Ochiai, T.
Osaka J. Math.
16 (1979), 605-616

CONE STRUCTURES AND POSITIVE CURRENTS

TAKUSHIRO OCHIAI

(Received July 24, 1978)

Introduction. P. Lelong has introduced the notion of *positivity* in the space of currents[(1)] on a complex manifold M. Given an irreducible analytic set V in M, he showed that V defines a closed positive current [V] by integrating forms on the regular points of V. Such a current as $[V]$ is now called an *analytic cycle*. Thus he has enabled us to study some properties of analytic sets from the view point of real analysis ([5]). Then using geometric measure theory of H. Federer ([1]), J. King proved his characterization theorem of analytic cycles among closed positive currents on M ([4]). Since then, many interesting results have been obtained on closed positive currents. For the detail we refer readers to the excellent survey article by R. Harvey ([2]).

On the other hand, D. Sullivan has introduced the notion of a *cone structure* C on C^∞ manifold M, and that of *C-structure currents* in the space of currents on M. He proved several general properties of C-structure currents with compact support, and obtained beautiful applications to the study of leaves of foliations on compact manifolds ([6]).

A complex manifold M has the natural cone structures C_k $(k=1,2,\cdots)$ defined by its almost complex structure and the notion of C_k-structure currents is exactly the same as that of positive currents of dimension k. Thus, as remarked in [6], a cone structure C and C-structure currents generalize positive currents on complex manifolds. However it seems to us that general properties of C-structure currents have not been exploited so much as we would expect from our knowledge on positive currents on complex manifolds. Therefore in this paper we shall study cone structures and structure currents in more general setting. For this purpose, first of all, we shall define our cone structures which are more general than those defined in [6]. Namely in our definition of a cone structure Ω, we only need a closedness of Ω, while the "continuity" of Ω is necessary for the definition in [6]. Then closely following the case of complex manifolds, we shall define the notion of *Ω-positive currents*[(2)] and that of

(1) in the sense of G. de Rham.

(2) In conjunction with complex case, we prefer this terminology to that of a *Ω-structure current.*

Ω*-sets* which is an analogue of analytic set in complex case. In the last section, we shall study their general properties.

Finally we should record our debt to papers [2] and [3] as well as to [6]. Several proofs adopted here are modifications of those given in [2]. The interesting example guaranteed by Lemma 3–5 is just a slight change of ϕ-geometries in [3], and our Theorem 5–6 is a minor generalization of Theorem in [3].

1. Convex cones in vector spaces. Let V be a real vector space of finite dimension. We write V^* for its dual space. For a non empty convex cone Ω in V, its *dual cone* Ω^* is defined by

$$\Omega^* := \{\alpha \in V^*;\ \alpha(u) > 0 \ \text{ for any } \ u \in \bar{\Omega},\ u \neq 0\}\ ,$$

where $\bar{\Omega}$ is the topological closure of Ω in V. Then the following is well known ([7]).

Lemma 1–1. *Let Ω be a non empty open convex cone in V.*
(i) *If $\Omega^* \neq \emptyset$, then Ω^* is also an open convex cone and we have $(\Omega^*)^* = \Omega$.* (ii) *We have $\Omega^* \neq \emptyset$ if and only if Ω contains no straight line.*

We write $\{\Omega\}$ for the linear subspace of V spanned by Ω. Regarding Ω as a subset of $\{\Omega\}$, we write Ω^0 for the set of the interior points in Ω. Then Ω^0 is a non empty open convex cone in $\{\Omega\}$.

Lemma 1–2. *Let Ω be a non empty convex cone in V.*
(i) *We have $\Omega^* \neq \emptyset$ if and only if Ω^0 containsno straight line.*
(ii) *If $\Omega^* \neq \emptyset$, then we have*
$$\overline{\Omega^*} = \{\alpha \in V^*;\ \alpha(u) \geqq 0 \ \textit{for any} \ u \in \bar{\Omega}\}$$
$$= \{\alpha \in V^*;\ \alpha(u) \geqq 0 \ \textit{for any} \ u \in \Omega\}\ .$$
(iii) *If $\Omega^* \neq \emptyset$, then we have*
$$\bar{\Omega} = \{u \in V;\ \alpha(u) \geqq 0 \ \textit{for any} \ \alpha \in \overline{\Omega^*}\}$$
$$= \{u \in V;\ \alpha(u) \geqq 0 \ \textit{for any} \ \alpha \in \Omega^*\}\ .$$

Proof(3). Fixing an inner product (,) on V, we identify V^* with V.
(i) Put $W_1 = \{\Omega\}$. We write W_2 for the orthogonal complement of W_1. Set

$$\Omega^\perp := \{u \in W_1;\ (u, v) > 0 \ \text{ for any } \ v \in \bar{\Omega},\ v \neq 0\}\ .$$

Then $\Omega^\perp$ is the dual cone of Ω^0 in W_1. A vector $v = w_1 + w_2 (w_1 \in W_1,\ w_2 \in W_2)$ is in Ω^* if and only if $w_1 \in \Omega^\perp$. Thus $\Omega^* \neq \emptyset$ if and only if $\Omega^\perp \neq \emptyset$. Thus our assertion follows from (ii) of Lemma 1–1.
(ii) We will only have to show

(3) Especially (ii) and (iii) are well known.

$$\overline{\Omega^*}\supset\{\alpha\in V^*;\ \alpha(u)\geqq 0\ \text{ for any }\ u\in\Omega\}\ .$$

Take any $v\in V$ satisfying

$$(v, u)\geqq 0\ \text{ for any }\ u\in\Omega\ .$$

Choose any $w\in\Omega^*$. Then for any $t>0$, we have $v+tw\in\Omega^*$. Thus $v=\lim_{t\to 0}(v+tw)\in\overline{\Omega^*}$.

(iii) We will only have to show

$$\bar{\Omega}\supset\{u\in V;\ \alpha(u)\geqq 0\ \text{ for any }\ \alpha\in\Omega^*\}\ .$$

Suppose a vector $u=u_1+u_2$ ($u_1\in W_1$, $u_2\in W_2$) satisfies

$$(u, v)\geqq 0\ \text{ for any }\ v\in\Omega^*\ .$$

Since $\Omega^*=\{v_1+v_2;\ v_1\in\Omega^\perp \text{ and } v_2\in W_2\}$, we have $(u_1, v)\geqq 0$ for any $v\in\Omega^\perp$ and $u_2=0$. From (ii) we have $u_1\in\overline{(\Omega^\perp)^*}$, where $(\Omega^\perp)^*$ is the dual cone of $\Omega^\perp$ in W_1. Since $(\Omega^\perp)^*=\Omega^0$ from (i) of Lemma 1–1, we have $u_1\in\overline{\Omega^0}=\bar{\Omega}$. q.e.d.

2. Convex cones in vector bundles. Let M be a connected C^∞ manifold of dimension m. Let $\pi\colon E\to M$ be a real vector bundle over M of fibre dimension r. For $x\in M$, we write E_x for the fibre of E over x. We write $\pi^*\colon E^*\to M$ for the dual vector bundle of E. By a *convex cone* in E, we mean a non empty subset Ω of E such that for any $x\in M$, $\Omega_x:=\Omega\cap E_x$ is a non empty convex cone in E_x. Wen Ω is moreover closed (resp. open) in E, we call Ω a *closed* (resp. *open*) *convex cone* in E. We set

$$\Omega^*:=\bigcap_{x\in M}(\Omega_x)^*\ .$$

Lemma 2–1. *Suppose Ω is a closed convex cone in E. Then Ω^* is an open subset of E^*.*

Proof. Suppose Ω^* were not open in E^*. Then there exist $\alpha\in\Omega^*$ and a sequence $\{\alpha_j\}\subset E^*-\Omega^*$ such that $\lim_{j\to\infty}\alpha_j=\alpha$. Set $x_j=\pi^*(\alpha_j)$ and $\pi^*(\alpha)=x$. Then we have $\lim_{j\to\infty}x_j=x$. We may assume that there exists a compact subset K with $\{x_j\}\subset K$. Choose a fibre metric on $E|_K$ and denote by $S(E|_K)$ its unit sphere bundle. Since $\alpha_j\notin\Omega^*$, there exists $u_j\in\overline{\Omega_{x_j}}$ ($=\Omega_{x_j}$) such that $\|u_j\|=1$ and $\alpha_j(u_j)\leqq 0$. Since $\{u_j\}$ is in the compact subset $\Omega\cap S(E|_K)$, we may assume $\{u_j\}$ converges to $u\in\Omega$. Then $u\in\Omega_x$ and $\alpha(u)=\lim_{j\to\infty}\alpha_j(u_j)\leqq 0$. This is a contradiction. q.e.d.

Corollary 2–2. *Let Ω be a closed convex cone in E. Suppose $\Omega^*_x:=(\Omega_x)^*$ is not empty for any $x\in M$. Then Ω^* is an open convex cone in E^*.*

3. Cone structures. We write TM (resp. T^*M) for the tangent (resp. cotangent) bundle of M. For a point $x\in M$, we write T_xM (resp. T_x^*M) for the tangent (resp. cotangent) space of M at x. For a non negative integer k, set

$$\wedge_k TM := TM\wedge\cdots\wedge TM \quad (k\text{-times})$$

and

$$\wedge^k TM := T^*M\wedge\cdots\wedge T^*M \quad (k\text{-times}).$$

The fibre of $\wedge_k TM$ (resp. $\wedge^k TM$) over $x\in M$ is denoted by $\wedge_k T_xM$ (resp. $\wedge^k T_xM$). Thus $\wedge_x T_xM$ (resp. $\wedge^k T_xM$) is the vector space of k-vectors (resp. k-covectors) of M at x. We remark $\wedge^k TM$ is the dual bundle of $\wedge_k TM$.

DEFINITION. By a *cone structure of dimension* k on M, we mean a closed convex cone Ω in $\wedge_k TM$ satisfying one of the following mutually equivalent conditions (cf. (i) of Lemma 1–2 and Corollary 2–2):

(3–1) $\Omega_x^* := \Omega^*\cap\wedge^k T_xM \neq \emptyset$ *for any* $x\in M$;

(3–2) Ω_x^0 *contains no straight line for any* $x\in M$;

(3–3) Ω^* *is an open convex cone in* $\wedge^k TM$.

A k-dimensional oriented regular submanifold N of M is called an Ω-*submanifold* if the following two conditions are satisfied:

(3–4) *for each* $x\in N$, *we have* $\wedge_k T_xN\cap\Omega_x=$ *the half line which defines the orientation of* N,

and

(3–5) N *has a locally finite volume.*

The following three examples are well known.

EXAMPLE 3–1. Let E be an involutive oriented k dimensional subbundle of TM. For each $x\in M$, set $\Omega_x=$the half line in $\wedge_k E_x$ defining the orientation. Then $\Omega := \bigcup_{x\in M}\Omega_x$ is a cone structure of dimension k on M. Then a closed connected Ω-submanifold is nothing but a closed leaf with locally finite volume.

EXAMPLE 3–2. Let M be an m dimensional connected complex manifold, and J the natural almost complex structure on M. For each $x\in M$, we write $SP_{k,k}(T_xM)$ for the closed convex cone in $\wedge_{2k}T_xM$ generated by $\{u_1\wedge Ju_1\wedge\cdots\wedge u_k\wedge Ju_k;\, u_1,\cdots,u_k\in T_xM\}$. Then $SP_{k,k}(M) := \bigcup_{x\in M} SP_{k,k}(T_xM)$ is a cone structure of dimension $2k$ on M. A closed $SP_{k,k}(M)$-submanifold is exactly the same as a non singular analytic set of pure dimension k (cf. [2]).

EXAMPLE 3–3. Let the notation be as in Example 3–2. Choose k and p so that $k+p=m$. For each $x\in M$, we write $\wedge_{k,k}T_xM$ for the subspace of $2k$-

vectors of type (k, k) at x. Set

$$WP_{k,k}(T_xM) := \{u \in \wedge_{k,k}(T_xM);\ u \wedge v \in SP_{m,m}(T_xM) \text{ for any } u \in SP_{p,p}(T_xM)\}\ .$$

Then $WP_{k,k}(M) = \bigcup_{x \in M} WP_{k,k}(T_xM)$ is a cone structure of dimension $2k$ on M. We have $SP_{k,k}(M) \subset WP_{k,k}(M)$. A closed $WP_{k,k}(M)$-submanifold is also a $SP_{k,k}(M)$-submanifold.

The following example will show that our cone structure is more general than those in [6].

EXAMPLE 3–4. For $(x, y) \in \boldsymbol{R}^2$, set

$$\Omega_{(x,y)} := \begin{cases} \{(a, b);\ 0 \leqq b \leqq a\} & \text{if } x > 0 \\ \{(a, b);\ 0 \leqq a, b\} & \text{if } x = 0 \\ \{(a, b);\ 0 \leqq a \leqq b\} & \text{if } x < 0\ . \end{cases}$$

Then $\Omega = \bigcup_{(x,y) \in \boldsymbol{R}^2} \Omega_{(x,y)}$ (disjoint) is a cone structure of dimension 1 on $\boldsymbol{R}^2$. This Ω is not cone structure in the sense of [6], because Ω has no "continuity"-property.

Fix a continuous Riemannian metric g on M. Then it definens the natural norm $|u|$ on $\wedge_k T_xM$. Set

$$G_k^+(T_xM) := \{u \in \wedge_k T_xM;\ |u| = 1 \text{ and } u \text{ is a simple vector}\}.$$

For a k-covector $\alpha \in \wedge^k T_xM$, the *comass* $||\alpha||^*$ is defined by

$$||\alpha||^* := \sup\ \{\alpha(u);\ u \in G_k^+(T_xM)\}\ .$$

For a k-vector $u \in \wedge_k T_xM$, the *mass* $||u||$ is defined by

$$||u|| := \sup\{\alpha(u);\ \alpha \in \wedge^k T_xM \text{ and } ||\alpha||^* = 1\}\ ,$$

(cf. [1]). Now let ϕ be a nowhere zero continuous k-form on M. For each $x \in M$, define $\Omega_{\phi,x}$ by

$$\Omega_{\phi,x} := \text{the closed convex cone in } \wedge_k T_xM \text{ generated by } \{u \in G_k^+(T_xM);\ \phi_x(u) = ||\phi_x||^*\}\ ,$$

and set $\Omega_\phi = \bigcup_{x \in M} \Omega_{\phi,x}$.

Lemma 3–5. *The notation being as above, Ω_ϕ is a cone structure of dimension k on M.*

Proof. Let $G_k^+(T_xM)^\wedge$ be the closed convex closure of $G_k^+(T_xM)$ in $\wedge_k T_xM$. We remark that

(3–6) $$G_k^+(T_xM)^\wedge = \{u \in \wedge_k T_xM;\ \|u\| = 1\}\ ,$$

and that for $\alpha \in \wedge^k T_xM$, we have

(3–7) $$\|\alpha\|^* = \sup\{\alpha(u);\ u \in G_k^+(T_xM)^\wedge\}\ .$$

Now $G_k^+(M)^\wedge := \bigcup_{x\in M} G_k^+(T_xM)^\wedge$ is a closed subset of $\wedge_k T_xM$ and $\phi\colon G_k^+(TM)^\wedge \to \boldsymbol{R}$ is continuous. Since $G_k^+(T_xM)^\wedge$ is compact, (3–7) implies that the mapping

$$\bar{\phi}\colon u \in G_k^+(M)^\wedge \mapsto \|\phi_{\pi(u)}\|^*$$

is continuous. Therefore the subset

$$X := \{u \in G_k^+(M)^\wedge;\ \phi(u) = \bar{\phi}(u)\}$$

is closed in $G_k^+(M)^\wedge$ and hence in $\wedge_k TM$. Since

$$\Omega_\phi = \{au;\ a \geq 0 \ \text{ and } \ u \in X\}\ ,$$

Ω_ϕ is a closed convex cone in $\wedge_k TM$. For any $u \in X \cap \wedge_k T_xM$, we have $\phi(u) = \|\phi_x\|^* > 0$. Therefore $\phi(u) > 0$ for any $u \in \Omega_{\phi,x}$, $u \neq 0$. Thus Ω_ϕ is a cone structure of dimension k on M. q.e.a.

REMARK 3–6. (i) The cone structure Ω_ϕ constructed as above is inspired by the ϕ-geometry in [3]. In fact, in [3], the set $\Phi := \{u \in G_k^+(M);\ \phi(u) = \|\phi\|^*\}$ is considered, where $\|\phi\|^* = \sup\{\|\phi_x\|^*;\ x \in M\}$. Therefore a ϕ-submanifold in [3] is in particular an Ω_ϕ-submanifold. In general the converse is not true.

(ii) Suppose M is a Kähler manifold with a Kähler form ω. Then the cone structure Ω_{ω^k} of dimension $2k$ defined as above is exactly that of Example 3–2. This is a direct consequence of Wirtinger's Inequality (cf. [2]).

(iii) In general the definition of Ω_ϕ depends upon the choice of a continuous Riemannian metric g on M. However a conformal change of g does not affect Ω_ϕ. Changing g with $\|\phi_x\|^{*2/k} g$, we may assutme $\|\phi_x\|^* = 1$ for any $x \in M$.

4. Current.. In this section we fix some terminology from current theory (e.g. [1]). We assume that M is a connected C^∞ manifold with a C^0 Riemannian metric.

Let $\mathcal{K}^k(M)$ denote the real vector space of continuous k-forms with compact support on M with the usual inductive limit topology. Let $\mathcal{D}^k(M)$ be the real vector space of C^∞ k-forms with compact support with the usual inductive limit topology. The space $\mathcal{D}_k'(M)$ of *currents of dimension k on M* is by definition the topological dual space of $\mathcal{D}_k(M)$. The topological dual space $\mathcal{K}_k'(M)$ of $\mathcal{K}^k(M)$ is a subset of $\mathcal{D}_k'(M)$ and by definition the space of *currents of dimension k* *representable by integration.*(4)

For $T\in\mathcal{K}_k'(M)$, the *total variation measure* $||T||$ is defined by

$$||T||(f):=\sup\ \{|T(\phi)|\,;\ \phi\in\mathcal{D}^k(M),\ ||\phi_x||^*\leqq f(x)\ \text{ for any }\ x\in M\}\ ,$$

where $f\in\mathcal{K}^0(M), f\geqq 0$. Then the following is well known (e.g. [1]).

Lemma 4–1. *Let $T\in\mathcal{K}_k'(M)$. Then there exists a $||T||$-measurable k-vector field.*

$$\vec{T}\colon x\in M\mapsto \wedge_k T_x M$$

satisfying the following three conditions:

(4–1) *for any $\phi\in\mathcal{D}^k(M)$, we have*

$$\phi_x(\vec{T}(x))=\lim_{\varepsilon\to 0}\frac{T(\chi_{B(x,\varepsilon)}\phi)}{||T||(B(x,\varepsilon))}$$

for $||T||$ almost all x in M, where $B(x,\varepsilon)$ is the ε-ball with center x and $\chi_{B(x,\varepsilon)}$ is its characteristic function;

(4–2) $||\vec{T}(x)||=1$ *for $||T||$ almost all x in M*;

and

$$T(\phi)=\int_M\phi_x(\vec{T}(x))d||T||(x) \tag{4–3}$$

for any $\phi\in\mathcal{K}^k(M)$.

Moreover if there exists another $||T||$-measurable k-vector field $\vec{S}$ satisfying (4–1), (4–2) *and* (4–3), *then*

$$\vec{T}(x)=\vec{S}(x)\ \textit{for}\ ||T||\ \textit{almost all}\ x\ \textit{in}\ M.$$

DEFINITION. Let $T\in\mathcal{K}_k'(M)$ and $\vec{T}$ be the k-vector field as in Lemma 4–1. By abuse of language, we call $\vec{T}$ the *density k-vector field* of $\vec{T}$.

5. Ω-positive currents. In this section we fix an m-dimensional connected C^∞ manifold with a C^0 Riemannian metric g and a cone structure Ω of dimension k on M.

DEFINITION. A k-form $\phi\in\mathcal{D}^k(M)$ is called *Ω-positive* if $\phi(u)\geqq 0$ for any $u\in\Omega$. A current $T\in\mathcal{D}_k'(M)$ is called *Ω-positive* if $T(\phi)\geqq 0$ for any Ω-positive k-form $\phi\in\mathcal{D}^k(M)$.

EXAMPLE 5–1. Let N be an Ω-submanifold of M (cf. section 3). Define

(4) For any $T\in K_R'(M)$, using Riesz representation theorem, the domain of definition T can be naturally extended to all bounded Borel measurable k-forms with compact support (cf. 4–3).

a current $[N]\in\mathcal{D}'_k(M)$ by

$$[N](\phi):=\int_N \phi$$

for any k-form $\phi\in\mathcal{D}^k(M)$. Then $[N]$ is Ω-positive. Moreover if N is closed, then $[N]$ is a closed current.

DEFINITION. A C^∞ k-form ω on M is called a *Ω-transversal k-form* (according to [6]) if $\omega(u)>0$ for any $u\in\Omega$.

Lemma 5–1. *There exists an Ω-transversal k-form on M.*

Proof. From Corollary 2–2, we know Ω^* is an open convex cone in $\wedge^k TM$. Thus for each point $x\in M$, there exist an open neighbourhood U and a C^∞ k-form α on U such that $\alpha(u)>0$ for any $u\in\Omega_y$ $(y\in U)$. Therefore there exist a locally finite open covering $\{U_j\}$ of M and C^∞ k-forms α_j on U_j with $\alpha_j(u)>0$ for any $u\in\Omega_y$ $(y\in U_j)$. Let $\{\rho_j\}$ be a partition of unity subordinating to $\{U_j\}$. Then $\omega=\sum\rho_j\alpha_j$ is a Ω-transversal k-form q.e.d.

Theorem 5–1. *If T is an Ω-positive current in $\mathcal{D}'_k(M)$, then T is representable by integration* (i.e., $T\in\mathcal{K}'_k(M)$).

Proof. Choose $x\in M$ arbitrarily and fix it. From Corollary 2–2, we know Ω^* is an open convex cone in $\wedge^k TM$. Therefore if we choose a sufficiently small open neighbourhood U of x, there exist Ω-positive k-forms $\phi^1,\cdots,\phi^l\in\mathcal{D}^k(M)$ such that $\{\phi^1_y,\cdots,\phi^l_y\}$ is a basis for $\wedge^k T_yM$ for any $y\in U$. Choosing U sufficiently small, we may assume U is oriented. Then $\wedge^k T_yM$ and $\wedge^{m-k}T_yM$ are dual to each other. Therefore, choosing U sufficiently small, there exist $(m-k)$-forms $\theta^1,\cdots,\theta^l\in\mathcal{D}^{m-k}(M)$ such that $\{\theta^1_y,\cdots,\theta^l_y\}$ is the dual basis for $\wedge^{m-k}T_yM$ for any $y\in U$. On U, we can write T as

$$T=\sum_{j=1}^{l} T_j\theta^j$$

where $T_j\in\mathcal{D}'_0(U)$ $(\underset{id}{=}\mathcal{D}'_m(U))$. Then for any $f\in\mathcal{D}^0(U)$ with $f\geqq 0$, $f\phi^i$ is Ω-positive. So we have

$$0\leqq T(f\phi^i)=\sum_{j=1}^{l} T_j(f\theta^j\wedge\phi^i)=T_i(f\theta^i\wedge\phi^i)=T_i(f)\,.$$

Therefore T_i is a positive Radom measure. Therefore $T|_U\in\mathcal{K}'_k(U)$. Since x is arbitrary, we can prove $T\in\mathcal{K}'_k(M)$. q.e.d.

Theorem 5–2. *Let $\vec{T}\in\mathcal{K}'_k(M)$ and $\vec{T}$ the density k-vector field of T (cf. section 4). Then T is Ω-positive if and only if $\vec{T}(x)\in\Omega_x$ for $||T||$ almost all x in M.*

Proof. Suppose T is Ω-positive. Then for any Ω-positive k-form

$\phi \in \mathcal{D}^k(M)$, we have $T(\chi_{B(x,\varepsilon)}\phi) \geqq 0$. Thus (4–1) implies $\phi_x(\vec{T}(x)) \geqq 0$ for any Ω-positive k-form $\phi \in \mathcal{D}^k(M)$. Since Ω^* is open in $\wedge^k TM$, we see

$$\Omega_x^* = \{\phi_x;\ \phi \in \mathcal{D}^k(M) \text{ and } \Omega\text{-positive}\}. \tag{5–1}$$

Therefore we have $\alpha(\vec{T}(x)) \geqq 0$ for any $\alpha \in \Omega_x^*$. Then (iii) of Lemma 1–2 implies $\vec{T}(x) \in \bar{\Omega}_x = \Omega_x$. The "if"-part is trivial. q.e.d.

Proposition 5–3. *Let $\Omega P_k'(M)$ be the set of all Ω-positive currents in $\mathcal{D}_k'(M)$. Thcn $\Omega P_k'(M)$ is a closed convex cone in $\mathcal{D}_k'(M)$, and contains no straight line.*

Proof. Let $\{T_j\}$ be a sequence in $\Omega P_k'(M)$ such that $\lim_{j\to\infty} T_j = T \in \mathcal{D}_k'(M)$ with respect to the simple topology. Then for any Ω-positive k-form ϕ, we have $T(\phi) = \lim_{j\to\infty} T_j(\phi) \geqq 0$. Thus $T \in \Omega P_k'(M)$. Clearly $\Omega P_k'(M)$ is a convex cone. Suppose there existed $S \in \Omega P_k'(M)$ and $T \in \mathcal{D}_k'(M)$ such that

$$\{S + tT;\ t \in \boldsymbol{R}\} \subset \Omega P_k'(M).$$

Then we must have

(5–2) $T(\phi) = 0$ *for any Ω-positive k-form ϕ in $\mathcal{D}^k(M)$.*

From Theorem 5–1, we know $T \in \mathcal{K}_k'(M)$. Let $\vec{T}$ be the density k-vector field of T. Then from (4–1) we know that for $\|T\|$-almost all x in M, we have

(5–3) $\phi_x(\vec{T}(x)) = 0$ *for any Ω-positive k-form ϕ in $\mathcal{D}^k(M)$.*

Since Ω_x^* is open, (5–1) and (5–3) imply that $\vec{T}(x) = 0$ for $\|T\|$ almost all x in M. Then from (4–2) we can conclude that the support of $\|T\| = \emptyset$. Thus $T = 0$. q.e.d.

Proposition 5–4. *Let $\mathcal{E}_k'(M)$ be the set of currents with compact support on M. Then fon any Ω-transversal k-form ω, the subset*

$$A := \{T \in \Omega P_k'(M) \cap \mathcal{E}_k'(M);\ T(\omega) = 1\}$$

is relatively compact in $\mathcal{D}_k'(M)$.

Proof. It suffices to show that for any $\eta \in \mathcal{D}^k(M)$, $\{|T(\eta)|;\ T \in A\}$ is bounded. Let K be the support of η. Set $c_K := \inf\{\omega(u);\ u \in \Omega_x,\ \|u\| = 1,\ x \in K\}$. Then $c_K > 0$. Take any $T \in A$ and write $\vec{T}$ for its density k-vector field (cf. Theorem 5–1). Then we have

$$|T(\eta)| = \left|\int_M \eta_x(\vec{T}(x))\,d\|T\|(x)\right| \leqq \int_K |\eta_x(\vec{T}(x))|\,d\|T\|(x)$$

$$\leqq \sup_{x \in K} ||\eta_x||^* \int_K d||T|| .$$

On the other hand, we have

$$1 = T(\omega) = \int_M \omega_x(\vec{T}(x)) d||T||(x)$$
$$\geqq \int_K \omega_x(\vec{T}(x)) d||T||(x) \geqq c_K \int_K d||T|| .$$

Therefore we see

$$|T(\eta)| \leqq \frac{\sup_{x \in K} ||\eta_x||^*}{c_K}$$

for any $T \in A$. q.e.d.

DEFINITION. A closed subset X of M is called a Ω-*subset* if there exists an open dense subset U of X such that

(5–4) *U is a connected Ω-submanifold and the Hausdorff (k-1)-dimensional measure of $X \sim U$ is zero.*

Then we define the current $[X] \in \mathcal{D}'_k(M)$ by $[X] := [U]$ (cf. Example 5–1). This is independent of the particular choice of U satisfying (5–4).

Lemma 5–2. *The current $[X]$ is closed and Ω-positive.*

Proof. It remains to show $[X]$ is closed. We know the support of $d[X]$ is in $X \sim U$. Since $[X]=[U]$ is a locally flat current (cf. p. 316 in [2]), $d[X]$ is also a locally flat current of dimension (k-1). Then we know $d[X]=0$, because the Hausdorff (k-1)-dimensional measure of the support of $d[X]$ is zero (cf. 4.1. [1]). q.e.d.

Lemma 5–3. *Let T be an Ω-positive closed current. Suppose the suport of T is in an Ω-subset X. Then we have $T=a[X]$ ($a \geqq 0$).*

Proof. The proof goes similarly as in [2]. Since T is in $\mathcal{K}'_k(M)$ and $dT=0$, T is a locally flat current of dimension k. Remark $[X]$ is also a locally flat current of dimension k. Set $S=X \sim U$. Then U is a closed Ω-submanifold in $M \sim S$ and $T|_U$ is a closed locally flat current of dimension k with support in U. Thus there exists $a \in \boldsymbol{R}$ with $T|_U=a[U]$ on $M \sim S$. Therefore T-$a[X]$ is a locally flat current of dimension k with support in S. Since the Hausdorff k-dimensional measure of S is zero, we have T-$a[X]=0$. Clearly $a \geqq 0$. q.e.d.

Theorem 5–5. *Let X be an Ω-subset. Then $\{a[X]; a \geqq 0\}$ is an extreme ray in the closed convex cone of Ω-positive closed currents.*

Proof. The proof goes similarly as in [2]. Fix an Ω-transversal k-form

ω on M (cf. Lemma 5–1). Let K be any compact subset of M. Let $T\in\mathcal{K}'_k(M)$ and $\vec{T}$ the density k-vector field of T. Then we have

$$(5\text{–}5)\qquad T(\chi_K\omega)=\int_K\omega_x(\vec{T}(x))d\|T\|(x)$$
$$\leqq \sup_{x\in K}\|\omega_x\|^*\|T\|(K)\qquad \text{(cf. 4–2)}.$$

If T is moreover Ω-positive, then we have

$$(5\text{–}6)\qquad c_K\|T\|(K)\leqq T(\chi_K\omega)\,,$$

where $c_K=\inf\{\omega(u);u\in\Omega_x,\|u\|=1,x\in K\}>0$. From (5–5) and (5–6), we know that for an Ω-positive current T,

(5–7) *the support of T=the support of $T\wedge\omega$.*

Now suppose $[X]=S+T$ where S, T are closed Ω-positive currents. Then from (5–7), we have

$$\begin{aligned}\text{the support of } S &= \text{the upoort of } S\wedge\omega\\ &\subset \text{the support of } (S+T)\wedge\omega\\ &= \text{the support of } [X]\wedge\omega\\ &= \text{the support of } [X].\end{aligned}$$

From Lemma 5–3, we have $S=a[X]$, $a\geqq 0$. q.e.d.

Theorem 5–6. *Let ϕ be a nowhere zero closed k-form on M. Let g be a C^0-Riemannian metric on M such that $\|\phi_x\|^*=1$ for any $x\in M$*(5)*. Let T be a closed Ω_ϕ-positive current. Then T is homologically volume minimizing*(6)*.*

Proof. Let $\vec{T}$ be the density k-vector field of T. From Theorem 5–2, we have

(5–8) $\vec{T}(x)\in(\Omega_\phi)_x$ for $\|T\|$ almost all x in M.

On the other hand (3–6) and (4–2) imply

(5–9) $\vec{T}(x)\in G_k^+(T_xM)^\wedge$ for $\|T\|$ almost all x in M.

Then (5–8) and (5–9) imply

(5–10) $\phi_x(\vec{T}(x))=1$ for $\|T\|$ almost all x in M.

Let K be any compact subset of M. Remarking ϕ is a Ω_ϕ-transversal k-

(5) Confer (iii) of Remark 3–6.

(6) By this, we mean that for any current R in D'_{k+1} with compact support, we have $\|T\|(K)\leqq\|\chi_K T+dR\|$ for any compact subset K of M.

form, (5–10) implies

$$\chi_K T(\phi) = \int_K \phi_x(\vec{T}(x)) d\|T\|(x) = \|T\|(K) .$$

Let R be any current in $\mathcal{D}'_{k+1}(M)$ with compact support. Then we have

$$\begin{aligned} \|T\|(K) &= \chi_K T(\phi) \\ &= (\chi_K T + dR)(\phi) \qquad (\phi \text{ being closed}) \\ &\leqq \|\chi_K T + dR\|(M), \end{aligned}$$

where the last inequality follows from (5–5) applied to the compact set $K \cup$ the support of dR and we put $\|\chi_K T + dR\|(M) =$ if $dR \notin \mathcal{K}'_k(M)$. q.e.d.

UNIVERSITY OF TOKYO

Bibliography

[1] H. Federer: Geometric measure theory, Springer-Verlag, New York, 1969.

[2] R. Harvey: *Holomorphic chains and their boundaries,* Proc. Sym. Pure Math. Vol. 30, Amer. Math. Soc., Providence, R.I. (1977), 185–220.

[3] R. Harvey and B. Lawson: Abstracts US-Japan seminar on minimal submanifolds, including geodesics, September 19–22, 1977, Tokyo.

[4] J. King: *The currents defined by analytic varieties,* Acta. Math. **127** (1971), 185–220.

[5] P. Lelong: Plurisubharmonic functions and positive differential forms, Gordon and Breach, New York, 1969.

[6] D. Sullivan: *Cycles for the dynamical study of foliated manifolds and complex manifolds,* Invent. Math. **36** (1970), 225–255.

[7] E.B. Vinberg: *The theory of homogeneous convex cones,* Trudy Moskov. Mat. Obšč. 12 (1963), 303–358 (=Transactions of the Moscow Math. Soc., 1963, 340–403).