{\gamma}_i(\boldsymbol{x})$ becomes apparent everywhere in $U_1(\boldsymbol{x}^*)$ as far as K satisfies $1<K<1+c$ (Corollary 3 to Theorem 6). Take a point $\boldsymbol{y}$ on the portion of a fixed hyperplane through $\boldsymbol{x}^*$ comprised in $U_1(\boldsymbol{x}^*)$. Since $\tilde{\lambda}^i(\boldsymbol{x})$ is continuously differentiable function in $\boldsymbol{x} \in U_1(\boldsymbol{x}^*)$, $\tilde{\gamma}_i(\boldsymbol{x})$ is expressible by a system of differential equations of type

$$\frac{dx^1}{l_i^1(\boldsymbol{x})} = \frac{dx^2}{l_i^2(\boldsymbol{x})} = \cdots = \frac{dx^n}{l_i^n(\boldsymbol{x})} = dt$$

with smooth coefficients $l_i^j(\boldsymbol{x})$ $(j=1, 2, \cdots, n)$ normalized by the condition $\sum_{j=1}^{n} (l_i^j(\boldsymbol{x}))^2 = 1$, t being a real parameter ranging over the interval $(0, 1)$. The solution arc $\boldsymbol{x} = \Phi_K(t;\, \boldsymbol{y})$ through $\boldsymbol{y}$ turns out to be such that $d\Phi_K(t;\, \boldsymbol{y})/dt$ is continuously differentiable both in t and in $\boldsymbol{y}$ (Petrovski [2], p. 96). The family $\{d\Phi_K(t;\, \boldsymbol{y})/dt \mid 1<K<1+c\}$ is normal on the portion of the product $(0, 1) \times \{(y^1, \cdots, y^n) \mid y^n = x^{n*}\}$ comprised in $(0, 1) \times U_1(\boldsymbol{x}^*)$. For any sequence $\{K_n\}_{n=1,2,\cdots}$ such that $\lim K_n = 1$, $\{d\Phi_{K_n}(t;\, \boldsymbol{y})/dt\}_{n=1,2,\cdots}$ contains a subsequence which converges uniformly on the compact $[\eta, 1-\eta] \times [\mathrm{cl}\, V(\boldsymbol{x}^*) \cap \{(y^1, \cdots, y^n) \mid y^n = x^{n*}\}]$, where $\eta>0$ is any small number and $V(\boldsymbol{x}^*)$ a neighbourhood of $\boldsymbol{x}^*$ satisfying $\mathrm{cl}\, V(\boldsymbol{x}^*) \subset U_1(\boldsymbol{x}^*)$. Since the limit $\phi(t;\, \boldsymbol{y})$ does not vanish identically for $0<t<1$, the indefinite integral $\boldsymbol{x} = \Phi(t;\, \boldsymbol{y}) = \int^t \phi(u;\, \boldsymbol{y})du$ represents a smooth arc passing through $\boldsymbol{y}$. Let $\boldsymbol{y}$ vary on the hyperplane. Then we get a 1 system of orbital arcs defined everywhere in $U_1(\boldsymbol{x}^*)$ which coincides with the original i-th characteristic arcs of F in $U_1(\boldsymbol{x}^*) \cap G_1$ and expresses some i-th characteristic arcs (in the wider sense) of F for $U_1(\boldsymbol{x}^*) \cap G_1^c$ (A. (a), (b)). Hence the $(n-1)$-dimensional characteristic submanifold orthogonal to it can be constructed throughout $U_1(\boldsymbol{x}^*)$ and turns out a prolongation of the like $(n-1)$-dimensional characteristic submanifold in G_1 (Theorem 3), i.e., the orthogonal complement to $\gamma_i(\boldsymbol{x})$ has been continued up to $G_1 \cup U_1(\boldsymbol{x}^*)$.

(b) Let $\boldsymbol{x}_0 \in \partial O$. There exists a small neighbourhood $U_1(\boldsymbol{x}_0)$ in which a 1 system of the smooth i-th characteristic arcs $\{\gamma_i(\boldsymbol{x})\}$ in the wider sense is defined everywhere and coincides with original characteristic arc in the proper sense for $\boldsymbol{x} \in U_1(\boldsymbol{x}_0) \cap O$. The argument goes in quite the same way as in B. (a_1).

(c) Let $\boldsymbol{x}_0 \in (\mathrm{cl}\, O)^c$: $\lambda^i(\boldsymbol{x}_0)$ is a multiple root of the characteristic equation (10). Hence the eigen space belonging to $\lambda^i(\boldsymbol{x}_0)$ is of dimension ≥ 2. Since the i-th characteristic arc to start at $\boldsymbol{x}_0$ is indeterminate, we have the i-th characteristic arc $\gamma_i(\boldsymbol{x}_0)$ only in the wider sense. Consider the composite mapping $\widetilde{F}_{i,x_0,K}(\boldsymbol{x})$; the notations $\tilde{\lambda}^i$ for the i-th characteristic root and $\tilde{\gamma}_i$ for the i-th characteristic arc shall be the ones referred to the mapping $\widetilde{F}_{i,x_0,K}$. Given any small $\varepsilon>0$, there is a K in the interval $(1, 1+\varepsilon)$ which renders $\tilde{\lambda}^i(\boldsymbol{x}_0)$ simple (Lemma 2). Hence, for such a K, there is a neighbourhood $U_1(\boldsymbol{x}_0)$ where $\tilde{\lambda}^i(\boldsymbol{x})$ is simple (Corollary 1 to Theorem 6). Therefore we can find a constant $c>0$ such that $\tilde{\lambda}^i(\boldsymbol{x})$ is simple everywhere in $U_1(\boldsymbol{x}_0)$ for any $K<1+c$ in the same manner as in B. (a_1).

Applying the theorem concerning the normal family to the parametric representations for characteristic arcs belonging to such $\lambda^i(\boldsymbol{x})$, we obtain a 1 system of orbital arcs (the i-th characteristic arcs in the wider sense) in $U_1(\boldsymbol{x}_0)$, whose specific member through $\boldsymbol{x}_0$ coincides with $\gamma_i(\boldsymbol{x}_0)$, the i-th characteristic arc through $\boldsymbol{x}_0$ in the wider sense prescribed (A. (a), (b)). Thus the orthogonal complement to $\gamma_i(\boldsymbol{x}_0)$ has been defined well in $U_1(\boldsymbol{x}_0)$.

C. We shall show inductively the following: the $(n-1)$-dimensional characteristic submanifold $M_{1\cdots\check{i}\cdots n}(\boldsymbol{x})$ whose local existence around the initial point $\boldsymbol{x}_0$ has just been verified in B. admits a successive extension off the original neighbourhood. Starting from $\boldsymbol{x}_0$, we assume to have obtained an $(n-1)$-dimensional characteristic submanifold M together with a finite number of points $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_k$ satisfying the conditions: (i) M coincides with $M_{1\cdots\check{i}\cdots n}(\boldsymbol{x}_0)$ in $U(\boldsymbol{x}_0)$; (ii) $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_k \in M$; (iii) some neighbourhood $U_j(\boldsymbol{x}_{j-1})$ is assigned to every $\boldsymbol{x}_{j-1}$ such that $U_1(\boldsymbol{x}_0) \subseteq U(\boldsymbol{x}_0)$, $\boldsymbol{x}_j \in U_j(\boldsymbol{x}_{j-1})$ and that $U_{j+1}(\boldsymbol{x}_j) \not\subset U_j(\boldsymbol{x}_{j-1})$ $(j=1, \cdots, k, k+1)$; (iv) M is orthogonal to $\gamma_i(\boldsymbol{x})$ at every point $\boldsymbol{x} \in \{\bigcup_{j=1}^{k+1} U_j(\boldsymbol{x}_{j-1})\} \cap M$.

(a) When $\boldsymbol{x}_k$ belongs to O, the $(n-1)$-dimensional characteristic submanifold M is extensible not only beyond $U_{k+1}(\boldsymbol{x}_k)$ but also beyond the boundary of the component in which $\boldsymbol{x}_k$ lies, as we have seen in B. (a_1).

(b) Let $\boldsymbol{x}_k$ lie on ∂O. Obviously, M is extensible towards the interior to O. Hence we have only to concern ourselves with the extensiblilty towards the exterior to O. If we take a point $\boldsymbol{x}_{k+1}$ on $M \cap U_{k+1}(\boldsymbol{x}_k) \cap (\mathrm{cl}\, O)^c$ sufficiently near $\partial U_{k+1}(\boldsymbol{x}_k)$, there can be found a neighbourhood $U_{k+2}(\boldsymbol{x}_{k+1}) \not\subset U_{k+1}(\boldsymbol{x}_k)$ such that a 1 system of smooth i-th characteristic arcs $\{\gamma_i(\boldsymbol{x})\}$ (in the wider sense) is defined well in $U_{k+1}(\boldsymbol{x}_k) \cup U_{k+2}(\boldsymbol{x}_{k+1})$ and that it coincides with the original $\gamma_i(\boldsymbol{x})$ for all $\boldsymbol{x} \in U_{k+1}(\boldsymbol{x}_k) \cap U_{k+2}(\boldsymbol{x}_{k+1})$ (B. (c))

(c) Next let the point $\boldsymbol{x}_k$ be in $(\mathrm{cl}\, O)^c$. If we take a suitable point $\boldsymbol{x}_{k+1} \in M \cap U_{k+1}(\boldsymbol{x}_k)$, there is a neighbourhood $U_{k+2}(\boldsymbol{x}_{k+1}) \not\subset U_{k+1}(\boldsymbol{x}_k)$ such that the 1 system of characteristic arcs in $U_{k+1}(\boldsymbol{x}_k)$ can be extended up to $U_{k+1}(\boldsymbol{x}_k) \cup U_{k+2}(\boldsymbol{x}_{k+1})$ as a 1 system of characteristic arcs in the wider sense (B. (c)).

It will cause no misunderstanding if we denote the submanifold M by $M_{1\cdots\overset{\vee}{i}\cdots n}(x_0)$. Anyhow there exists an $(n-1)$-dimensional characteristic submanifold $M_{1\cdots\overset{\vee}{i}\cdots n}(x_0)$ as well as a chain of neighbourhoods $\{U_\nu(x_{\nu-1})\}_{\nu=1,\cdots}$ such that $x_\nu\in M_{1\cdots\overset{\vee}{i}\cdots n}(x_0)\cap U_\nu(x_{\nu-1})$, $U_{\nu+1}(x_\nu)\not\subset U_\nu(x_{\nu-1})$ and that $M_{1\cdots\overset{\vee}{i}\cdots n}(x_0)$ is orthogonal to $\gamma_i(x)$ everywhere.

D. Finally we prove that $M_{1\cdots\overset{\vee}{i}\cdots n}(x_0)$ is unlimited in R^n. Indeed, if we assume that $\partial M_{1\cdots\overset{\vee}{i}\cdots n}(x_0)$ contains a finite point x^*, it will lead us to a contradiction in the following manner:

(a) It is evedent that $x^*\notin O$.

(b) If $x^*\in(\text{cl } O)^c$, it contradicts C. (c).

(c) Hence the only one possibility left to us must be the case $x^*\in\partial O$.

There is a small disk neighbourhood $V(x^*)$ about x^* such that the tangential oscillation of the j-th characteristic arc $\gamma_j(x)$ in the wider sense $(j\neq i)$ in $V(x^*)$ is less than $\pi/8n$. We consider $\tilde{\gamma}_i(x)$ with respect to $\widetilde{F}_{i,x^*,K}(x^*)$ $(x\in V(x^*))$ and let $K\to 1$. Then the 1 system of smooth i-th characteristic arcs $\{\gamma_i(x)\}$ is well defined throughout $V(x^*)$, which coincides with the original i-th characteristic arcs in the proper sense in $V(x^*)\cap O$ and in the wider sense in $V(x^*)\cap O^c$ (A. (a), (b)). Therefore the original $M_{1\cdots\overset{\vee}{i}\cdots n}(x_0)$ turns out to be prolongable beyond x^* as an orthogonal complement to the i-th characteristic arcs contrary to the assumption.

Thus the proof of the theorem is completed.

4. Period-parallelepiped of cylindrical type

In this section and the following we shall concern ourselves exclusively with special n-tori, i.e., the tori with orthogonal period vectors. Let $b_i=b^ie_i$ (resp. $b_i'=b^{i\prime}e_i$) with $b^i\in R$ (resp. $b^{i\prime}\in R$) $(i=1, 2, \cdots, n)$. Consider the family $\mathcal{F}_0=\mathcal{F}_0(b_1, \cdots, b_n; b_1', \cdots, b_n')$ of mappings $x'=F(x)$ subject to the requirements:

1° F is a twice continuously differentiable, non-singular and orientation-preserving automorphism of R^n;

2° F satisfies the relationship

$$F(x+\sum_{i=1}^n m_ib_i)=x'+\sum_{i=1}^n m_ib_i' \tag{13}$$

for all $x\in R^n$ and all $m_i\in Z$.

We may assume without loss of generality that $b^i>0$, $b^{i\prime}>0$ $(i=1, 2, \cdots, n)$. The family is non-void, since we can construct a mapping $x^{i\prime}=\psi^i(x^i)$ $(i=1, 2, \cdots, n)$ by means of an n-tuple of monotone increasing functions ψ^i of class $C^2\,[0, b^i]$, such that $\psi^i(0)=0$, $\psi^i(b^i)=b^{i\prime}$ $(i=1, 2, \cdots, n)$.

Lemma 5. *Let Γ_i be any family of rectifiable paths in $Z^n(b_1, \cdots, b_n)$ which connect the i-th bases $H_i^\pm$. Then we have*

$$\operatorname{mod} \Gamma_i \leq \frac{b^1 b^2 \cdots b^n}{(b^i)^n}.$$

Proof. Let $\gamma \in \Gamma_i$ be arbitrary. Then the length of γ is at least b^i. Hence $\operatorname{mod} \Gamma^i \leq b^1 b^2 \cdots b^n/(b^i)^n$ (cf. Väisälä [5], p. 20).

Theorem 9. *Let F belong to $\mathcal{F}_0(\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n; \boldsymbol{b}'_1, \cdots, \boldsymbol{b}'_n)$. Then the characteristic arcs of F are necessarily periodic with respect to $\mathfrak{G}(\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n)$; the characteristic arcs in the wider sense can be made periodic with respect to $\mathfrak{G}(\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n)$.*

Proof. Let $\boldsymbol{y}_1^-$ be any point on the i-th lower base H_i^- and let $\boldsymbol{y}_1^+$ the point on the i-th upper base H_i^+ of $\boldsymbol{I}^n(\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n)$ equivalent to each other with respect to $\mathfrak{G}(\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n)$. At least one, say $\gamma_i^- = \gamma_i^-(\boldsymbol{y}_1^-)$ (resp. $\gamma_i^+ = \gamma_i^+(\boldsymbol{y}_1^+)$), of the n characteristic arcs through $\boldsymbol{y}_1^-$ (resp. $\boldsymbol{y}_1^+$) in the wider sense points to the direction which does not lie on H_i^- (resp. H_i^+). Starting from $\boldsymbol{y}_1^-$ (resp. $\boldsymbol{y}_1^+$) we continue this i-th characteristic arc γ_i^- (resp. γ_i^+) in the wider sense towards the interior of $\boldsymbol{I}^n(\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n)$. Let $S^- = M_{1\cdots\check{i}\cdots n}(\boldsymbol{y}_1^-)$ and $S^+ = M_{1\cdots\check{i}\cdots n}(\boldsymbol{y}_1^+)$ be the orthogonal complement to $\gamma_i^-(\boldsymbol{y}_1^-)$ and $\gamma_i^+(\boldsymbol{y}_1^+)$ respectively which are equivalent to each other in regard to $\mathfrak{G}(\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n)$. Although the arc γ_i^- (resp. γ_i^+) is prolongable unlimitedly in $\boldsymbol{R}^n$, we commit ourselves to obey the rule below in extending them so that they may be confined to the single fundamental region $\boldsymbol{Z}^n(\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n)$ bounded by the $n-1$ pairs of hyperplanes $H_\nu^\pm$ ($\nu = 1, 2, \cdots, n$; $\nu \neq i$) and a pair of hypersurfaces $S^\pm$. When γ_i^- (resp, γ_i^+) comes to a boundary point $\boldsymbol{y}_*^-$ (resp. $\boldsymbol{y}_*^+$) of $\boldsymbol{I}^n(\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n)$, we consider the other boundary point $\tilde{\boldsymbol{y}}_*^-$ (resp. $\tilde{\boldsymbol{y}}_*^+$) which is equivalent to $\boldsymbol{y}_*^-$ (resp. $\boldsymbol{y}_*^+$) with respect to $\mathfrak{G}(\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n)$ and restart anew without giong out of the interval $\boldsymbol{I}^n$. Notice

Proposition 1. *The directed path γ_i^- (resp. γ_i^+) neither closes nor intersects itself nor ends anywhere on $\boldsymbol{I}^n(\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n)$.*

We want to show that $\gamma_i = \gamma_i^- \cup \gamma_i^+$ connects the bases H_i^- and H_i^+ on the closed interval $\boldsymbol{I}^n(\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n)$ in the sense above generalized.

We parametrize the curve γ_i with a real parameter t varying over the interval $[0, 1]$ in such a way that the initial point $\boldsymbol{y}_1^- \in S^-$ corresponds to $t=0$ and that any point of γ_i is an image of some $t \in (0, 1]$. There is some $\delta > 0$ such that the i-th coordinate $x^i(t)$ of the point $\boldsymbol{x}(t) \in \gamma_i$ increases monotonically on the subinterval $[0, \delta]$. Let t run through $[0, 1]$. Then, if $\sup x^i(t) = b^i$, there rests nothing to prove. So we assume $\sup_t x^i(t) < b^i$, contrary to the assertion. On the torus generated by identifying the opposite faces of $\boldsymbol{I}^n$, however, the curve γ_i must be closed, because rectifibale. Hence there exists some value $t_0 > \delta$, such that $\boldsymbol{x}(t_0) \in S^-$ owing to Proposition 1. The point $\boldsymbol{y}_2^- = \boldsymbol{x}(t_0)$ lies on S^- but never coincides with $\boldsymbol{y}_1^-$ by definition of the i-th characteristic arc (in the wider sense). Taking arbitrary parameter values t_1, t_2 of the representation

$x=x(t)$ of γ_i such that $0<t_1<t_2<t_0$, we set $x_j=x(t_j)$ $(j=1, 2)$. We may suppose that S^- is equipped with an $(n-1)$-dimensional simplicial decomposition Ξ such that one $(n-1)$-dimensional simplex $\sigma_1^{(n-1)}$ belonging to Ξ has the vertex at y_1^- and another $\sigma_2^{(n-1)}$ the vertex at y_2^-. Let $\sigma_1^{(n)}$ (resp. $\sigma_2^{(n)}$) denote the n-dimensional simplex spanned by $\sigma_1^{(n-1)}$ and x_1 (resp. $\sigma_2^{(n-1)}$ and x_2). The orientation $\overrightarrow{y_1^-, x_1}$ on the 1-skeleton of $\sigma_1^{(n)}$ induces the orientation on $\sigma_1^{(n-1)}$. The orientation $\overrightarrow{x_2, y_2^-}$ on the 1-skeleton of $\sigma_2^{(n)}$ likewise induces the orientation on $\sigma_2^{(n-1)}$. But the orientations thus induced on $\sigma_1^{(n-1)}$ and $\sigma_2^{(n-1)}$ are incoherent. This is a contradiction. q.e.d.

It is in search of means to generalize the well known dilatation or Beltrami coefficient to higher dimensional case that we have hitherto devoted so many pages. It seems to the authors' knowledge that almost sole prevalent notion standing near our aim is *outer and inner dilatation* due to Gehring-Väisälä (cf. Väisälä [5], pp. 41–48), which has not only succeeded in defining a quasiconformality of homeomorphisms of a region in $\boldsymbol{R}^n$ but also sufficed to develop the qualitative theory of quasiconformal mappings rather extensively. On the contrary our purpose at hand is to set up an extremum problem to minimize *dilatations* in a certain sense within the family $\mathcal{F}_0(b_1, \cdots, b_n; b_1', \cdots, b_n')$ which admits a unique solution. We expect also the extremal mapping to reduce to Teichmüller's affine mapping when $n=2$.

Let F be of $\mathcal{F}_0(b_1, \cdots, b_n; b_1', \cdots, b_n')$ and let $x\in\boldsymbol{R}^n$ be any point. These data, of course, determine the characteristic ellipsoid $\mathfrak{E}(dx^1/\sqrt{\lambda^1(x)}, \cdots, dx^n/\sqrt{\lambda^n(x)})$ at x. It means no restriction of generality to assume that $\lambda^1(x)\leq\lambda^2(x)\leq\cdots\leq\cdots\leq\lambda^n(x)$ at least at the fixed point x. According to Väisälä [5], the outer (resp. inner) dilatation at x for such a smooth mapping F was $\{\lambda^n(x)\}^n/\lambda^1(x)\cdots\lambda^n(x)$ (resp. $\lambda^1(x)\cdots\lambda^n(x)/\{\lambda^1(x)\}^n$). Both of them, however, do not reach our demand of uniqueness when $n\geq 3$ because they pay no attention to the intermediate semi-axes $\lambda^2(x), \cdots, \lambda^{n-1}(x)$. More precisely, the numbering of semi-axes appears to play an essential part while the above numbering in the order of magnitudes may break down for the prolongation $\{\lambda^i(x^*)\}_{i=1,\cdots,n}$ at other point x^* if the path to x^* runs through a set of points of semi-conformality. The relation of order at various points is too complicated to handle. Then, there might be another way of ordering. Since we work only with the interval $\boldsymbol{I}^n(b_1, \cdots, b_n)$ with the identified faces and since the characteristic arcs necessarily reach the bounding faces (Theorem 9), we could try to number the characteristic arcs through the point x so that the specific one connecting the i-th faces $H_i^{\pm}$ may be the i-th, $\gamma_i(x)$. The eigen value corresponding to $\gamma_i(x)$ should be the i-th, $\lambda^i(x)$. But unfortunately, there exists some F in $\mathcal{F}_0(b_1, \cdots, b_n; b_1', \cdots, b_n')$ for which more than two characteristic arcs through x connect $H_i^{\pm}$. Thus we seem to be in a maze.

We shall be able to circumvent the difficulty by recalling the Lavrentiev-Ahlfors-Bers's approach in the case $n=2$, that is to say, it was not the *Dilatationsquotient* but the Beltrami coefficient that actually determined the conformal structure. We shall notice that we have had to take account of the inclination of characteristic ellipsoids to say nothing of their shape. It is seen by a simple computation that one needs more parameters than the number of dimensions to determine the characteristic ellipsoid at each point completely. But couldn't we find, for example, favourably n parameters which characterize the conformal modulus of the n-torus ?

5. Dilatation-vector at point

On the basis of the above examinations we propose to adopt the

DEFINITION 8. Let F belong to $\mathcal{F}_0(\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n; \boldsymbol{b}'_1, \cdots, \boldsymbol{b}'_n)$. The *i-th dilatation* of F at $\boldsymbol{x}\in\boldsymbol{R}^n$ is

$$D^i(\boldsymbol{x}; F) = \frac{\left\|\dfrac{\partial F}{\partial x^i}\right\|^n}{\det J(\boldsymbol{x}; F)}, \qquad i=1, 2, \cdots, n.$$

This definition, which fails to determine the characteristic ellipsoid at each point completely ($n\geq 3$), will be satisfactory and adequate for our present purpose as we shall see in all that follows.

EXAMPLE 2. When $n=2$, a pair of quantities $D^1(\boldsymbol{x}; F)$, $D^2(\boldsymbol{x}; F)$ is verified, by the chain rule in the partial differentiation, to determine the characteristic ellipse in full detail. In other words, the information of $(D^1(\boldsymbol{x}; F), D^2(\boldsymbol{x}; F))$ is equivalent to that of Beltrami coefficient of F.

REMARK 2. The i-th dilatations in our sense needs not necessarily be bounded below by 1 ($i=1, 2, \cdots, n$).

Theorem 10. *For any $F\in\mathcal{F}_0(\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n; \boldsymbol{b}'_1, \cdots, \boldsymbol{b}'_n)$ there hold the estimates*

$$\sup_x D^i(\boldsymbol{x}; F) \geq \frac{b^1 b^2 \cdots b^n}{b^{1\prime} b^{2\prime} \cdots b^{n\prime}} \left(\frac{b^{i\prime}}{b^i}\right)^n, \qquad i=1, 2, \cdots, n. \tag{14}$$

For all the n inequalities to hold with the equality signs it is necessary and sufficient that $F(\boldsymbol{x})=R\boldsymbol{x}+\boldsymbol{c}$ with

$$R = \begin{pmatrix} b^{1\prime}/b^1 & 0 & \cdots & 0 \\ 0 & b^{2\prime}/b^2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & b^{n\prime}/b^n \end{pmatrix}$$

and a constant vector

$$c = \begin{pmatrix} c^1 \\ c^2 \\ \vdots \\ c^n \end{pmatrix}.$$

Proof. We may suppose that $\boldsymbol{x}'=F(\boldsymbol{x})$ keeps the origin fixed and maps the interval $\boldsymbol{I}^n(\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n)$ onto a cylindrical region $\boldsymbol{Z}^n(\boldsymbol{b}_1', \cdots, \boldsymbol{b}_n')$. Here we use the notations below:

Γ_0: the family of all straight segements which connect the i-th bases $H_i^\pm$ of $\boldsymbol{I}^n$ and are orthogonal to $H_i^\pm$;

$\boldsymbol{I}^n(\boldsymbol{b}_1', \cdots, \boldsymbol{b}_n')$: the interval equivalent to $\boldsymbol{Z}^n(\boldsymbol{b}_1', \cdots, \boldsymbol{b}_n')$;

$\rho(\boldsymbol{x})$: a non-negative Borel function defined on $\boldsymbol{I}^n$;

$\rho'(\boldsymbol{x}')=\rho\circ F^{-1}(\boldsymbol{x}')$ and $L(\rho')$ is referred to the image path family $F(\Gamma_0)$.

First we suppose $\rho(\boldsymbol{x})$ to be bounded above by a constant $c>0$. A homothetic transformation on the cylindrical region $\boldsymbol{Z}(\boldsymbol{b}_1', \cdots, \boldsymbol{b}_n')$ permits us to assume $A(\rho')\geq A(c)$. We have then via Hölder's inequality that for any $\gamma\in\Gamma_0$

$$\begin{aligned}(15)\qquad [L(\rho')]^n &= \left[\inf_{F(\Gamma_0)}\int_{F(\gamma)}\rho'(\boldsymbol{x}')\|d\boldsymbol{x}\|\right]^n \leq \left[\int_\gamma \left\|\frac{\partial F}{\partial x^i}\right\| c\,dx^i\right]^n \\ &\leq \left[\int_\gamma \left\|\frac{\partial F}{\partial x^i}\right\|^n c\,dx^i\right]\left[\int_\gamma c\,dx^i\right]^{n-1} \\ &\leq \sup_x D^i(\boldsymbol{x};\, F)\,[L(c)]^{n-1}\int_0^{b_i}\det J(\boldsymbol{x};\, F)c\,dx^i .\end{aligned}$$

Integration of both end-sides over $\boldsymbol{I}^n$ yields

$$[L(\rho')]^n \operatorname{mes} \boldsymbol{I}^n \leq \sup_x D^i(\boldsymbol{x};\, F)[L(c)]^n \operatorname{mes} \boldsymbol{Z}^n ,$$

hence

$$(16)\qquad \frac{[L(\rho')]^n}{A(\rho')}\operatorname{mes} \boldsymbol{I}^n \leq \sup_x D^i(\boldsymbol{x};\, F)\frac{[L(c)]^n}{A(c)}\operatorname{mes} \boldsymbol{I}^{n\prime} .$$

Since the right-hand side in (16) is independent of ρ, we get by Lemma 5

$$\begin{aligned}(17)\qquad \frac{(b^{i\prime})^n}{b^{1\prime}b^{2\prime}\cdots b^{n\prime}}\operatorname{mes} \boldsymbol{I}^n &\leq \frac{\operatorname{mes} \boldsymbol{I}^n}{\operatorname{mod} F(\Gamma_0)} \\ &\leq \sup_x D^i(\boldsymbol{x};\, F)\operatorname{mes} \boldsymbol{I}^{n\prime}\frac{(b^i)^n}{b^1b^2\cdots b^n} .\end{aligned}$$

Every member of the inequalities (17) is invariant under homotheties of the $\boldsymbol{x}$- and $\boldsymbol{x}'$-spaces, so we conclude that

$$(18)\qquad K^i[F] = \sup_x D^i(\boldsymbol{x};\, F) \geq \frac{b^1b^2\cdots b^n}{b^{1\prime}b^{2\prime}\cdots b^{n\prime}}\left(\frac{b^{i\prime}}{b^i}\right)^n, \qquad i=1, 2, \cdots, n .$$

Next suppose that the equality sign holds in (18) for some $F_0 \in \mathcal{F}_0(\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n; \boldsymbol{b}_1', \cdots, \boldsymbol{b}_n')$. Then we must have $\|\partial F_0/\partial x^i\| = \text{const.}$, for otherwise the right member in (18) would be greater than the left by some positive quantity independent of ρ', hence we would have

$$\sup \frac{[L(\rho')]^n}{A(\rho')} = \frac{1}{\operatorname{mod} F_0(\Gamma_0)} < \frac{K^i[F_0]}{\operatorname{mod} \Gamma_0}$$

contrary to the assumption that the equality should hold in (18). Therefore the Euclidean length of all arcs of $F_0(\Gamma_0)$ are equal. If we put $\rho'(x') = c' = \text{const.}$ into (15), we have

$$\frac{(b^{i\prime})^n}{b^{1\prime} b^{2\prime} \cdots b^{n\prime}} \leq \frac{[L(c')]^n}{A(c')} \leq \sup \frac{[L(\rho')]^n}{A(\rho')} = \frac{1}{\operatorname{mod} F_0(\Gamma_0)} = K^i[F_0] \frac{(b^i)^n}{b^1 b^2 \cdots b^n}.$$

By the assumption that the equality holds in (18) for F_0, we see $L(1) = b^{i\prime}$. It follows that $x^{i\prime} = (b^{i\prime}/b^i) x^i$. Since the similar situation is caused by F_0 for all indices $i = 1, 2, \cdots, n$, we must have $F_0(\boldsymbol{x}) = R\boldsymbol{x}$. Conversely $F(\boldsymbol{x}) = R\boldsymbol{x}$ renders the equality in (14). q.e.d.

6. Extremal quasiconformal homeomorphism and conformal distance between tori

The group $\mathfrak{G}(\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n)$ (resp. $\mathfrak{G}(\boldsymbol{a}_1', \cdots, \boldsymbol{a}_n')$) of parallel displacement is described by means of n generators $\Theta_i: \boldsymbol{x} \mapsto \boldsymbol{x} + \boldsymbol{a}_i$ (resp. $\Theta_i': \boldsymbol{x} \mapsto \boldsymbol{x} + \boldsymbol{a}_i'$) ($i = 1, 2, \cdots, n$). Now given a couple of tori $\boldsymbol{T}^n(\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n) = \boldsymbol{R}^n/\mathfrak{G}(\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n)$ and $\boldsymbol{T}^{n\prime}(\boldsymbol{a}_1', \cdots, \boldsymbol{a}_n') = \boldsymbol{R}^n/\mathfrak{G}(\boldsymbol{a}_1', \cdots, \boldsymbol{a}_n')$, we consider a homeomorphism $\boldsymbol{x}' = F(\boldsymbol{x})$ which sends the former onto the latter under the following differential-topologic condition (D): F is twice continuously differentiable, non-singular and preserves orientation. Hereupon we shall commit ourselves to impose a further restriction on F

$$\text{(H)} \qquad F \circ \left(\prod_{i=1}^{n} \Theta_i^{m_i}\right)(\boldsymbol{x}) = \left(\prod_{i=1}^{n} \Theta_i'^{m_i}\right) \circ F(\boldsymbol{x}), \qquad (m_i \in \boldsymbol{Z}),$$

which specifies a certain homological nature thereof but affects no essential influence on the generality of our problem.

REMARK 3. F is an analogue to the quasiconformal diffeomorphism between so called *marked Riemann surfaces*. In order to emphasize this restriction we use the symbol $\dot{\boldsymbol{T}}^n$ (resp. $\dot{\boldsymbol{T}}^{n\prime}$) in place of $\boldsymbol{T}^n$ (resp. $\boldsymbol{T}^{n\prime}$).

Let $\dot{\mathcal{F}}(\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n; \boldsymbol{a}_1', \cdots, \boldsymbol{a}_n')$ denote the family of all such F, which turns out non-void. We want to obtain sharp estimates analogous to (14) in the preceding section also for mappings belonging to $\dot{\mathcal{F}}(\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n; \boldsymbol{a}_1', \cdots, \boldsymbol{a}_n')$. But to begin with, we explain our procedure in a simpler case.

Consider the family $\dot{\mathfrak{F}}(\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n; \boldsymbol{a}_1', \cdots, \boldsymbol{a}_n')$. We make of the column vectors $\boldsymbol{a}_i'$ $(i=1, 2, \cdots, n)$ the square matrix $A'=(\boldsymbol{a}_1', \cdots, \boldsymbol{a}_n')$, which induces a pair of orthogonal matrices T_1', T_2' such that $T_2'A'T_1'$ is equal to a diagonal matrix

$$B' = \begin{pmatrix} b^{1'} & 0 \cdots 0 \\ 0 & b^{2'} & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 \cdots & b^{n'} \end{pmatrix}$$

and that $\det B'=\det A$ (Lemma 1). If $b^{1'}=b^{2'}=\cdots=b^{n'}$, $\dot{\boldsymbol{T}}^{n'}$ is conformally equivalent to $\dot{\boldsymbol{T}}^n$. On setting $\boldsymbol{b}_i'=b^{i'}\boldsymbol{e}_i$ $(i=1, 2, \cdots, n)$, we remark that $T_2'\circ A'\circ T_1'(Q)$ is the interval $\boldsymbol{I}^n(\boldsymbol{b}_1', \cdots, \boldsymbol{b}_n')$. Now let F be any member of $\dot{\mathfrak{F}}(\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n; \boldsymbol{a}_1', \cdots, \boldsymbol{a}_n')$ and let $\boldsymbol{x}\in\boldsymbol{R}^n$ any point. If we set

$$\tilde{F}(\boldsymbol{x}) = (T_2'\circ A'\circ T_1')\circ A'^{-1}\circ F(\boldsymbol{x})\,,$$

$\tilde{F}(\boldsymbol{x})$ is everywhere defined in $\boldsymbol{R}^n$, satisfies the condition (D), (H) and consequently belongs to the family $\dot{\mathfrak{F}}_0(\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n; \boldsymbol{b}_1', \cdots, \boldsymbol{b}_n')$. The homomorphism

$$\dot{\mathfrak{F}}(\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n; \boldsymbol{a}_1', \cdots, \boldsymbol{a}_n')\ni F \mapsto \tilde{F}\in\dot{\mathfrak{F}}_0(\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n; \boldsymbol{b}_1', \cdots, \boldsymbol{b}_n')$$

is, by definition, injective and onto, namely an isomorphism. $\tilde{F}$ is affine linear if and only if F is affine linear.

For any $\boldsymbol{x}\in\boldsymbol{R}^n$ we set $\boldsymbol{x}'=F(\boldsymbol{x})$, $\tilde{\boldsymbol{x}}'=\tilde{F}(\boldsymbol{x})$. Then the counter-images by F^{-1} and $\tilde{F}^{-1}$ of the infinitesimal n-balls at $\boldsymbol{x}'$ and $\tilde{\boldsymbol{x}}'$ respectively coincide completely with one another owing to the definition of $\tilde{F}$, that is to say the characteristic ellipsoid of $F(\boldsymbol{x})$ and $\tilde{F}(\boldsymbol{x})$ are quite the same.

As to the general case we have only to superpose the procedure above exposed. In order to deal with the family $\dot{\mathfrak{F}}(\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n; \boldsymbol{a}_1', \cdots, \boldsymbol{a}_n')$ we set $A=(\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n)$ (resp. $A'=(\boldsymbol{a}_1', \cdots, \boldsymbol{a}_n')$), which again gives rise to a couple of orthogonal matrices T_1, T_2 (resp. T_1', T_2') such that $T_2AT_1=(b^i\delta_{ij})_{i,j=1,\cdots,n}$ (resp. $T_2'A'T_1'=(b^{i'}\delta_{ij})_{i,j=1,\cdots,n}$) and that $\det A=b^1b^2\cdots b^n$ (resp. $\det A'=b^{1'}b^{2'}\cdots b^{n'}$). Put $\boldsymbol{y}=T_2\circ A\circ T_1\circ A^{-1}(\boldsymbol{x})$ (resp. $\boldsymbol{y}'=T_2'\circ A'\circ T_1'\circ A'^{-1}(\boldsymbol{x}')$). As $\boldsymbol{x}$ varies throughout in the parallelepiped spanned by the n vectors $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n$, so does $\boldsymbol{y}$ in the interval $\boldsymbol{I}^n(\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n)$ and vice versa, where $\boldsymbol{b}_i=b^i\boldsymbol{e}_i$ $(i=1, 2, \cdots, n)$. Set

$$\boldsymbol{y}' = \tilde{F}(\boldsymbol{y}) = T_2'\circ A'\circ T_1'\circ A'^{-1}\circ F\circ A\circ T_1^{-1}\circ A^{-1}\circ T_2^{-1}(\boldsymbol{y})\,.$$

It defines a homeomorphism of the marked torus $\dot{\boldsymbol{T}}^n(\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n)$ onto the marked torus $\dot{\boldsymbol{T}}^n(\boldsymbol{b}_1', \cdots, \boldsymbol{b})$ satisfying the conditions (D), (H). The collection of all such $\tilde{F}$ constitutes the family $\dot{\mathfrak{F}}_0(\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n; \boldsymbol{b}_1', \cdots, \boldsymbol{b}_n')$. To the linear affinity $\boldsymbol{y}'=R\boldsymbol{y}$ with

$$R=\begin{pmatrix} b^{1\prime}/b^1 & 0 & \cdots & 0 \\ 0 & b^{2\prime}/b^2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & b^{n\prime}/b^n \end{pmatrix}$$

corresponds the same one

$$x' = F_0(x) = A'\circ T_1'^{-1}\circ A'^{-1}\circ T_2'^{-1}\circ R\circ T_2\circ A\circ T_1\circ A^{-1}(x)\,. \tag{19}$$

Referring to these notations and bearing the above facts in mind we pose and state

DEFINITION 9. Set $D^i(x;\ F)=D^i(y;\ \tilde{F})$ $(i=1, 2, \cdots, n)$ for $F\in\dot{\mathfrak{F}}(a_1, \cdots, a_n;\ a_1', \cdots, a_n')$. Given a marked torus $\dot{T}^n(a_1, \cdots, a_n)$, we name the interval $I^n(b_1, \cdots, b_n)$ to be the *canonical figure.*

Theorem 11. *Every class $\dot{\mathfrak{F}}(a_1, \cdots, a_n;\ a_1', \cdots, a_n')$ contains a unique homeomorphism F_0 which minimizes the n-tuple of maximal dilatations* $\{\sup_x D^i(x;F)\}_{i=1,\cdots,n}$. *The extremal mapping is the linear affinity between the marked tori $\dot{T}^n(a_1, \cdots, a_n)$ and $\dot{T}^n(a_1', \cdots, a_n')$. It amounts to the same thing, at least one index i exists $(i=1, 2, \cdots, n)$ such that*

$$\sup_x D^i(x;\ F)>\sup_x D^i(x;\ F_0)$$

if $F\neq F_0$.

In association with the fact that various conformal invariants were already obtained by solving relevant extremum problems, we wish to derive from Theorem 11 two kinds of conformal invariants. First of them is an analogue to the Teichmüller distance:

DEFINITION 10. The non-negative quantity

$$d(\dot{T}^n,\ \dot{T}^{n\prime}) = \sqrt{\sum_{i=1}^{n}[\log \min_F \sup_x D^i(x;\ F)]^2}$$

defined for $F\in\dot{\mathfrak{F}}(a_1, \cdots, a_n;\ a_1', \cdots, a_n')$ is termed *conformal distance* between the marked tori $\dot{T}^n=\dot{T}^n(a_1, \cdots, a_n)$ and $\dot{T}^n=\dot{T}^n(a_1', \cdots, a_n')$.

We have then

Theorem 12. *Two marked tori* $\dot{T}^n$ and $\dot{T}^{n\prime}$ *are conformally equivalent if and only if* $d(\dot{T}^n,\ \dot{T}^{n\prime})=0$.

Proof. If $\dot{T}^{n\prime}=\dot{T}^n(a_1', \cdots, a_n')$ is conformal with $\dot{T}^n=\dot{T}^n(a_1, \cdots, a_n)$, there is an F_0 in $\dot{\mathfrak{F}}(a_1, \cdots, a_n;\ a_1', \cdots, a_n')$ such that $D^i(x;\ F_0)=1$ $(i=1, 2, \cdots, n)$. Hence we have by Theorems 9, 10

$$1 \geq \frac{b^1 b^2 \cdots b^n}{b^{1\prime} b^{2\prime} \cdots b^{n\prime}} \left(\frac{b^{i\prime}}{b^i}\right)^n, \qquad i=1, 2, \cdots, n.$$

Since the inverse map F_0^{-1} must possess the same property, it follows that

$$1 \leq \frac{b^1 b^2 \cdots b^n}{b^{1\prime} b^{2\prime} \cdots b^{n\prime}} \left(\frac{b^{i\prime}}{b^i}\right)^n, \qquad i=1, 2, \cdots, n.$$

Therefore $\min_F \sup_x D^i(\boldsymbol{x}; F)=1 \quad (i=1, 2, \cdots, n)$.

Conversely if $d(\dot{\boldsymbol{T}}^n, \dot{\boldsymbol{T}}^{n\prime})=0$, we see by the same reason that

$$\frac{b^1 b^2 \cdots b^n}{b^{1\prime} b^{2\prime} \cdots b^{n\prime}} \left(\frac{b^{i\prime}}{b^i}\right)^n = 1, \qquad i=1, 2, \cdots, n.$$

Hence the respective canonical figures $\boldsymbol{I}^n(\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n)$ and $\boldsymbol{I}^n(\boldsymbol{b}_1', \cdots, \boldsymbol{b}_n')$ are homothetic to each other, so $\dot{\boldsymbol{T}}^{n\prime}$ is conformal with $\dot{\boldsymbol{T}}^n$ by Theorem 11. q.e.d.

7. Space of conformal moduli with real-analytic structure

Given a couple of real n-tori $\boldsymbol{T}^n(\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n)$ and $\boldsymbol{T}^n(\boldsymbol{a}_1', \cdots, \boldsymbol{a}_n')$, we observe an automorphism F of $\boldsymbol{R}^n$ possessing the properties $1^\circ \sim 3^\circ$:

1° F belongs to $C^2[\boldsymbol{R}^n]$;

2° F is non-singular and orientation preserving;

3° to any $m_i \in \boldsymbol{Z}$ $(i=1, 2, \cdots, n)$ there corresponds some $m_i' \in \boldsymbol{Z}$ $(i=1, 2, \cdots, n)$ such that

$$F(\boldsymbol{x}+\sum_{i=1}^n m_i \boldsymbol{a}_i) = F(\boldsymbol{x})+\sum_{i=1}^n m_i' \boldsymbol{a}_i'.$$

The homomorphism $(m_1, \cdots, m_n) \mapsto (m_1', \cdots, m_n')$ of the multi-indices induces a matrix $P=(\pi_{ij})_{i,j=1,2,\cdots,n}$, whose (i, j)-component π_{ij} satisfies the linear equation $\sum_{i=1}^n m_j \pi_{ij} = m_i'$ $(i=1, 2, \cdots, n)$. Then on setting

$$(\boldsymbol{a}_1'', \cdots, \boldsymbol{a}_n'') = (\boldsymbol{a}_1', \cdots, \boldsymbol{a}_n')P,$$

we see that F fulfills simpler periodicity relation

$$F(\boldsymbol{x}+\sum_{i=1}^n m_i \boldsymbol{a}_i) = F(\boldsymbol{x})+\sum_{i=1}^n m_i \boldsymbol{a}_i'' \qquad \text{for all} \quad m_i \in \boldsymbol{Z}$$

analogous to (13). Because the periods $\boldsymbol{a}_i''$ $(i=1, 2, \cdots, n)$ depend on the matrix P, we say that F satisfying $1^\circ \sim 3^\circ$ belongs to the class $\mathscr{H}(P)$. Obviously there are countably infinite numbers of classes $\mathscr{H}(P)$ but we may fix one of them once and for all without losing the essential generality. Henceforth we write again $\boldsymbol{a}_i'$ in place of $\boldsymbol{a}_i''$ $(i=1, 2, \cdots, n)$ for shortness' sake.

Let $\boldsymbol{I}^n(b^1\boldsymbol{e}_1, \cdots, b^n\boldsymbol{e}_n)$ (resp. $\boldsymbol{I}^n(b^{1\prime}\boldsymbol{e}_1, \cdots, b^{n\prime}\boldsymbol{e}_n)$) denote, as often cited, the canonical figure of the given torus $\boldsymbol{T}^n(\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n)$ (resp. $\boldsymbol{T}^n(\boldsymbol{a}_1', \cdots, \boldsymbol{a}_n')$). How should we measure deviation of $T^n(a_1', \cdots, a_n')$ from $T^n(a_1, \cdots, a_n)$ in reference to the conformality ? One of the answers was the conformal distance presented in the preceding section. Here is another one which appears to affect a farther-reaching influence than the continuity postulated in Introduction. We try to adopt the n-tuple itself of the smallest possible values of $\sup_x D^i(x; F)$ in $\mathcal{H}(P)$ $(i=1, 2, \cdots, n)$ as the measure looked for. According to Theorem 10, these infima read

$$p^i = \frac{b^1 b^2 \cdots b^n}{b^{1\prime} b^{2\prime} \cdots b^{n\prime}} \left(\frac{b'^i}{b^i}\right)^n, \qquad i=1, 2, \cdots, n.$$

In quite the same manner as in Theorem 12 we can prove

Theorem 13. *$\dot{T}^n(a_1', \cdots, a_n')$ is conformally equivalent to $\dot{T}^n(a_1, \cdots, a_n)$ if and only if $(p^1, \cdots, p^n)=(1, \cdots, 1)$.*

Theorem 14. *$T^n(a_1', \cdots, a_n')$ is conformally equivalent to $T^n(a_1, \cdots, a_n)$ if and only if the respective canonical figures $I^n(b^1 e_1, \cdots, b^n e_n)$ and $I^n(b^{1\prime} e_1, \cdots, b^{n\prime} e_n)$ fulfill the condition*

$$\frac{b^{1\prime}}{b^1} = \frac{b'^2}{b^2} = \cdots = \frac{b^{n\prime}}{b^n}$$

in a suitable class $\mathcal{H}(P)$.

The last theorem will motivate the following

DEFINITION 11. The homogeneous coordinate $(b^1, \cdots, b^n)$ in the n-dimensional real projective space $P^n(R)$ is called *conformal modulus* of a marked torus $\dot{T}^n(a_1, \cdots, a_n)$.

The space of our conformal moduli admits the real analytic structure to the effect that

Theorem 15. *When the periods $(a_1, \cdots, a_n)$ of a torus $T^n(a_1, \cdots, a_n)$ vary real-analytically in a real parameter t, the conformal modulus $(b^1, \cdots, b^n) \in P^n(R)$ of $T^n(a_1, \cdots, a_n)$ also varies real-analytically in t.*

Proof. Since $\lambda = |b^k|^2$ $(k=1, 2, \cdots, n)$ are positive and satisfy the algebraic equation

$$\det({}^tAA - \lambda E) = 0$$

with $A=(a_1, \cdots, a_n) = (a_{ij})_{i,j=1,2,\cdots,n}$, b^k is clearly real-analytic function of a_{ij} $(i, j, k=1, 2, \cdots, n)$, which proves the theorem.

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