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Osaka J. Math.
16 (1979), 775-795

ON MULTIPLY TRANSITIVE PERMUTATION GROUPS

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(Received January 9, 1978)

1. Introduction

In this paper we shall give some improvements of the following four results:

RESULT 1 (E. Bannai [5] Theorem 1). Let p be an odd prime. Let G be a permutation group on a set $\Omega=\{1, 2, \cdots, n\}$ which satisfies the following condition: For any p^2 elements $\alpha_1, \cdots, \alpha_{p^2}$ of Ω, a Sylow p-subgroup P of the stabilizer in G of the p^2 points $\alpha_1, \cdots, \alpha_{p2}$ is nontrivial and fixes p^2+r points of Ω, and moreover P is semiregular on the set $\Omega-I(P)$ of the remaining $|\Omega|-p^2-r$ points, where r is independent of the choice of $\alpha_1, \cdots, \alpha_{p^2}$ and $0\leqslant r\leqslant p-1$. Then $n=p^2+p+r$, and one of the following three cases holds: (1) There exists an orbit Ω_1 of G such that $|\Omega-\Omega_1|\leqslant r$ and $G^{\Omega_1}\geqslant A^{\Omega_1}$. Moreover, $(G_{\Omega-\Omega_1})^{\Omega_1}\geqslant A^{\Omega_1}$. (2) $r=p-1$, and G has just two orbits Ω and Ω_2 (with $|\Omega_1|\geqslant|\Omega_2|\geqslant p$) such that $G^{\Omega_1}\geqslant A^{\Omega_1}$. Moreover $(G_{\Omega_2})^{\Omega_1}\geqslant A^{\Omega_1}$ and G^{Ω_2} is primitive and contains an element of a p-cycle (therefore $G^{\Omega_2}\geqslant A^{\Omega_2}$ if $|\Omega_2|\geqslant p+3$). (3) $r=p-1$, and G is imprimitive on Ω with just two blocks Ω_1 and Ω_2. Moreover, $(G_{\Omega_1})^{\Omega_2}\geqslant A^{\Omega_2}$ and $(G_{\Omega_2})^{\Omega_1}\geqslant A^{\Omega_1}$.

RESULT 2 (E. Bannai [4] Theorem 1). Let p be an odd prime. Let G be a $2p$-transitive permutation group such that either (i) each element in G of order p fixes at most $2p+(p-1)$ points, or (ii) a Sylow p-subgroup of $G_{1,2,\cdots,2p}$ is cyclic. Then G is one of S_n $(2p\leqslant n\leqslant 4p-1)$ and A_n $(2p+2\leqslant n\leqslant 4p-1)$.

RESULT 3 (D. Livingstone and A. Wanger [10] Lemma 10). If G is a k-transitive group on a set Ω of n points, with $n>k\geqslant 4$, then there exists a subset Π of $k+1$ points such that $G^{\Pi}_{(\Pi)}\geqslant A^{\Pi}$.

RESULT 4 (H. Wielandt [13] Satz B). If G is a nontrivial t-transitive group on Ω of n points, and if t is sufficiently large, then $\log(n-t)>\dfrac{1}{2}t$.

In § 2 and § 3, we shall prove the following two theorems which improve Result 1 and Result 2.

Theorem A. *Let p be an odd prime. Let G be a permutation group on a* *set* $\Omega=\{1, 2, \cdots, n\}$ *which satisfies the following condition. For any* $2p$ *points* $\alpha_1, \cdots, \alpha_{2p}$ *of* Ω, *a Sylow* p*-subgroup* P *of the stabilizer in* G *of the* $2p$ *points* $\alpha_1, \cdots, \alpha_{2p}$ *is nontrivial and fixes exactly* $2p+r$ *points of* Ω, *and moreover* P *is semiregular on the set* $\Omega-I(P)$ *of the remaining* $n-2p-r$ *points, where* r *is independent of the choice of* $\alpha_1, \cdots, \alpha_{2p}$ *and* $0\leqslant r\leqslant p-2$. *Then* $n=3p+r$, *and there exists an orbit* Γ *of* G *such that* $|\Gamma|\geqslant 3p$ *and* $G^{\Gamma}\geqslant A^{\Gamma}$.

Theorem B. *Let* p *be an odd prime* $\geqslant 11$. *Let* G *be a permutation group on a set* $\Omega=\{1, 2, \cdots, n\}$ *which satisfies the following condition. For any* $2p$ *points* $\alpha_1, \cdots, \alpha_{2p}$ *of* Ω, *a Sylow* p*-subgroup* P *of the stabilizer in* G *of the* $2p$ *points* $\alpha_1, \cdots, \alpha_{2p}$ *is nontrivial and fixes exactly* $3p-1$ *points of* Ω, *and moreover* P *is semiregular on the set* $\Omega-I(P)$ *of the remaining* $n-3p+1$ *points. Then* $n=4p-1$, *and one of the following two cases holds*: (1) *There exists an orbit* Γ *of* G *such that* $|\Gamma|\geqslant 3p$ *and* $G^{\Gamma}\geqslant A^{\Gamma}$. (2) *G has just two orbits* Γ_1 *and* Γ_2 *with* $|\Gamma_1|\geqslant p$, $|\Gamma_2|\geqslant p$ *and* $|\Gamma_1|+|\Gamma_2|=4p-1$, *and* G^{Γ_i} *is* $(|\Gamma_i|-p+1)$*-transitive on* Γ_i $(i=1, 2)$. *Moreover,* $G^{\Gamma_i}\geqslant A^{\Gamma_i}$ *if* $|\Gamma_i|\geqslant p+3$.

REMARK. We note that T. Oyama proved:

RESULT 5 (T. Oyama [12] Theorem 1). Let G be a permutation group on $\Omega=\{1, 2, \cdots, n\}$. Assume that a Sylow 2-subgroup P of the stabilizer of any four points in G satisfies the following condition: P is a nonidentity semiregular group and P fixes exactly r points. Then (I) $r=4$, then $|\Omega|=6$, 8 or 12, and $G=S_6$, A_8 or M_{12} respectively. (II) If $r=5$, then $|\Omega|=7$, 9 or 13. In particular, if $|\Omega|=9$, then $G\leqslant A_9$, and if $|\Omega|=13$, then $G=S_1\times M_{12}$. (III) If $r=7$ and $N_G(P)^{I(P)}\leqslant A_7$, then $G=M_{23}$.

Theorem A and Theorem B might look to be too technical. However they are useful in applications. In § 4, we shall prove the following two consequences of them which improve Result 3 and Result 4 respectively.

Theorem C. *Let* p *be an odd prime. Let* G *be a nontrivial* $2p$*-transitive group on* $\Omega=\{1, 2, \cdots, n\}$. *Then there exists a subset* Γ *of* Ω *such that* $|\Gamma|\geqslant 3p-1$ *and* $G_{(\Gamma)}^{\Gamma}\geqslant A^{\Gamma}$.

Theorem D. *Let* G *be a nontrivial* t*-transitive group on* $\Omega=\{1, 2, \cdots, n\}$. *If* t *is sufficiently large, then* $\log(n-t)>\frac{3}{4}t$.

We give the outline of § 2. Let G be a group satisfying the assumption of Theorem A. Then, G has the only one orbit whose length is not less than p. So, we may assume that G is transitive on Ω. Moreover, we find that if $p\geqslant 5$, then G is $(p+3)$-transitive on Ω, and that if $p=3$, then G is 5-transitive on Ω. Suppose that $G\ngeqslant A^{\Omega}$. Similarly to Bannai [4, § 1], we get a contradiction by using the idea of Miyamoto and Nago which uses the formula of

Frobenius ingeniously (cf. [11, Lemma 1.1]).

Next we give the outline of § 3. Let G be a counter-example to Theorem B with the least degree. So, we may assume that G is transitive on Ω. Moreover, we find that G is $\left(p+\frac{p+1}{2}+2\right)$-transitive on Ω. Again by the similar argument to that of [4, § 1], we get a contradiction.

NOTATION. Our notation will be more or less standard. Let Ω be a set and Δ be a subset of Ω. If G is a permutation group on Ω, then G_Δ denotes the pointwise stabilizer of Δ in G, and $G_{(\Delta)}$ denotes the global stabilizer of Δ in G. When $\Delta=\{\alpha_1, \cdots, \alpha_k\}$, we also denote G_Δ by $G_{\alpha_1,\cdots,\alpha_k}$. The totality of points left fixed by a set X of permutations is denoted by $I(X)$, and if a subset Γ of Ω is fixed as a whole by X, then the restriction of X on Γ is denoted by X^Γ. For a permutation x, let $\alpha_i(x)$ denote the number of i-cycles of x and $\alpha(x)=\alpha_1(x)$. S^Ω and A^Ω denote the symmetric and alternating groups on Ω. If $|\Omega|$, the cardinality of Ω, is n, we denote them S_n and A_n instead of S^Ω and A^Ω.

Acknowledgement. The author would like to thank Professor E. Bannai for suggesting him the present research and giving him many advices.

2. Proof of Theorem A

Let G be a permutation group satisfying the assumption of Theorem A.

Step 1. *G has an orbit Γ such that $|\Gamma|\geqslant 3p$ and $|\Omega-\Gamma|<p$.*

Proof. Since a Sylow p-subgroup of the stabilizer in G of $2p$ points is nontrivial and fixes exactly $2p+r$ points, we have $|\Omega|\geqslant 3p+r$ and that G has an orbit Γ whose length is at least p. Set $|\Gamma|\equiv k \pmod p$ with $0\leqslant k\leqslant p-1$.

Suppose that $|\Gamma|=p+k$. We take $k+1$ points $\alpha_1, \cdots, \alpha_{k+1}$ from Γ and $2p-k-1$ points $\alpha_{k+2}, \cdots, \alpha_{2p}$ from $\Omega-\Gamma$. A Sylow p-subgroup of $G_{\alpha_1,\cdots,\alpha_{2p}}$ fixes at least $3p-1$ points, which contradicts the assumption of Theorem A. Hence we have $|\Gamma|\geqslant 2p+k$.

Suppose that $|\Omega-\Gamma|\geqslant p$. We take $p+k+1$ points $\alpha_1, \cdots, \alpha_{p+k+1}$ from Γ and $p-k-1$ points $\alpha_{p+k+2}, \cdots, \alpha_{2p}$ from $\Omega-\Gamma$. A Sylow p-subgroup of $G_{\alpha_1,\cdots,\alpha_{2p}}$ fixes at least $3p-1$ points, which contradicts the assumption of Theorem A. Hence we have $|\Omega-\Gamma|<p$. So, we have $|\Gamma|\geqslant 3p$. (q.e.d.)

By Step 1, from now on we may assume that G is transitive on Ω.

Step 2. *Let $1\leqslant t\leqslant p+2$. If G is t-transitive on Ω, then G is t-primitive on Ω.*

Proof. Suppose, by way of contradiction, that G is t-transitive on Ω, and that $G_{1,\cdots,t-1}$ is imprimitive on $\Omega-\{1, \cdots, t-1\}$. Let $\Gamma_1, \cdots, \Gamma_s$ be a system of imprimitivity of $G_{1,\cdots,t-1}$. Let $|\Gamma_1|\equiv k \pmod p$, where $0\leqslant k\leqslant p-1$. We divide the consideration into the following two cases: (I) $2p-(t-1)>k$. (II) $2p-(t-1)\leqslant k$.

Suppose that Case (I) holds. First assume that $|\Gamma_1|\geqslant 2p$. We take $k+1$ points $\alpha_t, \cdots, \alpha_{t+k}$ from Γ_1 and $2p-t-k$ points $\alpha_{t+k+1}, \cdots, \alpha_{2p}$ from Γ_2. A Sylow p-subgroup of $G_{1,\cdots,t-1,\alpha_t,\cdots,\alpha_{2p}}$ fixes at least $3p-1$ points, which is a contradiction. Next assume that $p\leqslant|\Gamma_1|<2p$. We take $k+1$ points $\alpha_t, \cdots, \alpha_{t+k}$ from Γ_1. Moreover, we are able to take $2p-t-k$ points $\alpha_{t+k+1}, \cdots, \alpha_{2p}$ from $\Omega-(\Gamma_1\cup\{1, \cdots, t-1\})$. A Sylow p-subgroup of $G_{1,\cdots,t-1,\alpha_t,\cdots,\alpha_{2p}}$ fixes at least $3p-1$ points, which is a contradiction. Hence we may assume that $|\Gamma_1|<p$. Let γ_i be a point of Γ_i $(i=1, \cdots, s)$. Assume $s\leqslant 2p-t+1$. Then a Sylow p-subgroup of $G_{1,\cdots,t-1,\gamma_1,\cdots,\gamma_s}$ is trivial, a contradiction. Hence $s>2p-t+1$. Since a Sylow p-subgroup of $G_{1,\cdots,t-1,\gamma_1,\cdots,\gamma_{2p+t-1}}$ fixes at most $3p-2$ points, we have $(k-1)\leqslant(2p-t+1)\leqslant p-2$. But, since $t\leqslant p+2$ and $k\geqslant 2$, we have a contradiction.

Suppose that Case (II) holds. In this case, we have $t=p+2$ and $k=p-1$. We take a point α from Γ_1 and $p-2$ points $\beta_1, \cdots, \beta_{p-2}$ from Γ_2. A Sylow p-subgroup of $G_{1,\cdots,p+1,\alpha,\beta_1,\cdots,\beta_{p-2}}$ fixes at least $3p-1$ points, which is a contradiction. (q.e.d)

Step 3. *G is $(p+3)$-transitive on Ω when $p\geqslant 5$, and G is 5-transitive on Ω when $p=3$.*

Proof. In order to prove Step 3, we show that if G is t-transitive on Ω then G is $(t+1)$-transitive on Ω, where $1\leqslant t\leqslant p+2$ when $p\geqslant 5$ and $1\leqslant t\leqslant 4$ when $p=3$. Suppose, by way of contradiction, that G is t-transitive on Ω, but G is not $(t+1)$-transitive on Ω. By Step 2, G is t-primitive on Ω. Let $\Delta_1, \cdots, \Delta_s$ be the orbits of $G_{1,\cdots,t}$ on $\Omega-\{1, \cdots, t\}$, where $s\geqslant 2$. By Theorem 18.4 in [14], $|\Delta_i|\geqslant p$ for every Δ_i $(i=1, \cdots, s)$. Let $|\Delta_i|\equiv u_i \pmod p$, where $0\leqslant u_i\leqslant p-1$ $(i=1,\cdots,s)$. By the assumption of t, we have that $p-2\leqslant 2p-t\leqslant 2p-1$ when $p\geqslant 5$, and $2\leqslant 2p-t\leqslant 5$ when $p=3$. We divide the consideration into the following two cases: (I) $2p-t\geqslant p$. (II) $2p-t<p$.

Suppose that Case (I) holds. First assume that $2p-t-u_1-1\leqslant p$. We take u_1+1 points $\alpha_1, \cdots, \alpha_{u_1+1}$ from Δ_1 and $2p-t-u_1-1$ points $\beta_1, \cdots, \beta_{2p-t-u_1-1}$ from Δ_2. A Sylow p-subgroup of $G_{1,\cdots,t,\alpha_1,\cdots,\alpha_{u_1+1}\beta_1,\cdots,\beta_{2p-t-u_1-1}}$ fixes at least $3p-1$ points, which is a contradiction. Next assume that $2p-t-u_1-1>p$ and $|\Delta_1|\geqslant 2p$. we take u_1+p+1 points $\alpha_1,\cdots,\alpha_{u_1+p+1}$ from Δ_1 and $p-t-u_1-1$ points $\beta_1, \cdots, \beta_{p-t-u_1-1}$ from Δ_2. A Sylow p-subgroup of $G_{1,\cdots,t,\alpha_1,\cdots,\alpha_{u_1+p+1},\beta_1,\cdots,\beta_{p-t-u_1-1}}$ fixes at least least $3p-1$ points, which is a contradiction. Hence we may assume that $2p-t-u_1-1>p$ and $|\Delta_1|<2p$. We take u_1+1 points $\alpha_1, \cdots, \alpha_{u_1+1}$ from Δ_1. Moreover we are able to take $2p-t-u_1-1$ points $\beta_1, \cdots, \beta_{2p-t-u_1-1}$ from $\Omega-(\{1, \cdots, t\}\cup\Delta_1)$. A Sylow p-subgroup of $G_{1,\cdots,\alpha_1,\cdots,\alpha_{u_1+1},\beta_1,\cdots,\beta_{2p-t-u_1-1}}$ fixes at least $3p-1$ points, which is a contradiction.

Suppose that Case (II) holds. In this case, we have that $2p-t=p-2$ or $p-1$ when $p\geqslant 5$, and $2p-t=2$ when $p=3$. Assume that there is an orbit Δ_i of $G_{1,\cdots,t}$ with $u_i<2p-t$. We take u_i+1 points $\alpha_1,\cdots,\alpha_{u_i+1}$ from Δ_i and $2p-t-u_i-1$ points $\beta_1,\cdots,\beta_{2p-t-u_i-1}$ from $\Omega-(\{1,\cdots,t\}\cup\Delta_i)$. A Sylow p-subgroup of $G_{1,\cdots,t,\alpha_1,\cdots,\alpha_{u_i+1},\beta_1,\cdots,\beta_{2p-t-u_i-1}}$ fixes at least $3p-1$ points, which is a contradiction. Hence $u_i\geqslant 2p-t$ for every Δ_i $(i=1,\cdots,s)$. Assume that $s\geqslant 3$ or $p=3$. We take a point α_1 from Δ_1 and a point α_2 from Δ_2. If $p=3$, then a Sylow p-subgroup of $G_{1,2,3,4,\alpha_1,\alpha_2}$ fixes at least 8 points, which is a contradiction. If $p\geqslant 5$, we take $2p-t-2$ points $\beta_1,\cdots,\beta_{2p-t-2}$ from Δ_3. Then a Sylow p-subgroup of $G_{1,\cdots t,\alpha_1,\alpha_2,\beta_1,\cdots,\beta_{2p-t-2}}$ fixes at least $3p-1$ points, which is a contradiction. Thus we have $p\geqslant 5$ and $s=2$. So, $\Omega=\{1,\cdots,t\}\cup\Delta_1\cup\Delta_2$. Hence $2p+r=t+\mu_1+\mu_2$. Let Q be a Sylow p-subgroup of $G_{1,\cdots,t}$. Then, $N_G(Q)^{I(Q)}$ is t-transitive and has an element of order p. Since $3p-2\geqslant|I(Q)|=t+u_1+u_2\geqslant t+2(2p-t)=2p+(2p-t)$, we have $|I(Q)|=3p-2$, and $N_G(Q)^{I(Q)}\geqslant A^{I(Q)}$ by [14, Theorem 13.10]. So, $N_G(Q)_{1,\cdots,t}^{I(Q)}$ has an element of order p. Hence Q is not a Sylow p-subgroup of $G_{1,\cdots,t}$, a contradiction. (q.e.d)

Step 4. *$G\geqslant A^{\Omega}$, or $\alpha_p(x)\geqslant 4$ for any element x of order p of G.*

Proof. Let us assume that $\min\{\alpha_p(X)\,|\,x$ is an element of order p of $G\}=m\leqslant 3$. Hence $|\Omega|\geqslant 2p+mp$. Since G is 5-transitive, we have $G\geqslant A^{\Omega}$ by [14, Theorem 13.10]. (q.e.d.)

From now on we assume that $G\ngeqslant A^{\Omega}$, and prove that this case does not occur.

Step 5. *Let a be an element of order p of G with $\alpha(a)=2p+r$. Then there exists an orbit Δ of $C_G(a)^{I(a)}$ such that $C_G(a)^{\Delta}\geqslant A^{\Delta}$ and $|\Delta|\geqslant 2p$.*

Proof. We may assume that

$$a=(1)(2)\cdots(2p+r)(2p+r+1,\cdots,3p+r)\cdots.$$

Set $T=C_G(a)_{2p+r+1,\cdots,3p+r}^{I(a)}$. For any p points $\alpha_1,\cdots,\alpha_p$ of $I(a)$, a normalizes $G_{\alpha_1,\cdots,\alpha_p,2p+r+1,\cdots,3p+r}$. Hence a centralizes an element of order p of $G_{\alpha_1,\cdots,\alpha_p,2p+r+1,\cdots,3p+r}$. So, $T_{\alpha_1,\cdots,\alpha_p}$ has an element of order p for any p elements $\alpha_1,\cdots,\alpha_p$ of $I(a)$. Thus T has an orbit Γ with $|\Gamma|\geqslant p$. Let $|\Gamma|=p+k$. Suppose that $0\leqslant k\leqslant p-1$. We take $k+1$ points $\delta_1,\cdots,\delta_{k+1}$ from Γ and $p-k-1$ points $\delta_{k+2},\cdots,\delta_p$ from $I(a)-\Gamma$. Then $T_{\delta_1,\cdots,\delta_p}$ has no element of order p, which is a contradiction. Therefore T has an orbit Γ whose length is at least $2p$. Since it is easily seen that T^{Γ} is primitive, we have $T^{\Gamma}\geqslant A^{\Gamma}$ by [14, Theorem 13.9]. Let Δ be an orbit of maximal length of $C_G(a)^{I(a)}$, then $C_G(a)^{\Delta}\geqslant A^{\Delta}$ and $|\Delta|\geqslant 2p$. (q.e.d.)

Step 6. *For any $2p$ points $\alpha_1, \cdots, \alpha_{2p}$ of Ω, the order of a Sylow p-subgroup of $G_{\alpha_1,\cdots,\alpha_{2p}}$ is p.*

Proof. Suppose, by way of contradiction, that for some $2p$ points $\alpha_1, \cdots, \alpha_{2p}$, the order of a Sylow p-subgroup P of $G_{\alpha_1,\cdots,\alpha_{2p}}$ is more than p. We may assume that $\{\alpha_1, \cdots, \alpha_{2p}\}=\{1, \cdots, 2p\}$ and $I(P)=\{1, \cdots, 2p, \cdots, 2p+r\}$. For any $2p$ points $\gamma_1, \cdots, \gamma_{2p}$ of $I(P)$, the order of a Sylow p-subgroup of $G_{\gamma_1,\cdots,\gamma_{2p}}$ is $|P|$. Let a be an element of order p of $Z(P)$. We may assume that

$$a=(1)(2)\cdots(2p+r)(2p+r+1, \cdots, 3p+r)\cdots.$$

Since a normalizes $G_{1,\cdots,p,2p+r+1,\cdots,3p+r}$, $G_{1,\cdots,p,2p+r+1,\cdots,3p+r}$ has an element b of order p commuting with a. We may assume that

$$b=(1)\cdots(p)(p+1, \cdots, 2p)(2p+1)\cdots(2p+r)(2p+r+1)\cdots(3p+r)\cdots.$$

Then we may assume that $P^b=P$. Since $C_P(b)$ is semiregular on $I(b)-(\{1, \cdots, p\}\cup\{2p+1, \cdots, 2p+r\})=\{2p+r+1, \cdots, 3p+r\}$, we have $|C_P(b)|=p$, and b does not centralize P. On the other hand, since $\langle P, b\rangle=P\cdot\langle b\rangle$, we have $\langle a\rangle\times\langle b\rangle\supseteq C_{\langle P,b\rangle}(b)\supseteq Z(\langle P, b\rangle)$. Hence $|Z(\langle P, b\rangle)|=|\langle a\rangle|=p$, since $[P, b]\neq 1$.

Now, since $I(a)=I(P)$, we have $C_G(a)\subseteq G_{(I(P))}=N_G(G_{I(P)})$. By the Frattini-Sylow argument, $N_G(G_{I(P)})=N_G(P)\cdot G_{I(P)}$. So, $C_G(a)\subseteq N_G(P)G_{I(P)}$. Hence $C_G(a)^{I(a)}=C_G(a)^{I(P)}\subseteq N_G(P)^{I(P)}$. Thus by Step 5, $N_G(P)^{I(P)}$ has an orbit Δ of maximal length such that $N_G(P)^{\Delta}\geqslant A^{\Delta}$ and $|\Delta|\geqslant 2p$. We may assume that $\Delta=\{1, 2, \cdots, |\Delta|\}$. Set $\Gamma=\{2, 3, \cdots, 2p\}$, then $N_G(P)_{(\Gamma)}^{\Gamma}\geqslant A^{\Gamma}$. Since $|I(P)-\Gamma|\leqslant p-1$, $|N_G(P)_\Gamma|_p$ (=the order of a Sylow p-subgroup of $N_G(P)_\Gamma)=|P|$. Moreover since $|N_G(P)_{(\Gamma)}^{\Gamma}|_p=p$, we have $N_G|(P)_{(\Gamma)}|_p=p\cdot|P|$. Thus $\langle P, b\rangle$ is a Sylow p-subgroup of $N_G(P)_{(\Gamma)}$.

Suppose that $C_G(P)_{(\Gamma)}^{\Gamma}=1$. Since $N_G(P)_{(\Gamma)}/C_G(P)_{(\Gamma)}\leqslant \mathrm{Aut}(P)$, A_{2p-1} is involved in $\mathrm{Aut}(P)$. But, we can easily seen that A_{2p-1} is not involved in $\mathrm{Aut}(P)$ (cf. [2. §2, (3)]), which is a contradiction. Therefore we have $C_G(P)_{(\Gamma)}^{\Gamma}\geqslant A^{\Gamma}$. Since the center of a Sylow p-subgroup of $N_G(P)_{(\Gamma)}$ is of order p, this is a contradiction. (q.e.d.)

Step 7. $|\Omega|-(2p+r)\not\equiv p \pmod{p^2}$.

(The proof of this step is the same as that of [4, §2], but we repeat it for the completeness.)

Proof. We may assume that there exist two elements a and b of order p which commute to each other such that

$$a=(1)\cdots(2p)(2p+1)\cdots(2p+r)(2p+r+1, \cdots, 3p+r)(3p+r+1, \cdots, 4p+r)\cdots,$$

and

$$b=(1, \cdots, p)(p+1, \cdots, 2p)(2p+1)\cdots(2p+r)(2p+r+1)\cdots$$
$$\cdots(3p+r)(3p+r+1)\cdots(4p+r)\cdots.$$

Since $\langle a, b\rangle$ normalizes $G_{p+1,\cdots,2p,2p+r+1,\cdots,3p+r}$, $G_G(\langle a, b\rangle)_{p+1,\cdots,2p,2p+r+1,\cdots,3p+r}$ has an element c of order p. The element c must be of the form

$$c=(1, \cdots, p)^{\alpha}(p+1) \cdots (2p) \cdots (2p+r) \cdots (3p+r)(3p+r+1, \cdots, 4p+r)^{\beta} \cdots,$$

where $1 \leqslant \alpha, \beta \leqslant p-1$. Suppose, by way of contradiction, that $|\Omega|-(2p+r) \equiv p \pmod{p^2}$. $\langle a, c\rangle$ has at least $p+2$ orbits of length p. Hence there is an integer γ $(1 \leqslant \gamma \leqslant p-1)$ such that $|I(ac^{\gamma})| \geqslant 3p$, which is a contradiction. (q.e.d)

From now on, let a be an element of order p of G such that

$$a=(1) \cdots (2p)(2p+1) \cdots (2p+r)(2p+r+1, \cdots, 3p+r)(3p+r+1, \cdots, 4p+r) \cdots .$$

By Step 5, $C_G(a)^{I(a)}$ has an orbit Δ such that $C_G(a)^{\Delta} \geqslant A^{\Delta}$ and $|\Delta| \geqslant 2p$. Hereafter we may assume that $\Delta=\{1, 2, \cdots, |\Delta|\}$.

Step 8. *Set $C_G(a)_0=C_G(a)$. If $p \geqslant 5$, then there is an integer $i (0 \leqslant i \leqslant 2)$ such that $C_G(a)_{0,\cdots,i}$ and $C_G(a)_{0,\cdots,i,i+1}$ have exactly m orbits on $\Omega-I(a)$, where m is at most three, and moreover m is at most two when $|\Omega|-(2p+r) \not\equiv 0 \pmod{p^2}$. If $p=3$, then there is an integer i $(0 \leqslant i \leqslant 1)$ such that $C_G(a)_i$ and $C_G(a)_{i,i+1}$ have exactly m orbits on $\Omega-I(a)$, where m is at most two, and moreover m is one when $|\Omega|-(2p+r) \not\equiv 0 \pmod{p^2}$.*

Proof. Suppose that $p \geqslant 5$. In order to prove Step 8 for $p \geqslant 5$, it is sufficient to show that $C_G(a)_{1,2,3}$ has at most three orbits on $\Omega-I(a)$, and that $C_G(a)_{1,2,3}$ has at most two orbits on $\Omega-I(a)$ when $|\Omega|-(2p+r) \not\equiv 0 \pmod{p^2}$.

Set $H=G_{1,2,3}$. Then H is p-transitive on $\Omega-\{1, 2, 3\}$ by Step 3. By the remark following Lemma 1.1 in [11], we get the following expression:

$$\frac{|H|}{p}=\sum_{x \in H} \alpha_p(x) \geqslant \sum_k \frac{|H|}{|C_H(u_k)|} \frac{1}{p} \sum_y{}' \alpha^*(y),$$

where u_k ranges all representatives of conjugacy classes (in H) of elements of order p, and y ranges all p'-elements in $C_H(u_k)$ and $\alpha^*(y)=\alpha(y^{\Omega-I(u_k)})$. Hence,

$$\frac{|H|}{p} \geqslant \frac{|H|}{|C_H(a)|} \frac{1}{p} \sum_y{}' \alpha^*(y).$$

Assume that $|\Omega|-(2p+r) \not\equiv 0 \pmod{p^2}$. Since a normalizes $G_{1,\cdots,p,2p+r+1,\cdots,3p+r}$, $G_{1,\cdots,p,2p+r+1,\cdots,3p+r}$ has an element b of order p with $ab=ba$. If $|I(X)|=2p+r$ for any nontrivial element x of $\langle a, b\rangle$, then $\langle a, b\rangle$ has just $p-1$ orbits of length p on $\Omega-\{1, \cdots, 3p+r\}$. So $|\Omega|-(2p+r) \equiv 0 \pmod{p^2}$, a contradiction. Hence H $(\supseteq \langle a, b\rangle)$ contains an element of order p which fixes less than $2p+r$ points, and so, the equality in the above expression does not hold. Now, assume that $x \in C_H(a)$ and $p \mid |x|$. Set $|x|=p \cdot s$. Since $|I(x^s)| \leqslant 2p+r$, we have $\alpha^*(x^s) \leqslant p \cdot \alpha_p((x^s)^{I(a)})$. So, $\alpha^*(x) \leqslant p \cdot \alpha_p(x^{I(a)})+2p \cdot \alpha_{2p}(x^{I(a)})$. Hence, we have that

$\sum_y' \alpha^*(y) \geqslant \sum_{y \in C_H(a)} (y) - p \cdot \sum_{y \in C_H(a)} \alpha_p(y^{I(a)}) - 2p \cdot \sum_{y \in C_H(a)} \alpha_{2p}(y^{I(a)})$. Since $C_H(a)^{\Delta - \{1,2,3\}} \geqslant A^{\Delta - \{1,2,3\}}$ and $|\Delta| \geqslant 2p$, we get $p \cdot \sum_{y \in C_H(a)} \alpha_p(y^{I(a)}) = p \cdot \sum_{y \in C_H(a)} \alpha_p(y^{\Delta - \{1,2,3\}}) = |C_H(a)|$ by the formula of Frobenius. Similarly, if $2p \cdot \sum_{y \in C_H(a)} \alpha_{2p}(y^{I(a)}) \neq 0$, then $2p \cdot \sum_{y \in C_H(a)} \alpha_{2p}(y^{I(a)}) = |C_H(a)|$. On the other hand, $\sum_{y \in C_H(a)} \alpha^*(y) = f \cdot |C_H(a)|$, where f is the number of orbits of $C_H(a)$ on $\Omega - I(a)$. Hence we get

$$\frac{|H|}{p} \geqslant \frac{|H|}{p}(f-2), \text{ and hence } f \leqslant 3 .$$

In the above expression, if $|\Omega| - (2p+r) \not\equiv 0 \pmod{p^2}$, the equality does not hold.

Suppose that $p=3$. Then $r=0$ or 1. If $r=0$, then G is 6-transitive on Ω by [10, Lemma 6]. So, we have $G \geqslant A^{\Omega}$ by [4, Theorem 1]. But this contradicts our assumption. Hence $r=1$. Since $\langle a \rangle \in \mathrm{Syl}_3(G_{1,2,3,4,5})$, we have $N_G(\langle a \rangle)^{I(a)} \geqslant A_7$ by Step 3. Hence $C_G(a)^{I(a)} \geqslant A_7$. Set $H = G_{1,2}$. Then H is 3-transitive on $\Omega - \{1, 2\}$, and $C_H(a)^{I(a) - \{1,2\}} \geqslant A_5$. By the similar argument as in the case $p \geqslant 5$, we have that $C_H(a)$ has at most two orbits on $\Omega - I(a)$, and that $C_H(a)$ is transitive on $\Omega - I(a)$ when $|\Omega| - 7 \not\equiv 0 \pmod 9$. Therefore, the consequences of Step 8 hold. (q.e.d.)

Step 9. *$C_G(a)_{1,2,\cdots,|\Delta|}$ has at most $2m$ orbits on $\Omega - I(a)$. Moreover $C_G(a)_{1,\cdots,p,\{p+1,p+2\},p+3,\cdots,|\Delta|}$ $(= C_{G_{(\{p+1,p+2\})}}(a)_{1,\cdots,p,p+3,\cdots,|\Delta|})$ has exactly m orbits on $\Omega - I(a)$.*

Proof. By Step 8, $C_G(a)_{0,\cdots,i}$ has exactly m orbits on $\Omega - I(a)$. Let $\Gamma_1, \cdots, \Gamma_m$ be the orbits. We take an arbitrarily fixed orbit Γ_j. Let $\Sigma_1, \cdots, \Sigma_k$ be the orbits of $C_G(a)_{1,\cdots,|\Delta|}$ on Γ_j. Since $C_G(a)_{0,\cdots,i} \triangleright C_G(a)_{1,\cdots,|\Delta|}$ and Γ_j is an orbit of $C_G(a)_{0,\cdots,i}$, $C_G(a)_{0,\cdots,i}^{\Delta - \{1,\cdots,i\}}$ acts on the set $\{\Sigma_1, \cdots, \Sigma_k\}$ transitively. Let $Y = C_{G_{0,\cdots,i}}(a)_{(\Sigma_1)}$. Then $|C_G(a)_{0,\cdots,i}^{\Delta - \{1,\cdots,i\}} : Y^{\Delta - \{1,\cdots,i\}}| = k$. Similarly, we have $|C_G(a)_{0,\cdots,i,i+1}^{\Delta - \{1,\cdots,i\}} : Y_{i+1}^{\Delta - \{1,\cdots,i\}}| = k$. Hence, $|C_G(a)_{0,\cdots,i}^{\Delta - \{1,\cdots,i\}} : C_G(a)_{0,\cdots,i,i+1}^{\Delta - \{1,\cdots,i\}}| = |Y^{\Delta - \{1,\cdots,i\}} : Y_{i+1}^{\Delta - \{1,\cdots,i\}}| = |\Delta| - i$. Therefore Y is transitive on $\Delta - \{1, \cdots, i\}$. Let $(\beta_1, \cdots, \beta_p)$ be a p-cycle of a such that $\{\beta_1, \cdots, \beta_p\} \subseteq \Sigma_1$. For any $p-i$ elements $\alpha_1, \cdots, \alpha_{p-i}$ of $\Delta - \{1, \cdots, i\}$, $G_{0,\cdots,i,\alpha_1,\cdots,\alpha_{p-i},\beta_1,\cdots,\beta_p}$ has an element b of order p commuting with a. Then $b \in Y$ and b^{Δ} is a p-cycle, and so, $Y_{\alpha_1,\cdots,\alpha_{p-i}}^{\Delta - \{1,\cdots,i\}}$ has the p-cycle. Since $\alpha_1, \cdots, \alpha_{p-i-1}, \alpha_{p-i}$ are any $p-i$ elements of $\Delta - \{1, \cdots, i\}$, we have $Y^{\Delta - \{1,\cdots,i\}} \geqslant A^{\Delta - \{1,\cdots,i\}}$ (cf. [14, Theorem 8.4, Theorem 13.9]). Therefore $k \leqslant 2$. If $k=2$, then $Y^{\Delta - \{1,\cdots,i\}} = A^{\Delta - \{1,\cdots,i\}}$ and $C_G(a)_{0,\cdots,i}^{\Delta - \{1,\cdots,i\}} = S^{\Delta - \{1,\cdots,i\}}$. Therefore Γ_j is an orbit of $C_G(a)_{1,\cdots,p,\{p+1,p+2\},p+3,\cdots,|\Delta|}$ on $\Omega - I(a)$, even if $k=2$. (q.e.d.)

Step 10. $|\Omega| - (2p+r) \equiv 2p$ *(mod p^2) and* $p \geqslant 5$.

Proof. Since a is an element of order p of the form

$$a=(1)\cdots(p)(p+1)\cdots(2p)(2p+1)\cdots(2p+r)(2p+r+1,\cdots,3p+r)$$
$$(3p+r+1,\cdots,4p+r)\cdots,$$

we may assume that $C_G(a)_{p+1,\cdots,2p,2p+r+1,\cdots,3p+r}$ has an element b of order p. By Step 7, we may assume that

$$b=(1,\cdots,p)(p+1)\cdots(2p)(2p+1)\cdots(2p+r)(2p+r+1)\cdots$$
$$(3p+r)(3p+r+1,\cdots,4p+r)\cdots.$$

Let $K=G_{1,\cdots,p\{p+1,p+2\}p+3,\cdots,|\Delta|}$ and $L=\langle b\rangle\cdot K$. Then $|C_L(a)\colon C_K(a)|=p$. By Step 9, $C_K(a)$ and $C_L(a)$ have exactly m orbits on $\Omega-I(a)$. Since $m|C_K(a)|=\sum_{y\in C_K(a)}\alpha^*(y)$ and $m|C_L(a)|=\sum_{y\in C_L(a)}\alpha^*(y)$, we have

$$m\frac{p-1}{p}|C_L(a)|=\sum_{y\in C_L(a)-C_K(a)}\alpha^*(y).$$

Next we show that the elements of order p of $\langle a, b\rangle$ are not conjugate to each other in $C_L(a)$. Suppose a^ib^j and $a^{i'}b^{j'}$ are conjugate to each other, where $0\leqslant i, j, i', j'\leqslant p-1$. If $j\neq j'$, then $(a^ib^j)^{\{1,\cdots,p\}}\neq(a^{i'}b^{j'})^{\{1,\cdots,p\}}$, which is a contradiction. Hence $j=j'$. Assume $i\neq i'$. There exists an element x in $C_L(a)$ such that $(a^ib^j)^x=a^{i'}b^j$. Then $(b^j)^x=a^{i'-i}b^j$. Since $(b^j)^{x^p}=a^{(i'-i)p}b^j=b^j$, we have $p\,\big|\,|x|$. Hence there exists a p-element x_0 in $C_L(a)\cap N_L(\langle a, b\rangle)$ such that $x_0\notin C_L(\langle a, b\rangle)$. Since $\langle a, b\rangle\in \mathrm{Syl}_p(C_L(a))$, this is a contradiction. Thus $i=i'$ and $j=j'$.

Let s be the number of orbits of length p of $\langle a, b\rangle$ on $\Omega-I(a)$. For each fixed j $(1\leqslant j\leqslant p-1)$, there are s elements $i_1,\cdots,i_s$ of $\{0, 1,\cdots,p-1\}$ such that $|I(a^{i_k}b^j)|=|I(a)|$ $(k=1,\cdots,s)$. Let i be an arbitrarily fixed element of $\{i_1,\cdots,i_s\}$, and let $\{\gamma_1,\cdots,\gamma_p\}=I(a^ib^j)\cap(\Omega-I(a))$. Since $\langle a, b\rangle$ is a Sylow P-subgroup of $C_L(\langle a, b\rangle)$, $C_L(\langle a, b\rangle)$ has the normal subgroup Y such that $C_L(\langle a, b\rangle)=\langle a, b\rangle\times Y$, where $(|Y|, p)=1$, and $Y\subseteq C_K(a)$. Since Y acts on $I(\langle a, b\rangle)=\{p+1,\cdots,2p, 2p+1,\cdots,2p+r\}$, Y acts on $\{\gamma_1,\cdots,\gamma_p\}$. Since $a^{\{\gamma_1\cdots\gamma_p\}}$ is a p-cycle and $[Y, a]=1$, we have $Y^{\{\gamma_1,\cdots,\gamma_p\}}=1$. Hence any element of $a^ib^j\cdot Y$ fixes at least p points of $\Omega-I(a)$. Moreover, it is clear that $a^ib^j\cdot Y\cap C_K(a)=\phi$. Therefore

$$\sum_{y\in C_L(\langle a,b\rangle)-C_K(a)}\alpha^*(y)\geqslant s(p-1)p|C_L(\langle a, b\rangle)\colon\langle a, b\rangle|.$$

Let d be any element of $C_L(a)$ such that d is conjugate to b in $C_L(a)$ and $d\neq b$. Then $\langle a, b\rangle\cap\langle a, d\rangle=\langle a\rangle$. Hence $C_L(\langle a, b\rangle)\cap C_L(\langle a, d\rangle)\subseteq C_K(a)$. Therefore, we have

$$\sum_{y\in C_L(a)-C_K(a)}\alpha^*(y)\geqslant s(p-1)p|C_L(a)\colon C_{C_L(a)}(b)|\,|C_L(\langle a, b\rangle)\colon\langle a, b\rangle|$$
$$=\frac{s(p-1)}{p}|C_L(a)|.$$

Hence, $\frac{m(p-1)}{p}|C_L(a)|\geqslant\frac{s(p-1)}{p}|C_L(a)|$. Then $m\geqslant s$. On the other hand, if $|\Omega|-(2p+r)\equiv hp \pmod{p^2}$, where $2\leqslant h\leqslant p$, then we have $s=h$. Therefore, we have that $|\Omega|-(2p+r)\equiv 2p \pmod{p^2}$ and $p\geqslant 5$, by Step 8. (q.e.d.)

Step 11. *We complete the proof.*

Proof. By Step 10, $\{2p+r+1,\cdots,3p+r\}$ and $\{3p+r+1,\cdots,4p+r\}$ are the orbits of length p of $\langle a, b\rangle$ on $\Omega-I(a)$, and $m=2$ and $p\geqslant 5$. By Step 4 we have $\alpha_p(a)\geqslant 4$, hence $|\Omega-I(a)|\geqslant p^2+2p$. Let $\Gamma_1,\cdots,\Gamma_l$ be the orbits of $C_G(a)_{1,2,\cdots,|\Delta|}$ on $\Omega-I(a)$, where $2\leqslant l\leqslant 4$ by Step 9. Since $|b|=p$, b acts on the set $\{\Gamma_1,\cdots,\Gamma_l\}$ trivially. If $l=2$, then Γ_1 and Γ_2 are the orbits of $C_G(a)_{1,\cdots,p\{p+1,p+2\}p+3,\cdots,|\Delta|}$ on $\Omega-I(a)$ by Step 9, and one of the following three cases holds: (i) $|\Gamma_1|\equiv 2p \pmod{p^2})$, $|\Gamma_2|\equiv 0 \pmod{p^2})$. (ii) $|\Gamma_1|\equiv 0 \pmod{p^2}$, $|\Gamma_2|\equiv 2p \pmod{p^2}$. (iii) $|\Gamma_1|\equiv|\Gamma_2|\equiv p \pmod{p^2}$. If $l=3$, then we may assume that $\Gamma_1\cup\Gamma_2$ and Γ_3 are the orbits of $C_G(a)_{1,\cdots,p,\{p+1,p+2\}p+3,\cdots,|\Delta|}$ on $\Omega-I(a)$, and one of the following two cases holds: (i) $|\Gamma_1|=|\Gamma_2|\equiv 0 \pmod{p^2}$, $|\Gamma_3|\equiv 2p \pmod{p^2}$. (ii) $|\Gamma_1|=|\Gamma_2|\equiv p \pmod{p^2}$, $|\Gamma_3|\equiv 0 \pmod{p^2}$. If $l=4$, then we may assume that $\Gamma_1\cup\Gamma_2$ and $\Gamma_3\cup\Gamma_4$ are the orbits of $C_G(a)_{1,\cdots,p\{p+1,p+2\}p+3,\cdots,|\Delta|}$ on $\Omega-I(a)$, and one of the following two cases holds: (i) $|\Gamma_1|=|\Gamma_2|\equiv 0 \pmod{p^2}$, $|\Gamma_3|=|\Gamma_4|\equiv p \pmod{p^2}$. (ii) $|\Gamma_1|=|\Gamma_2|\equiv p \pmod{p^2}$, $|\Gamma_3|=|\Gamma_4|\equiv 0 \pmod{p^2}$. We have the following for any value of l: There is a Γ_j $(1\leqslant j\leqslant 4)$ such that $|\Gamma_j|\equiv 0$ or $p \pmod{p^2}$ and $|\Gamma_j|\geq p^2$. Let $(\beta_1,\cdots,\beta_p)$ and $(\gamma_1,\cdots,\gamma_p)$ be two p-cycles of a such that $\{\beta_1,\cdots,\beta_p,\gamma_1,\cdots,\gamma_p\}\subseteq\Gamma_j$. $C_G(a)_{\beta_1,\cdots,\beta_p,\gamma_1,\cdots,\gamma_p}$ has an element c of order p. Hereafter we examine the relation between a and c. We may assume that

$$c=(1,\cdots,p)(p+1,\cdots,2p)(2p+1)\cdots(2p+r)(\beta_1)\cdots(\beta_p)(\gamma_1)\cdots(\gamma_p)\cdots.$$

Since $|\Gamma_j|\not\equiv 2p \pmod{p^2}$, $\langle a, c\rangle$ has at least $p+2$ orbits of length p on $\Omega-I(a)$. Let $K=G_{1,2,\cdots,|\Delta|}$, and $L=\langle c\rangle\cdot K$. By the same argument as in the proof of Step 10, we have that $l\cdot\frac{p-1}{p}|C_L(a)|=\sum_{y\in C_L(a)-C_K(a)}\alpha^*(y)$, and that the elements of $\langle a, c\rangle-\{1\}$ are not conjugate to each other in $C_L(a)$. For each fixed j $(1\leqslant j\leqslant p-1)$, there are at least $\frac{p+3}{2}$ elements $i_1,\cdots,i_{(p+3)/2}$ of $\{0, 1,\cdots,p-1\}$ such that $|I(a^{i_k}c^j)|\geqslant p+r$ $\left(k=1,\cdots,\frac{p+3}{2}\right)$. Let i be an arbitrarily fixed element of $\{i_1,\cdots,i_{(p+3)/2}\}$. Since $\langle a, c\rangle$ is a Sylow p-subgroup of $C_L(\langle a, c\rangle)$ there exists the normal subgroup M of $C_L(\langle a, c\rangle)$ such that $C_L(\langle a, c\rangle)=\langle a, c\rangle\times M$. First assume that a^ic^j fixes exactly p points $\delta_1,\cdots,\delta_p$ in $\Omega-I(a)$. Then, by the same argument as in the proof of Step 10, any element of $a^ic^j\cdot M$ fixes $\{\delta_1,\cdots,\delta_p\}$ pointwise. Next assume that a^ic^j fixes exactly $2p$ points $\eta_1,\cdots,\eta_{2p}$ in $\Omega-I(a)$

and a fixes $\{\beta_1, \cdots, \beta_p\}$ and $\{\gamma_1, \cdots, \gamma_p\}$ with $\{\beta_1, \cdots, \beta_p\} \cup \{\gamma_1, \cdots, \gamma_p\} = \{\eta_1, \cdots, \eta_{2p}\}$. If M fixes $\{\beta_1, \cdots, \beta_p\}$ and $\{\gamma_1, \cdots, \gamma_p\}$, then any element of $a^i c^j \cdot M$ fixes $\{\eta_1, \cdots, \eta_{2p}\}$ pointwise. And if M transposes $\{\beta_1, \cdots, \beta_p\}$ and $\{\gamma_1, \cdots, \gamma_p\}$ then there exists the subgroup M_0 of index two of M such that any element of $a^i c^j \cdot M_0$ fixes $\{\eta_1, \cdots, \eta_{2p}\}$ pointwise. Therefore, by the same argument as in the proof of Step 10, we have that

$$\sum_{y \in C_L(a) - C_K(a)} \alpha^*(y) \geqslant \frac{p+3}{2} \cdot (p-1) \cdot p \, |C_L(a) \colon C_{C_L(a)}(c)| \, |C_L(\langle a, c\rangle) \colon \langle a, c\rangle|$$
$$= \frac{(p+3)(p-1)}{2p} \cdot |C_L(a)| \, .$$

Hence $l \geqslant \frac{p+3}{2}$. So, we have $p=5$ and $l=4$.

We may assume that $|\Gamma_1| = |\Gamma_2| \equiv 0 \pmod{5^2}$. Let $(\delta_1, \cdots, \delta_5)$ and $(\eta_1, \cdots, \eta_5)$ be two 5-cycles of a such that $\{\delta_1, \cdots, \delta_5\} \subseteq \Gamma_1$ and $\{\eta_1, \cdots, \eta_5\} \subseteq \Gamma_2$. $C_G(a)_{\delta_1, \cdots, \delta_5, \eta_1, \cdots, \eta_5}$ has an element d of order 5. Since d acts on the set $\{\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4\}$ trivially, $\langle a, d\rangle$ has at least $2 \cdot 5 + 2$ orbits of length 5 on $\Omega - I(a)$. Hence, there exists an element x of order 5 of $\langle a, d\rangle$ such that $|I(x)| \geqslant 3 \cdot 5 + r$, which is a contradiction. (q.e.d.)

3. Proof of Theorem B

In the proof of Theorem B, we shall use the following Lemma.

Lemma. *There is no group satisfying the following condition: Let G be a 3-transitive group on Ω. Let α and β be two points of Ω. $G_{\alpha,\beta}$ is an imprimitive group on $\Omega - \{\alpha, \beta\}$ with two blocks Δ_1, Δ_2 of length $\frac{|\Omega|}{2} - 1$, and moreover, for any point γ of Δ_1 and any point δ of Δ_2, $G_{\alpha,\beta,\gamma,\delta}^{\Delta_1 - \{\gamma\}}$ and $G_{\alpha,\beta,\gamma,\delta}^{\Delta_2 - \{\delta\}}$ are 2-transitive groups.*

(I think that this lemma is esentially known already in [7, § 1, Proof of Theorem 1])

Proof of Lemma (cf. [7, § 1, Proof of Theorem 1]). Let G be a group satisfying the above condition.

Set $|\Omega| = n$ and $|\Delta_i| = v+1$ $(i=1, 2)$. Then $G_{\alpha\beta\gamma}$ has just two orbits Σ_1 and Σ_2 on $\Omega - \{\alpha, \beta, \gamma\}$ such that $|\Sigma_1| = v+1$ and $|\Sigma_2| = v$.

For any subset Δ of Ω with $|\Delta| = 4$, G_Δ has two orbits Π_1 and Π_2 on $\Omega - \Delta$ such that $|\Pi_1| = |\Pi_2|$ or $\big||\Pi_1| - |\Pi_2|\big| = 2$. In either case, G_Δ is a subgroup of $G_{\alpha_1\alpha_2\alpha_3}$ which satisfies the assumption of the Witt's Lemma [14, Theorem 9.4], where $\alpha_1, \alpha_2, \alpha_3$ are three elements of Δ. Hence $G_{(\Delta)}^{\Delta}$ is a 3-transitive group. Thus, $G_{(\Delta)}^{\Delta} = S_4$. Therefore, G acts on $\Omega^{(2)}$, the set of unordered pairs of elements of Ω, as a transitive permutation group of rank 4, where the orbitals, $\Gamma_0, \Gamma_1, \Gamma_2$ and Γ_3 of this permutation group are defined as follows: for $\{\alpha, \beta\} \in$

$\Omega^{(2)}$, $\Gamma_0(\{\alpha, \beta\})=\{\alpha, \beta\}$

$$\Gamma_1(\{\alpha, \beta\}) = \{\{\gamma, \delta\}\in\Omega^{(2)} \mid |\{\alpha, \beta\}\cap\{\gamma, \delta\}| = 1\}$$

$$\Gamma_2(\{\alpha, \beta\}) = \{\{\gamma, \delta\}\in\Omega^{(2)} \mid \{\alpha, \beta\}\cap\{\gamma, \delta\} = \phi\,. \ \delta \text{ is in the orbit of length } v \text{ of } G_{\alpha\beta\gamma} \text{ on } \Omega-\{\alpha, \beta, \gamma\}\}$$

$$\Gamma_3(\{\alpha, \beta\}) = \{\{\gamma, \delta\}\in\Omega^{(2)} \mid \{\alpha, \beta\}\cap\{\gamma, \delta\} = \phi\,. \ \delta \text{ is in the orbit of length } v+1 \text{ of } G_{\alpha\beta\gamma} \text{ on } \Omega-\{\alpha, \beta, \gamma\}\}\,.$$

The degrees corresponding to Γ_i ($i=0, 1, 2, 3$) are respectively

$$1,\ 2(n-2) = 4(v+1),\quad \frac{(n-2)v}{2} = v(v+1),\quad \frac{(n-2)(v+1)}{2} = (v+1)^2\,.$$

Moreover, these orbitals Γ_i ($i=0, 1, 2, 3$) are all self-paired.

Let us define the intersection matrices M_i ($i=0, 1, 2, 3$) for the permutation group G on $\Omega^{(2)}$ as follows:

$$M_i = (\mu_{jk}^{(i)}) \text{ with } 0\leqslant j\leqslant 3,\ 0\leqslant k\leqslant 3, \text{ where}$$
$$\mu_{jk}^{(i)} = |\Gamma_j(x)\cap\Gamma_i(y)| \text{ with } y\in\Gamma_k(x)$$
$$(\text{where } x, y\in\Omega^{(2)})\,.$$

Now we can obtain the intersection matrix M_2 (cf. [9, §4]). This is,

$$M_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & v & 2v-2 & 2v \\ v(v+1) & \dfrac{v(v-1)}{2} & -v+2 & v(v-1) \\ 0 & \dfrac{v(v+1)}{2} & v^2-1 & 0 \end{pmatrix}$$

By direct calculations, we obtain the eigenvalues θ_0, θ_1, θ_2 and θ_3 of M_2.

$$\theta_0 = v(v+1)\,,\quad \theta_1 = -v\,,\quad \theta_2 = \frac{-v^2+2+\sqrt{v^4+4v+4}}{2} \text{ and}$$
$$\theta_3 = \frac{-v^2+2-\sqrt{v^4+4v+4}}{2}\,.$$

Since $(v^2)^2<v^4+4v+4<(v^2+2)^2$, it is clear that θ_2 and θ_3 are irrational numbers.

Let us denote by $\pi^{(2)}$ the permutation character of G on $\Omega^{(2)}$. Then $\pi^{(2)}$ is multiplicity free and $\pi^{(2)}=1+X_1+X_2+X_3$, where $X_1=X^{(n-1,1)}|G$ and X_2 and X_3 are irreducible characters appearing in $X^{(n-2,2)}|G$ corresponding to θ_2 and θ_3 respectively. Since θ_2 and θ_3 are irrational, X_2 and X_3 are not rational characters (cf. [6, Lemma 1]), so X_2 and X_3 are algebraic conjugate and especially of the same degree. Therefore $X_2(1)=X_3(1)=n(n-3)/4$ and $X_1(1)=n-1$. By a theorem of Frame [14, Theorem 30.1 (A)], we obtain that the number

$$q=\left\{\frac{n(n-1)}{2}\right\}^2\frac{2(n-2)\cdot v(n-2)/2\cdot(n-2)(v+1)/2}{(n-1)\cdot n(n-3)/4\cdot n(n-3)/4}$$

must be an integer. But, since $n=2v+4$, we have a contradiction. (q.e.d.)

Proof of Theorem B. Let G be a counter-example to the theorem with the least possible degree.

Step 1. *The number of orbits of G on Ω is at most two.*

Proof. By Theorem A and the assumption for G, G has no orbit on Ω whose length is less than p.

Suppose, by way of contradiction, that G has three orbits Δ_1, Δ_2 and Δ_3 with $|\Delta_i|\geqslant p$ $(i=1, 2, 3)$. Set $|\Delta_i|\equiv k_i \pmod p$, where $0\leqslant k_i\leqslant p-1$ $(i=1, 2, 3)$. Assume that $2p-(k_1+k_2+2)\geqslant p$. We take k_1+p-1 points $\alpha_1, \cdots, \alpha_{k_1+p-1}$ from Δ_1, k_2+1 points $\beta_1, \cdots, \beta_{k_2+1}$ from Δ_2 and $p-k_1-k_2$ points $\gamma_1, \cdots, \gamma_{p-k_1-k_2}$ from Δ_3. A Sylow p-subgroup of $G_{\alpha_1,\cdots,\alpha_{k_1+p-1},\beta_1,\cdots,\beta_{k_2+1},\gamma_1,\cdots,\gamma_{p-k_1-k_2}}$ fixes at least $3p$ points, which contradicts the assumption of Theorem B. Hence $2p-(k_1+k_2+2)<p$. We take k_1+1 points $\alpha_1, \cdots, \alpha_{k_1+1}$ from Δ_1, k_2+1 points $\beta_1, \cdots, \beta_{k_2+1}$ from Δ_2 and $2p-k_1-k_2-2$ points $\gamma_1, \cdots, \gamma_{2p-k_1-k_2-2}$ from Δ_3. A Sylow p-subgroup of $G_{\alpha_1,\cdots,\alpha_{k_1+1},\beta_1,\cdots,\beta_{k_2+1},\gamma_1,\cdots,\gamma_{2p-k_1-k_2-2}}$ fixes at least $3p$ points, which is a contradiction. (q.e.d.)

Step 2. *We may assume that G is transitive on Ω.* ($|\Omega|\equiv p-1$ (*mod* p).)

Proof. Suppose that G is not transitive on Ω. By Step 1, G has two orbits Δ_1 and Δ_2 such that $\Delta_1\cup\Delta_2=\Omega$ and $|\Delta_i|\geqslant p$ $(i=1, 2)$. Set $|\Delta_i|=s_ip+k_i$, where $0\leqslant k_i\leqslant p-1$ $(i=1, 2)$. In this case $k_1+k_2=p-1$. By the assumption of Theorem B, $s_1\geqslant 2$ or $s_2\geqslant 2$. We may assume that $s_1\geqslant 2$ and $s_1\geqslant s_2$. We divide the consideration into the following three cases: (I) $s_1\geqslant 3$. (II) $s_1=s_2=2$. (III) $s_1=2$, $s_2=1$.

Suppose that Case (I) holds. By Theorem A and the assumption for G, $G^{\Delta_1}\geqslant A^{\Delta_1}$, and so, $s_1=3$. For k_2+1 points $\alpha_1, \cdots, \alpha_{k_2+1}$ of Δ_2, $G^{\Delta_1}_{\alpha_1,\cdots,\alpha_{k_2+1}}$ is $(p+k_1)$-transitive by [10, Lemma 6]. Since $G^{\Delta_1}_{\alpha_1,\cdots,\alpha_{k_2+1}}$ has an element x of order p with $\alpha_p(x)=2$, we have $G^{\Delta_1}_{\alpha_1,\cdots,\alpha_{k_2+1}}\geqslant A^{\Delta_1}$ by [14, Theorem 13.10]. This is a contradiction.

Suppose that Case (II) holds. We may assume that $k_1\geqslant k_2$. For $p+k_2+1$ points $\alpha_1, \cdots, \alpha_{p+k_2+1}$ of Δ_2, $G^{\Delta_1}_{\alpha_1,\cdots,\alpha_{p+k_2+1}}$ has an element of order p, and moreover $G^{\Delta_1}_{\alpha_1,\cdots,\alpha_{p+k_2+1}}$ is k_1-transitive by [10, Lemma 6]. Since $k_1\geqslant 5$, $G^{\Delta_1}_{\alpha_1,\cdots,\alpha_{p+k_2+1}}\geqslant A^{\Delta_1}$ by [14, Theorem 13.10]. This is a contradiction.

Suppose that Case (III) holds. By [10, Lemma 6] and [14, Theorem 13.10], G is a group satisfying the consequence (2) of Theorem B. This is a contradiction. (q.e.d.)

Step 3. *G is primitive on Ω. For any element x of order p of G, $\alpha_p(x) \geqslant 8$ holds.*

Proof. Suppose, by way of contradiction, that G is imprimitive on Ω. Let $\Delta_1, \cdots, \Delta_s$ be a system of imprimitivity of G. Set $|\Delta_i| \equiv k \pmod p$, where $0 \leqslant k \leqslant p-1$. First assume that $|\Delta_i| \leqslant p$. Then $s > 2p$ and we are able to take $2p$ points $\delta_1, \cdots, \delta_{2p}$ from Ω such that $\delta_i \in \Delta_i$ $(i=1, \cdots, 2p)$. A Sylow p-subgroup of $G_{\delta_1,\cdots,\delta_{2p}}$ fixes at least $4p$ points, which is a contradiction. Next assume that either $p < |\Delta_i| < 2p$, or $|\Delta_i| \geqslant 2p$ and $s \geqslant 3$. We take $k+1$ points $\alpha_1, \cdots, \alpha_{k+1}$ from Δ_1 and $k+1$ points $\beta_1, \cdots, \beta_{k+1}$ from Δ_2. We are able to take $2p-2k-2$ points $\gamma_1, \cdots, \gamma_{2p-2k-2}$ from $\Omega - (\Delta_1 \cup \Delta_2)$. A Sylow p-subgroup of $G_{\alpha_1,\cdots,\alpha_{k+1},\beta_1,\cdots,\beta_{k+1},\gamma_1,\cdots,\gamma_{2p-2k-2}}$ fixes at least $3p$ points, which is a contradiction. Therefore, we have that $|\Delta_i| \geqslant 2p$ and $s=2$. Then $\Omega = \Delta_1 \cup \Delta_2$ and $k = \frac{p-1}{2}$. By Theorem A, $|\Delta_i| = 3p + \frac{p-1}{2}$ or $2p + \frac{p-1}{2}$. By the similar argument to that of Case (II) of Step 2, we have a contradiction. Thus G is primitive on Ω. By [14, Theorem 13.10], for any element x of order p of G, we have $\alpha_p(x) \geqslant 8$. (q.e.d.)

Step 4. *Let $2 \leqslant t \leqslant p + \frac{p-1}{2} + 2$. If G is t-transitive on Ω, then G is t-primitive on Ω.*

Proof. Suppose, by way of contradiction, that G is t-transitive on Ω and $G_{1,\cdots,t-1}$ is imprimitive on $\Omega - \{1, \cdots, t-1\}$. Let $\Delta_1, \cdots, \Delta_s$ be a system of imprimitivity of $G_{1,\cdots,t-1}$ on $\Omega - \{1, \cdots, t-1\}$. Set $|\Delta_i| \equiv k \pmod p$ and $|\Delta_i| = lp+k$, where $0 \leqslant k \leqslant p-1$. In this case, $(t-1)+sk \equiv p-1 \pmod p$. We divide the consideration into the following two cases: (I) $2p-t+1 \geqslant p$. (II) $2p-t+1 < p$.

Suppose that Case (I) holds. First assume that $l=0$. Then $s > 2p-t+1$ and we are able to take $2p-t+1$ points $\delta_1, \cdots, \delta_{2p-t+1}$ of Ω such that $\delta_i \in \Delta_i$ $(i=1, \cdots, 2p-t+1)$. A Sylow p-subgroup of $G_{1,\cdots,t-1,\delta_1,\cdots,\delta_{2p-t+1}}$ fixes at least $3p$ points, which is a contradiction. Secondly assume that $l=1$. By Step 3, we get $s \geqslant 8$. Assume that $k \geqslant \frac{p-1}{2}$. We take a point α from Δ_1, a point β from Δ_2, a point γ from Δ_3 and $2p-t-2$ points $\delta_1, \cdots, \delta_{2p-t-2}$ from $\Delta_4 \cup \Delta_5$. A Sylow p-subgroup of $G_{1,\cdots,t-1,\alpha,\beta,\gamma,\delta_1,\cdots,\delta_{2p-t-2}}$ fixes at least $3p$ points, which is a contradiction. Hence we have $k \leqslant \frac{p-3}{2}$ when $l=1$. We take $k+1$ points $\alpha_1, \cdots, \alpha_{k+1}$

from Δ_1, $k+1$ points $\beta_1,\cdots,\beta_{k+1}$ from Δ_2 and $2p-t-2k-1$ points $\gamma_1,\cdots,\gamma_{2p-t-2k-1}$ from $\Delta_3\cup\Delta_4$. A Sylow p-subgroup of $G_{1,\cdots,t-1,\alpha_1,\cdots,\alpha_{k+1},\beta_1,\cdots,\beta_{k+1},\gamma_1,\cdots,\gamma_{2p-t-2k-1}}$ fixes at least $3p$ points, which is a contradiction. Thirdly assume that $l\geqslant 2$ and $2p-t-k\neq k,\ k+p$. We take $k+1$ points $\alpha_1,\cdots,\alpha_{k+1}$ from Δ_1 and $2p-t-k$ points $\beta_1,\cdots,\beta_{2p-t-k}$ from Δ_2. A Sylow p-subgroup of $G_{1,\cdots,t-1,\alpha_1,\cdots,\alpha_{k+1},\beta_1,\cdots,\beta_{2p-t-k}}$ fixes at least $3p$ points, which is a contradiction. Fourthly assume that $l\geqslant 2$ and $2p-t-k=k+p$. Assume that $s\geqslant 3$. We take $k+1$ points $\alpha_1,\cdots,\alpha_{k+1}$ from Δ_1, $k+1$ points $\beta_1,\cdots,\beta_{k+1}$ from Δ_2 and $p-1$ points $\gamma_1,\cdots,\gamma_{p-1}$ from Δ_3. A Sylow p-subgroup of $G_{1,\cdots,t-1,\alpha_1,\cdots,\alpha_{k+1},\beta_1,\cdots,\beta_{k+1},\gamma_1,\cdots,\gamma_{p-1}}$ fixes at least $3p$ points, which is a contradiction. Hence we have $\Omega=\{1,\cdots,t-1\}\cup\Delta_1\cup\Delta_2$ when $l\geqslant 2$ and $2p-t-k=k+p$. Since $k=\dfrac{p-t}{2}$ and $t\geqslant 2$, we get $t\geqslant 3$. Let γ be any point of Δ_1, and δ be any point of Δ_2. By [10, Lemma 6], it is easily seen that $G_{1,\cdots,t-1,\gamma,\delta}^{\Delta_1-\{\gamma\}}$ and $G_{1,\cdots,t-1,\gamma,\delta}^{\Delta_2-\{\delta\}}$ are $(k-1+p)$-transitive. By Lemma, we have a contradiction. Fifthly assume that $l\geqslant 2$ and $2p-t-k=k$. In this case, $k=\dfrac{2p-t}{2}\geqslant\dfrac{p-1}{2}$. Assume that $s\geqslant 3$. We take $k+1$ points $\alpha_1,\cdots,\alpha_{k+1}$ from Δ_1, $k-1$ points $\beta_1,\cdots,\beta_{k-1}$ from Δ_2 and a point γ from Δ_3. A Sylow p-subgroup of $G_{1,\cdots,t-1,\alpha_1,\cdots,\alpha_{k+1},\beta_1,\cdots,\beta_{k-1},\gamma}$ fixes at least $3p$ points, which is a contradiction. Hence, we have $\Omega=\{1,\cdots,t-1\}\cup\Delta_1\cup\Delta_2$ when $l\geqslant 2$ and $2p-t-k=k$. Let Q be a Sylow p-subgroup of $G_{1,\cdots,t}$. Then $N_G(Q)^{I(Q)}$ is a t-transitive group and $|I(Q)|\geqslant t-1+2k=2p-1$. Let x be an element of order p of Q with $|I(x)|=3p-1$, and $(\gamma_1,\cdots,\gamma_p)$ be a p-cycle of x. Let $\{\delta_1,\cdots,\delta_p\}$ be a subset of Ω such that if $|I(Q)|=2p-1$, then $\{\delta_1,\cdots,\delta_p\}=I(x)-I(Q)$, and if $|I(Q)|=3p-1$, then $x^{\{\delta_1,\cdots,\delta_p\}}$ is a p-cycle of x different from $(\gamma_1,\cdots,\gamma_p)$. $C_G(x)_{\gamma_1,\cdots,\gamma_p,\delta_1,\cdots,\delta_p}$ has an element y of order p. Since y fixes $I(Q)$, we may assume that $y\in N_G(Q)$. Then $y^{I(Q)}$ is an element of order p of $N_G(Q)^{I(Q)}$ which is 2-transitive on $I(Q)$ and we have $N_G(Q)^{I(Q)}\geqslant A^{I(Q)}$. Since $G_{1,\cdots,t-1}$ is imprimitive on $\Omega-\{1,\cdots,t-1\}$, this is a contradiction.

Suppose that Case (II) holds. In this case, $p+2\leqslant t\leqslant p+\dfrac{p-1}{2}+2$. Let Q be a Sylow p-subgroup of $G_{1,\cdots,t}$. Then $N_G(Q)^{I(Q)}$ is t-transitive on $I(Q)$. Since $|\Omega|\equiv p-1 \pmod p$, we have $|I(Q)|\equiv p-1 \pmod p$, and so, $|I(Q)|=2p-1$ or $3p-1$. Since $t\geqslant p+2$, $N_G(Q)^{I(Q)}$ has an element of order p, and so, we get $N_G(Q)^{I(Q)}\geqslant A^{I(Q)}$. We may assume that $\{\Delta_1,\cdots,\Delta_u\}$ is the subset of $\{\Delta_1,\cdots,\Delta_s\}$ such that $I(Q)\cap\Delta_i\neq\phi$ for $1\leqslant i\leqslant u$ and $I(Q)\cap\Delta_i=\phi$ for $u<i\leqslant s$. Since $G_{1,\cdots,t-1}$ is imprimitive on $\Omega-\{1,\cdots,t-1\}$, we have that $k\leqslant 1$ or $u=1$. Assume that $k\geqslant 2$. Then $u=1$, and so, $(t-1)+k\equiv p-1 \pmod p$. Hence $t-1+k=2p-1$. Then $p-\dfrac{p-1}{2}-2\leqslant k\leqslant p-2$. On the other hand, $(t-1)+sk\equiv p-1 \pmod p$. Then $(t+k)+(s-1)k\equiv 0 \pmod p$, and so, $p\,|\,s-1$. Hence

$s \geqslant p+1$. Let α_i be a point of Δ_i $(i=1, \cdots, s)$. A Sylow p-subgroup of $G_{1,\cdots,t-1,\alpha_1,\cdots,\alpha_{k+1}}$ fixes at least $2p+(k+1)(k-1)$ points. But, $(k+1)(k-1) \geqslant \left(p-\frac{p-1}{2}-1\right)\left(p-\frac{p-1}{2}-3\right) \geqslant p$, which is a contradiction. Therefore $k=0$ or 1. We take two points α_1, α_2 from Δ_1 and $2p-t-1$ points $\beta_1, \cdots, \beta_{2p-t-1}$ from Δ_2. A Sylow p-subgroup of $G_{1,\cdots,t-1,\alpha_1,\alpha_2,\beta_1,\cdots,\alpha_{2p-t-1}}$ fixes at least $3p$ points, which is a contradiction. (q.e.d.)

Step 5. *G is* $\left(p+\frac{p+1}{2}+2\right)$*-transitive on* Ω.

Proof. By Step 3 and Step 4, in order to prove Step 5 we show that if G is t-primitive on Ω then G is $(t+1)$-transitive on Ω, where $1 \leqslant t \leqslant p+\frac{p-1}{2}+2$. Suppose, by way of contradiction, that G is t-primitive on Ω, but G is not $(t+1)$-transitive on Ω. Let $\Delta_1, \cdots, \Delta_s$ be the orbits of $G_{1,\cdots,t}$ on $\Omega-\{1, \cdots, t\}$, where $s \geqslant 2$. We may assume that $|\Delta_1| \geqslant |\Delta_2| \geqslant \cdots \geqslant |\Delta_s| \geqslant p$ (cf. [14, Theorem 18.4]). Set $|\Delta_i| \equiv k_i \pmod p$ $(i=1, \cdots, s)$, then $t+k_1+\cdots+k_s \equiv p-1 \pmod p$. We divide the consideration into the following two cases: (I) $2p-t \geqslant p+1$. (II) $2p-t \leqslant p$.

Suppose that Case (I) holds. First assume that $|\Delta_1|=p$ or $p+1$. We take two points α_1, α_2 from Δ_1 and two points β_1, β_2 from Δ_2. We are able to take $2p-t-4$ points $\gamma_1, \cdots, \gamma_{2p-t-4}$ from $\Delta_3 \cup \cdots \cup \Delta_s$. A Sylow p-subgroup of $G_{1,\cdots,t-1,\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\cdots,\gamma_{2p-t-4}}$ fixes at least $3p$ points, which is a contradiction. Therefore $|\Delta_1| \geqslant p+2$. Secondly assume that $2p-t-k_1 \geqslant p$ and $|\Delta_1| \geqslant 2p+k_1$. We take $p-t-k_1$ points $\beta_1, \cdots, \beta_{p-t-k_1}$ from $\Delta_2 \cup \cdots \cup \Delta_s$. By [10, Lemma 6], $G_{1,\cdots,t,\beta_1,\cdots,\beta_{p-t-k_1}}^{\Delta_1}$ is $(p+k_1)$-transitive, which contradicts Theorem 17.7 in [14]. If $k_1=0$ or 1 then our assumptions are satisfied. Therefore $k_1 \geqslant 2$. Thirdly assume that either $2p-t-k_1 \geqslant p$ and $|\Delta_1|=p+k_1$, or $2p-t-k_1<p$. We are able to take $2p-t-k_1$ points $\beta_1, \cdots, \beta_{2p-t-k_1}$ from $\Delta_2 \cup \cdots \cup \Delta_s$. By [10, Lemma 6], $G_{1,\cdots,t,\beta_1,\cdots,\beta_{2p-t-k_1}}^{\Delta_1}$ is k_1-transitive, which contradicts Theorem 17.7 in [14].

Suppose that Case (II) holds. In this case, $p \leqslant t \leqslant p+\frac{p-1}{2}+2$. Let Q be a Sylow p-subgroup of $G_{1,\cdots,t}$, then $N_G(Q)^{I(Q)}$ is t-transitive, and $|I(Q)|=2p-1$ or $3p-1$. Since $t \geqslant p$, we have $N_G(Q)^{I(Q)} \geqslant A^{I(Q)}$. Hence, there is a unique orbit Δ_j such that $k_j \neq 0$. Since $t+k_j \equiv p-1 \pmod p$, we have that $k_j=2p-1-t \geqslant 3$. By [10, Lemma 6], $G_{1,\cdots,t}^{\Delta_j}$ is k_j-transitive, and so, we have $j \neq 1$ by [14, Theorem 17.7]. Assume that $s \geqslant 3$. We take a point α from Δ_1, $2p-t-2$ points $\beta_1, \cdots, \beta_{2p-t-2}$ from Δ_j and a point γ from Δ_i where $1<i \leqslant s$ and $i \neq j$. A Sylow p-subgroup of $G_{1,\cdots,t-1,\alpha,\beta_1,\cdots,\beta_{2p-t-2},\gamma}$ fixes at least $3p$ points, which is a contradiction. Therefore $s=j=2$. If $p \geqslant 13$, then $k_j=2p-1-t \geqslant 4$. This is a contradiction by [1]. Hence, we have $p=11$. Moreover, we have

$k_j=2p-1-t=3$ by [1]. By [8, Theorem 5], we have that either (i) $|\Delta_1|+|\Delta_2|+1=\frac{1}{2}(|\Delta_2|^2+|\Delta_2|+2)$, or (ii) $|\Delta_1|+|\Delta_2|+1=(\lambda+1)^2(\lambda+4)^2$, $|\Delta_2|=(\lambda+1)(\lambda^2+5\lambda+5)$, for some positive interger λ. Case (i) does not hold, since $3+1\not\equiv\frac{1}{2}(3^2+3+2)$ (mod 11). Moreover Case (ii) does not hold, since for every λ ($\lambda=0, 1, \cdots, 10$), we have $3+1\not\equiv(\lambda+1)^2(\lambda+4)^2$ (mod 11) or $3\not\equiv(\lambda+1)\cdot(\lambda^2+5\lambda+5)$ (mod 11). (q.e.d.)

Step 6. *Let a be an element of order p of the form*

$$a=(1)\cdots(p)\cdots(2p)\cdots(3p-1)(3p,\cdots,4p-1)\cdots.$$

Then one of the following holds for $C=C_G(a)^{I(a)}_{3p,\cdots,4p-1}$.

(i) *C has an orbit Δ such that $C^\Delta\geqslant A^\Delta$ and $|\Delta|\geqslant 2p$.*

(ii) *There exist two orbits Δ_1 and Δ_2 of C such that $|\Delta_i|\geqslant p$ and C^{Δ_i} is $(|\Delta_i|-p+1)$-transitive ($i=1, 2$), and $\Delta_1\cup\Delta_2=I(a)$. Moreover, if $|\Delta_i|\geqslant p+3$, then $C^{\Delta_i}\geqslant A^{\Delta_i}$.*

(iii) *C is an imprimitive group with two blocks Γ_1 and Γ_2 of length $p+\frac{p-1}{2}$ such that $C^{\Gamma_i}\geqslant A^{\Gamma_i}$ ($i=1, 2$).*

Proof. For any p points $\alpha_1, \cdots, \alpha_p$ of $I(a)$, $C_{\alpha_1,\cdots,\alpha_p}$ has an element of order p. Since C has an element of order p, it has an orbit whose length is at least p. Assume that C has two orbits Δ_1 and Δ_2 with $|\Delta_i|\geqslant p$ ($i=1, 2$). Set $|\Delta_i|=p+k_i$ ($i=1, 2$). If $\Delta_1\cup\Delta_2\neq I(a)$, then $k_1+k_2+2\leqslant p$. We take k_1+1 points $\alpha_1, \cdots, \alpha_{k_1+1}$ from Δ_1 and k_2+1 points $\beta_1, \cdots, \beta_{k_2+1}$ from Δ_2, so $C_{\alpha_1,\cdots,\alpha_{k_1+1},\beta_1,\cdots,\beta_{k_2+1}}$ has no element of order p, a contradiction. Hence $\Delta_1\cup\Delta_2=I(a)$. By [10, Lemma 6], we have that C is a group satisfying (ii). Assume that C has a unique orbit Δ with $|\Delta|\geqslant p$. Then we have $|\Delta|\geqslant 2p$. If C^Δ is pritmitive, by [14, Theorem 13.9] we have that C^Δ is a group satisfying (i) Assume that C^Δ is imprimitive. Let $\Gamma_1, \cdots, \Gamma_s$ be a system of imprimitivity of C^Δ. If $|\Gamma_1|<p$, then $|\Gamma_1|=2$. We take p points $\alpha_1, \cdots, \alpha_p$ with $\alpha_i\in\Gamma_i$ ($i=1, \cdots, p$), so $C_{\alpha_1,\cdots,\alpha_p}$ has no element of order p, a contradiction. Hence $|\Gamma_1|\geqslant p$, and so we have $s=2$ and $|\Gamma_1|=|\Gamma_2|=p+\frac{p-1}{2}$. By [10, Lemma 6], we have that C is a group satisfying (iii). (q.e.d.)

Step 7. *For any $2p$ points $\alpha_1, \cdots, \alpha_{2p}$ of Ω, the order of a Sylow p-subgroup of $G_{\alpha_1,\cdots,\alpha_{2p}}$ is p.*

Proof. Suppose, by way of contradiction, that for some $2p$ points $\alpha_1, \cdots, \alpha_{2p}$, the order of a Sylow p-subgroup P of $G_{\alpha_1,\cdots,\alpha_{2p}}$ is more than p. We may assume that $\{\alpha_1, \cdots, \alpha_{2p}\}=\{1, \cdots, 2p\}$ and $I(P)=\{1, \cdots, 2p, \cdots, 3p-1\}$. Let a be an element of order p of $Z(P)$. We may assume that

$$a=(1)\cdots(3p-1)(3p,\cdots,4p-1)\cdots.$$

Since $C_{G_1}(a)^{I(a)-\{1\}}$ is a permutation group of degree $3p-2$, one of the following two cases holds:

(I) $C_{G_1}(a)^{I(a)-\{1\}}$ has an orbit Δ such that $C_{G_1}(a)^{\Delta}\geqslant A^{\Delta}$ and $|\Delta|\geqslant 2p-1$.

(II) $C_{G_1}(a)^{I(a)-\{1\}}$ has two orbits Δ_1, Δ_2 such that $|\Delta_i|\geqslant p$ and $C_{G_1}(a)^{\Delta_i}$ is $(|\Delta_i|-p+1)$-transitive $(i=1,2)$, and $\Delta_1\cup\Delta_2=I(a)-\{1\}$. Moreover, if $|\Delta_i|\geqslant p+3$, then $C_{G_1}(a)^{\Delta_i}\geqslant A^{\Delta_i}$.

Suppose that Case (I) holds. We may assume that $\Delta=\{2,3,\cdots,|\Delta|,|\Delta|+1\}$. Let $\Gamma=\{2,3,\cdots,2p\}$, then $\Gamma\subseteq\Delta$. Since $C_{G_1}(a)^{\Delta}\geqslant A^{\Delta}$, we have $G_{1(\Gamma)}^{\Gamma}\geqslant A^{\Gamma}$. On the other hand, by the Frattini-Sylow argument, $G_{1(\Gamma)}=N_{G_{1(\Gamma)}}(G_{1\Gamma})=N_{G_{1(\Gamma)}}(P)\cdot G_{1\Gamma}$. Hence, $N_{G_1}(P)_{(\Gamma)}^{\Gamma}=G_{1(\Gamma)}^{\Gamma}\geqslant A^{\Gamma}$, so we have $|N_{G_1}(P)_{(\Gamma)}|_p$ (=the order of a Sylow p-subgroup of $N_{G_1}(P)_{(\Gamma)})=|P|\cdot p$. $C_G(a)_{1,2p+1,\cdots,3p-1,3p,\cdots,4p-1}$ has an element b of order p. Since $|\Gamma|<2p$, b^{Γ} is a p-cycle. Since b normalizes $G_{1,\cdots,3p-1}$, we may assume that $P^b=P$. Then $\langle b,P\rangle\in\mathrm{Syl}_p(N_{G_1}(P)_{(\Gamma)})$. Since $C_P(b)$ is semiregular on $(\Omega-I(P))\cap I(b)=\{3p,\cdots,4p-1\}$, we have $|C_P(b)|=p$. Hence, since $[P,b]\neq 1$ we have $|Z(\langle P,b\rangle)|=p$. Assume that $C_{G_1}(P)_{(\Gamma)}^{\Gamma}=1$. Since $N_{G_1}(P)_{(\Gamma)}/C_{G_1}(P)_{(\Gamma)}\leqslant\mathrm{Aut}(P)$, A_{2p-1} is involved in $\mathrm{Aut}(P)$. But, we can easily seen that A_{2p-1} is not involved in $\mathrm{Aut}(P)$ (cf. [2, §2. (3)]), which is a contradiction. Hence $C_{G_1}(P)_{(\Gamma)}^{\Gamma}\geqslant A^{\Gamma}$. Since the center of a Sylow p-subgroup of $N_{G_1}(P)_{(\Gamma)}$ is of order p, this is a contradiction.

Suppose that Case (II) holds. Then, one of the following two cases holds:

(i) $N_{G_1}(P)^{I(P)-\{1\}}\geqslant A^{I(P)-\{1\}}$.

(ii) Δ_1 and Δ_2 are the orbits of $N_{G_1}(P)^{I(P)-\{1\}}$. $N_{G_1}(P)^{\Delta_i}$ is $(|\Delta_i|-p+1)$-transitive $(i=1,2)$, and if $|\Delta_i|\geqslant p+3$, then $N_{G_1}(P)^{\Delta_i}\geqslant A^{\Delta_i}$.

If Case (i) holds, then we have a contradiction by the similar argument to that of Case (I). Hence we assume that Case (ii) holds. We may assume that $|\Delta_1|>|\Delta_2|$ and $\Delta_1=\{2,3,\cdots,|\Delta_1|,|\Delta_1|+1\}$. Let $\Gamma=\{2,3,\cdots,2p\}$. Since $|\Gamma\cap\Delta_2|\leqslant\frac{p-1}{2}$, we have $(C_{G_1}(a)_{\Gamma\cap\Delta_2})^{\Delta_1}\geqslant A^{\Delta_1}$ by [10, Lemma 6]. Then $N_{G_1}(P)_{(\Gamma)}^{\Delta_1}\geqslant A^{\Delta_1}$, and so, $|N_{G_1}(P)_{(\Gamma)}|_p=|P|\cdot p$. $C_G(a)_{1,2p+1,\cdots,3p-1,3p,\cdots,4p-1}$ has an element b of order p. Then b^{Δ_1} is a p-cycle, and we may assume that $P^b=P$. So $\langle b,P\rangle\in\mathrm{Syl}_p(N_{G_1}(P)_{(\Gamma)})$. By the same argument as in Case (I), we have $|Z(\langle b,P\rangle)|=p$. Assume that $C_{G_1}(P)_{(\Gamma)}^{\Delta_1}=1$. Then $C_{G_1}(a)_{\Delta_1}\geqslant C_{G_1}(a)_{(\Gamma)}$. Since $N_{G_1}(P)_{(\Gamma)}/C_{G_1}(P)_{(\Gamma)}\leqslant\mathrm{Aut}(P)$ and $N_{G_1}(P)_{(\Gamma)}/N_{G_1}(P)_{\Delta_1}\cong N_{G_1}(P)_{(\Gamma)}^{\Delta_1}\geqslant A^{\Delta_1}$, we have that $A_{(3p-1)/2}$ is involved in Aut (P). But, we can easily seen that $A_{(3p-1)/2}$ is not involved in Aut (P) (cf. [2, § 2. (3)]), which is a contradiction. Hence $C_{G_1}(P)_{(\Gamma)}^{\Delta_1}\geqslant A^{\Delta_1}$ Since the center of a Sylow p-subgroup of $N_{G_1}(P)_{(\Gamma)}$ is of order p, this is a contradiction. (q.e.d.)

By the same argument as in Step 7 in the proof of Theorem A, we have

Step 8. $|\Omega|-(3p-1)\not\equiv p\ (mod\ p^2)$.

From now on, let a be an element of order p of the form

$$a=(1)\cdots(2p)(2p+1)\cdots(3p-1)(3p,\cdots,4p-1)(4p,\cdots,5p-1)\cdots.$$

We divide the consideration into the following two cases:

(α) $C_G(a)^{I(a)}$ has an orbit Δ such that $|\Delta|\geqslant 2p$ and $C_G(a)^\Delta \geqslant A^\Delta$;

(β) otherwise.

When Case (α) holds, we may assume that $\Delta=\{1,\cdots,|\Delta|\}$. When Case ($\beta$) holds, we may assume that $\Delta_1=\{1,\cdots,w\}$ and $\Delta_2=\{w+1,\cdots,3p-1\}$ are the orbits or the blocks of $C_G(a)^{I(a)}$, and that $|\Delta_1|\geqslant|\Delta_2|\geqslant p$.

By the same argument as in Step 8, Step 9, Step 10 and Step 11 in the proof of Theorem A, we have

Step 9. *Case (α) does not hold.*

Hereafter we assume that Case (β) holds.

Step 10. *Set* $C_G(a)_{w+1,w+2,\cdots,2p,0}=C_G(a)_{w+1,w+2,\cdots,2p}$. *There is an integer* i $(0\leqslant i\leqslant 1)$ *such that* $C_G(a)_{w+1,w+2,\cdots,2p,i}$ *and* $C_G(a)_{w+1,w+2,\cdots,2p,i,i+1}$ *have exactly* m *orbits on* $\Omega-I(a)$, *where* m *is at most two, and moreover* $m=1$ *when* $|\Omega|-(3p-1)\not\equiv 0$ *(mod* p^2*).*

Proof. In order to prove Step 10, it is sufficient to show that $C_G(a)_{w+1,\cdots,2p,1,2}$ has at most two orbits on $\Omega-I(a)$, and is transitive on $\Omega-I(a)$ when $|\Omega|-(3p-1)\not\equiv 0 \pmod{p^2}$.

Set $H=G_{w+1,\cdots,2p,1,2}$. Then H is p-transitive on $\Omega-\{w+1,\cdots,2p,1,2\}$ by Step 5. By the remark following Lemma 1.1 in [11], we get the following expression:

$$\frac{|H|}{p}\geqslant\frac{|H|}{|C_H(a)|}\frac{1}{p}\sum_y{}' \alpha^*(y),$$

where y ranges all p'-elements in $C_H(a)$ and $\alpha^*(y)=\alpha(y^{\Omega-I(a)})$. Here the equality does not hold when $|\Omega|-(3p-1)\not\equiv 0 \pmod{p^2}$ (cf. Step 8 in the proof of Theorem A). Now, $\sum_y{}'\alpha^*(y)\geqslant\sum_{y\in C_H(a)}\alpha^*(y)-p\cdot\sum_{y\in C_H(a)}\alpha_p(y^{I(a)})$. Since $|\Delta_1-\{1,2\}|\geqslant p+\frac{p-1}{2}-2\geqslant p+3$, we have $C_H(a)^{\Delta_1-\{1,2\}}\geqslant A^{\Delta_1-\{1,2\}}$ by Step 6. Hence, $p\cdot\sum_{y\in C_H(a)}\alpha_p(y^{I(a)})=p\cdot\sum_{y\in C_H(a)}\alpha_p(y^{\Delta_1-\{1,2\}})=|C_H(a)|$ by the formula of Frobenius. On the other hand, $\sum_{y\in C_H(a)}\alpha^*(y)=f\cdot|C_H(a)|$, where f is the number of orbits of $C_H(a)$ on $\Omega-I(a)$. Hence we get

$$\frac{|H|}{p}\geqslant\frac{|H|}{p}(f-1), \qquad \text{and hence } f\leqslant 2.$$

In the above expression, if $|\Omega|-(3p-1)\not\equiv 0 \pmod{p^2}$, the equality does not hold. (q.e.d.)

Step 11. *$C_G(a)_{1,2,\cdots,2p}$ has at most $2m$ orbits on $\Omega-I(a)$. Moreover, $C_G(a)_{1,\cdots,p,\{p+1,p+2\}p+3,\cdots,2p}$ $(=C_{G(\{p+1,p+2\})}(a)_{1,\cdots,p,p+3,\cdots,2p})$ has exactly m orbits on $\Omega-I(a)$.*

Proof. By Step 10, $C_G(a)_{w+1,\cdots,2p,i}$ has exactly m orbits on $\Omega-I(a)$. Let $\Gamma_1, \cdots, \Gamma_m$ be the orbits. We take an arbitrarily fixed orbit Γ_j of $C_G(a)_{w+1,\cdots,2p,i}$ on $\Omega-I(a)$. Let $\Sigma_1, \cdots, \Sigma_k$ be the orbits of $C_G(a)_{1,2,\cdots,2p}$ on Γ_i. Since $C_G(a)_{w+1,\cdots,2p,i} \rhd C_G(a)_{1,2,\cdots,2p}$ and Γ_j is an orbit of $C_G(a)_{w+1,\cdots,2p,i}$, $C_G(a)^{\Delta_1-\{i\}}_{w+1,\cdots,2p,i}$ acts on the set $\{\Sigma_1, \cdots, \Sigma_k\}$ transitively. Let $Y=C_{G(\Sigma_1)}(a)_{w+1,\cdots,2p,i}$, then $|C_G(a)^{\Delta_1-\{i\}}_{w+1,\cdots,2p,i}: Y^{\Delta_1-\{i\}}|=k$. Similarly we have that $|C_G(a)^{\Delta_1-\{i\}}_{w+1,\cdots,2p,i,i+1}: Y^{\Delta_1-\{i\}}_{i+1}|=k$. Hence, $|C_G(a)^{\Delta_1-\{i\}}_{w+1,\cdots,2p,i}: C_G(a)^{\Delta_1-\{1\}}_{w+1,\cdots,2p,i,i+1}|=|Y^{\Delta_1-\{i\}}: Y^{\Delta_1-\{i\}}_{i+1}|=|\Delta_1|-i$. Therefore Y is transitive on $\Delta_1-\{i\}$. Let $(\beta_1, \cdots, \beta_p)$ be a p-cycle of a such that $\{\beta_1, \cdots, \beta_p\} \subseteq \Sigma_1$. For any $w-p-i$ elements $\alpha_1, \cdots, \alpha_{w-p-i}$ of $\Delta_1-\{i\}$, $C_G(a)_{i,\alpha_1,\cdots,\alpha_{w-p-i},w+1,\cdots,2p,\beta_1,\cdots,\beta_p}$ has an element b of order p. Then $b \in Y$ and b^{Δ_1} is a p-cycle, and so, $Y^{\Delta_1-\{i\}}_{\alpha_1,\cdots,\alpha_{w-p-i}}$ has the p-cycle. Since $\alpha_1, \cdots, \alpha_{w-p-i-1}, \alpha_{w-p-i}$ are any $w-p-i$ points of $\Delta_1-\{i\}$, we have $Y^{\Delta_1-\{i\}} \geqslant A^{\Delta_1-\{i\}}$ (cf. [14, Theorem 13.9]). Therefore $k \leqslant 2$. If $k=2$, then $Y^{\Delta_1-\{i\}}=A^{\Delta_1-\{i\}}$ and $C_G(a)^{\Delta_1-\{1\}}_{w+1,\cdots,2p,i}=S^{\Delta_1-\{i\}}$. Therefore Γ_j is an orbit of $C_G(a)_{1,\cdots,p\{p+1,p+2\}p+3,\cdots,2p}$ on $\Omega-I(a)$, even if $k=2$. (q.e.d.)

Step 12. *We complete the proof.*

Proof. Since a is an element of order p of the form

$$a=(1)\cdots(p)(p+1)\cdots(3p-1)(3p,\cdots,4p-1)(4p,\cdots,5p-1)\cdots,$$

$C_G(a)_{p+1,\cdots,2p,3p,\cdots,4p-1}$ has an element b of order p. By Step 8, we may assume that

$$b=(1,\cdots,p)(p+1)\cdots(3p-1)(3p)\cdots(4p-1)(4p,\cdots,5p-1)\cdots.$$

Let $K=G_{1,\cdots,p\{p+1,p+2\}p+3,\cdots,2p}$ and $L=\langle b\rangle \cdot K$. By the same argument as Step 10 in the proof of Theorem A, we have a contradiction. (q.e.d.)

4. Proofs of Theorem C and Theorem D

Proof of Theorem C. Let G be a nontrivial $2p$-transitive group on $\Omega=\{1, \cdots, n\}$. Let P be a Sylow p-subgroup of $G_{1,\cdots,2p}$, then $P \neq 1$ and P is not semiregular on $\Omega-I(P)$ by [3] and [4]. Moreover, $N_G(P)^{I(P)}$ is S_m $(2p \leqslant m \leqslant 3p-1)$ or A_m $(2p+2 \leqslant m \leqslant 3p-1)$. Hence, if $n\ (\equiv |I(P)|) \equiv p-1 \pmod{p}$, then Theorem C holds. Suppose that $n \not\equiv p-1 \pmod{p}$. Let Q be a subgroup of P such that the order of Q is maximal among all subgroups of P fixing more than $|I(P)|$ points. Set $N=N_G(Q)^{I(Q)}$, then N has an orbit Γ such that $N^\Gamma \geqslant A^\Gamma$ and $|\Gamma| \geqslant 3p$, by Theorem A. (q.e.d.)

Proof of Theorem D. Let G be a nontrivial t-transitive group on $\Omega=$

$\{1, \cdots, n\}$. Suppose that t is sufficiently large. By Satz B in [13], $\log(n-t)>\frac{t}{2}$. By the proof of [13, Satz B], we can see that $\log(n-t)>\left(\frac{1}{2}+\varepsilon_0\right)t$ for some $\varepsilon_0>0$. Moreover, we can see that, in the proof of [13, Satz B], it was only used that for any k-transitive group H on Σ, there exists a subset Π of Σ such that $|\Pi|=k$ and $H_{(\Pi)}^{\Pi}\geqslant A^{\Pi}$.

Let $p_1=2$, $p_2=3$, $\cdots$, and p_i be the *i-th* prime number. Then $\lim_{i\to\infty}\frac{p_{i+1}}{p_i}\to 1$. (This result is well known in the theory of numbers.)

Since t is sufficiently large, by the above remark and Theorem C, there exists a positive number ε which is sufficiently close to 0, and exists a subset Δ of Ω such that $|\Delta|\geqslant\left(\frac{3}{2}-\varepsilon\right)t$ and $G_{(\Delta)}^{\Delta}\geqslant A^{\Delta}$. Therefore we have $\log(n-t)>\frac{3}{4}t$. (q.e.d.)

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