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STANDARD COMPONENTS OF TYPE M_{12} AND $\cdot 3$

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Intensive activity in the course of the past few years has brought very close to completion the following problem.

PROBLEM. Let G be a finite group with $F^*(G)$ simple. Let T be a subgroup of G and L a subnormal subgroup of $C_G(T)$ with $L/O(L)$ isomorphic to a known quasisimple group. Identify G.

The main contribution to the solution of this problem is the Unbalanced Group Theorem, whose proof now appears to be nearing completion.

Theorem 1.1 (Unbalanced group theorem). *Let G be a finite group with $F^*(G)$ simple. Let t be an involution of G. Then either G is known or* $0(C_G(t))=1$.

We shall call a group G balanced if $0(C_G(t))\subseteq 0(G)$ for all involutions t of G. A crucial corollary to the unbalanced group theorem is the $B(G)$ theorem. Before stating this result, we must review some definitions. A perfect subnormal subgroup L of H is said to be a 2-component if $L/0(L)$ is quasisimple. We say that L is a component if $0(L)\subseteq Z(L)$. The 2-layer of H, denoted $L_{2'}(H)$ is the product of all 2-components of H. Similarly, the layer of H, denoted $L(H)$, is the product of all components of H.

Theorem 1.2 ($B(G)$ theorem). *Let G be a finite group with* $0(G)=1$. *Let t be an involution of L. Then every 2-component of $C_G(t)$ is a component of $C_G(t)$.*

The next major contribution to our problem is the Component theorem of Aschbacher and Foote. For G a finite group, let $\mathcal{L}(G)$ be the set of all components of $C_G(t)$ for t ranging over the involutions of G. We define a relation $<$ on $\mathcal{L}(G)$ as follows:

$K<L$ if there exists a pair (s,t) of commuting involutions with K a component of $C_G(s)$, L a component of $C_G(t)$ and $K\subseteq LL^s$.

We extend $<$ to a transitive relation $\ll$ on $\mathcal{L}(G)$. We say that K is maximal in $\mathcal{L}(G)$ if $K\ll L$ implies $K\cong L$. Finally we say that K is standard in

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G if $[K, K^g] \neq 1$ for all $g \in G$ and $|C_G(K) \cap C_G(K)^g|$ is odd for all $g \in G - N_G(K)$.

Theorem 1.3 (Component theorem of Aschbacher and Foote). *Let G be a finite group with $F^*(G)$ simple. Suppose that K is maximal in $\mathcal{L}(G)$. Then either K is standard in G or K has 2 rank 1 and $F^*(G)$ is isomorphic to $PSL(4, q)$, $PSU(4, q)$, $PSp(4, q)$ or $G_2(q)$ for odd q.*

REMARKS. This result is essentially contained in [3, Theorem 1] and [11, Theorem 1]. However certain discrepancies in the definition of maximal component and the hypotheses merit clarification.

In [3], Aschbacher defined a relation $\ll$ on $\mathcal{L}(G)$ as the transitive extension of the relation $<^*$ given by:

$$L <^* K \text{ if there exists an involution } t \text{ with}$$
$$L \trianglelefteq E(C(t)),\ K = [K, t] \text{ and } L \subseteq K\,.$$

Clearly if $L \ll K$ in Aschbacher's sense, then $L \ll K$ in our sense. Moreover, if $L \ll K$ in Aschbacher's sense, then $|K| > |L|$ or $K = L$. Hence Aschbacher's relation is a partial ordering on $\mathcal{L}(G)$ and it makes sense to speak of $\mathcal{L}^*(G)$ as the maximal elements of $\mathcal{L}(G)$ under this partial order.

Now if K is maximal in our sense and $K \ll L$ in Aschbacher's sense, then $K = L$ and so $K \in \mathcal{L}^*(G)$. Thus $\mathcal{L}^*(G)$ contains all of our maximal components.

Now Aschbacher's Theorem 1 is stated for those $K \in \mathcal{L}(G)$ such that if $L \in \mathcal{L}(G)$ and K is a homomorphic image of L, then $L \in \mathcal{L}^*(G)$. This hypothesis is very awkward to check. Fortunately, however, inspection of Aschbacher's proof reveals that only the following hypothesis is really used:

$$K \in \mathcal{L}(G) \text{ and if } K \ll L \in \mathcal{L}(G)\,,$$
$$\text{then } L \in \mathcal{L}^*(G)\,;$$

where $\ll$ is used in our sense. Now if K is maximal in our sense and if $K \ll L \in \mathcal{L}(G)$, then L is maximal in our sense whence, in particular, $L \in \mathcal{L}^*(G)$.

Thus Aschbacher's Theorem 1 is valid for all $K \in \mathcal{L}(G)$ which are maximal in our sense. If $K \in \mathcal{L}(G)$ has dihedral Sylow 2-subgroups and $K < L$ with $m_2(L) = 1$, then K is not maximal in our sense. Thus conclusion (3) of Aschbacher's theorem does not apply. Moreover, our hypothesis that $F^*(G)$ is simple rules out conclusion (4). Thus either K is standard in G or $m_2(K) = 1$ and conclusion (2) holds. In the latter case, Foote's Theorem 1 in [11] implies that $F^*(G)$ is isomorphic to $PSL(4, q)$, $PSU(4, q)$, $PSp(4, q)$ or $G_2(q)$ for q odd and K is isomorphic to $SL(2, q)$, as asserted.

Corollary 1.4. *Let G be a finite group with $F^*(G)$ simple. Let T_0 be a 2-subgroup of G and K a component of $C_G(T_0)$. Then there exists a chain*

$$K = L_0, L_1, L_2, \cdots, L_{n-1}, L_n = F^*(G)$$

satisfying

(1) *If* $L_i = L_j$, *then* $i=j$.
(2) L_i *is a component of* $C_G(T_i)$ *for some* 2-*subgroup* T_i *of* G.
(3) *For* $i \geq 1$, $T_i \subseteq S_{i-1} \in Syl_2(C_G(L_{i-1}))$ *and* L_{i-1} *is a component of* $C_G(N_{S_{i-1}}(T_i))$.
(4) $L_i \subseteq \langle L_{i-1}{}^{L(C_G(T_i))}\rangle$.
(5) *For each* i, $1 \leq i \leq n$, *one of the following hold*:
  (a) $L_i = \langle L_{i-1}{}^{L(C_G(T_i))}\rangle$ *and* $L_{i-1}C_G(L_i)/C_G(L_i)$ *is standard in some subgroup of* $N_G(L_i)/C_G(L_i)$ *containing* $L_iC_G(L_i)/C_G(L_i)$.
  (b) $L_i = \langle (L_{i-1})^{L(C_G(T_i))}\rangle$; $L_{i-1} \cong SL(2,q)$ *for some odd* q; $L_i/Z(L_i)$ *is isomorphic to* $PSL(4,q)$, $PSU(4,q)$, $PSp(4,q)$ *or* $G_2(q)$.
  (c) $L_i \neq \langle (L_{i-1})^{L(C_G(T_i))}\rangle$; $L_i/Z(L_i) \cong L_{i-1}/Z(L_{i-1})$.

Our proof of Corollary 1.4 requires two preliminary results.

Lemma 1.5. *Let* G *be a finite group and* S *a* 2-*subgroup of* G.
(i) *If* T *is a subgroup of* S, *then* $L_{2'}(C_G(S)) \subseteq L_{2'}(C_G(T))$.
(ii) *If* $0(G)=1$, *then* $L_{2'}(C_G(S)) = L(C_G(S))$.

Proof. (i) It is sufficient to consider the case where $[S:T]=2$. Let $C = C_G(T)S$ and $\bar{C} = C/T$. It is easy to see, using the 3-subgroup lemma, that $L_{2'}(C_G(S))T/T = L_{2'}(C_{\bar{C}}(\bar{S}))$. Similarly, $L_{2'}(C_G(T))T/T = L_{2'}(\bar{C})$. Then by the L-balance theorem of Gorenstein and Walter ([15], Proposition 4.2), we have $L_{2'}(C_G(S))T/T \subseteq L_{2'}(\bar{C})$. But then $L_{2'}(C_G(S)) \subseteq L_{2'}(C_G(T))T$ whereupon it follows that $L_{2'}(C_G(S)) \subseteq L_{2'}(C_G(T))$.

(ii) The proof is by induction on $|S|$. If $|S|=2$, then the result follows from Theorem 1.2. Assume now that T is a proper subgroup of S with $[S:T]=2$. By (i) and our inductive assumption, we have $L_{2'}(C_G(S)) \subseteq L_{2'}(C_G(T)) = L(C_G(T))$. Let $L = L(C_G(T))$, $C = LS$ and $\bar{C} = C/T\,0(L)$. Then as in (i), $\overline{L_{2'}(C_G(S))} = L_{2'}(C_{\bar{C}}(\bar{S}))$. But $|\bar{S}|=2$ and $0(\bar{C})=1$, hence by induction, $L_{2'}(C_{\bar{C}}(\bar{S})) = L(C_{\bar{C}}(\bar{S}))$. Therefore $[L_{2'}(C_G(S)), 0(L_{2'}(C_G(S)))] \subseteq T0(L)$ and we have that $0(L_{2'}(C_G(S))) \subseteq Z(L_{2'}(C_G(S)))$ by the 3-subgroup lemma. Thus $L_{2'}(C_G(S)) = L(C_G(S))$ as required.

Lemma 1.6. *Let* G *be a finite group with* $F^*(G)$ *simple such that Corollary* 1.4 *holds for all proper sections* Γ *of* G *with* $F^*(\Gamma)$ *simple. Let* V, W *be* 2-*subgroups with* $\langle 1\rangle \neq W \trianglelefteq V$. *Suppose that* L *is a component of* $C_G(V)$, M *is a component of* $C_G(W)$ *and* $M = \langle L^{L(C_G(W))}\rangle \neq L$. *Then there is a chain* $L = L_0, L_1, \cdots, L_n = M$ *satisfying* (1)-(5) *of Corollary* 1.4 *with* $L_i \subseteq M$ *for* $1 \leq i \leq n$.

Proof. Let $H = VM$ and $\bar{H} = H/C_H(M)$. Then $\bar{M} = F^*(\bar{H})$ and the conclusion of Corollary 1.4 holds in $\bar{H}$. Since $V \nsubseteq C_H(M)$ by assumption, we have that $\bar{V} \neq \langle 1 \rangle$ and $\bar{L}$ is a component of $C_{\bar{H}}(\bar{V})$.

Therefore, there exists a chain $\bar{L}=\bar{L}_0, \bar{L}_1, \cdots, \bar{L}_n=\bar{M}$ and 2-subgroups $\bar{T}_i$, $\bar{S}_i, 0 \leq i \leq n$ with $\bar{V}=\bar{T}_0$ such that (1)-(5) of Corollary 1.4 hold. Let L_i be the largest perfect normal subgroup of the preimage in H of $\bar{L}_i$. Let T_i and S_i be Sylow 2 subgroups respectively of the preimage in H of $\bar{T}_i$ and $\bar{S}_i$. As $C_H(M)/Z(M)$ is a 2-group, $C_H(M)$ has a normal Sylow 2-subgroup containing W. Thus $W \subseteq T_i \subseteq S_{i-1}$, L_i is quasisimple and $L_i \subseteq M$. Applying the 3-subgroup lemma, we then have that the chain $L=L_0, L_1, \cdots, L_n=M$ together with the 2-subgroups T_i, S_i, $0 \leq i \leq n$ satisfies (1)-(5) of Corollary 1.4 in H. We must show that the chain satisfies (1)-(5) of Corollary 1.4 in G.

First observe that $M \trianglelefteq \trianglelefteq C_G(W)$ and $C_G(T_i) \subseteq C_G(W)$ implies that $C_M(T_i) \trianglelefteq \trianglelefteq C_G(T_i)$. But L_i is a component of $C_M(T_i)$, hence L_i is a component of $C_G(T_i)$ as well. The same reasoning yields that L_{i-1} is a component of $C_G(N_{S_{i-1}}(T_i))$. Hence, if $S_i \subseteq S_i^* \in \mathrm{Syl}_2(C_G(L_i))$, then L_{i-1} is a component of $C_G(N_{S_{i-1}^*}(T_i))$. This shows that (1)-(4) of Corollary 1.4 hold. Consider the link L_{i-1}, L_i for $1 \leq i \leq n$. If $L_i \neq \langle L_{i-1}{}^{L(C_H(T_i))} \rangle$, then $L_i/Z(L_i) \cong L_{i-1}/Z(L_{i-1})$ and (5c) holds. Therefore, we may assume that $L_i = \langle L_{i-1}{}^{L(C_H(T_i))} \rangle$ so that $L_i = \langle L_{i-1}{}^{L(C_G(T_i))} \rangle$. If (5b) holds for L_{i-1}, L_i in H, then (5b) holds for L_{i-1}, L_i in G as well. Finally, if (5a) holds for L_{i-1}, L_i in H, set $Y=N_H(L_i)C_G(L_i)$ and $\bar{Y}=Y/C_G(L_i)$. Since $C_G(L_i) \subseteq C_G(L_{i-1})$, it follows from the 3-subgroup lemma that $C_{\bar{Y}}(\bar{L}_{i-1}) = \overline{C_Y(L_{i-1})}$. Hence we may use the corresponding result in H to easily verify that $\bar{L}_{i-1}$ is a standard component of some subgroup of $\bar{Y}$ containing $\bar{L}_i$. Thus (5a) holds and the proof is completed in all cases.

REMARK. Once Corollary 1.4 is proved the conclusion of Lemma 1.6 will hold for all finite groups G with $F^*(G)$ simple.

Proof of Corollary 1.4. Assume that G is a minimal counterexample and let L_0 be a counterexample subject to $|L_0/Z(L_0)|$ maximal and then $|C_G(L_0)|_2$ maximal. By our choice of L_0, we have that the following hold:

(i) If $L_0, L_1, \cdots, L_m$ is a chain satisfying (1)-(5), then $L_i/Z(L_i) \cong L_0/Z(L_0)$, $1 \leq i \leq m$.

(ii) Let V, W be 2-subgroups of G with $\langle 1 \rangle \neq W \trianglelefteq V$, L_0 a component of $C_G(V)$, M a component of $C_G(W)$ and $M = \langle L_0{}^{L(C_G(W))} \rangle$. Then $M=L_0$.

In order to prove (i), observe that if L_0 is a counterexample, then so is each L_i, $0 \leq i \leq m$. Hence by choice of L_0, (5c) is satisfied and $L_i/Z(L_i) \cong L_0/Z(L_0)$, $1 \leq i \leq m$. If the hypotheses of (ii) hold, then by Lemma 1.6, there exists a chain $L_0, L_1, \cdots, L_m=M$ satisfying (1)-(5). The result now follows from (i).

Let $S_0 \in \mathrm{Syl}_2(C_G(L_0))$ and let $s \in I(S_0)$. Then $L_0 \subseteq L(C_G(s))$ by Lemma 1.5. This leads to the following dichotomy.

(A) If $s \in I(S_0)$, then each component M of $\langle L_0{}^{L(C_G(s))} \rangle$ satisfies $M/Z(M)$

$\cong L_0/Z(L_0)$.

(B) For some $s\in I(S_0)$, there exists a component M of $\langle L_0^{L(C_G(s))}\rangle$ such that $M/Z(M)\not\cong L_0/Z(L_0)$.

Suppose first that (A) holds and let $s\in I(Z(S_0))$. By assumption, $\langle L_0^{L(C_G(s))}\rangle =M_1M_2\cdots M_r$ where $M_i/Z(M_i)\cong L_0/Z(L_0)$, $1\leq i\leq r$. We claim that up to reindexing, $L_0=M_1$, hence $L_0\in\mathcal{L}(G)$. If this is not the case, then we must have $r\geq 2$. Since S_0 centralizes L_0, $S_0/C_{S_0}(M_1M_2\cdots M_r)$ acts regularly on $\{M_1, M_2,\cdots, M_r\}$. An easy induction argument gives $|\sum_r|_2<4^{r-1}$, $r\geq 2$. Also $|M_i/Z(M_i)|_2 \geq 4$. Thus $|C_G(M_1)|_2\geq 4^{r-1}|C_{S_0}(M_1M_2\cdots M_r)|$ and we have

$$|C_G(M_1)|_2>|\textstyle\sum_r|_2|C_{S_0}(M_1M_2\cdots M_r)|\geq|S_0|\ .$$

But the chain L_0, M_1 satisfies (1)-(5), hence M_1 is a counterexample with $|M_1/Z(M_1)|=|L_0/Z(L_0)|$ and $|C_G(M_1)|_2>|C_G(L_0)|_2$ against the choice of L_0. This proves the claim.

Since $L_0\in\mathcal{L}(G)$, it follows from Theorem 1.3 and choice of L_0, that L_0 is not a maximal element of $\mathcal{L}(G)$. As $S_0\in \mathrm{Syl}_2(C_G(L_0))$, we may then find $t\in I(S_0)$ and a component M of $C_G(t)$ such that $M=\langle L_0^{LC_G(t))}\rangle\neq L_0$. But this contradicts (ii) with respect to $\langle t\rangle$, $\langle t, s\rangle$ and the components M of $C_G(t)$ and L_0 of $C_G(\langle t, s\rangle)$.

Finally, suppose (B) holds. Thus for some $s\in I(S_0)$, $L_0\subseteq L(C_G^{(s)})$ and $\langle L_0^{L(C_G(s))}\rangle$ has a component N with $N/Z(N)\not\cong L_0/Z(L_0)$. Let W_1 be a subgroup of S_0 containing s and of maximal order subject to $L_0\neq\langle L_0^{L(C_G(W_1))}\rangle$. Let $w_1\in N_{S_0}(W_1)-W_1$ with $w_1^2\in W_1$. By choice of W_1, L_0 is a component of $C_G(\langle W_1, w_1\rangle)$. Applying (ii), $\langle L_0^{L(C_G(W_1))}\rangle$ is not a component of $C_G(W_1)$, hence $\langle L_0^{L(C_G(W_1))}\rangle=M_1M_1^{w_1}$ where M_1 is a component of $C_G(W_1)$, $M_1\neq M_1^{w_1}$ and $M_1/Z(M_1)\cong L_0/Z(L_0)$. By Lemma 1.5, $L(C_G(W_1))\subseteq L(C_G(s))$, hence $\langle L_0^{L(C_G(s))}\rangle \subseteq\langle M_1^{L(C_G(s))}\rangle\langle (M_1^{w_1})^{L(C_G(s))}\rangle$. Without loss, we may assume that $N\subseteq\langle M_1^{L(C_G(s))}\rangle$. Now L_0, M_1 is a chain satisfying (1)-(5), hence M_1 is a counterexample as well. Repeating the analysis and using (i) and (ii), we may construct a chain of 2-groups $W_1\supseteq W_2\supseteq\cdots W_m\supseteq\langle s\rangle$ with $m\geq 2$ satisfying.

(a) M_j is a component of $C_G(W_j)$

(b) M_{j-1} is a component of $C_G(N_{W_{j-1}}(W_j))$

(c) $\langle M_{j-1}{}^{L(C_G(W_j))}\rangle=M_jM_j^{w_j}$ for some $w_j\in N_{W_{j-1}}(W_j)$ with $w_j^2\in W_j$ and $M_j\neq M_j^{w_j}$.

(d) $N\subseteq\langle M_j^{L(C_G(s))}\rangle$.

(e) $M_j/Z(M_j)\cong L_0/Z(L_0)$, $1\leq j\leq m$.

Evidently we may continue until M_m is a component of $L(C_G(s))$. But N is a component of $\langle L_0^{L(C_G(s))}\rangle$ with $N/Z(N)\not\cong L_0/Z(L_0)$ and this is incompatible with $N\subseteq M_m$ and $M_m/Z(M_m)\cong L_0/Z(L)_0$.

This final contradiction completes the proof of Corollary 1.4.

Corollary 1.7. *Let $\mathcal{K}$ be a set of isomorphism classes of finite quasisimple groups. Let the isomorphism classes be denoted by $[K]$ with representative K. Suppose that if L is a quasisimple group satisfying one of the following conditions then $[L]\in\mathcal{K}$.*

(1) *$L/Z(L)\cong K/Z(K)$ for some $[K]\in\mathcal{K}$.*

(2) *There is a standard component K in a subgroup of $Aut(L)$ containing $Inn(L)$ with $[K]\in\mathcal{K}$.*

(3) *$L/Z(L)$ is isomorphic to $PSL(4,q)$, $PSU(4,q)$, $PSp(4,q)$ or $G_2(q)$ and $[SL(2,q)]\in\mathcal{K}$ for some odd prime power q.*

Let G be a finite group with $F^*(G)$ simple, let T be a 2-subgroup of G and L a component of $C_G(T)$ with $[L]\in\mathcal{K}$. Then $[F^*(G)]\in\mathcal{K}$.

Proof. Let $L=L_0, L_1, L_2, \cdots, L_n=F^*(G)$ be a chain of quasisimple subgroups of G as given in Corollary 1.4. If $[L_{i-1}]\in\mathcal{K}$, then $[L_i]\in\mathcal{K}$ as well. Thus as $[L_0]\in\mathcal{K}$, $[L_n]\in\mathcal{K}$.

We shall call a family $\mathcal{K}$ which satisfies conditions (1)-(3) of Corollary 1.7 embedding-closed. We denote by Chev (5) the set of Chevalley groups over a finite field of characteristic 5. We now state our main theorem.

Theorem 1.8. *Let $\mathcal{A}$ be the set of all isomorphism classes $[A]$ such that either $A/Z(A)\in Chev(5)$ or $A/Z(A)$ is isomorphic to a member of*

$$\{A_{2n+1},\ n\geq 2;\ PSL(2,4^n),\ n=2^m,\ m\geq 0;\ PSU(3,4^n),\ n=2^m,\ m\geq 0;\ PSL(3,4^n),\ n=2^m,\ m\geq 0;\ M_{12},\ J_1,\ HJ,\ LyS,\ O'NS,\ He,\ Suz,\ \cdot 3\}\ .$$

Then $\mathcal{A}$ is embedding closed.

The work in this paper represents a brief coda to a vast symphony of theorems culminating in Theorem 1.8. We summarize the major antecedents below.

Theorem 1.9 (Aschbacher [1], [2], Gorenstein-Harada [14], Harris [20], Harris-Solomon [21], Solomon [26], [27], Walter [29]). *Let G be a finite group with $F^*(G)$ simple having a standard component A with $A/Z(A)\in Chev$ (5) or $A/Z(A)\cong A_{2n+1}$, $n\geq 2$, or $A\cong LyS$. Then $F^*(G)$ is isomorphic to some group in the following set.*

$$\{Chev\ (5),\ A_{2n+1},\ PSL(2,16)\ PSL(3,4),\ PSU(3,4),\ M_{12},\ J_1,\ HJ,\ LyS,\ He\}$$

Theorem 1.10 (Griess-Mason-Seitz [17], Nah [24], Seitz [25]). *Let G be a finite group with $F^*(G)$ simple having a standard component A with $A/Z(A)\cong PSL(2,4^n)$, $n\geq 2$, or $A/Z(A)\cong PSU(3,4^n)$, $n\geq 1$, or $A/Z(A)\cong PSL(3,4^n)$, $n\geq 1$. Then $F^*(G)$ is isomorphic to some group in the following set:*

$$\{PSL(2,4^n),\ n\geq 4;\ PSU(3,4^n),\ n\geq 2,\ PSL(3,4^n),\ n\geq 2,\ O'NS,\ He\ Suz\}$$

Theorem 1.11 (Finkelstein [8], [9]). *Let G be a finite group with $F^*(G)$ simple having a standard component A isomorphic to HJ or J_1. Then $F^*(G)$ is isomorphic to O'NS or Suz.*

Theorem 1.12 (Griess-Solomon [18], Solomon [28]). *Let G be a finite group with $F^*(G)$ simple. Then G does not have a standard component isomorphic to O'NS, He or Suz.*

Theorem 1.13 (Yoshida [32]). *Let G be a simple group having an involution t with $C_G(t)\cong Z_2\times M_{12}$. Then $G\cong \cdot 3$.*

We now examine how Theorem 1.8 could fail. By hypothesis, if $[SL(2,q)] \in \mathcal{A}$, then $q=5^n$. Also, $\mathcal{A}$ is closed under central quotients and central extensions and $\mathcal{A}$ contains $[K]$ whenever $K/Z(K)$ is isomorphic to $PSL(4,5^n)$, $PSU(4,5^n)$, or $PSp(4,5^n)$ or $G_2(5^n)$. The final condition requires that $[L]\in\mathcal{A}$ whenever there exists K standard in $G\leq \mathrm{Aut}(L)$ with $[K]\in\mathcal{A}$. This holds by Theorems 1.9–1.12 unless possibly if $K/Z(K)\cong M_{12}$, HJ, $\cdot 3$ or Suz. Thus Theorem 1.8 will be proved once the following result is established.

Theorem 1.14. *Let G be a finite group with $F^*(G)$ simple having a standard component K with $K/Z(K)$ isomorphic to M_{12}, HJ, $\cdot 3$ or Suz. Then $F^*(G)$ is isomorphic to Suz or $\cdot 3$.*

The remainder of the paper is devoted to the proof of Theorem 1.14.

2 Properties of M_{12}, *HJ*, *Suz* and $\cdot 3$

In this section, we enumerate those properties of M_{12}, HJ, Suz and $\cdot 3$ which are necessary for the proof of Theorem 1.14. In most cases, these are easily deduced from information given in ([5], [6], [7], [9], [23], [30], [31]). In what follows, K will be a proper 2-fold covering of M_{12}, HJ or Suz with $Z(K)=\langle t\rangle$, K^* a non-trivial extension of K by Z_2 and $\bar{K}^*=K/\langle t\rangle$. Note that for M_{12}, HJ and Suz, the outer automorphism group and a Sylow 2 subgroup of the Schur multiplier have order 2.

Lemma 2.1. *Let $\bar{K}\cong M_{12}$. Then*

(i) *$\bar{K}^*$ has 3 classes of involutions with representatives $\bar{z}$, $\bar{x}$ in $\bar{K}$ and $\bar{p}\in\bar{K}^* -\bar{K}$. Also $C_{\bar{K}}(\bar{z})\cong E_8\cdot S_4$, $C_{\bar{K}}(\bar{x})\cong Z_2\times S_5$ and $C_{\bar{K}}(\bar{p})\cong Z_2\times A_5$.*

(ii) *K has 3 classes of involutions with representatives t, z and zt.*

(iii) *For some $T\in Syl_2(K^*)$, $\langle z,t\rangle=Z(T)=Z(T\cap K)$. Furthermore, both $Aut(T)$ and $Aut(T\cap K)$ act trivially on $\langle z,t\rangle$.*

(iv) *All involution of K^*-K, if any exist, are conjugate. If p is such an involution, then $C_K(p)\cong Z_2\times A_5$.*

Proof Everything except part (iii) is clear. We shall prove that $\mathrm{Aut}(T\cap K)$

and Aut(T) act trivially on $\langle z, t\rangle$. It follows from the character table of K that z is a fourth power in $T\cap K$, zt is not a square in $T\cap K$ and t is a fourth power in T but not in $T\cap K$. This implies that Aut($T\cap K$) acts trivially on $\langle z,t\rangle$ and that $\langle zt\rangle$ is invariant under Aut(T). It suffices to prove that z does not fuse to t in Aut(T). Now K has an element δ of order 4 such that $|C_K(\delta)|=2^6$, $\delta^2=z$ and $\delta\not\sim\delta t$. Without loss, we may assume that $\delta\in T$ and $|C_T(\delta)|=2^6$. If $z^a=t$ for some $a\in \mathrm{Aut}(T)$, then $\lambda=\delta^a$ satisfies $\lambda^2=t$, $\lambda\not\sim\lambda t=\lambda^{-1}$ and $|C_T(\lambda)|=2^6$. This implies that $|C_{\bar{K}^*}(\bar{\lambda})|_2=2^5$ whereupon $\bar{\lambda}\sim\bar{x}$. But $x\sim xt=t^{-1}$ then gives a contradiction.

Lemma 2.2. *Let $\bar{K}\cong HJ$. Then*

(i) *K has 3 classes of involutions with representatives t, z and zt.*

(ii) *For some $T\in Syl_2(K^*)$, $\langle z,t\rangle=Z(T)=Z(T\cap K)$. Furthermore, both Aut($T$) and Aut($T\cap K$) act trivially on $\langle z,t\rangle$.*

(iii) *All involutions of K^*-K, if any exist are conjugate. If p is such an involution, then $C_K(p)\cong Z_2\times PSL(3,2)$.*

Proof Parts (i) and (iii) are easily deduced from the character table of K. In order to prove part (ii), we observe that z is a fourth power in $T\cap K$, zt is not a square in $T\cap K$ and t is a fourth power in T but not in $T\cap K$. This shows that Aut($T\cap K$) acts trivially on $\langle z,t\rangle$ and Aut(T) stabilizes $\langle zt\rangle$. Now K has an element δ of order 4 such that $|C_{K^*}(\delta)|_2=2^7$, $\delta^2=z$ and $\delta\not\sim\delta t$. Assuming that $\delta\in T$ with $|C_T(\delta)|=2^7$, it follows that if $a\in\mathrm{Aut}(T)$ with $z^a=t$, then $\lambda=\delta^a$ satisfies $\lambda^2=t$ and $|C_T(\lambda)|=2^7$. But then $\bar{\lambda}$ is an involution of $\bar{K}^*$ with $|C_{\bar{K}^*}(\bar{\lambda})|_2=2^6$ which is impossible.

Lemma 2.3. *Let $\bar{K}\cong Suz$. Then*

(i) *$\bar{K}$ has 2 classes of involutions with representatives $\bar{z}$ and $\bar{x}$. $0_2(C_{\bar{K}}(\bar{z}))=0_2(C_{\bar{K}^*}(\bar{z}))\cong Q_8{*}Q_8{*}Q_8$ and $C_{\bar{K}}(\bar{z})/(0_2(C_{\bar{K}}(\bar{z}))\cong\Omega_6^-(2)$. $C_{\bar{K}}(\bar{x})=(\bar{V}\times\bar{L})\langle\bar{\sigma}\rangle$ with $\bar{V}\cong E_4$, $\bar{L}\cong PSL(3,4)$, $\langle\bar{V},\bar{\sigma}\rangle\cong D_8$ and $\bar{\sigma}$ induces the unitary polarity on $\bar{L}$.*

(ii) *$\bar{K}^*-\bar{K}$ has 2 classes of involutions with representatives $\bar{p}_1$ and $\bar{p}_2$. $C_{\bar{K}}(\bar{p}_1)\cong Aut(M_{12})$ and $C_{\bar{K}}(\bar{p}_2)\cong Aut(HJ)$.*

(iii) *K has 3 classes of involutions with representatives t, z and zt.*

(iv) *K^*-K has exactly one class of involutions. If p is a representative, then $C_K(p)\cong\hat{M}_{12}$ or $\widehat{HJ}$.*

(v) *K^* has precisely 2 classes of elements of order 4 whose square is t. If δ is such an element, then either $\delta\in L$ and $\bar{\delta}\sim\bar{x}$ or $\delta\in K^*-K$ and $C_K(\delta)\cong\hat{M}_{12}$ or $\widehat{HJ}$.*

(vi) *K^* has no element δ of order 4 with $|C_{K^*}(\delta)|=2^{10}$.*

Proof. Parts (i)-(iii) are easily deduced from information given in ([30], [31]). Now K has an element γ of order 3 such that $C_{\bar{K}}(\bar{\gamma})=0(C_{\bar{K}}(\bar{\gamma}))\times\bar{B}$ with $0(C_{\bar{K}}(\bar{\gamma}))\cong E_9$ and $\bar{B}\cong A_6$. Now $C_{\bar{K}^*}(\bar{\gamma})/0(C_{\bar{K}}(\bar{\gamma}))\cong S_6$. Let $\bar{B}^*$ be an S_2

subgroup of $C_{\bar{K}^*}(\bar{\gamma})$ and assume, as we may, that $\bar{B}^* \supseteq \langle \bar{p}_1, \bar{p}_2, \bar{x} \rangle \cong E_8$ (see parts (i), (ii)). Now x has order 4, hence $B \cong SL(2,9)$, and since $B^* = \langle B, p_1 \rangle = \langle B, p_2 \rangle$, we conclude that

$$(*) \quad p_i \sim p_i t \text{ and } |p_i| \neq |p_j|,\ i \neq j\,.$$

An immediate consequence of (*) is that $|C_K(p_i)| = |C_{\bar{K}}(\bar{p}_i)|$, $i=1,2$. Also the fact that $E(C_{\bar{K}}(\bar{p}_i))$ contains conjugates of $\bar{x}$ implies that $C_K(p_1) \cong \hat{M}_{12}$ and $C_K(p_2) \cong \widehat{HJ}$. This proves part (iv).

Let δ be an element of order 4 of K^* with $|C_{K^*}(\delta)| = 2^{10}$. By (v), $\delta^2 = z$ or zt. Let $C = C_{K^*}(z)$ and $\bar{C} = C/\langle z, t \rangle$ so that $C_{K^*}(\delta) = C_C(\delta)$ and $\bar{\delta}$ is an involution of $\bar{C}$. Now $\bar{C} \cong \mathrm{Aut}(Q_8 * Q_8 * Q_8)$ and an easy computation (see [3], section 10) shows that each involution of $\bar{C}$ is centralized by some element of order 3. This, however, is incompatible with $|C_C(\delta)| = 2^{10}$ and the result is proved.

REMARK. It follows from Lemma 2.3 that every non-trivial extension of $\widehat{Suz}$ by Z_2 splits.

Lemma 2.4. *$\cdot 3$ has 2 classes of involutions with involutions of the two classes having centralizers isomorphic to $Z_2 \times M_{12}$ and $\widehat{Sp}(6,2)$ respectively. Also the Schur multiplier and outer automorphism group of $\cdot 3$ are trivial.*

Proof. See [16].

3 Proof of Theorem 1.14

Let G be a minimal counterexample to Theorem 1.14. Thus G is a finite group with $F^*(G)$ simple, G has a standard component K with $K/Z(K)$ isomorphic to M_{12}, HJ, Suz or $\cdot 3$ and G has minimal order subject to $F^*(G)$ not isomorphic to Suz or $\cdot 3$.

Proposition 3.1. *K is isomorphic to M_{12} or $\cdot 3$. Furthermore $|C_G(K)|_2 = 2$.*

Proof. We shall first show that $|C_G(K)|_2 = 2$ and then prove in a sequence of lemmas that K is isomorphic to M_{12} or $\cdot 3$.

It follows from the combined results of Aschbacher and Seitz ([1], [4]) that $C_G(K)$ has cyclic Sylow 2-subgroups. Applying [10, Theorem 2] in conjunction with the properties of M_{12}, HJ, Suz and $\cdot 3$ enumerated in section 2 and the Unbalanced Group Theorem gives $C_G(K) = \langle t, 0(C_G(K)) \rangle$ where $\langle t \rangle$ has order 2 and is self centralizing in $C_G(K)$. In particular, $C_G(t)/\langle t \rangle = \mathrm{Aut}_{C_G(t)}(K)$. Also $G = \langle F^*(G), t \rangle$.

In light of Theorems 1.11 and 1.12, it suffices to eliminate the cases where K is isomorphic to $\hat{M}_{12}$, $\widehat{HJ}$, or $\widehat{Suz}$. In the following lemmas, we employ the notation set up in Lemmas 2.1–2.3.

Lemma 3.2. *$K \not\cong \hat{M}_{12}$ or $\widehat{HJ}$.*

Proof. Assume not. Then $C_G(t)=K$ or K^* and t is not isolated in $C_G(t)$ by the Z^* theorem [12]. Suppose at first that $t^G \cap K \neq \{t\}$. Then by Lemma 2.1 (ii), t is conjugate to z or zt. Since $\langle z, t\rangle$ is the center of some Sylow 2 subgroup T of $C_G(t)$, t is conjugate to z or zt in $N_G(T)$. But by Lemma 2.1 (iii) or Lemma 2.2 (ii), $N_G(T)$ acts trivially on $\langle z, t\rangle$. Thus $t^G \cap K=\{t\}$. This implies that $C_G(t)=K^*$ and K^*-K contains a conjugate p of t. Let $V=\langle t,p\rangle$ so that $C_G(v)=\langle t,p\rangle \times L$ where $L\cong A_5$ if $K\cong \hat{M}_{12}$ by Lemma 2.1 (iii) and $L\cong PSL(3,2)$ if $K\cong \widehat{HJ}$ by Lemma 2.2 (iii). An easy argument shows that t must fuse to p in $N(V)$. Also $p\sim pt$ in $C_G(t)$, hence $N(V)$ acts as S_3 on V. In particular, there exists an element β of order 3 which acts regularly on V and centralizes L. Without loss, we may assume that $z\in L$ and $t^\beta=p$. But then $t\sim p\sim pz=t^\beta z=(tz)^\beta$, which gives $t\sim tz$, a contradiction.

Lemma 3.3. *$K \not\cong \widehat{Suz}$.*

Proof. Assume not. As in Lemma 3.2, we shall obtain a contradiction to $F^*(G)$ simple by showing that t is isolated in $C_G(t)$. Now $C_G(t)=K$ or K^*. By a result of D. Wright [31], we may assume that $C_G(t)=K^*$. If $t^G \cap K \neq \{t\}$, then by Lemma 2.3 (iii), $t^G \cap \{z,zt\} \neq \phi$. By extremal conjugation, we may find $g\in G$ with $z_1{}^g=t$ and $C_S(z_1)^g \subseteq S$ for some $z_1\in\{z,zt\}$ and $S\in Syl_2(K^*)$ with $z_1\in S$. Let $\delta\in S$ with $\delta^2=t$ and $|C_G(\delta)|=2^{10}\cdot 3^2\cdot 5\cdot 7$. Such a δ exists by Lemma 3.3 (iv) and $C_G(\delta)/0_2(C_G(\delta))\cong PSL(3,4)$. Also we may assume that $z_1\in E(C_G(\delta))\times\langle t\rangle$, whereupon $|C_G(\langle z_1,\delta\rangle)|=2^{10}$. Now $C_G(\langle z_1,\delta\rangle)^g=C_G(\langle t,\delta^g\rangle)=C_{K^*}(\delta^g)$. Hence δ^g is an element of order 4 of K^* with $|C_{K^*}(\delta^g)|=2^{10}$. This however, is in direct contradiction with Lemma 2.3 (vi). Therefore $t^G \cap K^* \subseteq \{t\} \cup (K^*-K)$. Let $S\in Syl_2(K^*)$, $p\in S-\{t\}$ and $g\in G$ with $p^g=t$ and $C_S(p)^g\subseteq S$. Then $C_{K^*}(p)^g=C_G(\langle p,t\rangle)^g=C_G(\langle t,t^g\rangle)=C_{K^*}(t^g)$. By Lemma 2.3 (iv), we may assume that $t^g=p$. This forces g to normalize $L=E(C_G(\langle t,p\rangle))$. But $L\cong\hat{M}_{12}$ or $\widehat{HJ}$ with $Z(L)=\langle t\rangle$ and hence $t^g=t$ against the choice of t.

With the completion of the proof of Proposition 3.1, we are therefore in the situation where K is isomorphic to M_{12} or $\cdot 3$ and $|C_G(K)|_2=2$. Let $\mathcal{C}$ be the set of all chains C of quasisimple groups:

$$C\colon L_0, L_1, \cdots, L_n = F^*(G)$$

constructed in Corollary 1.4 where $[L_0]\in\mathcal{A}$. Since

$$K = L_0, L_1 = F^*(G)$$

is such a chain and $[K]\in\mathcal{A}$, $\mathcal{C}$ is non-empty. We know a great deal about the quasisimple subgroups L_i of the chain C. In particular by Theorems 1.9–1.12 and induction, $[L_0]\in\mathcal{A}$ implies that $[L_i]\in\mathcal{A}$, $0\leq i\leq n-1$. Moreover, since $[L_n]\notin\mathcal{A}$, we must then have L_{n-1} standard in G, hence $L_{n-1}\cong M_{12}$ or $\cdot 3$ and $|C_G(L_{n-1})|_2=2$ by Proposition 3.1. We have proved that the following holds.

Lemma 3.4. *Let C: $L_0, L_1, \cdots, L_n=F^*(G)$ be a chain of $\mathcal{C}$. Then L_{n-1} is standard in G, $L_{n-1}\cong M_{12}$ or $\cdot 3$ and $|C_G(L_{n-1})|_2=2$.*

Now choose $C\in\mathcal{C}$ so that C has maximal length $n+1$ and for this fixed chain let $\langle t\rangle\in \mathrm{Syl}_2(C_G(L_{n-1}))$. Then $C_G(t)/\langle t\rangle=\mathrm{Aut}_G(L_{n-1})$ by the Unbalanced group theorem.

Lemma 3.5. *Let δ be an element of order 3 of L_{n-1} chosen so that $C_{L_{n-1}}(\delta)\cong Z_3\times Aut(PSL(2,8))$ if $L_{n-1}\cong\cdot 3$ and $C_{L_{n-1}}(\delta)\cong Z_3\times A_4$ if $L_{n-1}\cong M_{12}$. Such elements of order 3 exist by results in ([5], [7]). Let $\Delta=C_G(\delta)$ and $\bar{\Delta}=\Delta/0(\Delta)$. Then the following holds:*

(i) *If $L_{n-1}\cong\cdot 3$, then $L(\bar{\Delta})$ is isomorphic to $PSL(2,8)$, $PSL(2,8)\times PSL(2,8)$, $G_2(3)$, $PSL(2,64)$, $PSU(3,8)$, or $PSL(3,8)$.*

(ii) *If $L_{n-1}\cong M_{12}$ and $\bar{\Delta}$ is non-solvable, then either $F^*(\bar{\Delta})$ is isomorphic to A_6, A_7, $PSL(2,8)$, $PSL(3,3)$ or $PSU(3,3)$, or else $\bar{\Delta}$ is an extension of E_{16} by a subgroup of $N_{A_8}(\langle(123)\rangle)$ containing S_5.*

Proof. If $L_{n-1}\cong\cdot 3$, then $C_\Delta(t)=\langle t\rangle\times C_{L_{n-1}}(\delta)$. Thus $C_{\bar{\Delta}}(\bar{t})\cong Z_2\times \mathrm{Aut}\ PSL(2,8)$. Then (i) holds by [17].

Now suppose that $L_{n-1}\cong M_{12}$. Then $C_\Delta(t)=(\langle t\rangle\times C_{L_{n-1}}(\delta))\langle y\rangle$ where $y^2\in\langle t\rangle$, $C_G(t)=(\langle t\rangle\times L_{n-1})\langle y\rangle$ and either $y=1$ or $C_G(t)/\langle t\rangle\cong \mathrm{Aut}(M_{12})$ and $C_{\bar{\Delta}}(\bar{t})/\langle\bar{t}\rangle\cong S_4$. Hence $C_{\bar{\Delta}}(\bar{t})/\langle\bar{t}\rangle\cong A_4$ or S_4.

Let $\bar{C}=C_{\bar{\Delta}}(\bar{t})$, $\bar{Q}=0_2(\bar{C})$ and $\bar{E}=[\bar{Q},\bar{r}]$ for some $\bar{r}\in\bar{C}$ of order 3. Suppose that $\bar{H}\trianglelefteq\bar{\Delta}$ with $|\bar{H}|$ even. Then $\bar{Q}\cap\bar{H}\neq\langle 1\rangle$. Suppose that $\bar{Q}\cap\bar{H}=\langle\bar{t}\rangle$. Then $\langle\bar{t}\rangle=C_{\bar{H}}(\bar{t})$ and $0(\bar{H})=\langle 1\rangle$ implies $\bar{H}=\langle\bar{t}\rangle$ and $\bar{\Delta}=\bar{C}$, contrary to the non-solvability of $\bar{\Delta}$. Thus $\bar{E}\subseteq\bar{H}$ whenever $\bar{H}\trianglelefteq\bar{\Delta}$ with $|\bar{H}|$ even. In particular, $Z(\bar{\Delta})=\langle 1\rangle$, whence $\bar{\Delta}_1=0^2(\bar{\Delta})$ is fusion-simple. Moreover $\bar{\Delta}$ does not contain disjoint normal subgroups of even order. Finally, as $\bar{Q}$ is self-centralizing in $\bar{\Delta}$, $\bar{\Delta}$ has sectional 2-rank at most 4 by [19, Theorem 2]. Thus by [14, Corollary C] and the above, one of the following holds:

(a) $\bar{L}=L(\bar{\Delta})$ is a simple group of sectional 2-rank at most 4 and $\bar{\Delta}$ is isomorphic to a subgroup of $\mathrm{Aut}(L(\bar{\Delta}))$.

(b) $\bar{\Delta}$ is 2-constrained, $0_2(\bar{\Delta}_1')\cong E_8$ or E_{16} and $\bar{\Delta}_1'/0_2(\bar{\Delta}_1')\cong A_5, A_6, A_7, Z_3\times A_5$ or $L_3(2)$

Suppose that $\bar{T}=0_2(\bar{\Delta})\neq\langle 1\rangle$. Then $\bar{E}\subseteq\bar{T}$ and $\langle\bar{T},\bar{t}\rangle$ satisfies condition (*) of [22]. Then by Theorem A of [22], $\langle\bar{T},\bar{t}\rangle=\bar{T}_1\langle\bar{t}\rangle$ with $\bar{T}_1$ isomorphic to one of the following groups:

(i) E_{16}

(ii) $Z_{2^m}\times Z_{2^m}$ for some $m\geq 1$.

(iii) a Sylow 2-subgroup of $PSL(3,4)$.

(iv) a Sylow 2-subgroup of $PSU(3,4)$.

Moreover $\bar{r}$ acts fixed-point freely on $\bar{T}_1$. Thus $\bar{T}_1\subseteq 0_2(\bar{\Delta}_1')$. Hence $\bar{T}_1\cong E_{16}$ and $\bar{\Delta}_1'/\bar{T}_1\cong A_5, A_6, A_7$ or $Z_3\times A_5$. As $\bar{t}$ acts freely on $\bar{T}_1$, $C_{\bar{\Delta}/\bar{T}_1}(\bar{t})\cong Z_6$ or $Z_2\times S_3$. Hence $\bar{\Delta}/\bar{T}_1\cong S_5$ or $N_{A_8}(\langle(123)\rangle)$, as claimed.

Thus we may assume (a) holds whence by [14, Main Theorem], $\bar{L}$ is isomorphic to one of the following groups:

I. $PSL(n,q)$, $2\leq n\leq 5$; $PSU(n,q)$, $3\leq n\leq 5$; $G_2(q)$, ${}^2D_4(q)$, $PSp(4,q)$ or $Re(q)$ for some odd q.

II. $PSL(2,8)$, $PSL(2,16)$, $PSL(3,4)$, $PSU(3,4)$ or $Sz(8)$.

III. A_7, A_8, A_9, A_{10} *or* A_{11}.

IV. M_{11}, M_{12}, M_{22}, M_{23}, J_1, HJ, J_3, M^c or LyS.

By inspection of the information tabulated in [4, Table 1], $\bar{L}$ is not of type IV. Trivially if $\bar{L}$ is of type III, then $\bar{L}\cong A_7$. Suppose $\bar{L}$ is of type II. If $\bar{t}\in\bar{L}$, then $\bar{t}$ is 2-central and $\bar{L}\cong L_2(8)$. If $\bar{t}\notin\bar{L}$, then $C_{\bar{L}}(\bar{t})$ is non-solvable or isomorphic to $U_3(2)$, a contradiction.

Finally suppose that $\bar{L}$ is of type *I*. Let $\bar{u}$ be a 2-central involution of $\bar{L}$ centralized by $\bar{t}$. If $\bar{L}\cong PSL(5,q)$ or $PSU(5,q)$, then $\bar{t}$ normalizes $\bar{H}\trianglelefteq C_{\bar{L}}(\bar{u})$ with $\bar{H}\cong SL(4,q)$ or $SU(4,q)$. This is impossible by [13, (2.7) and (2.8)]. Moreover by [13, (2.5), (2.7) and (2.8)], $\bar{L}\not\cong PSp(4,q)$, $PSL(4,q)$ or $PSU(4,q)$. By definition, if $\bar{L}$ is of Ree type, then $C_{\bar{L}}(\bar{t})\cong Z_2\times PSL(2,q)$. Hence $\bar{L}=Re(3)$ $\cong \mathrm{Aut}\,(PSL(2,8))$. Thus $\bar{L}\cong PSL(2,q)$, $PSL(3,q)$, $PSU(3,q)$, ${}^2D_4(q)$ or $G_2(q)$. If $\bar{L}\cong PSL(2,q)$, then $\bar{t}$ is of field-type and $q=9$. If $\bar{L}\not\cong PSL(2,q)$ then $\bar{t}$ normalizes a subgroup $\bar{H}$ of $C_{\bar{L}}(\bar{u})$ with $\bar{H}\cong SL(2,3)$. If $\bar{L}\cong{}^2D_4(3)$, then $\bar{t}$ normalizes $\bar{H}_1\cong SL(2,3^3)$, which is impossible. If $\bar{L}\cong G_2(3)$, then $N_{\langle\bar{L},\bar{t}\rangle}(\bar{H})=C_{\bar{L}}(\bar{u})$. Hence $\bar{t}\in\bar{L}$. But then $\bar{t}\in\bar{u}^{\bar{L}}$, a contradiction. Thus $\bar{L}\cong PSL(3,3)$ or $PSU(3,3)$, as claimed.

Lemma 3.6. *The following conditions hold*:

(i) $n\geq 2$

(ii) $L_{n-2}\cong A_5$ *if* $L_{n-1}\cong M_{12}$

(iii) $L_{n-2}\cong M_{12}$ *if* $L_{n-1}\cong \cdot 3$

Let $\langle x\rangle=C_{L_{n-1}}(L_{n-2})\cong Z_2$. *Then*

(iv) *Either* $\langle L_{n-2}{}^{L(C_G(x))}\rangle=L_{n-2}$ *or* $\langle L_{n-2}{}^{L(C_G(x))}\rangle\cong L_{n-1}$ *and is a standard component of* G.

Proof. Suppose $n\geq 2$. Then by Lemmas 2.1 and 2.4, L_{n-2} is a standard component of L_{n-1} with $L_{n-2}\cong A_5$ if $L_{n-1}\cong M_{12}$ and $L_{n-2}\cong M_{12}$ if $L_{n-1}\cong\cdot 3$. Also $\langle x\rangle=C_{L_{n-1}}(L_{n-2})\cong Z_2$. In any event, $C_G(\langle t,x\rangle)$ has a component isomorphic to A_5 or M_{12} which is not standard in G and thus by Corollary 1.4, is a link in some chain of $\mathcal{C}$ of length at least 3. Thus $n\geq 2$ and (i)-(iii) hold.

In order to prove (iv), assume that $L_{n-2}\not\trianglelefteq L(C_G(x))$. Then by L-balance, $\langle L_{n-2}{}^{L(C_G(x))}\rangle=K_0K_0{}^t$ where K_0 is a component of $C_G(x)$ and either $K_0=K_0{}^t$ or else $K_0\neq K_0{}^t$ and $K_0/Z(K_0)\cong L_{n-2}$. If $K_0=K_0{}^t$, then applying Lemma 1.6 with respect to $\langle t,x\rangle$, $\langle x\rangle$ and the components L_{n-2} of $C_G(\langle t,x\rangle)$ and K_0 of $C_G(x)$, there exists a chain connecting L_{n-2} and K_0 such that each link satisfies (1)-(5) of Corollary 1.4. By maximal choice of n and the fact the $L_0\neq\langle L_0{}^{C_G(x)}\rangle=K_0$, $\mathcal{C}^1$: $L_0,L_1,\cdots,L_{n-2},K_0,L_n$ is a chain in $\mathcal{C}$. Therefore, K_0 is a standard component of G and $K_0\cong L_{n-1}$ by Lemma 3.4.

It remains for us to eliminate the case where $K_0\neq K_0{}^t$ and $K_0/Z(K_0)\cong L_{n-2}$. As $[K_0]\in\mathcal{A}$, it follows from Corollary 1.4 that there is a chain $C^*\in\mathcal{C}$ given by C^*: $K_0,K_1,\cdots,K_m=F^*(G)$. Since K_0 commutes with $K_0{}^t$, K_0 is not a standard component of G, hence $m\geq 2$. Consider the chain

$$L_0, L_1, \cdots, L_{n-2}, K_0, K_1, \cdots, K_m = F^*(G)$$

As $m\geq 2$, $m+n-1>n$. Hence by choice of n, $K_i=L_j$ for some i, j, $0\leq i<m$, $0\leq j\leq n-2$. We shall rule out this possibility and thus prove Lemma 3.6.

Suppose first that $L_{n-1}\cong\cdot 3$, $L_{n-2}\cong M_{12}$. As $C_G(\langle t,L_{n-2}\rangle)=\langle t,x\rangle$, $C_G(L_{n-2})$ has Sylow 2-subgroups of maximal class. In particular, L_{n-2} is the only component of $N_G(L_{n-2})$ isomorphic to M_{12}. Thus any predecessor of L_{n-2} in a chain must be isomorphic to A_5. In particular, $L_i\cong A_5$ for $0\leq i<n-2$. As $|K_j|\geq|M_{12}|$ for all j, we must have $K_j=L_{n-2}$ for some $j\geq 1$. But then K_{j-1} is a predecessor of L_{n-2} with $K_{j-1}\cong M_{12}$, a contradiction.

Suppose next that $L_{n-1}\cong M_{12}$, $L_{n-2}\cong A_5$. Clearly, if $K_i=L_j$ for some i, j, then we may assume that L_{n-2} has a predecessor $L_{n-3}\cong A_5$. If S_{n-3} and T_{n-2} are as in (3) of Corollary 1.4, then $L_{n-2}\neq\langle L_{n-3}{}^{L(C_G(T_{n-2}))}\rangle$ whereas L_{n-3} is a component of $C_G(N_{S_{n-3}}(T_{n-2}))$. This implies that $L_{n-2}\times L_{n-2}^s\subseteq L(C_G(T_{n-2}))$ for some $s\in N_{S_{n-3}}(T_{n-2})-T_{n-2}$. Now let $Y=C_G(L_{n-2})$ and $\bar{Y}=Y/0(Y)$. As L_{n-2}^s is a component of $C_G(T_{n-2})$ and $T_{n-2}\times L_{n-2}^s\subseteq Y$, $L(\bar{Y})\neq\langle 1\rangle$ by Lemma 1.5. Furthermore, $C_Y(t)=\langle t,x,y\rangle$ where $C_G(t)=(\langle t\rangle\times L_{n-1})\langle y\rangle$, $y^2\in\langle t\rangle$ and either $y=1$ or $C_G(t)/\langle t\rangle\cong \mathrm{Aut}(M_{12})$. Using the notation of Lemma 3.5, we may assume that $\delta\in L_{n-2}$. Therefore $Y\subseteq\Delta$ and we conclude from Lemma 3.5 that $F^*(\bar{Y})$ is isomorphic to A_5, A_6, A_7, $PSL(2,8)$, $PSL(3,3)$ or $PSU(3,3)$. But L_{n-2}^s is a component of $C_Y(T_{n-2})$, hence $\Delta=0(\Delta)Y$ with $\bar{Y}\cong S_7$. This, however, is incompatible with $C_Y(t)=\langle t,x,y\rangle$.

For convenience, set $K=L_{n-1}$, $J=L_{n-2}$. By Lemma 3.6 (iv), if $L=\langle J^{L(C_G(x))}\rangle$, then either $L=J$ or $L\cong K$ and L is a standard component of G.

Lemma 3.7. $K\not\cong M_{12}$.

Proof. Suppose by way of a contradiction that $K\cong M_{12}$. There are two cases to consider, namely $L=J$ or $L\cong K$.

Assume first that $L\cong K$. Then $C_G(t)=\langle t\rangle\times K$. In fact, if $C_G(t)/\langle t\rangle\cong\mathrm{Aut}(M_{12})$, then $x\in C_G(\langle t,x\rangle)'$ whereas $x\notin L=C_G(x)'$ by Lemma 3.4. Also $x\not\sim t$, since otherwise $G\cong\cdot 3$ by Theorem 1.13 against the choice of G. Now let $\langle\delta\rangle\in\mathrm{Syl}_3(J)$ and $\Delta=C_G(\delta)$. Then $C_\Delta(x)\cong C_\Delta(t)=\langle t\rangle\times\langle\delta\rangle\times H$ where $H\cong A_4$. We can choose $\langle x,x_1\rangle=0_2(H)$ and set $T=\langle t,x,x_1\rangle$. Clearly $t\notin Z^*(\Delta)$ and $t^\Delta\cap T\subseteq t\langle x,x_1\rangle$. It then follows using the action of H on T, that $N_\Delta(T)$ has orbits $t\langle x,x_1\rangle$ and $\langle x,x_1\rangle^\#$ on $T^\#$. This yields $|C_\Delta(x)\cap N_\Delta(T)|_2=2^5$ contradicting $|C_\Delta(x)|_2=2^3$.

We are therefore, in the situation where $L=J$. For $y\in I(G)$, let $J^*(y)$ be the product of all components of $C_G(y)$ isomorphic to A_5; if none, set $J^*(y)=1$. Suppose $J^*(x)\neq J$. Then as $C_G(\langle t,x\rangle)/J$ is a 2-group of rank at most 3, we have $J^*(x)=J\times J_1$ and t acts as an inner automorphism on J_1. Since $C_G(\langle J,t\rangle)\supseteq C_{J_1}(t)\times\langle x\rangle\cong E_8$, $C_G(t)=\langle t\rangle\times K(v)$ with v an involution chosen so that $[v,J]=1$ and $K\langle v\rangle\cong\mathrm{Aut}(M_{12})$. Also $C_G(\langle t,x\rangle)=\langle t\rangle\times(\langle x,v\rangle\times J)\langle x_1\rangle$ where $x_1\sim x$ in K and $[x_1,v]=x$. Now $\langle t,x,v\rangle\subseteq\langle x,J_1\rangle$ and x_1 normalizes J_1. Hence, $[x_1,v]=x$ then gives $x\in\langle x,x_1,J_1\rangle'$ contradicting $\langle x,x_1,J_1\rangle/J_1$ is abelian. We have thus shown that $J^*(x)=J$. In particular $J\trianglelefteq C_G(x)$. Therefore if $y\in I(G)$ and $J(y)$ is the product of all normal subgroups of $C_G(y)$ isomorphic to A_5, otherwise $J(y)=1$, then $J(x)=J$.

Let $z\in I(J)$ so that $C_K(z)\cong(Q_8*Q_8)S_3$ (split) with an S_3 subgroup acting faithfully on two central factors. Suppose that $J_0\trianglelefteq C_G(z)$, $J_0\cong A_5$. Since $C_{J_0}(t)\trianglelefteq C_G(\langle t,z\rangle)$, t acts as an inner automorphism on J_0. Now $C_{J_0}(t)\cong E_4$ and $C_{J_0}(t)$ contains an involution central in some Sylow 2-subgroup of $C_G(\langle t,z\rangle)$ implies that $C_{J_0}(t)\cap\langle t,z\rangle\neq 1$. At any rate, a Sylow 3-subgroup of $C_K(z)$ centralizes $C_{J_0}(t)$, hence $C_{J_0}(t)=\langle t,z\rangle$, a contradiction. So $J(z)=1$.

Let $\mathcal{W}=\{\langle x,z_1\rangle\mid z_1\in I(J)\}$ and if $W\in\mathcal{W}$, set $J(W)=\langle J(w)\mid w\in W^\#\rangle$. It follows from $J(x)=J$, $J(z)=1$ and the subgroup structure of K that $K=J(W)$ for each $W\in\mathcal{W}$. Thus $N_G(W)\subseteq N_G(J(W))=N_G(K)$ for each $W\in\mathcal{W}$. As $J\trianglelefteq C_G(x)$, $C_G(x)$ permutes the elements of $\mathcal{W}$. Therefore $C_G(x)\subseteq N_G(K)$ and by the Unbalanced Group Theorem, $C_G(x)\subseteq C_G(t)$.

Now $t^G\cap C_G(t)\neq\{t\}$ by the Z^*-Theorem. So $t^G\cap C_G(x)\neq\{t\}$ by Lemma 2.1. Let $w\in t^G\cap C_G(x)$ with $w\neq t$. Then $|C_G(x)|_2<|C_G(w)|_2$ implies that x induces a non-2-central involution on $L(C_G(w))$. By Lemma 2.1, $J=L(C_G(\langle w,x\rangle))$, hence $w\in C_G(\langle t,J\rangle)$. Since $w\neq t$, $w\sim wz$ in $N_G(K)$. But then $t\sim wz$ and repeating the argument with wz in place of w, we have $J\subseteq C_G(wz)$. This gives $J\subseteq C_G(\langle w,wz\rangle)\subseteq C_G(z)$ and provides us with the final contradiction.

Lemma 3.8. $K\not\cong\cdot 3$.

Proof. Suppose not. Again, there are two cases to consider, namely $L=J$ or $L\cong K$. The elimination of both cases is similar to but less complicated that in the proof of Lemma 3.7.

Let δ be an element of order 3 of J with $C_J(\delta)\cong Z_3\times A_4$. Then $C_K(\delta)\cong Z_3 \times \mathrm{Aut}(PSL(2,8))$ with $I(C_K(\delta))\subseteq x^K$. Let $\Delta=C_G(\delta)$ and $\bar{\Delta}=\Delta/0(\Delta)$. Since $C_{\bar{\Delta}}(\bar{t})=\overline{C_\Delta(t)}\cong Z_2\times \mathrm{Aut}(PSL(2,8))$, we have from Lemma 3.5 that $L(\bar{\Delta})$ is isomorphic to $PSL(2,8)$, $PSL(2,8)\times PSL(2,8)$, $G_2(3)$, $PSL(2,64)$, $PSU(3,8)$ or $PSL(3,8)$. Since $\bar{x}\in L(C_{\bar{\Delta}}(\bar{t}))\subseteq L(\bar{\Delta})$, $C_{\bar{\Delta}}(\bar{x})$ is solvable. An immediate consequence is that $L\not\cong K$. Otherwise, $C_\Delta(x)$ contains a subgroup isomorphic to $PSL(2,8)$.

Therefore, we have $L=J$. For $y\in I(G)$, let $J(y)$ be the product of all normal subgroups of $C_G(y)$ isomorphic to M_{12}, otherwise $J(y)=1$. Since $\delta\in J$ and $C_\Delta(x)$ is solvable, it follows from the structure of Δ that J is the unique component of $C_G(x)$ isomorphic to M_{12}. Thus $J(x)=J$. Let $\mathcal{W}$ be the set of all four subgroups W of $\langle x,J\rangle$ with $|C_{\langle x,J\rangle}(W)|_2=|\langle x,J\rangle|_2$. If $W=\langle x,w\rangle$ with $w\in J$, then $wx\sim w$ and $C_K(w)\cong \widehat{Sp(6,2)}$. Since $C_K(w)$ centralizes $J(w)$, x centralizes $J(w)$ and so $[J(w),J]=1$. But then $[\delta,J(w)]=1$ and the structure of Δ gives $J(w)=1$. As $C_G(x)$ is maximal in K, $K=\langle J(w)\,|\,w\in W^\#\rangle$ for all $W\in\mathcal{W}$ and since $C_G(x)$ permutes the members of $\mathcal{W}$, we conclude that $C_G(x)\subseteq N_G(K)$. Again, by the Unbalanced Group Theorem, this yields $C_G(x)\subseteq C_G(t)$.

Now $t\notin Z(G)$, hence by the Z^*-Theorem and inspection, there exists $t_1\in t^G \cap C_G(x)$ with $t_1\neq t$. Let $K_1=C_G(t_1)'$. Then x acts as a non-2-central involution on K_1 yields $J=L(C_{K_1}(x))$. Therefore $t_1\in C_G(\langle x,J\rangle)=\langle t,x\rangle$. We have shown that $\{t,tx\}=t^G\cap C_G(x)$. But if w is a 2-central involution of K centralizing x, then $x\sim xw$ in K. So, $t\sim tx\sim txw$ whereupon $\langle t,txw\rangle$ centralizes J. In particular xw centralizes J, a contradiction.

Lemma 3.6 completes the proof of Theorem 1.12.

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