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BOUNDARY POINTS OF THE DIRICHLET FUNDAMENTAL REGION

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1. Introduction

Recently many of the geometric aspects of the classical theory of Fuchsian groups have been extended by Eberlein and O'Neill [5] and by Eberlein [1]-[4] to a class of simply connected complete riemannian manifolds including those with sectional curvature $K \leqq c < 0$. In [2] and [5] it was shown how to compactify any n-manifold in this class to obtain a topological n-manifold $\bar{M} = M \cup M(\infty)$ homeomorphic to D^n. Since the ideal boundary $M(\infty)$ consists of classes of asymptotic geodesic rays, any group of isometries Γ acting on M extends naturally to a group of homeomorphisms of $\bar{M}$.

In this paper we consider discrete groups Γ of isometries possibly containing elliptic elements acting on manifolds in this class. Given a non fixed point p_0 of Γ, let F_0 denote the open Dirichlet fundamental region for Γ based at p_0 and let $G_0 = cl(F_0)$ where the closure is taken in M. Let $\partial^\infty(F_0)$ denote those points of the cone closure of G_0 in $\bar{M}$ lying in the ideal boundary $M(\infty)$. We first prove (Theorem 3.2) that $x \in \partial^\infty(F_0)$ if and only if all the points of the orbit $\Gamma(p_0)$ lie on or outside the horosphere $L(p_0, x)$ passing through p_0 determined by x. One consequence of this result is that axial fixed points of Γ cannot lie in $\partial^\infty(F_0)$. Thus the union of the cone closures of the translates of the Dirichlet fundamental region need not pave $\bar{M}$.

Motivated by recent results in the theory of Fuchsian groups, we then restrict our attention to geometrically finite discrete isometry groups. Here we will say that Γ is *geometrically finite at* p_0 if the Dirichlet fundamental region F_0 for Γ based at p_0 has only finitely many sides. If Γ is geometrically finite at p_0 and $x \in L(\Gamma) \cap \partial^\infty(F_0)$, then x in fact lies in $\partial^\infty(bd\ F_0)$.

It is a classical result of automorphic function theory ([8], p. 133) that if the fundamental polygon for a Fuchsian group is finite sided, then the boundary points of the fundamental polgyon in S^1 are either ordinary points or para-

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bolic vertices. Since the closure in $M(\infty)$ of $bd\ F_0$ in the cone topology is a finite set when dim $M=2$, using a result of Eberlein [1], Lemma 4.2b, we are able to show (Theorem 5.4) that this classical result is valid for discrete orientation preserving geometrically finite groups of isometries acting on uniform visibility surfaces without conjugate points.

2. Visibility manifolds and the Dirichlet fundamental region

The purpose of this section is to fix some necessary notation and to recall some concepts from [1]-[5] that will be needed in the sequel. First let M be a complete simply connected riemannian manifold without conjugate points and let $d: M\times M\to \boldsymbol{R}$ denote the distance function induced on M by the riemannian metric. The hypothesis M has no conjugate points holds here iff every pair of distinct points of M may be joined by a unique geodesic segment (up to parametrization). Tesselations of such manifolds induced by discrete subsets have been studied by Ehrlich-Im Hof [6], [7]. Let Γ be a discrete group of isometries for M. Then Γ acts discontinuously and properly discontinuously on M and is also countable. Thus we may fix the notation $\Gamma=\{\varphi_i\,;\,i\in I\}$ for a suitable set of nonnegative indices I and will in addition set $\varphi_0=Id$ and $\varphi_{-i}=\varphi_i^{-1}$ for each $i\neq 0$. Let $p_i=\varphi_i(p_0)$ for $i\neq 0$. The (open) Dirichlet region F_0 for Γ based at p_0 is then defined by

$$F_0=\{p\in M\,;\,d(p,p_0)<d(p,p_i)\,\forall i\neq 0\}\,.$$

Let G_0 denote the closure of F_0 in M and let $bd\ F_0=G_0-F_0$. The sides of the Dirichlet region may then be defined as follows (cf. [7]). Set

$$S_i=\{p\in M\,;\,d(p,p_0)=d(p,p_i)<d(p,p_k)\ \ \forall k\neq 0, i\}\,.$$

If S_i is nonempty, we call S_i a *side* of F_0. It was shown in [6], Theorem 3.19 and [7], Corollaire 2, that $bd\ F_0=\bigcup_i cl\ S_i$.

We will adopt the usual convention in this paper that all geodesics will be parametrized with unit speed. It is also shown in [6], Prop. 2.4, [7], section 2, that if $c:\boldsymbol{R}\to M$ is any geodesic with $c(0)=p_0$, then c intersects every bisector

$$M(p_0,p_i)=\{p\in M\,;\,d(p,p_0)=d(p,_ip)\}$$

transversely and at most once. This fact will be necessary for the proof of Theorem 3.2.

For the purpose of extending results from automorphic function theory to riemannian manifolds, it has been realized that the class of simply connected complete riemannian manifolds without conjugate points is too large but also that the class of simply connected complete riemannian manifolds with negative sectional curvature is smaller than needed, cf. Eberlein-O'Neill [5]. A basic problem with the class of manifolds without conjugate points is that the trichotomy between elliptic, axial and parabolic elements of a Fuchsian group does not extend to all manifolds of this class. This trichotomy does however hold for a class of manifolds including those with sectional curvature $K \leqq c < 0$ which Eberlein has defined and studied in a series of papers following [5]; see for instance [1] and [5] for a detailed discussion of this trichotomy and also an investigation of axial and parabolic isometries extending the classical results for Fuchsian groups to this class of manifolds.

We briefly recall some notation for and properties of manifolds in this class needed in the sequel. In [1] and [3], a *visibility Hadamard manifold* is defined to be a complete simply connected riemannian manifold with sectional curvature $K \leqq 0$ everywhere satisfying the Visibility Axiom of Eberlein-O'Neill [5], Defn. 4.2, p. 61. A *uniform visibility manifold without conjugate points* is defined to be a complete simply connected riemannian manifold without conjugate points satisfying the Uniform Visibility Axiom of Eberlein [2], Defn. 1.3, p. 153. Eberlein and O'Neill have shown how to compactify such manifolds by adding an ideal boundary $M(\infty)$ to M consisting of asymptotic classes of geodesic rays. They have defined a topology called the *cone topology* ([2], p. 155) for $\bar{M} = M \cup M(\infty)$ which makes $\bar{M}$ compact. Another topology, in which $\bar{M}$ is noncompact, called the *horocycle topology* has also been defined by Eberlein and O'Neill [5], p. 55, and used by Eberlein in studying the geodesic flow. We will let $\gamma_{p,x}$ denote the unique unit speed geodesic ray with $\gamma_{p,x}(0) = p$ and $\gamma_{p,x}(\infty) = x \in M(\infty)$ as in [3], p. 5. In [1], Prop. 2.8, Eberlein has shown that the function $\alpha\colon M \times M \times M \to \boldsymbol{R}$ defined by $\alpha(p, m, q) = d(m, q) - d(m, p)$ has a continuous extension $\alpha\colon M \times \bar{M} \times M \to \boldsymbol{R}$ using the cone topology on $\bar{M}$. The *horosphere $L(p, x)$ through p determined by $x \in M(\infty)$* may then be defined as

$$L(p, x) = \{m \in M \,;\, \alpha(p, x, m) = 0\} ,$$

cf. Eberlein [3], Defn. 1.10, p. 6. The function $f_x = \alpha(p, x, \cdot)$ is called the *Busemann function* determined by $\gamma_{p,x}$ and may be calculated as $f_x(m) = \lim_{t \to \infty}(d(\gamma_{p,x}(t), m) - t)$. Thus horospheres are level sets of Busemann functions.

One advantage of the cone compactification of M as $\bar{M} = M \cup M(\infty)$ is that a group of isometries Γ for M may be extended to act as a group of homeomorphisms of $\bar{M}$. Given $x \in M(\infty)$ and $\varphi \in \Gamma$, if x is represented by the geodesic ray γ, then $\varphi(x)$ is represented by the geodesic ray $\varphi \circ \gamma$, Using this extension, Eberlein and O'Neill [5] defined elliptic, axial and parabolic isometries for visibility manifolds. Following Eberlein [3], Defn. 1.6, we may classify a nontrivial element φ of a discrete group of isometries for a visibility Hadamard manifold (or a uniform visibility manifold without conjugate points) as *elliptic* if φ has a fixed point in M, *axial* if φ has no fixed points in M and two fixed points in $M(\infty)$, and *parabolic* if φ has no fixed points in M and one fixed point in $M(\infty)$. For a Fuchsian group acting on the Poincairé unit disk, this is equivalent to the usual definition of automorphic function theory.

Finally we need some notation for the boundary points of the Dirichlet fundamental region in the ideal boundary $M(\infty)$. Given a subset K of M, we will denote by $cl_\infty(K)$ the set of all $x \in M(\infty)$ that are in the cone closure of K in $\bar{M}$. We will denote the boundary of the Dirichlet fundamental region in $M(\infty)$ by $\partial^\infty(F_0)=cl_\infty(G_0)$ and also set $\partial^\infty(bd\ F_0)=cl_\infty(bd\ F_0)$.

3. The Busemann function and limit points of the Dirichlet region

In this section we assume that M^n, $n \geqslant 2$, is either a Hadamard manifold or a uniform visibility manifold without conjugate points. This insures that if γ is any geodesic ray in M and $p \in M$ is any point, then there is a unique geodesic ray σ asymptotic to γ with $\sigma(0)=p$. This then allows us to utlize the Eberlein-O'Neill ideal boundary $M(\infty)$ and compactification $\bar{M}$ of M and, moreover, the function $\alpha : M \times \bar{M} \times M \to \boldsymbol{R}$ defined in section 2 is continuous ([1], Prop. 2.8).

An immediate consequence of the transversality of intersection of geodesics through p with the bisector $M(p, q)$ noted in Ehrlich-Im Hof [6], Prop. 2.4 or [7] is the following proposition. Let Γ be a discrete group of isometries acting on M and let p_0 and F_0 be chosen as in section 2.

Proposition 3.1. *Let $c : [0, \infty) \to M$ be a geodesic ray with $c(0)=p_0$ and $c(\infty)=x \in M(\infty)$. Let $f_x=\alpha(p_0, x, \cdot) : M \to \boldsymbol{R}$ denote the Busemann function defined by the ray c. Either*

(i) *$c[0, \infty)$ is contained; F_0, or*

(ii) *there exists a $t_0>0$ such that $c(s) \notin \bar{F}_0$ for all $s>t_0$. Moreover if c meets the closure $\bar{S}_i$ of the side S_i of F_0, then $f_x(p_i)<0$.*

Proof. This follows easily from the fact that any geodesic through P_0 intersects with $M(p_0, p_i)$ transversally and at most once, together with the standard fact that the function $t \mapsto d(p_i, c(t))-t$ is strictly monotone decreasing for $c \neq \gamma_{p_0, p_i}$ since M has no conjugate points.

With the aid of Proposition 3.1 we now prove

Theorem 3.2. *The following are equivalent :*

(1) $x \in \partial^\infty(F_0)$

(2) *$\gamma_{p_0, x}[0, \infty)$ is contained in F_0*

(3) *$f_x(p_i)=\alpha(p_0, x, p_i) \geqq 0$ for all $i \in I$.*

Proof. It is convenient to prove the theorem by showing (2)$\Leftrightarrow$(3) and (1)$\Leftrightarrow$(3). Let $\gamma=\gamma_{p_0, x}$ during the course of the proof. First (3)$\Rightarrow$(2) is immediate from Prop. 3.1. Conversely if $\gamma[0, \infty)$ is contained in F_0, then $t=d(p_0, \gamma(t))<d(p_i, \gamma(t))$ for all $t\geqq 0$ so that $f_x(p_i)=\alpha(p_0, x, p_i)\geqq 0$. Thus (2)$\Rightarrow$(3).

If (3) holds, then $\gamma[0, \infty)$ is contained in F_0 by (2) so $x\in\partial^\infty(F_0)$ as the cone limit point of the sequence $\{\gamma(n) ; n\in \boldsymbol{Z}^+\}$. Thus (3)$\Rightarrow$(1). Now suppose $x\in\partial^\infty(F_0)$. Then there exists an infinite sequence $\{q_n\}\subset\bar{F}_0$ converging to x in the cone topology. Since $q_n\in\bar{F}_0$, we have $\alpha(p_0, q_n, p_i)\geqq 0$ for all n and i. Thus $f_x(p_i)=\lim_{n\to\infty} \alpha(p_0, q_n, p_i)\geqq 0$ for all $i\in I$ thus establising (1)$\Rightarrow$(3).

Condition (3) of Theorem 3.2 means geometrically that all the lattice points $\Gamma(p_0)$ lie on or outside the horosphere $L(p_0, x)$ when $x\in\partial^\infty(F_0)$. It is thus natural to consider the consequences of p_i lying on $L(p_0, x)$ when $x\in\partial^\infty(F_0)$. One consequence needed for section 5 is

Corollary 3.3. *If $x\in\partial^\infty(F_0)$ and $p_i\in L(p_0, x)$, then $\varphi_{-i}(x)\in\partial^\infty(F_0)$.*

Proof. Since $p_i\in L(p_0, x)$, it is known that $\alpha(p_0, x, \cdot)=\alpha(p_i, x, \cdot)$. On the other hand, $\alpha(p_i, x, \cdot)=\alpha(\varphi_i(p_0), x, \cdot)=\alpha(p_0, \varphi_{-i}(x), \varphi_{-i}(\cdot))$. The corollary is then immediate from (1)$\Leftrightarrow$(3) of Theorem 3.2.

4. Limit points of Γ and the boundary of the Dirichlet region

Let M be a visibility Hadamard manifold or a uniform visibility manifold without conjugate points. Let Γ be a discrete group of isometries (elliptic isometries allowed) for M. We may define the cone and horocycle limit sets of Γ, denoted by $L(\Gamma)$ and $L_h(\Gamma)$ just as in [5] even though Γ is allowed to contain elliptic elements. The discreteness of Γ implies that all the limit points must be in $M(\infty)$ even in the presence of elliptic isometries.

DEFINITION 4.1. Let $x\in M(\infty)$. Then $x\in L(\Gamma)$ (resp. $x\in L_h(\Gamma)$) iff there is an infinite sequence $\{p_{i(n)}\}\subset\Gamma(p_0)$ with $p_{i(n)}\to x$ in the cone (resp. horocycle) topology for $\bar{M}$.

An immediate consequence of (1)$\Leftrightarrow$(3) of Theorem 3.2 is the following

Proposition 4.2. *If $x\in\partial^\infty(F_0)$, then $x\notin L_h(\Gamma)$.*

Proof. If $x\in\partial^\infty(F_0)$, then all the lattice points p_i, $i\in I$, lie on or outside the horosphere $L(p_0, x)$ by Theorem 3.2. Thus no subsequence of lattice points can converge to x in the horocycle topology.

Corollary 4.3. *If x is an axial fixed point of Γ, then $x\notin\partial^\infty(F_0)$.*

Proof. This follows from Prop. 4.2 and the easily established fact that if x is an axial fixed point, then $x\in L_h(\Gamma)$.

REMARK 4.4. Since $\alpha(p_i, x, \cdot)=\alpha(p_0, \varphi_{-i}(x), \varphi_{-i}(\cdot))$, a stronger result is easily established using Theorem 3.2—(3). Namely if x is an axial fixed point of Γ, then $x \notin \bigcup_{i\in I}\partial^\infty(\varphi_i(\bar{F}_0))$. Thus the closures of the Dirichlet regions is $\bar{M}$ do not necessarily pave $\bar{M}$.

We now make a definition abstracted from Kleinian group theory. First set $\mathrm{Fix}(\Gamma)=\{m\in M\ ;\ \varphi_i(m)=m$ for some $i\neq 0\}$. Then $M-\mathrm{Fix}(\Gamma)$ is known to be dense in M (cf. [7], [6] Lemma 4.2). Consequently it makes sense to define the following concept.

DEFINITION 4.5. The discrete group Γ *is geometrically finite at* p_0 iff $p_0 \notin \mathrm{Fix}(\Gamma)$ and the Dirichlet region for Γ based at p_0 has only finitely many sides.

An important consequence of geometric finiteness is that if infinitely many of the geodesic segements γ_{p_0, p_n} intersect $bd\ F_0$, then infinitely many points $\gamma_{p_0, p_n} \cap bd\ F_0$ lie on a single side of F_0. Eberlein [1] has utilized this most effectiviely in his study of finitely connected surfaces, cf. [1], Theorem 4.2. In the present context, we need the following result for use in section 5. A proof may be found implicitly in chapter 4 of Eberlein [1] for finitely connected surfaces.

Proposition 4.6. *Let* Γ *be geometrically finite at* p_0. *Suppose* $x\in L(\Gamma)\cap \partial^\infty(F_0)$. *Then* $x\in\partial^\infty(bd\ F_0)$.

Proof. Since $x\in L(\Gamma)$, we may find $p_{i(n)}\rightarrow x$ in the cone topology. Since $p_{i(n)}\notin \bar{F}_0$, we have $\#(\gamma_{p_0, p_{i(n)}}\cap bd\ F_0)=1$ for all n. Then setting $\{r_{i(n)}\}=\gamma_{p_0, p_{i(n)}}\cap bd\ F_0$, since Γ is geometrically finite at p_0, we may assume all the $r_{i(n)}$'s lie on the closure $\bar{S}_j$ of a single side S_j of F_0. If $\{r_{i(n)}\}$ contained a subsequence convergent in M, it would follow that $\gamma_{p_0, x}\cap \bar{S}_j\neq\emptyset$ whence $f_x(p_j)<0$. Since $x\in\partial^\infty(F_0)$, this violates (1)$\Rightarrow$(3) of Theorem 3.2. Thus some subsequence of $\{r_{i(n)}\}$ converges to x in the cone topology. Therefore $x\in\partial^\infty(bd\ F_0)$.

5. Geometrically finite Dirichlet regions and parabolic fixed points on Hadamard surfaces

It is classically known in automorphic function theory that if the fundamental region for a Fuchsian group is finite sided, then the boundary points of the fundamental region are either ordinary points or parabolic vertices (Lehner [8], 4J, p. 133). In this section we extend this result to discrete orientation preserving groups of isometries acting on uniform visibility surfaces without conjugate points. Some preliminary results are in order.

Lemma 5.1. *Let M be a Hadamard surface or a uniform visibility surface without conjugate points. Let Γ be a discrete group of isometries acting on M. If Γ is geometrically finite at p_0, then $\partial^\infty(bd\ F_0)$ is a finite set.*

Proof. It is known [4], Prop. 2.7, that each bisector $M(p_0, p_i)$ has exactly two cone limit points in $M(\infty)$. Since Γ is geometrically finite at p_0, only finitely many bisectors determine the sides of F_0.

The following proposition may be implicitly found in Eberlein [1], Lemma 4.2b, stated for finitely connected surfaces from the viewpoint of (2) of our Theorem 3.2. Eberlein's proof however works for arbitrary discrete groups of isometries.

Proposition 5.3 (Eberlein). *Let M be a visibility Hadamard surface or a uniform visibility surface without conjugate points. Let Γ be a discrete group of isometries for M. Suppose that Γ is geometrically finite at p_0. If $x \in L(\Gamma) \cap \partial^\infty(bd\ F_0)$, then infinitely many points of the orbit $\Gamma(p_0)$ lie on the horosphere $L(p_0, x)$.*

Proof. Just use (2)$\Leftrightarrow$(3) of Theorem 3.2 and Lemma 4.2b of [1].

We are now ready to prove along the lines of Theorem 4.2 of [1]

Theorem 5.4. *Let M be a visibility Hadamard surface or a uniform visibility surface without conjugate points. Let Γ be a discrete orientation preserving group of isometries for M. If Γ is geometrically finite at p_0, then every point of $\partial^\infty(F_0)$ is either a cone ordinary points of Γ, i.e., lies in $M(\infty) - L(\Gamma)$, or is a parabolic fixed point of Γ.*

Proof. Let $A = L(\Gamma) \cap \partial^\infty(F_0)$. In view of Prop. 4.4, the set $A = L(\Gamma) \cap \partial^\infty(bd\ F_0)$. Since Γ is orientation preserving, no elliptic isometry in Γ can fix any point of $M(\infty)$. Also no axial isometry can fix any point of A by Cor. 4.3. Thus it is enough to show that given any $x \in A$, we have $\varphi(x) = x$ for some $\varphi \in \Gamma - \{Id\}$ to establish the theorem. By Prop. 5.3 and Cor. 3.3, the set $B = \{\varphi_i \in \Gamma\ ;\ \varphi_{-i}(x) \in \partial^\infty(F_0)\}$ has infinite cardinality. On the other hand, $B = \{\varphi_i \in \Gamma\ ;\ \varphi_{-i}(x) \in \partial^\infty(bd\ F_0)\}$ so that the finitensess of $\partial^\infty(bd\ F_0)$ implies that $\varphi_{-i}(x) = \varphi_{-j}(x)$ some nonzero $i, j \in I$, $i \neq j$. Thus x is fixed by a nontrivial element of Γ.

Finally we note that Proposition 4.3 of Eberlein [1] establishing the existence of a nonexpanding nonparabolic geodesic ray on infinitely connected visibility Hadamard surface shows that the requirement "Γ is geometrically finite at p_0" cannot be ommited from the hypothesis of Theorem 5.4.

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References

[1] P. Eberlein: *Geodesics and ends in certain surfaces without conjugate points,* Mem. Amer. Math. Soc. 199 (1978).

[2] P. Eberlein: *Geodesic flow in certain manifolds without conjugate points,* Trans. Amer. Math. Soc. **167** (1972), 151–170.

[3] P. Eberlein: *Horocycle flows on certain surfaces without conjugate points,* Trans. Amer. Math. Soc. **233** (1977), 1–36.

[4] P. Eberlein: *The cut locus of noncompact finitely connected surfaces without conjugate points,* Comment. Math. Helv. **51** (1976), 23–44.

[5] P. Eberlein and B. O'Neill: *Visibility manifolds,* Pacific J. Math. **46** (1973), 45–109.

[6] P. Ehrlich and H.-C. Im Hof: *Dirichlet regions in manifolds without conjugate points,* preprint, Berkeley, 1977.

[7] P. Ehrlich and H.-C. Im Hof: *Sur la géométrie des domaines de Dirichlet,* C.R. Acad. Sci. Paris, Sér. A **286** (1978), 457–458.

[8] J. Lehner: Discontinuous groups and automorphic functions, Math. Surveys ♯ 8, Amer, Math. Soc. 1964.