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ON THE INTERMEDIATE COHOMOLOGY GROUP OF A HOLOMORPHIC LINE BUNDLE OVER A COMPLEX TORUS

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Let $E=V/L$ be a complex torus, where V is an n-dimensional complex vector space and L a lattice of V. Let H be a Hermitian form on V and A the imaginary part of H. Then A is an $\boldsymbol{R}$-bilinear alternating form on V. We say that H is *a Riemann form of signature* (s, r) for the torus E if

(a) H is non-degenerate and of signature (s, r);
(b) A is integral valued on the lattice L.

To a Riemann form H we associate a factor

$$J_{H,\psi}(g, z) = \psi(g)\varepsilon\left[\frac{1}{2i}H(z, g)+\frac{1}{4i}H(g, g)\right] \tag{1}$$

with $g \in L$, $z\in V$, where $\varepsilon[\cdot]=\exp 2\pi i\cdot$ and ψ is a map from L to $\boldsymbol{C}_1^*=\{z\in\boldsymbol{C}\,|\,|z|=1\}$ satisfying $\psi(g+h)=\psi(g)\psi(h)\varepsilon\left[\frac{1}{2}A(g, h)\right]$; the function ψ being called a semi-character of L for A.
The factor $J_{H,\psi}$: $L\times V\rightarrow\boldsymbol{C}^*$ satisfies the equation

$$J_{H,\psi}(g+f, z) = J_{H,\psi}(g, h+z)J_{H,\psi}(h, z)\,,$$

where $g, h\in L$, $z\in V$.

Given the factor $J_{H,\psi}$ we define an action of the lattice group L on $V\times\boldsymbol{C}$ by the rule

$$(z, \xi)\cdot g = (z+g, J_{H,\psi}(g, z)\xi)\,,$$

where $z\in V$, $\xi\in\boldsymbol{C}$ and $g\in L$. The action of L on $V\times\boldsymbol{C}$ is free and the quotient of $V\times\boldsymbol{C}$ by L has a natural structure of a holomorphic line bundle over the complex torus $E=V/L$. We shall denote this line bundle by $F(H, \psi)$.

The following vanishing theorem for the cohomology of $F(H, \psi)$ is well-known [2, 4]: If H is a Riemann form of signature (s, r), then we have

$$H^q(E, F(H, \psi)) = 0$$

for $q \neq r$ (for a proof see Appendix 2 of this paper).

In particular if $r=0$, namely if H is positive definite, $H^q(E, F(H, \psi))=0$ except for $q=0$ and $H^0(E, F(H, \psi))$ is identified with the space of all holomorphic theta functions on V for the factor $J_{H,\psi}$. Replacing H by a suitable positive interger multiple of H, if necessary, these theta functions define a holomorphic imbedding of E into a complex projective space. A complex torus which admits a positive definite Riemann form is called an abelian variety. There exist complex tori which are not abelian varieties but which admit Riemann forms of signature (s, r) with $s>0$ and $r>0$ (see Appendix 1). For such a complex torus E, there exists no non-trivial theta function. However, there exists the non-trivial intermediate conomology group $H^r(E, F(H, \psi))$ with $0<r<n$.

The purpose of this paper is to give an interpretation of the intermediate cohomology group $H^r(E, F(H, \psi))$. Namely we associate to a Riemann form H of signature (s, r) a family of polarized abelian varieties (E_b, H_b) parametrized by elements b of the Hermitian symmetric space $\mathfrak{B}=U(H)/K$, where $U(H)$ is the unitary group of the Hermitian form H and K is a maximal compact subgroup of $U(H)$. Here E_b is of the form $E_b=V_b/L$, where V_b is an n-dimensional complex vector space with the same underlying real vector space as V and with a complex structure J_b distinct from that of V and parametrized by $b \in \mathfrak{B}$ and H_b is a positive definite Riemann form for E_b whose imaginary part is equal to A. We then assign a family of line bundles $F(H_b, \psi)$ over E_b for each b. Finally we shall show that there exists a canonically defined isomorphism from $H^r(E, F(H, \psi))$ to $H^0(E_b, F_b, \psi))$ for each b. We also see that there exists a family Φ_b of differentiable imbedding of E into a complex projective space which is partially holomorphic and partially antiholomorphic.

It should be mentioned that C.L. Siegel [3] has associated to an indefinite quadratic form a family of theta series parametrized by a symmetric space. It is possible to interpret Siegel's family of theta series as a subfamily of the family $\{H^0(E_b, F(H_b, \psi))\}$ of theta functions attached to a certain complex torus E and a Riemann form H related with the given indefinite quadratic form.

1. A Riemann form of signature (s, r) and a family of polarized abelian varieties

Let $E=V/L$ be a complex tours, where V is an n-dimensional complex vector space and L a lattice of V. We shall denote by W the underlying $2n$-dimensional real vector space of V and by J the complex structure of W defining the complex vector space V.

Let H be a non-degenerate Hermitian form on V of signature (s, r), where $s+r=n$. We denote by A the imaginary part of H. Then we have

$$H(u, v) = A(Ju, v)+iA(u, v), \ u, v \in W.$$

We assume that the alternating $\boldsymbol{R}$-bilinear form A to be integral valued on $L\times L$ and we call H a Riemann form of signature (s, r) for the complex tours E.

We shall denote by $U(H)$ the unitary group of the Hermitian form H. A basis $B=\{v_1, \cdots, v_n\}$ of V is said to be a *privileged basis* for H if the matrix of H with respect to the basis B is of the form

$$1_{s,r} = \begin{pmatrix} 1_s & 0 \\ 0 & -1_r \end{pmatrix}$$

where 1_s and 1_r denote the unit matrix of size s and r respectively.

The group $U(H)$ acts simply transitively on the set of all privileged bases for H. We denote by $V_1(B)$ and $V_2(B)$ the subspaces of V spanned by $\{v_1, \cdots, v_s\}$ and $\{v_{s+1}, \cdots, v_n\}$ respectively. Then we have

$$W = V_1(B) \oplus V_2(B)\,. \tag{2}$$

We say that two privileged bases B and B' are equivalent, $B\sim B'$, if $V_i(B)=V_i(B')$ for $i=1, 2$. We shall denote by $\mathfrak{B}$ the set of equivalence classes of priveleged bases for H. Then the group $U(H)$ acts transitively on $\mathfrak{B}$ and $\mathfrak{B}$ is identified with the Hermitian symmetric space $U(H)/K$, where K is a maximal compact subgroup of $U(H)$.

Let $b\in\mathfrak{B}$ and let B be a privileged basis representing b. We define a linear transformation J_b of W by requiring

$$J_b = J \quad \text{on} \quad V_1(B)$$

and

$$J_b = -J \quad \text{on} \quad V_2(B)\,.$$

We have $J_b^2=-1$ and hence J_b defines a complex structure on W. We shall denote by V_b the complex vector space defined by W and J_b.
Define the symbol $\varepsilon_k (k=1, 2, \cdots, n)$ by

$$\varepsilon_k = \begin{cases} 1, & k\in[1, s]\,, \\ -1, & k\in[s+1, n]\,. \end{cases}$$

If $B=\{v_1, \cdots, v_n\}$ is a privileged basis representing b, then we have

$$J_b v_k = \varepsilon_k J v_k$$

We also have $H(v_k, v_j)=\varepsilon_k\cdot\delta_{kj}$ and since $H(v_k, v_j)=A(Jv_k, v_j)+iA(v_k, v_j)$, we get $A(v_k, v_j)=0$ and $A(Jv_k, v_j)=\varepsilon_k\cdot\delta_{kj}$. It follows from these that the decomposition (2) is orthogonal for A and also for H. We have also $A(J_b u, J_b v)=A(u, v)$ for $u, v\in W$. For let $u=u_1+u_2$, $v=v_1+v_2$ with $u_1, v_1\in V_1(B)$ and $u_2, v_2\in V_2(B)$. Then $Ju_b=Ju_1-Ju_2$ and $J_b v=Jv_1-Jv_2$ and $Ju_1, Jv_1\in V_1(B)$ and $Ju_2, Jv_2\in V_2(B)$. Hence $A(J_b u, J_b v)=A(Ju_1, Jv_1)+A(Ju_2, Jv_2)=A(u_1, v_1)+A(u_2, v_2)$. We can then define a Hermitian form H_b on the complex vector space V_b by

$$H_b(u, v) = A(J_b u, v) + iA(u, v) .$$

Then the imaginary part of H_b is A and we have $H_b(v_k, v_j) = A(J_b v_k, v_j) + iA(v_k, v_j) = \varepsilon_k A(Jv_k, v_j) = \varepsilon_k^2 \delta_{kj} = \delta_{kj}$. This means that B is an orthonormal basis of V_b for the Hermitian form H_b and in particular H_b is positive definite and the decomposition (2) of W is also orthogonal for H_b.

Let now

$$E_b = V_b/L .$$

Then H_b is a positive definite Riemann form for E_b and hence E_b is an abelian variety. Thus we have associated to a complex torus E and a Riemann form H of signature (s, r) a family of polarized abelian varieties (E_b, H_b) parametrized by $b \in \mathfrak{B} = U(H)/K$.

We need the following lemma in the next section.

Lemma. *We have*

$$H(u, v) = H_b(u, v) \quad \textit{for} \quad u \in V_1(B)$$

and

$$H(u, v) = -H_b(v, u) \quad \textit{for} \quad u \in V_2(B) .$$

For we have $H_b(u, v) = A(J_b u, v) + iA(u, v)$ and $J_b u = Ju$ or $J_b u = -Ju$ according as $u \in V_1(B)$ or $u \in V_2(B)$.

2. The cohomology group $H^r(E, F(H, \psi))$

We associate to the Riemann form H of signature (s, r) for E the factor $J_{H,\psi}$ defined by (1) and the line bundle $F(H, \psi)$ over E. For the cohomology groups of $F(H, \psi)$ we have the following theorem.

Theorem 1. (i) *We have $H^q(E, F(H, \psi)) = 0$ for $q \neq r$.*

(ii) *Let $(z_1, \cdots, z_n)$ be the coordinates of the complex vector space V determined by a privileged basis B of V for the Hermitian form H. Then $H^r(E, F(H, \psi))$ is identified with the complex vector space of all C^∞ functions f on V satisfying the following conditions*:

1) *f is a differentiable theta function for the factor $J_{H,\psi}$; namely we have*

$$f(z+g) = J_{H,\psi}(z, g) \cdot f(z), \quad z \in V, \ g \in L .$$

2) $\dfrac{\partial f}{\partial \bar{z}_k} = 0$ *for* $k \in [1, s]$

and

$$\frac{\partial f}{\partial z_{s+j}} + \pi \bar{z}_{s+j} \cdot f = 0 \quad \textit{for} \quad j \in [1, r] .$$

The assertion (i) in Theorem 1 is a well-known vanishing theorem due to Mumford. We shall give a proof of Theorem 1 in the Appendix 2 based on the harmonic theory.

We denote by $H(B)$ the space of C^∞ functions f on W satisfying the above conditions (1) and (2) to make explicit its dependence of the condition (2) on the choice of the privileged basis B. We show that if B and B' are equivalent, then we have $H(B)=H(B')$. In fact, let $(z_1', \cdots, z_n')$ be the coordinates of V determined by B'. Then we have

$$z_i' = \sum_{j=1}^{s} a_{ij} z_j \qquad (i=1, \cdots, s)$$

and

$$z_{s+i}' = \sum_{j=1}^{r} b_{ij} z_{s+j} \qquad (i=1, \cdots, r)$$

where the matrices (a_{ij}) and (b_{ij}) are both unitary. We get

$$(*) \qquad \frac{\partial f}{\partial \bar{z}_k} = \sum_{i=1}^{s} \bar{a}_{ik} \frac{\partial f}{\partial \bar{z}_i'} \qquad (k=1, \cdots, s)$$

and

$$\frac{\partial f}{\partial z_{s+k}} + \pi \bar{z}_{s+k} f = \sum_{i=1}^{r} b_{ik} \frac{\partial f}{\partial z_{s+i}'} + \pi \Big(\sum_{i=1}^{r} \bar{b}_{ki}' \bar{z}_{s+i}' \Big) f,$$

where (b_{ki}') is the inverse matrix of (b_{ki}). Since (b_{ki}) is unitary, we have $(b_{ki}')={}^t(\bar{b}_{ki})$ and hence $\bar{b}_{ki}'=b_{ik}$ Hence we get

$$(**) \qquad \frac{\partial f}{\partial z_{s+k}} + \pi \bar{z}_{s+k} f = \sum_{i=1}^{r} b_{ik} \Big(\frac{\partial f}{\partial z_{s+i}'} + \pi \bar{z}_{s+i}' f \Big)$$

From (*) and (**) we get $H(B)=H(B')$. Hence we can denote the space of C^∞ functions f satisfying (1) and (2) by $H(b)$, $b \in \mathfrak{B}$.

Consider now the family of polarized abelian varieties (E_b, H_b) $(b \in \mathcal{B})$ defined in §1. We have the factor $J_{H_b,\psi}\colon L \times V_b \to \boldsymbol{C}^*$ defined by

$$J_{H_b,\psi}(g, u) = \psi(g) \varepsilon \Big[\frac{1}{2i} H_b(u, g) + \frac{1}{4i} H_b(g, g) \Big]$$

where $g \in L$ and $u \in V_b$; this is because the imaginary part of H_b is equal to A for any b. Let $F(H_b, \psi)$ be the line bundle over E_b associated with the factor $J_{H_b,\psi}$. Since H_b is positive definite, we have $H^q(E_b, F(H_b, \psi))=0$ for $q \neq 0$ and $H^0(E_b, F(H_b, \psi))$ is identified with the complex vector space of all holomorphic theta functions on V_b for the factor $J_{H_b,\psi}$.

Let $p_i\colon W \to V_i(B)$ $(i=1, 2)$ be the projection of W onto $V_i(B)$ with respect to the decomposition (2) of W and let

$$\phi_b(u) = \exp\,[-\pi H_b(p_2(u), p_2(u))]\,.$$

We have

$$\phi_b(u+g) = \phi_b(u) \exp L(u, g), \quad u \in W, \quad g \in L\,,$$

where

$$L(u, g) = -\pi[H_b(p_2(u), p_2(g)) + H_b(p_2(g), p_2(u)) + H_b(p_2(g), p_2(g))]\,.$$

Let θ be a holomorphic theta function on V_b for the factor $J_{H_b,\psi}$ and let

$$f = \phi_b \cdot \theta$$

We show that the function f satisfies the conditions (1) and (2) in Theorem 1, *i.e.* $f \in H(b)$. We have

$$f(u+g) = f(u)\psi(g) \exp\,[L(u, g) + \pi H_b(u, g) + \frac{\pi}{2} H_b(g, g)]\,.$$

Since the decomposition (2) is orthogonal for H_b we get

$$\pi H_b(u, g) + \frac{\pi}{2} H_b(g, g) = \pi H_b(p_1(u), p_1(g)) + \pi H_b(p_2(u), p_2(g)) + \frac{\pi}{2} H_b(p_1(g), p_1(g))$$

$$+ \frac{\pi}{2} H_b(p_2(g), p_2(g)) \quad \text{and hence} \quad L(u, g) + \pi H_b(u, g) + \frac{\pi}{2} H_b(g, g)$$

$$= \pi[H_b(p_1(u), p_1(g)) - H_b(p_2(g), p_2(u))] + \frac{\pi}{2} [H_b(p_1(g), p_1(g)) - H_b(p_2(g), p_2(g))]\,.$$

From Lemma at the end of §1 and from the orthogonality of the decomposition (2) for H we see that the left hand side of the above equality is equal to $\pi H(u, g) + \frac{\pi}{2} H(g, g)$. Hence we get $f(u+g) = f(u) \cdot \psi(g) \exp\,[\pi H(u, g) + \frac{\pi}{2} H(g, g)] = f(u) J_{H,\psi}(g, u)$ which shows that f is a differentiable theta function for the factor $J_{H,\psi}$.

Now let B be any privileged basis representing b and let $(z_1, \cdots, z_n)$ be the coordinates of V determined by B. Then B is also an orthonormal basis of V_b for the Hermitian form H_b and let $(w_1, \cdots, w_n)$ be the coordinates of V_b determined by B. Then as functions on W we have

$$z_i = w_i \quad \text{for} \quad i \in [1, s]\,,$$
$$\bar{z}_{s+i} = w_{s+i} \quad \text{for} \quad i \in [1, r]\,.$$

Since θ is a holomorphic function on V_b we have

$$\frac{\partial \theta}{\partial \overline{w}_k} = 0\,, \quad \text{for} \quad k \in [1, n]$$

and hence

$$\frac{\partial\theta}{\partial\bar{z}_i}=0\,,\quad i\in[1,s];\ \frac{\partial\theta}{\partial z_{s+i}}=0\,,\quad i\in[1,r]\,,$$

If $u=\sum_{k=1}^{n}z_kv_k$, then $p_2(u)=\sum_{i=1}^{r}z_{s+i}v_{s+i}$ and hence $H_b(p_2(u),p_2(u))=\sum_{i=1}^{r}|z_{s+i}|^2$ and so

$$\phi_b=\exp\,[-\pi\sum_{i=1}^{r}|z_{s+i}|^2]\,.$$

We see easily that we have $\frac{\partial f}{\partial\bar{z}_i}=0$ for $i\in[1,s]$ and $\frac{\partial f}{\partial\bar{z}_{s+i}}+\pi\bar{z}_{s+i}f=0$ and hence f belongs to $H(b)$. Analogously we can see that if f is a function belonging to $H(b)$, then the function θ defined by $\theta(u)=f(u)\cdot\phi_b(u)^{-1}$ is a holomorphic theta function on V_b for the factor $J_{H_b\,\psi}$ and moreover the map $f\to\theta$ defines a bijection of $H(b)$ onto the space $H^0(E_b,F(H_b,\psi))$ of holomorphic theta functions on V_b for the factor $J_{H_b,\psi}$. Since $H(b)$ is canonically isomorphic to $H^r(E,F(H,\psi))$ by Theorem 1, we obtain the following theorem.

Theorem 2. *Let H be a Riemann form of signature (s,r) for a complex torus E and let $F(H,\psi)$ be the holomorphic line bundle over E associated with the factor $J_{H,\psi}$ defined by* (1). *Let (E_b,H_b) and $(F(H_b,\psi))$ be the family of polarized abelian varieties and the family of line bundles over each E_b parametrized by $b\in\mathfrak{B}$. Then there exists a canonical isomorphism of $H^r(E,F(H,\psi))$ onto $H^0(E_b,F(H_b,\psi))$.*

In particular, we have

$$\dim H^r(E,\ F(H,\ \psi))=\dim H^0(E_o,\ F(H_b,\ \psi))$$

and since the imaginary part of H_b is equal to the imaginary part A of H, we have $\dim H^0(E_b,F(H_b,\psi))=e_1,\cdots,e_n$, where $e_1,\cdots,e_n$ are the elementary divisors of the integral valued alternating form A on $L\times L$. Thus we get also

$$\dim H^r(E,\ F(H,\ \psi))=e_1\cdots e_n\,.$$

Let $N+1=\dim H^r(E,F(H,\psi)$ and let $(f_0,f_1,\cdots,f_N)$ be a basis of the complex vector space $H(b)$ which is canonically isomorphic to $H^r(E,F(H,\psi)$. The map $u\to[f_0(u):\cdots:f_N(u)]$ defines a differentiable map $\tilde{\Phi}$ from W into the complex projective space P^N. Since each f_i is a differentiable theta function on W for the factor $J_{H,\psi}$, the map $\tilde{\Phi}$ defines a map Φ from $E=V/L$ into P^N.

Let $\theta_i=\phi_b\cdot f_i$ for $i\in[0,N]$. Then we have:

$$[\theta_0(u):\,\theta_1(u):\,\cdots\,:\theta_N(u)]=[f_0(u):f_1(u):\,\cdots\,:f_N(u)]\,.$$

It follows from this that Φ defines a holomorphic map from $E_b=V_b/L$ to P^N. We may assume without loss of generality that Φ is a holomorphic imbedding (this can be achieved by replacing H by $3H$ and ψ by ψ^3). Then Φ defines a differentiable imbedding of E into P^N. Let $(z_1, \cdots, z_n)$ be the coordinates on V corresponding to a privileged basis of V for H. Then these coordinates define local coordinates of the complex torus E at each point of E. Since Φ is holomorphic as a map from E_b into P^N, we see that Φ is holomorphic in $z_1, \cdots, z_s$ and anti-holomorphic in $z_{s+1}, \cdots, z_n$. Thus we get the following theorem.

Theorem 3. *Let H be a Riemann form of signature (s, r) for a complex torus E. Then the cohomology group $H^r(E, F(3H, \psi^3))$ of the holomorphic line bundle $F(3H, \psi^3)$ defines a differentiable imbedding of E into the complex projective space P^N with $N+1=\dim H^r(E, F(3H, \psi^3))$ which is holomorphic in $z_1, \cdots, z_s$ and anti-holomorphic in $z_{s+1}, \cdots, z_n$, where $(z_1, \cdots, z_n)$ are the coordinates of the complex vector space V determined by a privileged basis for H.*

Appendix 1. We give here an example of a complex 2-torus which is not an abelian variety and which admits a Riemann form or signature (1,1). Let

$$\omega_1=\begin{pmatrix}1\\0\end{pmatrix},\quad \omega_2=\begin{pmatrix}0\\1\end{pmatrix},\quad \omega_3=\begin{pmatrix}i\sqrt{2}\\ i\sqrt{3}\end{pmatrix},\quad \omega_4=\begin{pmatrix}i\sqrt{3}\\ -i\sqrt{5}\end{pmatrix}.$$

These vectors are linearly independent over $\boldsymbol{R}$ and they generate a lattice L of $\boldsymbol{C}^2$.

The matrix J_0 of the complex structure of $\boldsymbol{C}^2$ with respect to the basis $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ of $\boldsymbol{C}^2$ over $\boldsymbol{R}$ is of the form

$$J_0=\begin{pmatrix}0 & J_1\\ J_2 & 0\end{pmatrix}$$

where

$$J_1=\begin{pmatrix}-\sqrt{2} & -\sqrt{3}\\ -\sqrt{3} & \sqrt{5}\end{pmatrix}$$

and

$$J_2=\frac{1}{d}\begin{pmatrix}-\sqrt{5} & -\sqrt{3}\\ -\sqrt{3} & \sqrt{2}\end{pmatrix},\quad d=-\sqrt{10}-3\,.$$

Let A be an alternating $\boldsymbol{R}$-bilinear form on $\boldsymbol{C}^2\times\boldsymbol{C}^2$ which is integral valued on $L\times L$ and let A_0 be the matrix of A with respect to the basis $\{\omega_i\}$ and write

$$A_0=\begin{pmatrix}P_1 & P_2\\ -{}^tP_2 & P_3\end{pmatrix}\quad P_1=\begin{pmatrix}0 & p\\ -p & 0\end{pmatrix}\quad P_3=\begin{pmatrix}0 & q\\ -q & 0\end{pmatrix},$$

where p and q are integers and P_2 is an integral 2×2 matrix.

The alternating form A is the imaginary part of a Riemann form if and only if the $\boldsymbol{R}$-bilinear form S on $\boldsymbol{C}^2 \times \boldsymbol{C}^2$ defined by $S(u, v) = A(iu, v)$ is symmetric and non-degenerate. Let S_0 be the matrix of S with respect to $\{\omega_i\}$. Then we have $S_0 = {}^tJ_0 \cdot A_0$. We see easily that the condition that S_0 is symmetric is equivalent to the set of the following three conditions: (a) $P_1 J_1 = -{}^tJ_2 P_3$; (b) $P_2 J_2$ is symmetric; (c) ${}^tP_2 J_1$ is symmetric. The conditions (b) and (c) are both equivalent to the single condition that P_2 is to be of the form

$$P_2 = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \quad a \in \boldsymbol{Z}.$$

The condition (a) is equivalent to the condition $pd = q$, where $d = -\sqrt{10} - 3$ and p and q are integers and so (a) is equivalent to the condition $p = q = 0$.

Thus we reached the conclusion that S_0 is symmetric if and only if A_0 is the form

$$(*) \qquad A_0 = \begin{pmatrix} 0 & a1_2 \\ -a1_2 & 0 \end{pmatrix}$$

where $a \neq 0$ is an integer and 1_2 denote the 2×2 unit matrix. Then S_0 takes the form

$$S_0 = \begin{pmatrix} -aJ_2 & 0 \\ 0 & aJ_1 \end{pmatrix}$$

and S_0 is a non-singular matrix. However S_0 is not definite because the symmetric matrix J_1 is not definite. Thus A can be the imaginary part of a Riemann form if and only if A_0 is of the form (*) and when this is the case, the corresponding Riemann form is not definite but of signature (1,1). Hence $E = \boldsymbol{C}^2/L$ provides an example of a complex torus which is not an abelian variety and which admits a Riemann form of signature (1,1).

Appendix 2. Since the second assertion in Theorem 1 is an essential part of this article we give a proof of Theorem 1 in this appendix.

Let H be a Riemann form of signature (s, r) for a complex torus $E = V/L$ and $J_{H,\psi}$ the factor defined by (1) and $F(H, \psi)$ the holomorphic line bundle associated with $J_{H,\psi}$. Let D^q be the vector space of all $F(H, \psi)$-valued differential forms of type $(0, q)$ on E. Then the cohomology group $H(E, F(H, \psi))$ of E with coefficient in the sheaf of germs of holomorphic sections of $F(H, \psi)$ is isomorphic to the cohomology group of the complex $D = \sum_{q=0}^{n} D^q$, where the coboundary operator is given by d''(or $\bar{\partial}$). On the other hand, there exists a canonical identification of an $F(H, \psi)$-valued $(0, q)$-form α on E with a $(0, q)$-form φ on V (of class $\boldsymbol{C}^\infty$) satisfying the condition that

$$(*) \qquad T_g^* \varphi = J_{H,\psi}(g, \cdot)\varphi$$

for $g \in L$, where T_g denotes the translation of V by g. Then $d''\varphi$ also satisfies the same condition (*) and $d''\alpha$ is identified with $d''\varphi$. Denote by A^q the vector space of all $(0, q)$-form on V (of class C^∞) satisfying the condition (*). Then the cohomology group $H(E, F(H, \psi))$ is isomorphic to the cohomology group of the complex $A = \sum_{q=0}^{n} A^q$, where the coboundary operator is given by d''. Notice that A^0 is the vector space of all differentiable theta functions on V. Let $(z_1, \cdots, z_n)$ be coordinates on V. Then a $(0, q)$-form is expressed uniquely in the form

$$\varphi = \frac{1}{q!} \sum_J \varphi_J d\bar{z}_J ,$$

where $J = (j_1, \cdots, j_q)$ is a multi-index and each φ_J is alternating in the indices and $d\bar{z}_J = d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$. Since $d\bar{z}_J$ is invariant by translation, φ satisfies the condition (*) if and only if each component φ_J belongs to A^0.

Lemma 1. *Let $f, g \in A^0$ and let $\langle f, g \rangle$ be defined by*

$$\langle f, g \rangle(u) = f(u)g(u) \exp[-\pi H(u, u)]$$

for $u \in V$. Then the function $\langle f, g \rangle$ is invariant by the translation T_g for any $g \in L$.

We can verify the lemma by a straightforward computation.

We may consider $\langle f, g \rangle$ as function on the torus $E = V/L$.

Corollary of Lemma 1. *If $f, g \in A^0$, then*

$$|f(u)|\,|g(u)| < C \exp \pi H(u, u)$$

for any $u \in V$, where C is a positive constant.

Let us choose a *positive definite* Hermitian form G and let

$$G = \sum_{i,j} g_{ij} z_i \bar{z}_j .$$

Let

$$dV = \left(\frac{i}{2}\right)^n \det(g_{ij}) dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$$

the volume element on V determined by G. The volume element dV is invariant by translation and so it defines a translation invariant volume element dv on E such that $\pi^* dv = dV$, where $\pi: V \to E$ is the canonical projection. We define the inner product (f, g), where $f, g \in A^0$, by

$$(f, g) = \int_P \langle f, g\rangle dV\,,$$

where P is a fundamental paralleltop for the lattice L, or equivalently by

$$(f, g) = \int_E \langle f, g\rangle dv$$

regarding $\langle f, g\rangle$ as function on E.

Let us write

$$H = \sum_{i,j} h_{ij} z_i \bar{z}_j$$

and introduce covariant derivations D_i', D_i'' $(i=1, \cdots, n)$ by the formula

$$(D_i' f)(z) = \frac{\partial f}{\partial z_i}(z) - \pi(\sum_k h_{ik}\bar{z}_k) f(z)\,,$$
$$(D_i'' f)(z) = \frac{\partial f}{\partial \bar{z}_i}(z)\,.$$

We can show without difficulty that if $f \in A^0$, then we have $D_i' f$, $D_i'' f \in A^0$ for $i=1, \cdots, n$. We have also the following formulas:

$$\langle D_i' f, g\rangle + \langle f, D_i'' g\rangle = \frac{\partial}{\partial z_i}\langle f, g\rangle\,,$$
$$\langle D_i'' f, g\rangle + \langle f, D_i' g\rangle = \frac{\partial}{\partial \bar{z}_i}\langle f, g\rangle\,.$$

Integralating both sides of the equalities and using the Green's theorem we obtain

$$(D_i' f, g) + (f, D_i'' g) = 0\,,$$
$$(D_i'' f, g) + (f, D_i' g) = 0\,,$$

where $f, g \in A^0$ and $i \in [1, n]$.

Denote by g^{ij} the (i, j)-entry of the inverse matrix of the Hermitian matrix (g_{ij}) and let

$$g^{IJ} = g^{i_1 j_1} \cdots g^{i_q j_q}$$

where $I=(i_1, \cdots, i_q)$ and $J=(j_1, \cdots, j_q)$.

For φ, $\psi \in A^q$, we define the function $\langle \varphi, \psi\rangle$ by

$$\langle \varphi, \psi\rangle = \frac{1}{q!} \sum_{I,J} g^{IJ} \langle \varphi_I, \psi_J\rangle$$

Then $\langle \varphi, \psi\rangle$ is invariant by the translation $T_g (g \in L)$ and we define the inner product (φ, ψ) by

$$(\varphi, \psi)=\int_P \langle\varphi, \psi\rangle dV=\int_B \langle\varphi, \psi\rangle dv .$$

There exists the adjoint operator $\mathfrak{d}$ for the operator $d'': A^q \rightarrow A^{q+1}$ so that we have

$$(d''\varphi, \psi)=(\varphi, \mathfrak{d}\psi)$$

for $\varphi \in A^q$ and $\psi \in A^{q+1}$

We define the Laplacian $\square$ by

$$\square=d''\mathfrak{d}+\mathfrak{d}d''$$

Then $\square$ is an operator from A^q to A^q for all q and a $(0, q)$-form $\varphi \in A^q$ is said to be harmonic if $\square\varphi=0$. Each element of the cohomology group $H^q(A)$ of the complex A is represented by a unique harmonic form. In this sense we can say that each element of the cohomology group $H^q(E, F(H, \psi)$ is represented by a unique harmonic form. *Thus we may identify $H^q(E, F(H, \psi))$ with the vector space of all harmonic forms in A^q.* We now introduce the following notation. For $I=(i_1, \cdots, i_{q+1})$, I_u will denote the multi-index $(i_1, \cdots, \hat{i}_u, \cdots, i_{q+1})$, where the index i_u under $\wedge$ is omitted. We also introduce the operator D''^i by

$$D''^i=\sum_j g^{ij} D'_j$$

We can prove the following three formulas.

A) Let $\varphi \in A^q$. Then the components $(d''\varphi)_I$, $I=(i_1, \cdots, i_{q+1})$ of $d''\varphi \in A^{q+1}$ is given by the formula

$$(d''\varphi)_I=\sum_{u=1}^{q+1}(-1)^{u+1} D''_{i_u}\varphi_{I_u}$$

B) Let $\psi \in A^{q+1}$. Then the component $(\mathfrak{d}\psi)_J$, $J=(j_1, \cdots, j_q)$, of $\mathfrak{d}\psi$ is given by the formula

$$(\mathfrak{d}\psi)_J=-\sum_{j=1}^{n} D''^j \psi_{jJ} ,$$

where $jJ=(j, j_1, \cdots, j_q)$.

C) Let $\varphi \in A^q$. Then the component $(\square\varphi)_I$ of $\square\varphi \in A^q$ is given by the formula

$$(\square\varphi)_I=-\left(\sum_{i=1}^{n} D''^i D''_i\right)\cdot\varphi_I+\pi \sum_{u=1}^{q}(-1)^{u+1} \sum_{i=1}^{n}\left(\sum_{k=1}^{n} g^{ik} h_{k i_u}\right)\varphi_{iI_u}$$

where $iI_u=(i, i_1, \cdots, \hat{i}_u, \cdots, i_q)$.

We omit the proof of these formulas. Similar formulas had been proved in [1] in a somewhat different context, but the proof can be carried out quite similarly.

Up to this point the choices of the coordinates $(z_1, \cdots, z_n)$ and the positive definite Hermitian form $G=\sum_{i,j} g_{ij} z_i \bar{z}_j$ are arbitrary. From now on we choose privileged coordinates $(z_1, \cdots, z_n)$ for the Hermitian form H so that we have

$$H=\sum_{i=1}^{s}|z_i|^2-\sum_{i=1}^{r}|z_{s+i}|^2$$

and hence we have $h_{ij}=0$ for $i\neq j$ and $H_{ii}=\varepsilon_i$ (the symbol ε_i being defined in §1). We choose G such that

$$G=\frac{1}{a}(|z_1|^2+\cdots+|z_s|^2)+|z_{s+1}|^2+\cdots+|z_n|^2,$$

where $a>0$ (cf. [4]). Then we have $g^{ij}=0$ for $i\neq i$ and

$$g^{ii}=\begin{cases} a & \text{for } i\in[1, s] \\ 1 & \text{for } i\in[s+1, n]. \end{cases}$$

Then we have

$$\sum_k g^{ik}h_{kj}=\begin{cases} 0, & i\neq j \\ a, & i=j \text{ and } i\in[1, s] \\ -1, & i=j \text{ and } i\in[s+1, n]. \end{cases}$$

Let

$$\alpha_i=\begin{cases} a & \text{for } i\in[1, s] \\ -1 & \text{for } i\in[s+1, n]. \end{cases}$$

Then we have $\sum_{i=1}^{n}(\sum_{k=1}^{n} g^{ik}h_{ki_u})\varphi_{iI_u}=(-1)^{u+1}\alpha_{i_u}\varphi_I$ and we get

$$(\Box\varphi)_I=-(\sum_i D''^i D_i'')\varphi_I+\pi(\sum_{u=1}^{q}\alpha_{i_u})\varphi_I,$$

where

$$D''^i=g^{ii}D_i' \quad \text{(not summed)}.$$

From this we obtain

$$((\Box\varphi)_I, \varphi_I)=\sum_{i=1}^{n} g^{ii}(D_i''\varphi_I, D_i''\varphi_I)+\pi(\sum_{u=1}^{q}\alpha_{i_u})(\varphi_I, \varphi_I).$$

Since the first term of the right hand side is non-negative, we get

$$((\Box\varphi)_I, \varphi_I)\geqq \pi\cdot\alpha(I)\cdot(\varphi_I, \varphi_I).$$

where we put

$$\alpha(I)=\sum_{u=1}^{q}\alpha_{i_u}.$$

Let us denote by N (resp. M) the number of indices i_u such that $i_u \leq s$ (resp. $i_u > s$). Then by the definition of α_k, we have

$$\alpha(I) = a \cdot N - M .$$

For the multi-index I we may assume that these q indices are distinct, otherwise we get $\varphi_I = 0$. Suppose $q > r$. Then at least one of the indices i_u must be less than or equal to s and hence $N \geqq 1$. Then we get

$$\alpha(I) \geqq a - r$$

Choose a such that $a > r$. Then we have $\alpha(I) > 0$ for $q > r$.

Suppose the $\Box\varphi = 0$, where $\varphi \in A^q$ with $q > r$. Then

$$0 = ((\Box\varphi_I), \varphi_I) \geqq \pi\alpha(I) \cdot (\varphi_I, \varphi_I)$$

with $\alpha(I) > 0$. Hence we must have $(\varphi_I, \varphi_I) = 0$ and hence $\varphi_I = 0$ for any I and this means $\varphi = 0$. This shows that $H^q(A) = 0$ and hence $H^q(E, F(H, \psi)) = 0$ for $q > r$.

On the other hand, by the Serre duality, we have

$$H^q(E, F(H, \psi)) \cong H^{n-q}(E, K \otimes F^*) ,$$

where K is the canonical line bundle of E and F^* is the dual of $F(H, \psi)$. It is easy to see that $F^* \cong F(-H, \varphi^{-1})$ and $-H$ is of signature (r, s). Moreover since E is a complex torus, K is a trivial bundle and so we get

$$H^q(E, F(H, \psi)) \cong H^{n-q}(E, F(-H, \varphi^{-1})) .$$

Since $-H$ is of signature (r, s), we get from what we have already proved that $H^{n-q}(E, F(-H, \varphi^{-1})) = 0$ whenever $n - q > s$ or whenever $n - s = r > q$. Thus we get $H^q(E, F(H, \psi)) = 0$ for $q < r$ and these prove the first assertion in Theorem 1.

Consider now that case $q = r$, $\varphi \in A^r$ and $\Box\varphi = 0$. Even in this case we get $\alpha(I) \geqq a - r > 0$ except in the case where all of the r indices in I are greater than s, namely except in the case where I is a permutation of $(s+1, \cdots, n)$. Then we get $\varphi_I = 0$ for each I which is not a permutation of $(s+1, \cdots, n)$ and φ is of the form

$$(**) \qquad \varphi = f d\bar{z}_{s+1} \wedge \cdots \wedge d\bar{z}_n ,$$

where $f = \varphi_{s+1, \cdots, n}$.

Conversely let φ be a $(0, r)$-form on V of the type (**) belonging to A^r. Then $f \in A^0$ and we have

$$d''\varphi = 0 \rightleftarrows \frac{\partial f}{\partial \bar{z}_i} = 0 \quad \text{for} \quad i \in [1, s]$$

and

$$(\mathfrak{d}\varphi)_I = -\sum_{i=1}^{n} g^{ii} D_i' \varphi_{iI}$$

where $I=(i_1, \cdots, i_{r-1})$. If $I \neq (s+1, \cdots, \hat{u}, \cdots, n)$ for some u, (i, I) cannot be a permutation of $(s+1, \cdots, n)$ and $\varphi_{iI}=0$ and hence $(\mathfrak{d}\varphi)_I=0$. If $I=(s+1, \cdots, \hat{u}, \cdots, n)$ for some u, then

$$(\mathfrak{d}\varphi)_I = \pm g^{uu} D_u' f$$

Therefore we have

$$\mathfrak{d}\varphi = 0 \rightleftarrows D_u' f = 0 \quad \text{for} \quad u = s+1, \cdots, n\,.$$

It follows from our definition of the operator D_u' and from the fact $h_{ij}=\delta_{ij}\cdot\varepsilon_j$, we see that

$$D_u' f = \frac{\partial f}{\partial z_u} + \pi \bar{z}_u f\,.$$

We have thus proved that the space of harmonic $(0, r)$-form φ in A^r consists of all the $(0, r)$-form φ on V of the form

$$\varphi = f d\bar{z}_{s+1} \wedge \cdots \wedge d\bar{z}_n\,,$$

where

1) f is a differentiable theta function for the factor $J_{H,\psi}$,

2) $\dfrac{\partial f}{\partial \bar{z}_i}=0$ for $i \in [1, s]$

and

$$\frac{\partial f}{\partial z_i} + \pi \bar{z}_i f = 0 \quad \text{for} \quad i \in [s+1, n]\,.$$

Then we can identify the cohomology group $H^r(A)$ with the vector space of functions f satisfying the conditions 1) and 2) and this proves the second assertion in Theorem 1.

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