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THE INVERSE SCATTERING PROBLEM FOR THE DIRAC OPERATOR AND THE MODIFIED KORTEWEG-DE VRIES EQUATION

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The main purpose of the present paper is to construct the solution of the initial value problem for the modified Korteweg-de Vries (*KdV*) equation

$$v_t-6v^2v_x+v_{xxx}=0\,,\qquad -\infty<x,\, t<\infty\,. \tag{0.1}$$

The subscripts x, t denote partial differentiations. We study smooth real valued solutions which tend to $\pm m$ as $x\to\pm\infty$ for a positive constant m.

As an analogue of the method of Gardner, Greene, Kruskal and Miura (*GGKM*) [3], we construct these solutions in terms of the scattering data of the one dimensional Dirac operator

$$L_{iv}=i\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}D+i\begin{bmatrix}0 & -v\\ v & 0\end{bmatrix},\quad D=d/dx\,.$$

In [9], Zakharov and Shabat have studied the initial value problem for the non-linear Schrödinger equation

$$iu_x+u_{xx}-|u|^2u=0 \tag{0.2}$$

with the step type initial data as above. They have developed the inverse scattering theory of

$$L_u=i\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}D+\begin{bmatrix}0 & u^*\\ u & 0\end{bmatrix}$$

on formal basis, where u^* is the complex conjugate of u. They have constructed the exact solutions of (0.2) in terms of the scattering data of L_u, assuming that the reflection coefficient identically vanishes.

Now, L_{iv} can be obtained from L_u by putting $u=iv$, where v is a real valued function. By virtue of this restriction, the argument can be considerably simplified and, in the sequel, we can complete the inverse scattering theory of L_{iv}. This result enables us to construct the solutions with general step type initial data.

In § 1, we describe preliminary materials which concern the Jost solutions and the scattering data of L_u. In § 2, we derive the fundamental integral equation. In § 3, the solvability of the fundamental integral equation is established. In § 4, the inverse scattering problem for L_{iv} are discussed. Finally, in § 5, the solutions of the initial value problem for the modified *KdV* equation (0.1) are constructed.

Throughout the paper, c^* denotes the complex conjugate of c.

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1. Scattering data

In this section, we expose the generality of the scattering data of L_u without the assumption $u=iv$. In deriving the following results, methods developed for the Schrödinger operator and other operators have been used in modified form. For these results, we refer to [1], [2] [4], [6], [8] and [9].

Let m be a positive real number. Put

$$m_\pm = m \exp(i\alpha_\pm)\,, \qquad -\pi \leqslant \alpha_\pm \leqslant \pi\,.$$

For a complex valued measurable function $u=u(x)$ which tends to $m_\pm$ as $x \to \pm\infty$, consider the eigenvalue problem

$$L_u y = \lambda y\,, \quad y = {}^t(y_1, y_2)\,, \quad \lambda = \xi + i\kappa\,, \tag{1.1}$$

on the real axis $(-\infty, \infty)$.

Let $\zeta=\zeta(\lambda)$ be the two-valued algebraic function defined by

$$\zeta^2 = \lambda^2 - m^2$$

and R be the upper leaf of the two-sheeted Riemann surface associated with ζ. We assume $\operatorname{Im}\zeta > 0$ for $\lambda \in R$. For $\xi \in \boldsymbol{R}_m = \boldsymbol{R} \setminus [-m, m]$, put

$$\sigma = \sigma(\xi) = (\operatorname{sgn} \xi)(\xi^2 - m^2)^{1/2}\,.$$

For a two-dimensional vector $y={}^t(y_1, y_2)$ and a matrix $A=(a_{ij})$ of order 2, put

$$y^\# = {}^t(y_2^*, y_1^*)\,, \qquad y^\tau = {}^t(y_2, y_1)\,,$$
$$A^\# = \begin{bmatrix} a_{22}^* & a_{21}^* \\ a_{12}^* & a_{11}^* \end{bmatrix}, \qquad A^\tau = \begin{bmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix}.$$

If $y=y(x)$ is a solution of (1.1), then $y^\#$ is a solution of (1.1), λ being replaced by λ^*.

For solutions $y(x)$ and $z(x)$ of (1.1), the Wronskian

$$[y; z] = y_1 z_2 - y_2 z_1$$

is constant.

Put

$$f^0_+(x, \lambda) = {}^t(m_+^{-1}(\lambda-\zeta), 1) \exp(i\zeta x)$$
$$f^0_-(x, \lambda) = {}^t(1, m_-^{*^{-1}}(\lambda-\zeta) \exp(-i\zeta x) .$$

They are solutions of (1.1) for $u(x)\equiv m_\pm$ respectively.

Gasymov [4; Theorem 1.2.1] has shown the following.

Theorem 1.1 (Gasymov [4]). *If we assume*

$$\sigma_\pm(x) = \pm\int_x^{\pm\infty}(1+|y|)|u(y)-m_\pm|dy+\sup_{\pm y>\pm x}|u(y)-m_\pm|<\infty,$$

then there exist unique solutions $f_\pm(x, \lambda)$ of (1.1) such that

$$f_\pm(x, \lambda) = f^0_\pm(x, \lambda)+o(1)$$

as $x\to\pm\infty$. $f_\pm(x, \lambda)$ are analytic in $\lambda\in R$. Moreover there exist matrix functions $A_\pm(x, y)=((A_{\pm ij}(x, y))_{i,j=1,2}$ such that

$$f_\pm(x, \lambda) = f^0_\pm(x, \lambda)\pm\int_x^{\pm\infty}A_\pm(x, y)f^0_\pm(y, \lambda)dy . \tag{1.2}$$

Furthermore

$$|A_{\pm ij}(x, y)|\leqslant C_\pm\sigma_\pm(x+y)$$

and

$$A^\sharp_\pm(x, y) = A_\pm(x, y)$$

are valid. We have

$$u(x) = -2iA_{+21}(x, x)+m_+ .$$

Proof. Put

$$E(x, \lambda) = (f^0_-, (x, \lambda), f^0_+(x, \lambda)) ,$$

then we have

$$f_+(x, \lambda) = f^0_+(x, \lambda)-iE(x, \lambda)\int_x^\infty E(y, \lambda)^{-1}\begin{bmatrix} 0 & u^*(y)-m^*_+ \\ -u(y)+m_+ & 0 \end{bmatrix}f_+(y, \lambda)dy . \tag{1.3}$$

This integral equation can be solved by successive approximation which leads to the existence of the solution and its analyticity.

We refer to [4; pp53–63] for the existence of kernels $A_\pm$. Q.E.D.

The functions $f_\pm(x, \lambda)$ are called the Jost solutions.

If we assume that $u=iv$ and $\alpha_\pm=\pm 2^{-1}\pi$, where v is real, then the proof of this theorem can be considerably simplified as follows. Put

$$E(\lambda)=\begin{bmatrix} 1 & im^{-1}(\zeta-\lambda) \\ im^{-1}(\zeta-\lambda) & 1 \end{bmatrix} \tag{1.4}$$

If we set

$$h_{\pm}(x, \zeta) = E(\lambda)^{-1} f_{\pm}(x, \lambda) \exp(\mp i\zeta x) \qquad (\lambda \neq 0),$$

then $h_{\pm}(x, \zeta)$ are analytic in ζ, $\mathrm{Im}\,\zeta > 0$. Assuming

$$(1.5) \quad h_{+}(x, \zeta) = {}^{t}(0, 1) + \int_{0}^{\infty} K_{+}(x, y) \exp(2i\zeta y) dy, \; K_{+} = {}^{t}(K_{+1}, K_{+2}),$$

put (1.5) into (1.3). And we have

$$(1.6) \qquad K_{+1}(x, y) + \int_{x}^{x+y} (v(z)-m) K_{+2}(z, x+y-z) dz = -v(x+y)+m$$

$$(1.7) \qquad K_{+2}(x, y) + \int_{x}^{\infty} (v(z)+m) K_{+1}(z, y) dz = 0 .$$

These integral equations can be solved by successive approximation. From this, $K_{\pm}$ are real vectors. We have

$$v(x) = -K_{+1}(x, 0) + m = K_{-2}(x, 0) - m .$$

The matrix

$$2^{-1} \begin{bmatrix} K_{\pm 2}(x, 2^{-1}(y-x)) & K_{\pm 1}(x, 2^{-1}(y-x)) \\ K_{\pm 1}(x, 2^{-1}(y-x)) & K_{\pm 2}(x, 2^{-1}(y-x)) \end{bmatrix}$$

coincides with the kernels $A_{\pm}(x, y)$ in Theorem 1.1.

Returning to the case of general complex potential, put

$$f_{\pm}(x, \xi) = f_{\pm}(x, \xi + i0), \qquad \xi \in \boldsymbol{R}_{m} .$$

We have

$$[f_{+}(x, \xi); f_{+}^{\#}(x, \xi)] = 2\sigma(\sigma-\xi)/m^{2} .$$

Since $\sigma(\sigma-\xi)$ does not vanish for $\xi \in \boldsymbol{R}_{m}$, $f_{+}(x, \xi)$ and $f_{+}^{\#}(x, \xi)$ are linearly independent solutions of (1.1). Therefore one can express

$$(1.8) \qquad f_{-}(x, \xi) = a_{+}(\xi) f_{+}^{\#}(x, \xi) + b_{+}(\xi) f_{+}(x, \xi) .$$

Similarly, we have

$$f_{+}(x, \xi) = a_{-}(\xi) f_{-}^{\#}(x, \xi) + b_{-}(\xi) f_{-}(x, \xi) .$$

We have

$$a_{+}(\xi) = a_{-}(\xi) = a(\xi) = m^{2}[f_{+}; f_{-}]/2\sigma(\sigma-\xi)$$

and

$$(1.9) \qquad b_{+}(\xi) = -b_{-}(\xi) = m^{2}[f_{-}; f_{+}^{\#}]/2\sigma(\sigma-\xi) .$$

We have

$$(1.10) \qquad |a(\xi)|^{2} = 1 + |b_{\pm}(\xi)|^{2} .$$

This implies that $a(\xi)$ does not vanish for $\xi \in \boldsymbol{R}_m$.

The coefficient $a(\xi)$ can be extended to the analytic function

$$a(\lambda) = m^2[f_+(x, \lambda); f_-(x, \lambda)]/2\zeta(\zeta-\lambda), \qquad \lambda \in \boldsymbol{R}. \tag{1.11}$$

Put (1.2) into (1.9) and (1.11) and calculate the Wronskians, and we can obtain the integral representations of $a(\lambda)$ and $b_\pm(\xi)$. For instance, we have

$$a(\lambda) = \frac{(\zeta-\lambda)^2 - m^2 \exp\{i(\alpha_+ - \alpha_-)\}}{2\zeta(\zeta-\lambda)\exp\{i(\alpha_+ - \alpha_-)\}} + \frac{1}{2\zeta(\zeta-\lambda)}\int_0^\infty \{\alpha_1(x) + (\zeta-\lambda)\alpha_2(x) + (\zeta-\lambda)^2\alpha_3(x)\} \exp(2i\zeta x)dx,$$

where $\alpha_j(x)$ $(j=1, 2, 3)$ which are integrable can be expressed explicitly in terms of the kernels $A_\pm$.

Because $f_\pm$ are linearly dependent at the zero of $a(\lambda)$, they are square integrable by their asymptotic property. By virtue of formal selfadjointness of L_u, zeros $a(\lambda)$ belong to $(-m, m)$. Let λ^0 be one of zeros of $a(\lambda)$. Then

$$f_-(x, \lambda^0) = d^0 f_+(x, \lambda^0)$$

is valid for some constant d^0. We have

$$a'(\lambda^0) = -i(2\eta^0)^{-1} m_- d^{0*} \int_{-\infty}^\infty |f_+(x, \lambda^0)|^2 dx, \tag{1.12}$$

where $\eta^0 = (m^2 - \lambda^{02})^{1/2}$. Hence λ^0 is a simple zero of $a(\lambda)$.

Similarly to [6; pp133–134], we can show that $a(\lambda)$ has only finite number of zeros. We denote them by $\lambda_1, \lambda_2, \cdots, \lambda_N$. Put

$$r_\pm(\xi) = b_\pm(\xi)/a(\xi), \qquad \xi \in \boldsymbol{R}_m,$$

which are called reflection coefficients. We have

$$r_\pm(\xi) = O(\xi^{-1}), \qquad |\xi| \to \infty,$$

and

$$|r_\pm(\xi)| < 1, \qquad \xi \in \boldsymbol{R}_m. \tag{1.13}$$

Put

$$n_{\pm j} = \left\{\int_{-\infty}^\infty f_\pm^*(x, \lambda_j) f_\pm(x, \lambda_j) dx\right\}^{-1}, \qquad j = 1, 2, \cdots, N.$$

We call the collection

$$\{r_\pm(\xi), n_{\pm j}, \lambda_j, j = 1, 2, \cdots, N\} \tag{1.14}$$

the scattering data of L_u.

In the following, we assume that $u=iv$ and $\alpha_{\pm}=\pm 2^{-1}\pi$, where v is real. Putting (1.5) into (1.9) and (1.1), we have

$$a(\lambda)=\lambda\left(1+\int_0^\infty \alpha(x)\exp(2i\zeta x)\,dx\right)\Big/\zeta \tag{1.15}$$

and

$$b_+(\xi)=(2i\sigma)^{-1}\int_{-\infty}^\infty \beta(x)\exp(-2i\sigma x)dx\,, \tag{1.16}$$

where $\alpha(x)$ and $\beta(x)$ are real valued integrable functions which can be expressed explicitly in terms of kernels $K_{\pm}$. By (1.14) and (1.15), we have

$$a(-\lambda)=-a(\lambda)$$

and

$$b_+(\xi)=O(\xi^{-1})\,.$$

Hence, if λ^0 is a zero of $a(\lambda)$, then $a(\lambda)$ vanishes also at $\lambda=-\lambda^0$. Therefore zeros of $a(\lambda)$ consist of $\pm\kappa_j$, where

$$0=\kappa_0<\kappa_1<\cdots<\kappa_n<m\,.$$

The linear dependence of $f_{\pm}$ implies that of $h_{\pm}\exp(\pm i\zeta x)$. Therefore we have

$$h_-(x, i\eta_j)\exp(\eta_j x)=d_j h_+(x, i\eta_j)\exp(-\eta_j x)\,,\qquad j=0, 1, \cdots, n\,,$$

for some real number d_j, where $\eta_j=(m^2-\kappa_j^2)^{1/2}$. Put

$$c_{+0}=\left\{\int_{-\infty}^\infty |f_+(x, 0)|^2dx\right\}^{-1}=id_0/2a'(0)\,,$$
$$c_{+j}=2\left\{\int_{-\infty}^\infty |f_+(x, \pm\kappa_j)|^2dx\right\}^{-1}=imd_j/\eta_j a'(\pm\kappa_j)\,,\qquad j=1, 2, \cdots, n\,.$$

Define c_{-j} by

$$\begin{aligned} c_{+0}c_{-0}&=-(2a'(0))^{-2}\,,\\ c_{+j}c_{-j}&=-m^2(\eta_j^2 a'(\kappa_j))^{-2}\,,\qquad j=1, 2, \cdots, n\,. \end{aligned} \tag{1.17}$$

By (1.12), $c_{\pm j}$ are positive numbers.

In place of (1.14), we call the collection

$$\{r_{\pm}(\xi),\ c_{\pm j},\ \kappa_j\,,\ j=0, 1, 2, \cdots, n\}$$

the scattering data of L_{iv}.

By the similar arguments as in [2, p 149], we can show that the condition

$$r(\xi)\to\mp i\qquad(\xi\to\pm m) \tag{1.18}$$

are valid, if and only if

$$1+\int_0^\infty \alpha(y)dy \neq 0 .$$

Moroever the condition

(1.19) $$r(\xi)<\delta<1 , \qquad \xi\in \boldsymbol{R}_m ,$$

is valid, if and only if

$$1+\int_0^\infty \alpha(y)dy = 0 .$$

Put

$$B_1(\lambda) = \lambda^{-1}\zeta \prod_{j=1}^n (\zeta-i\eta_j)^{-1}(\zeta+i\eta_j)$$

and

$$B_2(\lambda) = \lambda^{-1}(\zeta+im) \prod_{j=1}^n (\zeta-i\eta_j)^{-1}(\zeta+i\eta_j) .$$

If the condition (1.18) holds, then $B_1(\lambda)a(\lambda)$ is analytic in ζ, $\operatorname{Im}\zeta>0$, and has no zero. If we set

$$a_0(\zeta) = B_1(\lambda)a(\lambda)$$

and

$$g(x) = \pi^{-1}\int_{-\infty}^\infty \log a_0(\sigma) \exp(-2i\sigma x)d\sigma ,$$

where integration is taken in L^2-sense, then, by (1.15) and the Payley-Wiener's theorem, $g(x)$ is a real valued function which vanishes for $x<0$. Hence we have

(1.20) $$g(x)+g(-x) = \pi^{-1}\int_{-\infty}^\infty \log|a_0(\sigma)|^2 \exp(-2i\sigma x)\, d\sigma$$

and

(1.21) $$\log a_0(\zeta) = 2^{-1}\left\{\int_0^\infty g(x)\exp(2i\zeta x)dx+\int_{-\infty}^0 g(-x)\exp(-2i\zeta x)dx\right\} .$$

Eliminating $g(x)$ in (1.21) by (1.20), we have

$$\log a_0(\zeta) = (2\pi i)^{-1}\int_{-\infty}^\infty (\sigma-\zeta)^{-1}\log|a_0(\sigma)|^2 d\sigma .$$

Hence, by (1.10), we obtain

(1.22) $$a(\lambda) = B_1(\lambda)^{-1}\exp\left\{(2\pi i)^{-1}\int_{-\infty}^\infty (\sigma-\zeta)^{-1}\log[\xi^{-2}\sigma^2(1-|r(\xi)|^2)]^{-1}d\sigma\right\} .$$

Similarly to above, we have

(1.23) $$a(\lambda) = B_2(\lambda)^{-1}\exp\left\{(2\pi i)^{-1}\int_{-\infty}^\infty (\sigma-\zeta)^{-1}\log(1-|r(\xi)|^2)^{-1}d\sigma\right\} ,$$

if (1.18) holds. Thus we can reconstrct $a(\lambda)$ from the reflection coefficient $r(\xi)$.

2. The fundamental integral equation

In this and subsequent sections, we assume that $u=iv$ and $\alpha_{\pm}=\pm 2^{-1}\pi$, where v is real.

In [8], Zakharov and Shabat have derived integral equations which connect kernels $A_{\pm}$ with the scattering data of L_u. In this sectiontwe derive similar integral equations which connect kernels $K_{\pm}$ with the scattering data of L_{iv}.

By (1.8) we have

$$a(\xi)^{-1}J(\xi)h_{-}(x,\sigma)-{}^t(1,0)=\{h_{+}(x,\sigma)-{}^t(0,1)\}^{\#}\\ +r_{+}(\xi)J(\xi)\exp(2i\sigma x)h_{+}(x,\sigma)\,,$$

where

$$J(\xi)=E(\xi+i0)^{\#-1}E(\xi+i0)=\xi^{-1}\begin{bmatrix}\sigma & -im\\ -im & \sigma\end{bmatrix}$$

Now, multiply $\pi^{-1}\exp(2i\sigma y)$ on the above identity and integrate over $(-\infty,\infty)$ with respect to σ, where integrations are taken in L^2-sense. We have

$$\pi^{-1}\int_{-\infty}^{\infty}\{a(\xi)^{-1}J(\xi)h_{-}(x,\sigma)-{}^t(1,0)\}\exp(2i\sigma y)d\sigma=2i\sum_{j=0}^{n}R_j\,,$$

where R_j is the residue at $\zeta=i\eta_j$ of

$$a(\lambda)^{-1}J(\lambda)h_{-}(x,\zeta)\exp(2i\zeta y)$$

which is a meromorphic function in ζ, $\operatorname{Im}\zeta>0$, with simple poles $i\eta_j$. We have

$$R_j=ic_{+j}\exp(-2\eta_j(x+y))\begin{bmatrix}-\eta_j/m & 1\\ 1 & -\eta_j/m\end{bmatrix}h_{+}(x,i\eta_j)\,.$$

Hence we have

$$K_{+}^{\tau}(x,y)+F_{+}(x+y)^t(0,1)+\int_0^{\infty}F_{+}(x+y+z)K_{+}(x,z)dz=0\quad(y>0)\,,\tag{2.1+}$$

where

$$F_{+}(x)=2\sum_{j=0}^{n}c_{+j}\begin{bmatrix}-\eta_j/m & 1\\ 1 & -\eta_j/m\end{bmatrix}\exp(-2\eta_j x)\\ +\pi^{-1}\int_{-\infty}^{\infty}r(\xi)J(\xi)\exp(2i\sigma x)d\sigma\,.\tag{2.2+}$$

Similarly we have

$$K_{-}^{\tau}(x,y)+F_{-}(x+y)^t(1,0)+\int_{-\infty}^{0}F_{-}(x+y+z)K_{-}(x,z)dz=0\quad(y<0)\,,\tag{2.1−}$$

where

$$(2.2-)\qquad F_-(x) = 2\sum_{j=0}^{n} c_{-j}\begin{bmatrix} -\eta_j/m & 1 \\ 1 & -\eta_j/m \end{bmatrix} \exp(2\eta_j x)$$
$$+\pi^{-1}\int_{-\infty}^{\infty} r(\xi)J(\xi)\,\mathrm{epx}\,(-2i\sigma x)d\sigma\,.$$

By (1.15) and (1.16), we have

$$r(\xi)^* = r(-\xi)\,.$$

This shows that $F_\pm(x)$ are real matrices.

We call (2.1$\pm$) the fundamental integral equations.

3. Solvability of the fundamental equation

In this section we discuss the solvability of the fundamental equation (2.1) as an integral equation for K.

Assuming that G is bounded integrable in (a, ∞) for any a, put

$$(T_{G,x}f)(y) = \int_0^\infty G(x+y+z)f(z)dz$$

for $f \in L^1(0, \infty)$. Then $T_{G,x}$ is a completely continuous operator as an operator on $L^1(0, \infty)$.

We have

Theorem 3.1. *If $F(x)$ defined by (2.2) is bounded integrable in (a, ∞) for any a, then $I+T_{F^T,x}$ has the bounded inverse for any x, where I is the identity.*

Proof. Suppose ϕ is a solution of

$$(I+T_{F^T,x})\phi = 0$$

in $L^1(0, \infty)$. By the boundedness of F, that of ϕ follows. So ϕ belongs to $L^2(0, \infty)$. Put

$$h(\zeta) = {}^t(h_1(\zeta), h_2(\zeta)) = \int_0^\infty \phi(x)\exp(2i\zeta x)dx\,, \qquad \mathrm{Im}\,\zeta>0\,,$$
$$X(\zeta) = {}^t(h_1(\zeta), h_2(\zeta), h_1^*(\zeta), h_2^*(\zeta))\,,$$
$$R(x, \sigma) = r(\xi)J(\xi)^T\exp(2i\sigma x)\,,$$
$$H(x, \sigma) = \begin{bmatrix} E & R(x, \sigma)^* \\ R(x, \sigma) & E \end{bmatrix}$$

and

$$H_j(x) = 2c_j\exp(-2\eta_j x)\begin{bmatrix} 1 & -\eta_j/m \\ -\eta_j/m & 1 \end{bmatrix},$$

where E is the unit matrix of order 2. Then we have

$$(3.1) \qquad 0 = \int_0^\infty \phi(y)^*(I+T_{F^T,x})\phi(y)dy$$
$$= \pi^{-1}\int_{-\infty}^\infty X(\sigma)^*H(x, \sigma)X(\sigma)d\sigma+\sum_{j=0}^n h(i\eta_j)^*H_j(x)h(i\eta_j)\,.$$

H_j are nonnegative definite real symmetric matrices. On the other hand, the Hermitian matrix H is unitarily equivalent to the diagonal matrix

$$\begin{pmatrix} 1+|r(\xi)| & 0 & 0 & 0 \\ 0 & 1+|r(\xi)| & 0 & 0 \\ 0 & 0 & 1-|r(\xi)| & 0 \\ 0 & 0 & 0 & 1-|r(\xi)| \end{pmatrix}.$$

Hence, by (1.14), the right hand side of (3.1) contains only positive terms. Therefore we have

$$X(\sigma)^*H(x, \sigma)X(\sigma) = 0$$

for any x, σ. Therefore $h(\sigma)=0$ follows. This shows $\phi(x)=0$. Q.E.D.

By Theorem 3.1, the operator equation

$$(3.2) \qquad (I+T_{F^T,x})\phi = \psi_x$$

is uniquely solvable for a continuous L^1-valued function ψ_x. We denote the unique solution by ϕ_x. Then, by Theorem 3.1, ϕ_x is a continuous L^1-valued function. Moreover we have

Lemma 3.2. *Suppose that F is absolutely continuous and F, F' are in $L^1(a, \infty)$ for any a. Let ψ_x be continuously differentiable in x as a L^1-valued function, then the solution ϕ_x is differentiable in x and*

$$(I+T_{F^T,x})\,\phi' = \psi_x'-T_{F^{T\prime},x}\phi_x$$

holds.

A proof for this Lemma is completely parallel to [7; Lemma 4.3, pp 342–343].

Put $\psi_x=-F(x+y)^{Tt}(0, 1)$ and the equation (3.2) coincides with the fundamental equation (2.1). By Theorem 3.1 and Lemma 3.2, $K(x, y)$ is differentaible in the ordinary sense. Put

$$(3.3) \qquad v(x) = -K_1(x, 0)+m$$

and

$$(3.4) \qquad f(x, \lambda) = \exp(i\zeta x)E(\lambda)\left\{{}^t(0, 1)+\int_0^\infty K(x, y)\exp(2i\zeta y)dy\right\},$$

where $E(\lambda)$ is the matrix defined by (1.4). Then we have

Theorem 3.3. *If F is absolutely continuous and F, F' are in $L^1(a, \infty)$ for any a, then f defined by* (3.4) *is differentiable in x and satisfies*

$$L_{iv}f = \lambda f \tag{3.5}$$

for $v=v(x)$ defined by (3.3).

Proof. Put

$$J(x, y) = {}^t(K_{2x}(x, y)-(v(x)+m)K_1(x, y),\ K_{1x}(x, y)-K_{1y}(x, y) \\ -(v(x)-m)K_2(x, y)) .$$

Then, (3.5) holds if and only if $J(x, y)=0$. We have

$$F_2'(x) = 2mF_1(x) ,$$

where

$$F(x) = \begin{bmatrix} F_1(x) & F_2(x) \\ F_2(x) & F(x)_1 \end{bmatrix} .$$

By this relation, we have

$$J(x, y)^\tau + \int_0^\infty F(x+y+z)J(x, z)dz = 0 .$$

Hence, by Theorem 3.1, $J(x, y)=0$ follows. Q.E.D.

4. The inverse problem

Let n be a nonnegative integer, κ_j ($j=0, 1, \cdots, n$) be nonnegative numbers such that

$$0 = \kappa_0 < \kappa_1 < \cdots < \kappa_n < m$$

and c_j ($j=0, 1, \cdots, n$) be positive numbers. Suppose $r(\xi)$ ($\xi \in \boldsymbol{R}_m$) be a function which satisfies the conditions

$$r(-\xi) = r(\xi)^* , \qquad |r(\xi)| < 1,\ \xi \in \boldsymbol{R}_m , \\ r(\xi) = O(\xi^{-1}) \qquad (\xi \to \pm\infty) .$$

Moreover we assume that either

$$r(\xi) \to \mp i \qquad (\xi \to \pm m) ,$$

or

$$|r(\xi)| < \delta < 1 , \qquad \xi \in \boldsymbol{R}_m .$$

Determine $a(\xi)$ from $r(\xi)$ by (1.22) and (1.23) respectively. Put

$$a(\xi) = a(\xi + i0) \\ r_+(\xi) = r(\xi), \quad r_-(\xi) = -a(\xi)^{-1}a(-\xi)r_+(\xi-)$$

and define c_{-j} from $c_{+j}=c_j$ according to (1.16).

Put

$$F_{\pm}(x) = 2\sum_{j=0}^{n} c_{\pm j}\begin{bmatrix} -\eta_j/m & 1 \\ 1 & -\eta_j/m \end{bmatrix} \exp(\mp 2\eta_j x) + \pi^{-1}\int_{-\infty}^{\infty} r_{\pm}(\xi)J(\xi)\exp(\pm 2i\sigma x)d\sigma .$$

We assume that $F_{\pm}(x)$ are absolutely continuous and $F_{\pm}(\pm x)$, $F'_{\pm}(\pm x)$ belong to $L^1(a, \infty)$ for any a.

Let $K_{\pm}(x, y)$ be the unique solutions of the fundamental equations (2.1$\pm$) whose kernels $F_{\pm}$ are defined above.

Put

$$v_+(x) = -K_{+1}(x, 0)+m$$

and

$$v_-(x) = K_{-2}(x, 0)-m .$$

By Theorem 3.3,

$$f_+(x, \lambda) = \exp(i\zeta x)E(\lambda)\left\{{}^t(0, 1)+\int_0^{\infty} K_+(x, y)\exp(2i\zeta y)dy\right\}$$

and

$$f_-(x, \lambda) = \exp(-i\zeta x)E(\lambda)\left\{{}^t(1, 0)+\int_{-\infty}^{0} K_-(x, y)\exp(-2i\zeta y)dy\right\}$$

satisfy (1.1) for $v=v_{\pm}$ respectively.

Next we show that $v_{\pm}(x)$ coincide. This follows immediately, once the equality

$$a(\xi)^{-1}f_-(x, \xi) = f_+^{\#}(x, \xi)+r_+(\xi)f_+(x, \xi), \qquad \xi\in\mathbf{R}, \tag{4.1}$$

is established, where

$$f_{\pm}(x, \xi) = f_{\pm}(x, \xi+i0), \qquad \xi\in\mathbf{R}_m .$$

Put

$$g(x, \sigma) = h_+^{\#}(x, \sigma)+\exp(2i\sigma x)r_+(\xi)J(\xi)h_+(x, \sigma)$$

and

$$G(x, y) = \pi^{-1}\int_{-\infty}^{\infty}\{g(x, \sigma)-{}^t(1, 0)\}\exp(2i\sigma y)d\sigma ,$$

where

$$h_+(x, \sigma) = {}^t(1, 0)+\int_0^{\infty} K_+(x, y)\exp(2i\sigma y)dy .$$

Then we have

$$G(x, y) = K_+(x, y)+F_+^0(x+y){}^t(0, 1)+\int_0^{\infty} F_+^0(x+y+z)K_+(x, z)dz ,$$

where

$$F_+^0(x) = \pi^{-1}\int_{-\infty}^{\infty} r_+(\xi)J(\xi)\exp(2i\sigma x)d\sigma\,.$$

Lemma 4.1. *The function $g(x, \sigma)$ can be extended to the domain,* $\operatorname{Im}\zeta>0$, *as a meromorphic function $g(x, \zeta)$ whose poles are simple and exhausted by $i\eta_j$* $(j=0, 1, 2, \cdots, n)$.

Proof. Putting

$$q_j(x, \zeta) = -ic_{+j}(\zeta-i\eta_j)^{-1}\begin{bmatrix}\zeta/im & -1\\ -1 & \zeta/mi\end{bmatrix}\exp(2i\zeta x)\Big\{{}^t(0, 1) + \int_0^{\infty} K_+(x, z)\exp(2i\zeta z)dz\Big\}$$

and

$$g_1(x, \sigma) = g(x, \sigma) - {}^t(0, 1) - \textstyle\sum_{j=0}^{n} q_j(x, \sigma)\,, \qquad \sigma\in \boldsymbol{R}\,.$$

We have

$$\pi^{-1}\int_{-\infty}^{\infty} q_j(x, \sigma)\exp(2i\sigma y)d\sigma$$
$$= 2c_{+j}\exp(-2\eta_j(x+y))\begin{bmatrix}\eta_j/m & -1\\ -1 & \eta_j/m\end{bmatrix}\Big\{{}^t(0, 1)+\int_0^{\infty} K_+(x, z)\exp(2i\eta_j z)dz\Big\}\,.$$

By the fundamental equation,

$$G(x, y) = \pi^{-1}\textstyle\sum_{j=0}^{n}\displaystyle\int_{-\infty}^{\infty} q_j(x, \sigma)\exp(2i\sigma y)d\sigma\,, \qquad (x+y,\ y>0),$$

follows. Therefore, we have

$$\int_{-\infty}^{\infty} g_1(x, \sigma)\exp(2i\sigma y)d\sigma = 0\,, \qquad (x+y,\ y>0)\,. \tag{4.2}$$

So, $g_1(x, \sigma)$ can be extended to the analytic function $g_1(x, \zeta)$, $\operatorname{Im}\zeta>0$. Q.E.D.

Put

$$J(\lambda) = \lambda^{-1}\begin{bmatrix}\zeta & -im\\ -im & \zeta\end{bmatrix}, \qquad \lambda\in R\,,$$

$$h(x, \zeta) = a(\lambda)J(\lambda)^{-1}g(x, \zeta) \tag{4.3}$$

and

$$f(x, \lambda) = \exp(-i\zeta x)J(\lambda)h(x, \zeta)\,.$$

By Lemma 4.1, $f(x, \lambda)$ is holomorphic in $\lambda\in R$.

We have

Theorem 4.2. *The function $h(x, \zeta)$ defined by* (4.3) *is represented as*

$$(4.4)\qquad h(x, \zeta) = {}^t(0, 1)+\int_{-\infty}^{0} K(x, y) \exp(-2i\zeta y)dy\,,$$

where $K(x, y)$ *is the unique solution of the fundamental equation* (2.1–).

Proof. By the absolute continuity of F and the integrability of F', the existence and integrability of $K_{+y}(x, y)$ follows. Hence $\sigma g_1(x, \sigma)$ is bounded as a function of σ. By (4.2), we can apply the Phragmén-Lindelöf type argument (see [6;pl68, problem 32]) and conclude that $\zeta g_1(x, \zeta)$ is bounded in the domain $\mathrm{Im}\,\zeta>0$ for $x>0$. This implies that as $|\zeta|\to\infty(\mathrm{Im}\,\zeta\geqslant 0)$

$$h(x, \zeta)-{}^t(1, 0)\to 0\,,$$

where convergence is uniform. Hence we have

$$\int_{-\infty}^{\infty}\{h(x, \sigma)-{}^t(1, 0)\} \exp(2i\sigma y)d\sigma = 0\,, \qquad (y>0)\,.$$

Therefore, the representation (4.4) holds.

By direct calculation, we have

$$a^{-1}(\xi)J(\xi)h_+(x, \sigma) = h^\sharp(x, \sigma)+\exp(-2i\sigma x)r_-(\xi)J(\xi)h(x, \sigma)\,.$$

Hence the kernel $K(x, y)$ satisfies the fundamental equation (2.1–). Q.E.D.

By this Theorem, the equality

$$K(x, y) = K_-(x, y)$$

follows. This shows that

$$f(x, \lambda) = f_-(x, \lambda)\,, \qquad x>0\,.$$

So we have shown the fulfillment of the equality (4.1). Therefore $v_\pm(x)$ coincide for $x>0$.

From the fundamental equation, the estimates

$$|K_\pm(x, y)| < C_\pm \sup_{\pm z\geqslant\pm(x+y)}|F_\pm(z)|$$

follows. Hence, we have finally

Theorem 4.3. *Let* $r(\xi)$ *satisfy the conditions formulated at the beginning of this section and also we assume that* $m_\pm(\pm x)$ *belong to* $L^1(a, \infty)$ *for any* a, *where*

$$m_\pm(x) = \sup_{\pm z\geqslant\pm x}|F_\pm(x)|\,.$$

Then

$$\{r_\pm(\xi), c_{\pm j}, \kappa_j, j = 0, 1, \cdots, n\}$$

are the scattering data of L_{iv}.

For the application of this result to the construction of the solution of the modified *KdV* equation (0.1), we need the relation between the smoothness of the potential v and that of the reflection coefficient $r(\xi)$.

Let S be the space of C^∞-functions which are rapidly decreasing together with all their derivatives and D_m be the set of C^∞-functions which tend to $\pm m$ as $x \to \pm\infty$ and whose derivatives belong to S.

We have

Lemma 4.4. *Suppose that the potential v is n-times continuously differentiable function with integrable derivatives. Then $K_+^{(j,k)}(x, y) = (\partial/\partial x)^j (\partial/\partial y)^k K_+(x, y)$ exist for j, k; $1 \leqslant j+k \leqslant n$ and the estimates*

$$|K_{+1}^{(j,k)}(x, y) + v^{(j+k)}(x+y)| + |K_{+2}^{(j,k)}(x, y)| \leqslant C_+ \sigma_+(x+y)$$

hold.

The proof of this Lemma is completely parallel to that of [7; Lemma 1.3, p 334].

Next we have

Theorem 4.6. *The potential v belongs to D_m if and only if $\xi^{-1} r(\xi)$ belongs to S as the function of a variable σ.*

Proof. If we express $\alpha(x)$ and $\beta(x)$ defined by (1.15) and (1.16) in terms of $K_{\pm j}$ by calculating the Wronskians in (1.8) and (1.9), then, by Lemma 4.4, $\alpha(x)$ and $\beta(x)$ are infinitely differentiable except at $x=0$ and rapidly decreasing together with all derivatives.

By (2.1), we have

$$h_-(x, \sigma) = a(\xi) J(\xi) h_+^\sharp(x, \sigma) + b(\xi) h_+(x, \sigma) \exp(2i\sigma x) .$$

Multiply $\pi^{-1} \exp(2i\sigma y)(-|x| < y < 0)$ on the second component of the above relation, integrate over $(-\infty, \infty)$ with respect to σ, differentiate with respect to y and let $y \uparrow 0$. Then we have an explicit representation for $\beta(x)$

$$\beta(x) = v'(x) - (v(x) - m) \int_{-\infty}^{x} (v^2(z) - m^2) dz + 2m \int_{x}^{\infty} (v^2(z) - m^2) dz$$
$$+ \int_0^\infty \alpha'(z) K_{+1}(x, z) + (2m\alpha(z) - \beta(x+z)) K_{+2}(x, z) dz .$$

Hence $\beta(x)$ is infinitely differentiable even at $x=0$, *i.e*, $\beta(x)$ belongs to S. Next we assume

$$1 + \int_0^\infty \alpha(x) dx \neq 0 .$$

Then, by Lemma 4.4, $(2i\sigma\xi a(\xi))^{-1}$ is a C^∞-function of σ. As mentioned above, $2i\sigma b(\xi)$ belongs to S. Hence

$$\xi^{-1}r(\xi) = 2i\sigma b(\xi)/2i\sigma\xi a(\xi)$$

belongs to S.

On the other hand if we assume

$$1+\int_0^\infty \alpha(x)dx = 0\,, \tag{4.5}$$

then we have

$$\int_{-\infty}^\infty \beta(x)dx = 0\,.$$

This implies that there exists $\gamma(x)\in S$ such that

$$\gamma'(x) = \beta(x)\,.$$

This shows

$$b(\xi) = \int_{-\infty}^\infty \gamma(x)\exp(-2i\sigma x)\,dx\,.$$

The condition (4.5) implies that $(\xi a(\xi))^{-1}$ is a C^∞-function with bounded derivatives. Therefore $\xi^{-1}r(\xi)$ belongs to S.

The proof for the converse statement can be obtained by induction based on Lemma 3.2. Q.E.D.

5. Construction of the solution of the modified KdV equation

Put

$$B_{v(t)} = -4D^3+3\begin{bmatrix} v^2 & v_x \\ v_x & v^2 \end{bmatrix}D+3D\begin{bmatrix} v^2 & v_x \\ v_x & v^2 \end{bmatrix}.$$

Then, by direct calculation, the modified KdV equation (0.1) is equivalent to

$$dL_{iv(t)}/dt = [B_{v(t)}, L_{iv(t)}] = B_{v(t)}L_{iv(t)}-L_{iv(t)}B_{v(t)}\,. \tag{5.1}$$

Let $v=v(t)=v(x, t)$ be a smooth solution of (0.1). Suppose

$$L_{iv(t)}f_\pm = \lambda f_\pm\,. \tag{5.2}$$

Differentiate this with respect to t, then, by (5.1),

$$df_\pm/dt-B_{v(t)}f_\pm$$

satisfy the differential equation (5.2). Hence if v belongs to D_m for each t, then, by the asymptotic property and the uniqueness of the Jost solution, we have

$$df_\pm/dt-B_{v(t)}f_\pm = (\mp 4i\zeta^3\mp 6i\zeta m^2)f_\pm\,. \tag{5.3}$$

Differentiating (1.8) with respect to t and eliminating $df_\pm/dt$ by (5.3), we have

$$da/dtf^*_\pm + \{db_\pm/dt \mp (8i\sigma^3 + 12m^2 i\sigma)b_\pm\} f_\pm = 0\,.$$

So we have

$$a(\xi, t) = a(\xi, 0)$$

and

$$b_\pm(\xi, t) = b_\pm(\xi, 0) \exp\{\pm(8i\sigma^3 + 12m^2 i\sigma)t\}\,. \tag{5.4}$$

Hence $a(\lambda, t)$ is independent of t and so are its zeros $\pm\kappa_j$ $(j=0, 1, \cdots, n)$. Similarly we have

$$c_{\pm j}(t) = c_{\pm j}(0) \exp\{\pm(8\eta_j^3 - 12m^2\eta_j\})t\}\,. \tag{5.5}$$

Conversely, suppose that

$$\{r_\pm(\xi), c_{\pm j}, \kappa_j, j = 0, 1, \cdots, n\}$$

are the scattering data of the operator L_{iv}, $v \in D_m$. Define $r_\pm(\xi, t) = b_\pm(\xi, t)/a(\xi)$ and $c_{\pm j}(t)$ by (5.4) and (5.5). Put

$$F_\pm(x, t) = 2\sum_{j=0}^n c_{\pm j}(t)\begin{bmatrix} -\eta_j/m & 1 \\ 1 & -\eta_j/m \end{bmatrix} \exp(\mp 2\eta_j x)$$
$$+\pi^{-1}\int_{-\infty}^{\infty} r_\pm(\xi, t) J(\xi) \exp(\pm 2i\sigma x) d\sigma\,.$$

Then, by Theorem 3.1, the fundamental equations (2.1$\pm$) with the kernels $F_\pm(x, t)$ are uniquely solvable. We denote the solutions by $K_\pm(x, y, t)$. Put

$$\begin{aligned} v_+(x, t) &= -K_{+1}(x, 0, t) + m \\ v_-(x, t) &= K_{-2}(x, 0, t) - m\,. \end{aligned} \tag{5.6}$$

As $r(\pm m, t) = r(\pm m)$, the condition required to show $v_+(x, t) = v_-(x, t)$ is clearly satisfied. Thus, by Theorem 4.3 and 4.5, we have

Theorem 5.1. *If $v(x)$ belongs to D_m, then there exists the unique potential $v(x, t) \in D_m$ whose scattering data is*

$$\{r_\pm(\xi, t), c_{\pm j}(t), \kappa_j, j = 0, 1, \cdots, n\}$$

for each t.

We have finally

Theorem 5.2. *The potential $v(x, t)$ defined by* (5.6) *satisfies the modified KdV equation* (0.1).

Proof. It is sufficient to show that the relation (5.3) holds. Infact, differentiate (5.2) with respect to t and eliminate $df_{\pm}/dt$ by (5.3). Then we have

$$(dL_{iv(t)}/dt-[B_{v(t)},\ L_{iv(t)}])f=0\,.$$

By direct calculation, the relation (5.3) is equivalent to

$$dh_{\pm}/dt=g_{\pm}\,, \tag{5.7}$$

where

$$h_{+}(x,\ \zeta,\ t)={}^{t}(0,\ 1)+\int_{0}^{\infty}K_{+}(x,\ y,\ t)\exp(2i\zeta y)dy\,,$$

$$h_{-}(x,\ \zeta,\ t)={}^{t}(1,\ 0)+\int_{-\infty}^{0}K_{-}(x,\ y,\ t)\exp(-2i\zeta y)dy$$

and

$$g_{\pm}(x,\ \zeta,\ t)=12\zeta^{2}h_{\pm x}\mp 12i\zeta h_{\pm xx}-4h_{\pm xxx}$$
$$+6\begin{bmatrix} v^{2} & v_{x} \\ v_{x} & v^{2}\end{bmatrix}(\pm i\zeta h_{\pm}+h_{\pm x})+3\begin{bmatrix} 2vv_{x} & v_{xx} \\ v_{xx} & 2vv_{x}\end{bmatrix}h_{\pm}\mp 6i\zeta m^{2}h_{\pm}\,.$$

Substitute (5.8) into this and integrate by part. Then we have

$$g_{+}(x,\ \zeta,\ t)=\int_{0}^{\infty}J(x,\ y,\ t)\exp(2i\zeta y)dy\,,$$

where

$$J(x,\ y,\ t)=-K_{+xxx}+3\begin{bmatrix} v^{2}+m^{2} & v_{x} \\ v_{x} & v^{2}+m^{2}\end{bmatrix}K_{+x}\,.$$

As $F(x,\ y)$ is differentiable with respect to t, so is K_{+}. The relation

$$F_{t}+F_{xxx}-6m^{2}F_{x}=0$$

is valid. Hence we have

$$K_{+t}^{T}(x,\ y,\ t)+\int_{0}^{\infty}F(x+y+z,\ t)K_{+t}(x,\ z,\ t)dz=D(x,\ y,\ t)\,, \tag{5.9}$$

where

$$D(x,\ y,\ t)=\int_{0}^{\infty}(F_{xxx}(x+y+z,\ t)-6m^{2}F_{x}(x+y+z,\ t)K_{+}(x,\ z,\ t)dz$$
$$+(F_{xxx}(x+y+z,\ t)-6m^{2}F_{x}(x+y,\ t))^{t}(0,\ 1)\,.$$

By direct calculation, we can show that $J(x,\ y,\ t)$ satisfies (5.9). Therefore, by Theorem 3.1, $K_{+t}=J$ follows. Q.E.D.

Next we discuss the reflectionless solution which can be obtained under the assumption $r(\xi)\equiv 0$. This implies

$$F_{\pm}(x) = 2 \sum_{j=0}^{n} c_{\pm j} \begin{bmatrix} -\eta_j/m & 1 \\ 1 & -\eta_j/m \end{bmatrix} \exp(\mp 2\eta_j x).$$

This shows that we can express the unique solution $K(x, y)$ of the fundamental equation as

$$K(x, y) = 2 \sum_{j=0}^{n} c_j \begin{bmatrix} -\eta_j/m & 1 \\ 1 & -\eta_j/m \end{bmatrix} f_j(x) \exp(-2\eta_j(x+y)),$$

where $f_j(x) = {}^t(f_{1j}(x), f_{2j}(x))$. Substitute this into the fundamental equation (2.1), and we have the system of the $2(n+1)$ linear algebraic equations

$$(5.10) \qquad f_j(x) + \sum_{j=0}^{n} c_j \begin{bmatrix} -\eta_j/m & 1 \\ 1 & \eta_j - /m \end{bmatrix} (\eta_i + \eta_j)^{-1} \exp(-2\eta_j x) f_j(x) = -{}^t(1, 0), \qquad (i = 0, 1, \cdots, n),$$

whose coefficient matrix is easily seen to be nondegenerate. Let $f_{ij}(x)$ ($i=1, 2$ and $j=0, 1, \cdots, n$) be the unique solutions of (5.10). Then we have the reflectionless potential

$$(5.11) \qquad v_n^0(x) = 2 \sum_{j=1}^{n} c_j (m^{-1}\eta_j f_{1j}(x) - f_{2j}(x)) \exp(-2\eta_i x) + m.$$

Put

$$h_{\pm j}(x) = c_j(1 \mp m^{-1}\eta_j) \exp(-\eta_j x)(f_{1j}(x) \pm f_{2j}(x)),$$

where $j=1, 2, \cdots, n$ for $+$ and $j=0, 1, \cdots, n$ for $-$. Then we can rewrite the formula (5.11) as

$$(5.12) \qquad v_n^0(x) = \sum_{j=1}^{n} h_{+j}(x) \exp(-\eta_j x) - \sum_{j=0}^{n} h_{-j}(x) \exp(-\eta_j x) + m.$$

The functions $h_{\pm j}$ satisfy the linear algebraic equations

$$h_{\pm i}(x) + a_{\pm i} \exp(-\eta_i x) \sum_j (\eta_i + \eta_j)^{-1} h_{\pm j}(x) \exp(-\eta_j x) = -a_{\pm i} \exp(-\eta_i x),$$

where $a_{\pm i} = c_i(1 \mp m^{-1}\eta_i)$. Put

$$A_+ = (a_{+i} \exp(-(\eta_i + \eta_j)x)(\eta_i + \eta_j)^{-1})_{i,j=1,2 \cdots,n}$$

and

$$A_- = (a_{-i} \exp(-(\eta_i + \eta_j)x)(\eta_i + \eta_j)^{-1})_{i,j=0,1,\cdots,n}$$

Then $E_n + A_+$ and $E_{n+1} + A_-$ are positive definite, where E_k is the unit matrix of order k. (See [5; Lemma 1].)

We have

Proposition 5.3. *The equality*

$$v_n^0(x) = d\{\log(\det(E_n+A_+)/\det(E_{n+1}+A_-))\}/dx+m$$

holds.

Proof. By the Cramer's formula, we have

$$h_{+i}(x) = D_i/\det(E_n+A_+)\,,$$

where D_i is the determinant obtained by replacing the *i-th* column of $\det(E_n+A_+)$ by| ${}^t(-a_{+1}\exp(-\eta_1 x), -a_{+2}\exp(-\eta_2 x), \cdots, -a_{+n}\exp(-\eta_n x))$. On the other we have

$$d\{\log\det(E_n+A_+)\}/dx = \sum_{i=1}^n \Delta_i/\det(E_n+A_+)\,,$$

where Δ_i is the determinant obtained by replacing the i-th column of $\det(E_n+A_+)$ by ${}^t(-a_{+1}\exp(-(\eta_1+\eta_i)x), -a_{+2}\exp(-(\eta_2+\eta_i)x), \cdots, -a_{+n}\exp(-(\eta_n+\eta_i)x))$. Hence we have

$$\Delta_i = \exp(-\eta_i x)D_i\,.$$

Therefore we have

$$d\{\log\det(E_n+A_+)\}/dx = \sum_{i=0}^n h_{-i}(x)\exp(-\eta_i x)\,.$$

Completely pallalel to above, we have

$$d\{\log\det(E_{n+1}+A_-)\}/dx = \sum_{i=0}^n h_{-i}(x)\exp(-\eta_i x)\,. \qquad \text{Q.E.D.}$$

If the reflectionless scattering data $S_0=\{0, c_j(t), \kappa_j, j=0, 1, \cdots, n\}$ depend on t as (5.5), we denote the unique solutions of (5.10) which correspond to S_0 by $f_{ij}(x, t)$ ($i=1$, 2 and $j=0, 1, \cdots, n$). Then we have the explicit formula of the reflectionless solutions

$$v(x, t) = 2\sum_{j=0}^n c_j(m^{-1}\eta_j f_{1j}(x, t)-f_{2j}(x, t))\exp(-2\eta_j z_j)+m\,, \tag{5.13}$$

where $z_j=x-(4\eta_j^2-6m^2)t$.

Now suppose $n=0$ in (5.13), and we have

$$v_0^0(x, t) = m\tanh(m(x+2m^2t+\delta))\,,$$

where $\delta=(2m)^{-1}\log(c^{-1}m)$. Thus the reflectionless solutions (5.13) contain the traveling wave solution $v_0^0(x, t)$.

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