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FORGETFUL SPECTRAL SEQUENCES

Dedicated to Professor A. Komatu on his 70th birthday

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In [1] we descussed general properties of τ-cohomology theories. One of the basic tools in studying a τ-cohomology theory is the forgetful exact sequences which form a natural exact couple. Hence it provides a natural spectral sequence which we call the forgetful spectral sequences. An analysis of this spectral sequence will provide a deeper insight to the structure of the forgetful exact sequence, which we do for $MR^{*,*}(pt)$ in a forth-coming work. These spectral sequences are used partly by Landweber [4] for $MR^{*,*}(pt)$ and by Seymour [9] for $KR^{*,*}(X)$.

In the present work we study basic properties of these spectral sequences. In §1 we study elementary properties of them and show that a forgetful spectral sequence converges to the fixed-point cohomology under certain conditions (Theorem 1.14 and Proposition 1.16). In §2 we see that they have analogies with Bockstein spectral sequences with respect to differentials. In §3 we discuss periodicities which come essentially from Clifford modules. In §4 we study multiplicative properties of them for multiplicative τ-cohomology theories.

1. Definitions and elementary properties

In the present work every τ-cohomology theory is considered on pairs of *finite* τ-complexes for the sake of simplicity. Notations and terminologies of [1] are used freely.

Let $h^{*,*}$ be a τ-cohomology theory. There holds the following exact sequence

$$\cdots \to h^{p-1,q}(X, A) \xrightarrow{\chi} h^{p,q}(X, A) \xrightarrow{\psi} \psi h^{p+q}(X, A) \xrightarrow{\delta} h^{p-1,q+1}(X, A) \to \cdots,$$

called the forgetful exact sequence, for any pair (X, A) of finite τ-complexes [1], (5.1). Set

$$\begin{aligned} &D_1^{p,q} = h^{p,q}(X, A), \quad E_1^{p,q} = \psi h^{p+q}(X, A), \\ &i_1 = \chi\colon D_1^{p,q} \to D_1^{p+1,q}, \quad j_1 = \psi\colon D_1^{p,q} \to E_1^{p,q}, \\ &k_1 = \delta\colon E_1^{p,q} \to D_1^{p-1,q+1}. \end{aligned}$$

We get a bigraded exact couple

$$\langle D_1^{*,*}, E_1^{*,*}; i_1, j_1, k_1\rangle, D_1^{*,*} = \sum_{p,q} D_1^{p,q}, E_1^{*,*} = \sum_{p,q} E_1^{p,q},$$

in the sense of Massey [7], which yields a bigraded spectral sequence $E_r^{*,*} = \sum_{p,q} E_r^{p,q}$, $r \geqq 1$, in the standard way. This will be called the *forgetful spectral sequence* associated with $h^{*,*}(X, A)$. To make the meaning of bigrading of $E_1^{*,*}$ more precise, we remark that we are rather regarding as

$$E_1^{p,q} = h^{p,q}(S^{1,0} \times X, S^{1,0} \times A)$$

which is identified with the above setting through the isomorphism

$$h^{p,q}(S^{1,0} \times X, S^{1,0} \times A) \approx \psi h^{p+q}(X, A)$$

([1], (3.3)). The forgetful spectral sequence is natural with respect to (X, A) because of the naturality of the forgetful exact sequence. The forgetful spectral sequence associated with $MR^{*,*}(pt)$ was partially discussed by Landweber [4] and will be completely computed in a forth-coming paper of the author.

Put

$$D_r^{p,q} = \operatorname{Im}[\chi^{r-1}\colon h^{p-r+1,q}(X, A) \to h^{p,q}(X, A)]$$

for $r \geqq 2$ as usual. Then

$$\langle D_r^{*,*}, E_r^{*,*}; i_r, j_r, k_r\rangle, D_r^{*,*} = \sum_{p,q} D_r^{p,q}, E_r^{*,*} = \sum_{p,q} E_r^{p,q},$$

$r \geqq 2$, are successive derived couples. Since

$$\deg i_1 = (1, 0), \deg j_1 = (0, 0) \text{ and } \deg k_1 = (-1, 1).$$

we see that

$$\deg i_r = (1, 0), \deg j_r = (1-r, 0) \text{ and } \deg k_r = (-1, 1).$$

Hence the r-th differential of the spectral sequence is the following type of homomorphism:

$$d_r\colon E_r^{p,q} \to E_r^{p-r,q+1}. \tag{1.1}$$

The modules $D_r^{p,q}$ give the following decreasing filtration

$$h^{p,q}(X, A) = D_1^{p,q} \supset D_2^{p,q} \supset \cdots \supset D_r^{p,q} \supset D_{r+1}^{p,q} \supset \cdots$$

of $h^{p,q}(X, A)$. Put

$$D_\infty^{p,q} = \bigcap_r D_r^{p,q}. \tag{1.2}$$

A non-zero element of $D_\infty^{p,q}$ is called an *element of infinite filtration* of $h^{p,q}(X, A)$.

Thus $h^{p,q}(X, A)$ contains no elements of infinite filtration iff $D_\infty^{p,q}=\{0\}$. We say that $h^{*,*}(X, A)$ contains no elements of infinite filtration iff $h^{p,q}(X, A)$ contains no elements of infinite filtration for every $(p, q)\in Z\times Z$.

Put

$$(1.3)\qquad K_r^{p,q} = \operatorname{Ker}[\chi^{r-1}\colon h^{p,q}(X, A) \to h^{p+r-1,q}(X, A)]$$

for $1\leqq r<\infty$. We get an increasing sequence

$$\{0\} = K_1^{p,q}\subset K_2^{p,q}\subset\cdots\subset K_r^{p,q}\subset K_{r+1}^{p,q}\subset\cdots\subset h^{p,q}(X, A).$$

We define

$$(1.4)\qquad K_\infty^{p,q} = \bigcup_r K_r^{p,q}.$$

Express

$$E_r^{p,q} = Z_r^{p,q}/B_r^{p,q}$$

as the sub-quotient of $E_r^{p,q}$, $1\leqq r<\infty$. Then

$$(1.5)\qquad Z_r^{p,q} = k_1^{-1}D_r^{p-1,q+1} \text{ and } B_r^{p,q} = j_1K_r^{p,q}$$

for $1\leqq r<\infty$, as is well-known (e.g., cf., [6], p. 336, (5.4)).

Put

$$(1.6)\qquad Z_\infty^{p,q} = \bigcap_r Z_q^{p,q} \text{ and } B_\infty^{p,q} = \bigcup_r B_r^{p,q}.$$

Or equivalently,

$$(1.6')\qquad Z_\infty^{p,q} = k_1^{-1}D_\infty^{p-1,q+1} \text{ and } B_\infty^{p,q} = j_1K_\infty^{p,q}.$$

We define the E_∞-term by

$$E_\infty^{*,*} = \sum_{p,q} E_\infty^{p,q},\quad E_\infty^{p,q} = Z_\infty^{p,q}/B_\infty^{p,q}.$$

Observe the homomorphism

$$j_1\colon D_1^{p,q} \to E_1^{p,q}.$$

By (1.6′) we see that

$$j_1D_1^{p,q}\subset Z_\infty^{p,q}\subset Z_r^{p,q},\qquad 1\leqq r<\infty.$$

Since $\operatorname{Ker} j_1=D_2^{p,q}$, we see by (1.5) and (1.6′) that

$$j_1^{-1}B_r^{p,q} = D_2^{p,q}+K_r^{p,q},\qquad 1\leqq r\leqq\infty.$$

Thus j_1 induces injections

$$(1.7)\qquad l_r\colon D_1^{p,q}/(D_2^{p,q}+K_r^{p,q})\subset Z_\infty^{p,q}/B_r^{p,q}\subset E_r^{p,q}$$

for $1 \leqq r \leqq \infty$.

REMARK 1.8. If $h^{p-1,q+1}(X, A)$ contains no elements of infinite filtration, then $Z_\infty^{p,q} = \operatorname{Ker} k_1 = \operatorname{Im} j_1$ by (1.6′), whence l_∞ becomes an isomorphism in deg (p, q).

We shall see that $E_\infty^{p,q}$ is related with the direct limit $\varinjlim_s \{h^{s,q}(X, A), \chi\}$. Let

$$\kappa = \kappa_{p,q} \colon h^{p,q}(X, A) \to \varinjlim_s h^{s,q}(X, A)$$

be the canonical map and put

$$F^{p,q} = \operatorname{Im} \kappa_{p,q}. \tag{1.9}$$

We get an increasing filtration

$$\cdots \subset F^{p,q} \subset F^{p+1,q} \subset \cdots$$

of $\varinjlim_s h^{s,q}(X, A)$ such that $\bigcup_p F^{p,q} = \varinjlim_s h^{s,q}(X, A)$. By definitions we see that $\kappa_{p,q} \colon D_1^{p,q} \to F^{p,q}$ is surjective and $\operatorname{Ker} \kappa_{p,q} = K_\infty^{p,q}$. That is, we get an isomorphism

$$D_1^{p,q} / K_\infty^{p,q} \approx F^{p,q}$$

induced by $\kappa_{p,q}$. Let

$$\mathcal{G}(\lim_s h^{s,*}(X, A)) = \sum_{p,q} F^{p,q} / F^{p-1,q}$$

be the bigraded module associated with the filtration (1.9) of $\varinjlim_s h^{s,*}(X, A) = \sum_q \varinjlim_s h^{s,q}(X, A)$. Since the diagram

$$\begin{array}{ccc} D_1^{p,q} & \xrightarrow{\kappa_{p,q}} & \\ \uparrow \chi & & \varinjlim_s h^{s,q}(X, A) \\ D_1^{p-1,q} & \xrightarrow{\kappa_{p-1,q}} & \end{array}$$

is commutative, we get an isomorphism

$$\tilde{\kappa}_{p,q} \colon D_1^{p,q} / (D_2^{p,q} + K_\infty^{p,q}) \approx F^{p,q} / F^{p-1,q} \tag{1.10}$$

induced by $\kappa_{p,q}$.

Composing (1.10) with (1.7) for $r = \infty$ and making use of Remark 1.8 we obtain

Proposition 1.11. *The E_∞-term of the forgetful spectral sequence associated with $h^{*,*}(X, A)$ contains the bigraded module $\mathcal{G}(\varinjlim_s h^{s,*}(X, A))$ as a bigraded* *submodule. If $h^{*,*}(X, A)$ contains no elements of infinite filtration, then there holds the bigraded isomorphism $E_\infty^{*,*} \approx \mathcal{G}(\varinjlim_s h^{s,*}(X, A))$.*

When the forgetful cohomology theory ψh^* is connective, the forgetful spectral sequence behaves more conveniently for applications.

Lemma 1.12. *If the forgetful cohomology theory ψh^* is connective, then i_ϕ^* induces an isomorphism*

$$\varinjlim_s h^{s,q}(X, A) \approx \phi h^q(\phi X, \phi A)$$

for every $q \in Z$.

Proof. In the exact sequence of $h^{*,*}$ for the triple $(X, \phi X \cup A, A)$,

$$h^{s,q}(X, \phi X \cup A) \approx \tilde{h}^{s,q}(X/(\phi X \cup A)) \approx 0$$

for large s since $X/(\phi X \cup A)$ is almost free, [1], Lemma 5.3; and

$$h^{s,q}(\phi X \cup A, A) \approx h^{s,q}(\phi X, \phi A)$$

by excision isomorphism. Hence

$$i_\phi^*: h^{s,q}(X, A) \to h^{s,q}(\phi X, \phi A)$$

is isomorphic for large s, and

$$\varinjlim_s h^{s,q}(X, A) \approx \varinjlim_s h^{s,q}(\phi X, \phi A)$$

for all $q \in Z$, in which the right hand side is the same as $\phi h^q(\phi X, \phi A)$ by the definition of the fixed-point cohomology theory ϕh^*, [1], (4.4). q.e.d.

Thus, when ψh^* is connective, the filtration (1.9) in $\varinjlim_s h^{s,q}(X, A)$ can be regarded as a filtration in $\phi h^q(\phi X, \phi A)$ by the above lemma and the associated bigraded module $\mathcal{G}(\phi h^*(\phi X, \phi A))$ is isomorphic to $\mathcal{G}(\varinjlim_s h^{s,*}(X, A))$.

Lemma 1.13. *If the forgetful cohomology theory ψh^* is connective, then there exists an integer $r_0 = r_0(p, q) > 0$ for each $(p, q) \in Z \times Z$ such that*

$$K_{r_0}^{p,q} \approx K_{r_0+1}^{p,q} \approx \cdots \approx K_\infty^{p,q}.$$

Proof. $\psi h^{p+q}(X, A) \approx 0$ for large p. Hence, by the forgetful exact sequence χ: $h^{p,q}(X, A) \to h^{p+1,q}(X, A)$ is isomorphic for large p, which shows the lemma.

By Proposition 1.11 and Lemmas 1.12 and 1.13 we obtain

Theorem 1.14. *Let $h^{*,*}$ be a τ-cohomology theory such that the forgetful* *theory ψh^* is connective, and (X, A) be a pair of finite τ-complexes.* i) *The E_∞-term of the forgetful spectral sequence associated with $h^{*,*}(X, A)$ contains the bigraded module $\mathcal{G}(\phi h^*(\phi X, \phi A))$ as a bigraded submodule. If $h^{*,*}(X, A)$ contains no elements of infinite filtration, then there holds the bigraded isomorphism $E_\infty^{*,*} \approx \mathcal{G}(\phi h^*(\phi X, \phi A))$.* ii) *For each $(p, q) \in Z \times Z$ there exists an integer $r_0 = r_0(p, q) > 0$ such that*

$$B_{r_0}^{p,q} \approx B_{r_0+1}^{p,q} \approx \cdots \approx B_\infty^{p,q}$$

and there holds the decreasing sequence

$$E_{r_0}^{p,q} \supset E_{r_0+1}^{p,q} \supset \cdots \supset E_\infty^{p,q} .$$

When ψh^* is connective, we get an inclusion

$$\bigcap_{r > r_0} E_r^{p,q} \supset E_\infty^{p,q}$$

for each $(p, q) \in Z \times Z$. And the convergence problem of the forgetful spectral sequence becomes to find conditions which makes the above inclusion an actual isomorphism. Our spectral sequence is called *finitely convergent* if there exists an integer $r_1 = r_1(p, q) > 0$ for each $(p, q) \in Z \times Z$ such that

$$E_{r_1}^{p,q} \approx E_{r_1+1}^{p,q} \approx \cdots \approx E_\infty^{p,q} .$$

The finite convergence is of course the convergence in the strongest sense.

Lemma 1.15. *If ψh^* and ϕh^* are of finite type, and if ψh^* is connective, then $h^{p,q}(X, A)$ is finitely generated for any $(p, q) \in Z \times Z$ and any pair (X, A) of finite τ-complexes.*

Proof. Recall that a cohomology theory k^* is of finite type iff $k^n(pt)$ is finitely generated for every integer n. Then, by induction on dim Y we see that $k^n(Y, B)$ is finitely generated for any integer n and any pair (Y, B) of finite CW-complexes.

Now, i_ϕ^*: $h^{p,q}(X, A) \approx \phi h^q(\phi X, \phi A)$ for large p (cf., the proof of Lemma 1.12), which is finitely generated. Making use of the forgetful exact sequence and the fact that $\psi h^n(X, A)$ is finitely generated, an induction on p in decending order completes the proof.

We remark that the above lemma can be applied to $SR^{*,*}$ and $MR^{*,*}$.

Proposition 1.16. *The foregtful spectral sequence associated with $MR^{*,*}$ (X, A) is finitely convergent.*

Proof. Im $\chi = D_2^{p,q}$ consists of elements of order 2 by [1], Corollary 12.13. Hence it is a finite group for every $(p, q) \in Z \times Z$ by the above lemma, and the sequence

$$D_1^{p,q} \supset D_2^{p,q} \supset \cdots \supset D_\infty^{p,q}$$

becomes stationary after a finite step, which, combined with (1.5) and Theorem 1.14, implies the proposition. q.e.d.

The forgetful spectral sequence associated with $MR^{*,*}(X, A)$ converges to $MO^*(\phi X, \phi A)$ if $MR^{*,*}(X, A)$ contains no elements of infinite filtration. This assumption is true in case $(X, A)=pt$. But we have no good general characterization for vanishing of elements of infinite filtration at the moment.

2. Differentials

In this section we will see an analogy between forgetful and Bockstein spectral sequences with respect to differentials. In a Bockstein spectral sequence successive differentials are related to higher Bockstein operations. Similar phenomena can be found also in forgetful spectral sequences.

Let $h^{*,*}$ be a τ-cohomology theory. All discussions in the present work is valid for any pair (X, A) of finite τ-complexes unless otherwise stated. But we discuss mainly for a finite τ-complex X to simplify notations. The readers may replace X by (X, A), whereby X_+ by X/A of course.

The smash product of the τ-cofibration

$$S_+^{r,0} \xrightarrow{\iota_r} B_+^{r,0} \xrightarrow{\pi_r} B_+^{r,0}/S_+^{r,0} = \Sigma^{r,0} \tag{2.1}$$

with X_+ induces the exact sequence

$$\cdots \to \tilde{h}^{p,q}(\Sigma^{r,0}\wedge X_+) \xrightarrow{\pi_r^*} \tilde{h}^{p,q}(B_+^{r,0}\wedge X_+) \xrightarrow{\iota_r^*} \tilde{h}^{p,q}(S_+^{r,0}\wedge X_+)$$
$$\xrightarrow{\delta_r^*} \tilde{h}^{p,q+1}(\Sigma^{r,0}\wedge X_+) \to \cdots$$

for each integer $r>0$. Here

$$\tilde{h}^{p,q}(S_+^{r,0}\wedge X_+) = h^{p,q}(S^{r,0}\times X),$$
$$\tilde{h}^{p,q}(\Sigma^{r,0}\wedge X_+) \approx h^{p-r,q}(X)$$

by $\sigma^{r,0}$-suspension isomorphism, and

$$\tilde{h}^{p,q}(B_+^{r,0}\wedge X_+) \approx h^{p,q}(X)$$

since $B^{r,0}$ is τ-contractible. Thus we get the following exact sequence

$$\cdots \to h^{p-r,q}(X) \xrightarrow{\alpha_r} h^{p,q}(X) \xrightarrow{\beta_r} h^{p,q}(S^{r,0}\times X) \xrightarrow{\delta_r} h^{p-r,q+1}(X) \to \cdots \tag{2.2}$$

for each $r>0$. Particularly when $r=1$. (2.2) gives our fundamental exact couple to define forgetful spectral sequence, i.e., (2.3) $$\alpha_1 = i_1,\ \ \beta_1 = j_1 \ \text{ and } \ \delta_1 = k_1 .$$

Identify $B_+^{r,0} = B_+^{1,0} \wedge B_+^{r-1,0}$ and $\Sigma^{r,0} = \Sigma^{1,0} \wedge \Sigma^{r-1}$ for $r > 1$, then $\pi_r = \pi_1 \wedge \pi_{r-1}$, which implies that $\alpha_r = \alpha_1 \circ \alpha_{r-1}$. Thus, inductively on r, we see that

(2.4) $$\alpha_r = i_1^r\colon h^{p,q}(X) \to h^{p+r,p}(X) .$$

Therefore

$$D_r^{p,q} = \text{Im } \alpha_{r-1} \ \text{ and } \ K_r^{p,q} = \text{Ker } \alpha_{r-1}$$

and, by (1.5)

(2.4) $$Z_r^{p,q} = k_1^{-1}(\text{Im } \alpha_{r-1}) \ \text{ and } \ B_r^{p,q} = j_1(\text{Ker } \alpha_{r-1}) .$$

Let

$$\gamma_{r,s}\colon B^{r,0} \subset B^{s,0}$$

be the τ-inclusion such that $\gamma_{r,s}(t_1, \cdots, t_r) = (t_1, \cdots, t_r, 0, \cdots, 0)$ for $0 < r \leqq s$, which is an inclusion $(B^{r,0}, S^{r,0}) \subset (B^{s,0}, S^{s,0})$ or τ-pairs and hence induces τ-inclusions

(2.6) $$\eta_{r,s}\colon S^{r,0} \subset S^{s,0} \ \text{ and } \ \zeta_{r,s}\colon \Sigma^{r,0} \subset \Sigma^{s,0}$$

by restrictions and passing to quotients. The commutative diagram

(2.7) $$\begin{array}{ccccc} S^{r,0} & \xrightarrow{\iota_r} & B^{r,0} & \xrightarrow{\pi_r} & \Sigma^{r,0} \\ \downarrow{\scriptstyle \eta_{r,s}} & & \downarrow{\scriptstyle \gamma_{r,s}} & & \downarrow{\scriptstyle \zeta_{r,s}} \\ S^{s,0} & \xrightarrow{\pi_s} & B^{s,0} & \xrightarrow{\pi_s} & \Sigma^{s,0} \end{array}$$

induces a commutative diagram involving the exact sequences
(2.2) for r and s. Here

$$\gamma_{r,s}^* = id \,|\, h^{p,q}(X)$$

since $B^{r,0}$ and $B^{s,0}$ are τ-contractible, where $\gamma_{r,s}^* = (\gamma_{r,s} \wedge 1)^*$. $\zeta_{r,s}$ can be factorized as the composition of the sequence

$$\Sigma^{r,0} = \Sigma^{r,0} \wedge B_+^{0,0} \xrightarrow{1 \wedge \gamma_{0,s-r}} \Sigma^{r,0} \wedge B_+^{s-r,0} \xrightarrow{1 \wedge \pi_{s-r}} \Sigma^{s,0} .$$

Thus we see that

(2.8) $$\zeta_{r,s}^* = \alpha_{s-r} = i_1^{s-r}\colon h^{p,q}(X) \to h^{p+s-r,q}(X) .$$

Thereby no troubles arise from permutations of parameters because of [1], Proposition 4.2. Now we get the following commutative diagram:

(2.9) $$\begin{array}{ccccccccc} \to h^{p-s,q}(X) & \xrightarrow{\alpha_s} & h^{p,q}(X) & \xrightarrow{\beta_s} & h^{p,q}(S^{s,0} \times X) & \xrightarrow{\delta_s} & h^{p-s,q+1}(X) & \to \\ \downarrow{\scriptstyle i_1^{s-r}} & & \| & & \downarrow{\scriptstyle \eta_{r,s}^*} & & \downarrow{\scriptstyle i_1^{s-r}} & \\ \to h^{p-r,q}(X) & \xrightarrow{\alpha_r} & h^{p,q}(X) & \xrightarrow{\beta_r} & h^{p,q}(S^{r,0} \times X) & \xrightarrow{\delta_r} & h^{p-r,q+1}(X) & \to \end{array}$$

for $0<r\leqq s$.

In particular, putting $r=1$ and replacing s by r, we get the following commutative diagram:

(2.10)
$$\begin{tikzcd}
\rightarrow h^{p-r,q}(X) \arrow[r, "\alpha_r"] \arrow[d, "i_1^{r-1}"] & h^{p,q}(X) \arrow[r, "\beta_r"] \arrow[d, equal] & h^{p,q}(S^{r,0}\times X) \arrow[r, "\delta_r"] \arrow[d, "\eta^*_{1,r}"] & h^{p-r,q+1}(X) \rightarrow \arrow[d, "i_1^{r-1}"] \\
\rightarrow D_1^{p-1,q} \arrow[r, "i_1"] & D_1^{p,q} \arrow[r, "j_1"] & E_1^{p,q} \arrow[r, "k_1"] & D_1^{p-1,q+1} \rightarrow .
\end{tikzcd}$$

(2.10) yields the following commutative diagram:

(2.10′)
$$\begin{tikzcd}
h^{p,q}(S^{r,0}\times X) \arrow[rr, "\delta_r"] \arrow[d, "\eta^*_{1,r}"] & & h^{p-r,q+1}(X) \arrow[r, "\beta_r"] \arrow[d, equal] & h^{p-r,q+1}(S^{r,0}\times X) \arrow[d, "\eta^*_{1,r}"] \\
E_1^{p,q} \arrow[r, "k_1"] & D_1^{p-1,q+1} & D_1^{p-r,q+1} \arrow[l, "i_1^{r-1}"'] \arrow[r, "j_1"] & E_1^{p-r,q+1}
\end{tikzcd}$$

on one hand. Chasing the diagram (2.10) we see easily that

(2.11)
$$\mathrm{Im}\ \eta^*_{1,r} = Z_r^{p,q}$$

on the other hand. Hence, passing to sub-quotient of range, $\eta^*_{1,r}$ induces an *epimorphism*

(2.12)
$$\lambda_r\colon h^{p,q}(S^{r,0}\times X) \rightarrow E_r^{p,q}$$

for $r>0$. (2.10′) and (2.12) imply the following commutative diagram

(2.13)
$$\begin{tikzcd}
h^{p,q}(S^{r,0}\times X) \arrow[r, "\delta_r"] \arrow[d, "\lambda_r"] & h^{p-r,q+1}(X) \arrow[r, "\beta_r"] \arrow[d, "i_1^{r-1}"] & h^{p-1,q+1}(S^{r,0}\times X) \arrow[d, "\lambda_r"] \\
E_r^{p,q} \arrow[r, "k_r"] & D_r^{p-1,q+1} \arrow[r, "j_r"] & E_r^{p-r,q+1}
\end{tikzcd}$$

for $r>0$, where vertical maps are all epimorphic. As to the corresponding diagram of Bockstein spectral sequences we refer to [2], Proposition 11.1.

Obviously

$$\eta^*_{r,t} = \eta^*_{r,s}\circ\eta^*_{s,t}$$

for $0<r\leqq s\leqq t$. Since Ker $d_r=Z_{r+1}^{p,q}/B_r^{p,q}$ and $\lambda_r\circ\eta^*_{r,r+1}$ is induced from $\eta^*_{1,r+1}$ (2.11) implies that

(2.14)
$$\mathrm{Ker}\ d_r = \mathrm{Im}\ (\lambda_r\circ\eta^*_{r,r+1})$$

for $r>0$.

To discuss further relations we need some preparations. Let (X, A) and (Y, B) be compact τ-pairs. As quotient spaces of $X\cup X\times I\times Y\cup Y$, routine checks of identifying relations show the following equalities of τ-spaces:

(2.15) $$X*Y/(A*Y\cup X*B)=(X/A)\wedge\Sigma^{0,1}\wedge(Y/B)\,,$$

(2.16) $$(X*Y)/(X*B)=(C(X_+)/X)\wedge(Y/B)\,,$$

where "$*$" denote unreduced joins. These relations are well-known in non-equivariant case. Particularly when $B=\phi$, we get

(2.16′) $$X*Y/X=(C(X_+)/X)\wedge(Y_+)$$

from (2.16) because $X*\phi=X$.

Proposition 2.17. *There holds the τ-homeomorphism*

$$S^{r+s,0}/S^{r,0}\approx\Sigma^{r,0}(S_+^{s,0})$$

for $r>0$ and $s>0$.

Proof. Identify $S^{r,0}*S^{s,0}=S^{r+s,0}$ by the corresopndence "$(x,t,y)\mapsto(\sqrt{t}\,x,\sqrt{1-t}\,y)$", $x\in S^{r,0}$, $y\in S^{s,0}$, $0\leqq t\leqq 1$, and $C(S_+^{r,0})=B^{r,0}$ so that cone-parameters correspond to radial lengths. Then we get the proposition from (2.16′). q.e.d.

The τ-homeomorphism (2.17) may be given by the correspondence

(2.17′) $$(x,y)\mapsto(x,(1/\|y\|)y)$$

for $(x,y)\in S^{r+s,0}$, $x=(x_1,\cdots,x_r)$ and $y=(y_1,\cdots,y_s)$.

Let X be a finite τ-complex. The cofibration

$$S_+^{r,0}\xrightarrow{\eta_{r,r+s}}S_+^{r+s,0}\xrightarrow{\xi_{r+s,s}}S^{r+s,0}/S^{r,0}\approx\Sigma^{r,0}(S_+^{s,0})\,,$$

smashed with X_+, induces the following exact sequence

(2.18)
$$\begin{aligned}\cdots\to h^{p-r,q}(S^{s,0}\times X)&\xrightarrow{\xi^*_{r+s,s}}h^{p,q}(S^{r+s,0}\times X)\\&\xrightarrow{\eta^*_{r,r+s}}h^{p,q}(S^{r,0}\times X)\xrightarrow{\delta_{s,r}}h^{p-r,q+1}(S^{s,0}\times X)\to\cdots\,,\end{aligned}$$

where

(2.18′) $$\xi^*_{r+s,s}\colon h^{p,q}(S^{s,0}\times X)\to h^{p+r,q}(S^{r+s,0}\times X)$$

is the composition of the following sequence

$$\begin{aligned}&h^{p,q}(S^{s,0}\times X)=\tilde{h}^{p,q}(S_+^{s,0}\wedge X_+)\\&\xrightarrow{\sigma^{r,0}}\tilde{h}^{p+r,q}(\Sigma^{r,0}(S_+^{s,0})\wedge X_+)\approx\tilde{h}^{p+r,q}(S^{r+s,0}/S^{r,0}\wedge X_+)\ \text{by (2.17)}\\&\xrightarrow{(\xi_{r+s,s}\wedge 1)^*}\tilde{h}^{p+r,q}(S_+^{r+s,0}\wedge X_+)=h^{p+r,q}(S^{r+s,0}\times X)\,,\end{aligned}$$

and

(2.18″) $$\delta_{s,r}\colon h^{p,q}(S^{r,0}\times X)\to h^{p-r,q+1}(S^{s,0}\times X)$$

is the composition of the sequence

$$\begin{aligned} &h^{p,q}(S^{r,0}\times X)=\tilde{h}^{p,q}(S_+^{r,0}\wedge X_+)\\ &\xrightarrow{\delta^*}\tilde{h}^{p,q+1}(S^{r+s,0}/S^{r,0}\wedge X_+)\approx\tilde{h}^{p,q+1}(\Sigma^{r,0}(S_+^{s,0})\wedge X_+)\text{ by (2.17)}\\ &\xrightarrow{\sigma^{-r,0}}\tilde{h}^{p-r,q+1}(S_+^{s,0}\wedge X_+)=h^{p-r,q+1}(S^{s,0}\times X)\,. \end{aligned}$$

The projection $B^{r+s,0}\to B^{r,0}$ to the first r coordinates induces the following commutative diagram:

$$\begin{tikzcd} S^{r,0} \arrow[r, "\eta_{r,r+s}"] \arrow[d, equal] & S^{r+s,0} \arrow[r, "\xi_{r+s,s}"] \arrow[d, "\pi'_{r+s}"] & S^{r+s,0}/S^{r,0} \approx \Sigma^{r,0}(S_+^{s,0}) \arrow[d] \arrow[dr, "\Sigma^{r,0}\iota_s"] & \\ S^{r,0} \arrow[r, "\iota_r"] & B^{r,0} \arrow[r, "\pi_r"] & \Sigma^{r,0} & \Sigma^{r,0}(B_+^{s,0})\,, \arrow[l, "\Sigma^{r,0}\pi'"'] \end{tikzcd}$$

where $\pi'\colon B_+^{s,0}\to\Sigma^{0,0}$ is the τ-homotopy equivalence and π'_{r+s} is τ-equivalent to ι_{r+s} through the τ-homotopy equivalence $B^{r,0}\simeq_\tau B^{r+s,0}$. The commutativity of the right square follows from (2.17′). Hence it induces the following commutative diagram:

(2.19) $$\begin{tikzcd} \to h^{p-r,q}(X) \arrow[r, "\alpha_r"] \arrow[d, "\beta_s"] & h^{p,q}(X) \arrow[r, "\beta_r"] \arrow[d, "\beta_{r+s}"] & h^{p,q}(S^{r,0}\times X) \arrow[r, "\delta_r"] \arrow[d, equal] & h^{p-r,q+1}(X)\to \arrow[d, "\beta_s"] \\ \to h^{p-r,q}(A) \arrow[r, "\xi^*"] & h^{p,q}(B) \arrow[r, "\eta^*"] & h^{p,q}(S^{r,0}\times X) \arrow[r, dashed, "\delta_{s,r}"] & h^{p-r,q+1}(A)\to\,. \end{tikzcd}$$

where $A=S^{s,0}\times X$, $B=S^{r+s,0}\times X$, $\xi^*=\xi^*_{r+s\ s}$, $\eta^*=\eta^*_{r,r+s}$ for simplicity, and the horizontal sequences are exact sequences (2.2) and (2.18).

Proposition 2.20. *$Z_r^{p,q}=\mathrm{Ker}\ \delta_{r-1,1}$ and $B_r^{p,q}=\mathrm{Im}\ \delta_{1,r-1}$ for $r>1$.*

Proof. Replace (r, s) by $(1, r-1)$ in the diagram (2.19). By (2.3) and (2.4) we see that

$$\delta_{r-1,1}=\beta_{r-1}\circ\delta_1=\beta_{r-1}\circ k_1$$

up to signs and

$$Z_r^{p,q}=k_1^{-1}(\mathrm{Im}\ \alpha_{r-1})=\mathrm{Ker}\ (\beta_{r-1}\circ k_1)=\mathrm{Ker}\ \delta_{r-1\ l}\,.$$

Next, replace (r, s) by $(r-1, 1)$ in the diagram (2.19). Then

$$\delta_{1,r-1}=\beta_1\circ\delta_{r-1}=j_1\circ\delta_{r-1}$$

and

$$B_r^{p,q}=j_1(\mathrm{Ker}\ \alpha_{r-1})=\mathrm{Im}\ (j_1\circ\delta_{r-1})=\mathrm{Im}\ \delta_{1,r-1}\,. \qquad \text{q.e.d.}$$

(2.19) for $s=1$, combined with (2.13), implies the following commutative diagram:

$$(2.21)\qquad \begin{array}{ccc} h^{p,q}(S^{r,0}\times X) & \xrightarrow{\delta_{1,r}} & Z_r^{p-r,q+1}\subset E_1^{p-r,q+1} \\ \downarrow{\lambda_r} & & \downarrow{p_r} \\ E_r^{p,q} & \xrightarrow{d_r} & E_r^{p-r,q+1}, \end{array}$$

where p_r is the canonical projection, which will be used in the next section.

3. Periodicities

As usual, a map

$$(3.1)\qquad \mu\colon R^k\times R^n \to R^n$$

is called an orthogonal multiplication iff μ is bilinear and norm-preserving, i.e., $||\mu(x, y)||=||x||\cdot||y||$ for any $x\in R^k$ and $y\in R^n$. The orthogonal multiplication (3.1) will be called to be *of type* (k, n). As is well-known, an orthogonal multiplication of type (k, n) exists iff R^n becomes a C_{k-1}-module, where C_k denotes the Clifford algebra generated by R^k with a negative definite quadratic form and $C_0=R$. Hence it exists iff n is a multiple of a_k, where $a_k=2^{\varphi(k-1)}$, $\varphi(k)$=the number of integers s such that $0<s\leqq k$ and $s\equiv 0,1,2$ or 4 mod 8. (Cf., e.g., [3]).

Let μ be an orthogonal multiplication of type (k, n). Denote $\mu(x, y)=xy$ for simplicity and define a map

$$\omega_{k,n}\colon R^{k,0}\times R^{n,0} \to R^{k,0}\times R^{0,n}$$

by $\omega_{k,n}(x, y)=(x, xy)$. Then $\omega_{k,n}$ is a τ-map and induces a τ-homeomorphism (denoted by the same letter)

$$(3.2)\qquad \omega_{k,n}\colon S^{k,0}\times(B^{n,0}, S^{n,0}) \approx S^{k,0}\times(B^{0,n}, S^{0,n})$$

of τ-pairs by restricting domain and range, which is called a *periodicity map of type* (k, n).

Let $h^{*,*}$ be a τ-cohomology theory and X a finite τ-complex. The τ-homeomorphism (3.2) induces the isomorphism

$$(3.3)\qquad \omega_{k,n}^*\colon h^{p,q}(S^{k,0}\times X) \approx h^{p-n,q+n}(S^{k,0}\times X)$$

for any $(p, q)\in Z\times Z$ as the composition of the following sequence of isomorphisms:

$$\begin{aligned} h^{p,q}(S^{k,0}\times X) &\overset{\sigma^{0,n}}{\approx} h^{p,q+n}(S^{k,0}\times(B^{0,n}, S^{0,n})\times X) \\ &\overset{(\omega_{k,n}\wedge 1)^*}{\approx} h^{p,q+n}(S^{k,0}\times(B^{n,0}, S^{n,0})\times X) \\ &\overset{\sigma^{-n,0}}{\approx} h^{p-n,q+n}(S^{k,0}\times X)\,. \end{aligned}$$

The isomorphism (3.3) is of course natural with respect to X, and hence we may regard it as a kind of periodicity. Since a_k is the minimal n for a fixed k such that $\omega_{k\ n}$ is defined, we obtain

Proposition 3.4. *$h^{*,*}$ with coefficients $S^{k,0}$ admits a periodic isomorphism of period $(-a_k, a_k)$ for any τ-cohomology theory $h^{*,*}$.*

Let r and s be positive integers and

$$\omega_{r+s}\colon S^{r+s,0}\times(B^{n,0}, S^{n,0}) \approx S^{r+s,0}\times(B^{0,n}, S^{0,n})$$

be a periodicity map of type $(r+s, n)$. Let

$$S^{r,0}\subset S^{r+s,0} \text{ and } S^{s,0}\subset S^{r+s,0}$$

be τ-inclusions to the first r and last s coordinates respectively. By restricting ω_{r+s} we get induced periodicity maps

$$\omega_r\colon S^{r,0}\times(B^{n,0}, S^{n,0}) \approx S^{r,0}\times(B^{0,n}, S^{0,n})$$

and

$$\omega_s\colon S^{s,0}\times(B^{n,0}, S^{n,0}) \approx S^{s,0}\times(B^{0,n}, S^{0,n})$$

respectively. The cofibration $S^{r,0}_+\subset S^{r+s,0}_+\to S^{r+s,0}/S^{r,0}$, smashed with $\Sigma^{n,0}$ and $\Sigma^{0,n}$, yields the following commutative diagram:

$$\begin{array}{ccccc}
S^{r,0}_+\wedge\Sigma^{n,0} & \to & S^{r+s,0}_+\wedge\Sigma^{n,0} & \to & (S^{r+s,0}/S^{r,0})\wedge\Sigma^{n,0} \\
\downarrow{\scriptstyle \omega_r} & & \downarrow{\scriptstyle \omega_{r+s}} & & \downarrow{\scriptstyle \omega'} \\
S^{r,0}_+\wedge\Sigma^{0,n} & \to & S^{r+s,0}_+\wedge\Sigma^{0,n} & \to & (S^{r+s,0}/S^{r,0})\wedge\Sigma^{r,0}\,,
\end{array}$$

where ω' is the induced τ-homeomorphism. Replace $S^{r+s,0}/S^{r,0}$ by $\Sigma^{r,0}(S^{s,0}_+)$, Proposition 2.17. Then ω' will be replaced by the τ-homeomorphism

$$\omega''\colon \Sigma^{r,0}(S^{s,0}_+)\wedge\Sigma^{n,0} \approx \Sigma^{r,0}(S^{s,0}_+)\wedge\Sigma^{0,n}\,.$$

Lemma 3.5. $\omega''\simeq_\tau \Sigma^{r,0}\omega_s$.

Proof. Indentify

$$\Sigma^{r,0}(S^{s,0}_+) = (S^{r,0}\times I\times S^{s,0}/S^{r,0}\times 1\times S^{s,0})\cup_{\pi_0} S^{s,0}\,,$$

where $\pi_0\colon S^{r,0}\times 0\times S^{s,0}\to S^{s,0}$ is the projection. Then ω'' is given by

$$\omega''((u, t, v), y) = ((u, t, v), (\sqrt{t}\,u, \sqrt{1-t}\,v)y)\,,$$

$u\in S^{r,0}$, $v\in S^{s,0}$, $0\leqq t\leqq 1$, and $y\in\Sigma^{n,0}$. Define a τ-homotopy

$$\Omega_\theta\colon S^{r,0}\times I\times S^{s,0}\times(B^{n,0}, S^{n,0}) \to S^{r,0}\times I\times S^{s,0}\times(B^{0,n}, S^{0,n})$$

by $\Omega_\theta((u, t, v), y)=((u, t, v), (\sqrt{t\theta}u, \sqrt{1-t\theta}\, v)y)$, $0\leqq\theta\leqq 1$. Since $\Omega_\theta(S^{r,0}\times 1\times S^{s,0}\times B^{n,0})\subset S^{r,0}\times 1\times S^{s,0}\times B^{0,n}$, passing to quotients we get a τ-homotopy

$$\tilde{\Omega}_\theta\colon (S^{r,0}\times I\times S^{s,0}/S^{r,0}\times 1\times S^{s,0})\wedge\Sigma^{n,0}\to(S^{r,0}\times I\times S^{s,0}/S^{r,0}\times 1\times S^{s,0})\wedge\Sigma^{0,n},$$

$0\leqq\theta\leqq 1$. $\tilde{\Omega}_\theta|S^{r,0}\times 0\times S^{s,0}\times\Sigma^{n,0}$ does not depend on θ and transforms to ω_s through projection $\pi_0\times 1$. Thus, putting $\omega_\theta=\tilde{\Omega}_\theta\cup_{\pi_0\times 1}\omega_s$ we get a τ-homotopy

$$\omega_\theta\colon \Sigma^{r,0}(S^{s,0}_+)\wedge\Sigma^{n,0}\to\Sigma^{r,0}(S^{s,0}_+)\wedge\Sigma^{0,n},$$

$0\leqq\theta\leqq 1$. By construction we see easily that

$$\omega_0=\Sigma^{r,0}\omega_s \text{ and } \omega_1=\omega''. \qquad \text{q.e.d.}$$

By the above lemma we get the following τ-homotopy-commutative diagram of cofibrations:

$$\begin{array}{ccccc} S^{r,0}_+\wedge\Sigma^{n,0} & \to & S^{r+s,0}_+\wedge\Sigma^{n,0} & \to & \Sigma^{r,0}(S^{s,0}_+)\wedge\Sigma^{n,0} \\ \downarrow{\scriptstyle\omega_r} & & \downarrow{\scriptstyle\omega_{r+s}} & & \downarrow{\scriptstyle\Sigma^{r,0}\omega_s} \\ S^{r,0}_+\wedge\Sigma^{0,n} & \to & S^{r+s,0}_+\wedge\Sigma^{0,n} & \to & \Sigma^{r,0}(S^{s,0}_+)\wedge\Sigma^{0,n}. \end{array}$$

Thus we get

Proposition 3.6. *Let*

$$\omega_{r+s,n}\colon S^{r+s,0}\times(B^{n,0}, S^{n,0})\approx S^{r+s,0}\times(B^{0,n}S^{0,n})$$

be a periodicity map of type $(r+s, n)$, *and*

$$\omega_{r,n}\colon S^{r,0}\times(B^{n,0}, S^{n,0})\approx S^{r,0}\times(B^{0,n}, S^{0,n}),$$
$$\omega_{s,n}\colon S^{s,0}\times(B^{n,0}, S^{n,0})\approx S^{s,0}\times(B^{0,n}, S^{0,n})$$

be periodicity maps defined by restricting $\omega_{r+s\,n}$, *i.e.*, $\omega_{r,n}=\omega_{r+s,n}|S^{r,0}\times(B^{n,0}, S^{n,0})$ *and* $\omega_{s,n}=\omega_{r+s,n}|S^{s,0}\times(B^{n,0}, S^{n,0})$. *Then homomorphisms in the exact sequence* (2.18) *commute with periodicity isomorphisms* $\omega^*_{r,n}$, $\omega^*_{s,n}$ *and* $\omega^*_{r+s,n}$ *up to signs.*

The signs in the above proposition come from permutations of suspension parameters and depend only on r, s and n.

Let

$$\text{(3.7)} \qquad \omega\colon S^{1,0}\times(B^{n,0}, S^{n,0})\approx S^{1,0}\times(B^{0,n}, S^{0,n})$$

be the periodicity map defined by

$$\text{(3.7')} \qquad \omega(-1, x)=(-1, x) \text{ and } \omega(1, x)=(1, -x),$$

$x\in B^n$, which is defined for all $n\geqq 1$ and will be called the *canonical* periodicity map of type $(1, n)$.

Let

$$\omega_{1,n}\colon S^{1,0}\times(B^{n,0}, S^{n,0}) \approx S^{1,0}\times(B^{0,n}, S^{0,n})$$

be an arbitrary periodicity map of type $(1, n)$. Since $\omega_{1,n}$ is defined by making use of an orthogonal multiplication, we see that the assignment "$x\mapsto\pi_2\circ\omega_{1,n}(1, x)$", $x\in R^n$, is an orthogonal map, where $\pi_2\colon S^{1,0}\times R^n\to R^n$ is the 2-nd projection. Hence there exists $A\in 0(n)$ such that $\omega_{1,n}(-1, x)=(-1, xA)$ for all $x\in B^n$. Then $\omega_{1,n}(1, x)=(1, -xA)$ by bilinearlity. Thus we get the following commutative diagram of τ-homeomorphisms:

$$\begin{tikzcd} & S^{1,0}\times(B^{0,n}, S^{0,n}) \arrow[d, "1\times A"] \\ S^{1,0}\times(B^{n,0}, S^{n,0}) \arrow[ur, "\omega"] \arrow[r, "\omega_{1,n}"'] & S^{1,0}\times(B^{0,n}, S^{0,n}), \end{tikzcd}$$

which implies that

$$\omega^*_{1,n}=(\det A)\circ\omega^*\,.$$

And we obtain

Proposition 3.8. *Every periodicity isomorphism $\omega^*_{1,n}$ coincides to each other up to ±1 times for the same n.*

In the sense of this proposition we may regard that $\omega^*_{1,n}$ is essentially unique.

Take an integer $r>0$. Let $\omega_{r,n}$ be a periodicity map of type (r, n), and ω' and ω'' be periodicity maps of types $(1, n)$ and $(r-1, n)$ respectively induced by $\omega_{r,n}$ restricted. By Proposition 3.6 we get the (up to signs) commutative diagram

$$\begin{tikzcd} h^{p,q}(S^{1,0}\times X) \arrow[r, "\delta_{r-1,1}"] \arrow[d, "\wr\ \omega'^*"] & h^{p-1,q+1}(S^{r-1,0}\times X) \arrow[d, "\wr\ \omega''^*"] \\ h^{p-n,q+n}(S^{1,0}\times X) \arrow[r, "\delta_{r-1,1}"] & h^{p-n-1,q+n+1}(S^{r-1,0}\times X), \end{tikzcd}$$

which implies that

$$\omega'^*\colon Z_r^{p,q}\approx Z_r^{p-n,q+n}$$

by Proposition 2.20. Similarly we get the commutative diagram

$$\begin{tikzcd} h^{q,q}(S^{r-1,0}\times X) \arrow[r, "\delta_{1\ r-1}"] \arrow[d, "\wr\ \omega''^*"] & h^{p-r+1,q+1}(S^{1,0}\times X) \arrow[d, "\wr\ \omega'^*"] \\ h^{p-n,q+n}(S^{r-1,0}\times X) \arrow[r, "\delta_{1,r-1}"] & h^{p-n-r+1,q+n+1}(S^{1,0}\times X), \end{tikzcd}$$

which implies that

$$\omega'^{*}: B_r^{p,q} \approx B_r^{p-n,q+n}$$

by Proposition 2.20. Now, passing to quotients we get an induced periodic isomorphism

$$\omega_{r,n}^{*}: E_r^{p,q} \approx E_r^{p-n,q+n} \tag{3.9}$$

which is unique up to ± 1 times by Proposition 3.8. Note that (3.9) is essentially induced by $\omega_{r,n}$ so that we denoted it by $\omega_{r,n}^{*}$.

The above periodicity map $\omega_{r,n}$ induces a periodicity map of type (s, n) by restriction for $0<s\leqq r$ and hence gives rise to a periodic isomorphism

$$\omega_{r,n}^{*}: E_s^{p,q} \approx E_s^{p-n,q+n} \tag{3.9$'$}$$

for $0<s\leqq r$, which is denoted also by $\omega_{r,n}^{*}$.

Using a periodicity map of type (r, a_r) we obtain

Theorem 3.10. *Let $h^{*,*}$ be a τ-cohomology theory and X a finite τ-complex. In the forgetful spectral sequence associated with $h^{*,*}(X)$ there holds a periodic isomorphism*

$$\varpi_r: E_r^{p,q} \approx E_r^{p-a_r,q+a_r}$$

of period $(-a_r, a_r)$ for each r, $0<r<\infty$, and any $(p, q)\in Z\times Z$. This isomorphism is unique up to ± 1 times and natural with respect to X.

Remark that the epimorphism λ_r, (2.12), is induced by $\eta_{1,r}^{*}$ and hence commutes with $\omega_{r,n}^{*}$ by Proposition 3.6.

Next, all maps in the diagram (2.21) except d_r commute up to signs with periodic isomorphisms induced by a periodicity map of type $(r+1, n)$. Since λ_r and p_r are surjective in (2.21), it follows that d_r commutes also with such a periodic isomorphism up to signs ± 1. Thus we obtain

Theorem 3.11. *Let $\omega_{r+1,n}$ be a periodicity map of type $(r+1, n)$. In the forgetful spectral sequence associated with $h^{*,*}(X)$, let $\omega_{r+1,n}^{*}: E_r^{p,q}\approx E_r^{p-n,q+n}$ be the induced periodic isomorphism. Then the following diagram is commutative up to ± 1 times:*

$$\begin{array}{ccc} E_r^{p,q} & \xrightarrow{d_r} & E_r^{p-r,q+1} \\ \wr\!\wr\ \omega_{r+1,n}^{*} & & \wr\!\wr\ \omega_{r+1,n}^{*} \\ E_r^{p-n,q+n} & \xrightarrow{d_r} & E_r^{p-n-r,q+n+1} \end{array}.$$

In particular, the periodic isomorphism

$$\omega_{r+1,a_{r+1}}^{*}: E_r^{p,q} \approx E_r^{p-a_{r+1},q-a_{r+1}}$$

commutes with d_r up to sign ± 1.

Let $\omega_{1,m}$ and $\omega_{1,n}$ be periodicity maps of types $(1, m)$ and $(1, n)$ respectively. $\omega_{1,m}\times\omega_{1,n}$, restricted to the diagonal of $S^{1,0}\times S^{1,0}$, gives a periodicity map $\omega_{1,m+n}$. Obviously we can decompose as

$$\omega_{1,m+n} = (\omega_{1,m}\times 1)\circ(1\times\omega_{1,n}),$$

which implies that

$$\omega^*_{1,m+n} = \pm\omega^*_{1,m}\circ\omega^*_{1,n} \tag{3.12}$$

because $\rho=-1$ on $h^{*,*}(S^{1,0}\times X)$, [1], Proposition 3.6. (3.12) and Proposition 3.8 imply that

Proposition 3.13. *Let $\omega_{r,n}$ be a periodicity map of type (r, n). Then the induced periodic isomorphism $\omega^*_{r,n}$: $E_r^{p,q}\approx E_r^{p-n,q+n}$ is equal to an iterated composition of ϖ_r up to ± 1 times.*

4. Multiplicative structures

Here we discuss multiplicative properties of forgetful spectral sequences.

Let $h^{*,*}$ be a multiplicative τ-cohomology theory, [1], §6, and X and Y be finite τ-complexes. Let

$$\mu\colon h^{p,q}(X)\otimes h^{p',q'}(Y)\to h^{p+p',q+q'}(X\times Y)$$

be the multiplication in $h^{*,*}$, $(p, q, p', q')\in Z\times Z\times Z\times Z$, which determines the following natural homomorphisms:

$$\begin{aligned}
\mu^D &= \mu\colon D_1^{p,q}(X)\otimes D_1^{p',q'}(Y)\to D_1^{p+p',q+q'}(X\times Y)\\
\mu_1 &= (d\times 1)^*\circ\mu\colon E_1^{p,q}(X)\otimes E_1^{p',q'}(Y)\to E_1^{p+p',q+q'}(X\times Y),\\
\mu' &= \mu\colon E_1^{p,q}(X)\otimes D_1^{p',q'}(Y)\to E_1^{p+p',q+q'}(X\times Y),\\
\mu'' &= \mu\colon D_1^{p,q}(X)\otimes E_1^{p',q'}(Y)\to E_1^{p+p',q+q'}(X\times Y)
\end{aligned}$$

after obvious identifications by switching maps, where $d\colon S^{1,0}\subset S^{1,0}\times S^{1,0}$ is the diagonal inclusion.

First we observe the relations of these pairings with i_1, j_1 and k_1. By [1], (6.4), we obtain the relations

$$i_1\circ\mu^D(x\otimes y) = \mu^D(i_1x\otimes y) = \mu^D(x\otimes i_1y). \tag{4.1}$$

Let $\pi\colon S_+^{1,0}\to\Sigma^{0,0}$ be the map to collapse $S^{1,0}$ to 0. The commutative diagrams,

$$\begin{tikzcd}
 & S_+^{1,0}\wedge S_+^{1,0} \arrow[dr, "\pi\wedge 1"] & \\
S_+^{1,0} \arrow[ur, "d"] \arrow[rr, "1"] & & \Sigma^{0,0}\wedge S_+^{1,0} = S_+^{1,0},
\end{tikzcd}
\quad
\begin{tikzcd}
 & S_+^{1,0}\wedge S_+^{1,0} \arrow[dr, "1\wedge\pi"] & \\
S_+^{1,0} \arrow[ur, "d"] \arrow[rr, "1"] & & S_+^{1,0}\wedge\Sigma^{0,0} = S_+^{1,0},
\end{tikzcd}$$

imply the following relations

$$\begin{aligned}\mu'(x\otimes y)&=\mu_1(x\otimes j_1 y)\,,\\ \mu''(x\otimes y)&=\mu_1(j_1 x\otimes y)\,.\end{aligned}\tag{4.2}$$

And the naturality of μ implies the relations

$$j_1\circ\mu^D(x\otimes y)=\mu'(j_1 x\otimes y)=\mu''(x\otimes j_1 y)\,.\tag{4.2'}$$

By naturality of μ and the relations of μ with suspensions, [1], §6, $M3$), a routine diagram implies the relation

$$k_1\circ\mu'(x\otimes y)=\mu^D(k_1 x\otimes y)\,.\tag{4.3}$$

And, by a parallel diagram we see that $k_1\circ\mu''(x\otimes y)$ and $\mu^D(x\otimes k_1 y)$ coincide except the sign $(-\rho)^q$ for $x\in D_1^{p,q}(X)$. But $\rho=-1$ on $E_1^{*,*}(Y)=h^{*,*}(S^{1,0}\times Y)$, [1], Proposition 3.6, and hence we get the relation

$$k_1\circ\mu''(x\otimes y)=\mu^D(x\otimes k_1 y)\,.\tag{4.3'}$$

(4.2) and (4.2′) imply relations

$$\mu_1(j_1 x\otimes j_1 y)=\mu''(x\otimes j_1 y)=\mu'(j_1 x\otimes y)=j_1\circ\mu^D(x\otimes y)\,.\tag{4.4}$$

In particular, $j_1\colon D_1^{*,*}\to E_1^{*,*}$ is multipliative, which is but already observed in [1], §6.

(4.1) implies that

$$\mu^D(i_1^{r-1}x\otimes i_1^{s-1}y)=i_1^{r+s-2}\circ\mu^D(x\otimes y)\tag{4.5'}$$

for $r>0$ and $s>0$. Hence we get homomorphisms

$$\mu_{r,s}^D\colon D_r^{p,q}(X)\otimes D_s^{p',q'}(Y)\to D_{r+s-1}^{p+p',q+q'}(X\times Y)\tag{4.5}$$

by restricting μ^D. Off course $\mu_{1,1}^D=\mu^D$.

(4.3) and (4.3′) imply that

$$\begin{aligned}\mu'(Z_r^{p,q}(X)\otimes D_s^{p',q'}(Y))&\subset Z_{r+s-1}^{p+p',q+q'}(X\times Y)\,,\\ \mu''(D_r^{p,q}(X)\otimes Z_s^{p',q'}(Y))&\subset Z_{r+s-1}^{p+p',q+q'}(X\times Y)\,.\end{aligned}\tag{4.6}$$

Next, let $x\in B_r^{p,q}(X)$ and $y\in D_s^{p',q'}(Y)$. Express as $x=j_1x'$, $i_1^{r-1}x'=0$ by (1.5). Then

$$\mu'(x\otimes y)=j_1\circ\mu^D(x'\otimes y)$$

by (4.2′), and

$$i_1^{r-1}\circ\mu^D(x'\otimes y)=\mu^D(i_1^{r-1}x'\otimes y)=0\,.$$

Thus $\mu'(x\otimes y)\in B_r^{p+p',q+q'}(X\times Y)$ by (1.5). Furthermore, if $s>1$ then $j_1\circ\mu^D(x'\otimes y)=\mu''(x'\otimes j_1 y)=0$. And we get

$$(4.6') \qquad \begin{aligned} &\mu' (B_r^{p,q}(X) \otimes D_1^{p',q'}(Y)) \subset B_r^{p+p',q+q'}(X \times Y)\,, \\ &\mu' (B_r^{p,q}(X) \otimes D_s^{p',q'}(Y)) = 0 \quad \text{for } s > 1\,. \end{aligned}$$

Similarly we get

$$(4.6'') \qquad \begin{aligned} &\mu''(D_1^{p,q}(X) \otimes B_s^{p',q'}(Y)) \subset B_s^{p+p',q+q'}(X \times Y)\,, \\ &\mu''(D_r^{p,q}(X) \otimes B_s^{p',q'}(Y)) = 0 \quad \text{for } r > 1\,. \end{aligned}$$

By (4.6), (4.6′) and (4.6″), μ' and μ'' indue homomorphisms

$$(4.7) \qquad \begin{aligned} &\mu'_{r,s}\colon E_r^{p,q}(X) \otimes D_s^{p',q'}(Y) \to E_{r+s-1}^{p+p',q+q'}(X \times Y)\,, \\ &\mu''_{r,s}\colon D_r^{p,q}(X) \otimes E_s^{p',q'}(Y) \to E_{r+s-1}^{p+p',q+q'}(X \times Y) \end{aligned}$$

by passing to sub-quotients.

Observe $\mu^D_{r,1}$, $\mu^D_{1,r}$, $\mu'_{r,1}$ and $\mu''_{1,r}$ by (4.5) and (4.7), then we obtain

Proposition 4.8. *$D_r^{*,*}$ and $E_r^{*,*}$ are functors of bilateral bigraded $h^{*,*}$-modules.*

By definitions, (4.1) and (4.2′), we see easily the relations

$$(4.9) \qquad \begin{aligned} j_r \circ \mu^D_{r,1}(x \otimes y) &= \mu'_{r,1}(j_r x \otimes y)\,, \\ j_s \circ \mu^D_{1,s}(x \otimes y) &= \mu''_{1,s}(x \otimes j_s v)\,. \end{aligned}$$

And, (4.3) and (4.3′) imply

$$(4.10) \qquad \begin{aligned} k_{r+s-1} \circ \mu'_{r,s}(x \otimes y) &= \mu^D_{r,s}(k_r x \otimes y)\,, \\ k_{r+s-1} \circ \mu''_{r,s}(x \otimes y) &= \mu^D_{r,s}(x \otimes k_s y)\,. \end{aligned}$$

Then, (4.9) and (4.10) yield

$$(4.11) \qquad \begin{aligned} d_r \circ \mu'_{r,1}(x \otimes y) &= \mu'_{r,1}(d_r x \otimes y)\,, \\ d_s \circ \mu''_{1,s}(x \otimes y) &= \mu''_{1,s}(x \otimes d_s y)\,. \end{aligned}$$

By (4.1), (4.9), (4.10) and (4.11) we obtain

Proposition 4.12. *i_r, j_r, k_r and d_r are homomorphisms of bilateral $h^{*,*}$-modules.*

Corollary 4.13. *The forgeftul spectral sequences are spectral sequences of bilateral $h^{*,*}$-modules whenever $h^{*,*}$ is multiplicative.*

Before going into discussions of multiplications in the forgetful spectral sequences we need to prepare an involution in each spectral sequence arising from its structure.

Let $\tau\colon S^{1,0} \to S^{1,0}$ be the involution of $S^{1,0}$. Since τ is a τ-map we have an induced homomorphism

$$(\tau\times 1)^*\colon E_1^{p,q}\to E_1^{p,q}$$

for each $(p, q)\in Z\times Z$. Put

$$(4.14)\qquad (\tau\times 1)^* x = \bar{x}$$

for each $x\in E_1^{p,q}$. Clearly, the map "$x\mapsto\bar{x}$" is involutive.

Let $\pi\colon S_+^{1,0}\to\Sigma^{0,0}$ be the τ-map to collapse $S^{1,0}$ to 0. The commutativity of the following diagram

$$\begin{tikzcd} S_+^{1,0} \arrow[d, "\tau"] \arrow[dr, "\pi"] & \\ S_+^{1,0} \arrow[r, "\pi"'] & \Sigma^{0,0} \end{tikzcd}$$

implies

$$(4.15)\qquad \bar{x} = x \qquad \textit{whenever}\ \ x\in \operatorname{Im} j_1.$$

The involution of the pair $(B^{1,0}, S^{1,0})$ is also a τ-map. Then the naturality of the connecting morphisms and A4) of [1], §2, imply that

$$k_1\bar{x} = \rho\circ k_1 x\,.$$

On the otherhand, k_1 is a Λ-module map, whence

$$\rho\circ k_1 x = k_1(\rho\circ x) = k_1(-x)\,.$$

Hence

$$(4.16)\qquad k_1(x+\bar{x}) = 0$$

for any $x\in E_1^{p,q}$.

The present involution of $E_1^{p,q}$ induces involutions of $E_r^{p,q}$ for all $r>1$. However,

$$\eta_{1,r}\circ\tau \simeq_\tau \eta_{1\,r}$$

for $r>1$ by a rotation is $S^{r,0}$. Hence, by (2.11),

$$(4.17)\qquad \bar{x} = x \qquad \textit{whenever}\ x\in Z_r^{p,q}\ \textit{and}\ r>1.$$

Therefore the involutions induced into higher terms are trivial, i.e.,

$$(4.18)\qquad \bar{x} = x \qquad \textit{for}\ x\in E_r^{p,q}\ \textit{with}\ r>1.$$

The following proposition shows that the condition $[\mu_n]$, $n\geqq 1$, of Massey [8] is satisfied in our spectral sequence.

Proposition 4.19. *Let $x\in Z_r^{p,q}(X)$, $y\in Z_r^{p',q'}(Y)$ and $1\leqq r<\infty$. There exist elements $a\in D_1^{p-r,q+1}(X)$, $b\in D_1^{p'-r,q'+1}(Y)$ and $c\in D_1^{p+p'-r,q+q'+1}(X\times Y)$ such that*

$k_1 x = i_1^{r-1}(a)$, $k_1 y = i_1^{r-1}(b)$ and $k_1 \circ \mu_1(x \otimes y) = i_1^{r-1}(c)$, and satisyfing

$$j_1 c = \mu''(a \otimes y) + (-1)^{(p+q)(r-1)} \mu'(x \otimes b)\,.$$

Once this proposition is established, the standard argument of [8] implies that

$$\begin{aligned} &\mu_1(Z_r^{p,q}(X) \otimes Z_r^{p',q'}(Y)) \subset Z_r^{p+p',q+q'}(X \times Y)\,, \\ (4.20) \qquad &\mu_1(B_r^{p,q}(X) \otimes Z_r^{p',q'}(Y)) \subset B_r^{p+p',q+q'}(X \times Y)\,, \\ &\mu_1(Z_r^{p,q}(X) \otimes B_r^{p',q'}(Y)) \subset B_r^{p+p',q+q'}(X \times Y)\,. \end{aligned}$$

And μ_1 induces the multiplication

$$(4.21) \qquad \mu_r\colon E_r^{p,q}(X) \otimes E_r^{p',q'}(Y) \to E_r^{p+p',q+q'}(X \times Y)$$

by passing to quotients for every $1 \leqq r \leqq \infty$.

Now we go to discuss Proposition 4.19. Identify $B^{a,0} \times B^{b,0} = B^{a+b,0}$ and $\partial B^{a+b,0} = S^{a+b,0}$. Then we get the equality

$$S_+^{a+b,0}/(S_+^{a,0} \wedge S_+^{b,0}) = \Sigma^{a,0} \wedge S_+^{b,0} \vee S_+^{a,0} \wedge \Sigma^{b,0}$$

of τ-spaces. Let

$$i'_a\colon \Sigma^{a,0} \wedge S_+^{b,0} \subset S_+^{a+b,0}/(S_+^{a,0} \wedge S_+^{b,0})$$

and

$$i''_b\colon S_+^{a,0} \wedge \Sigma^{b,0} \subset S_+^{a+b,0}/(S_+^{a,0} \wedge S_+^{b,0})$$

be the τ-inclusions to wedge summands. Then we get the direct sum decomposition

$$\begin{aligned} &\tilde{h}^{c,d}((S_+^{a+b,0}/S_+^{a,0} \wedge S_+^{b,0}) \wedge W_+) \\ &= \tilde{h}^{c,d}(\Sigma^{a,0} \wedge S_+^{b,0} \wedge W_+) \oplus \tilde{h}^{c,d}(S_+^{a,0} \wedge \Sigma^{b,0} \wedge W_+) \end{aligned}$$

induced by $(i'_a \wedge 1)^* \oplus (i''_b \wedge 1)^*$, where $W = X \times Y$. The following commutative diagram of τ-cofibrations

$$\begin{tikzcd} & B_+^{a,0} \wedge S_+^{b,0} \arrow[r] \arrow[d] & \Sigma^{b,0} \wedge S_+^{b,0} \arrow[d, "i'_a"] \\ S_+^{a,0} \wedge S_+^{b,0} \arrow[r] \arrow[ur] \arrow[dr] & S_+^{a+b,0} \arrow[r] & S_+^{a+b,0}/S_+^{a,0} \wedge S_+^{b,0} \\ & S_+^{a,0} \wedge B_+^{b,0} \arrow[r] \arrow[u] & S_+^{a,0} \wedge \Sigma^{b,0} \arrow[u, "i''_b"'] \end{tikzcd}$$

imply the following commutative diagram

$$\begin{tikzcd}[column sep=small] & \tilde{h}^{c,d+1}(\Sigma^{a,0}\wedge B_{+}^{b,0}\wedge W_{+}) \\ \tilde{h}^{c,d}(S_{+}^{a,0}\wedge S_{+}^{b,0}\wedge W_{+}) \arrow[ur, "\delta'_a"] \arrow[r, "\Delta_{a,b}"] \arrow[dr, "\delta''_b"'] & \tilde{h}^{c,d+1}((S_{+}^{a+b,0}/S_{+}^{a,0}\wedge S_{+}^{b,0})\wedge W_{+}) \arrow[u, "(i'_a\wedge 1)^*"'] \arrow[d, "(i''_b\wedge 1)^*"] \\ & \tilde{h}^{c,d+1}(S_{+}^{a,0}\wedge\Sigma^{b,0}\wedge W_{+}) , \end{tikzcd}$$

where $\Delta_{a,b}$, δ'_a and δ''_b are connecting morphisms of the corresponding τ-cofibrations. Thus

$$\Delta_{a,b}=(\pi'_a\wedge 1)^*\circ\delta'_a\oplus(\pi''_b\wedge 1)^*\circ\delta''_b\,, \tag{4.22}$$

where

$$\pi'_a\colon S_{+}^{a+b,0}/(S_{+}^{a,b}\wedge S_{+}^{b,0})\to\Sigma^{a,0}\wedge S_{+}^{b,0}$$

and

$$\pi''_b\colon S_{+}^{a+b,0}/(S_{+}^{a,0}\wedge S_{+}^{b,0})\to S_{+}^{a,0}\wedge\Sigma^{b,0}$$

are τ-projections to collapse the other wedge summand.

Consider the multiplication

$$\mu\colon h^{p,q}(S^{a,0}\times X)\otimes h^{p',q'}(S^{b,0}\times Y)\to h^{p+p',q+q'}(S^{a,0}\times S^{b,0}\times X\times Y)\,.$$

By naturality and compatibility of μ with suspensions, [1], §6, we see that

$$\begin{aligned} \delta'_a\circ\mu(u\otimes v)&=\mu(\delta_a^*u\otimes v)\,,\\ \delta''_b\circ\mu(u\otimes v)&=\rho^p(-1)^q\mu(u\otimes\delta_b^*v) \end{aligned} \tag{4.23}$$

for $u\in h^{p,q}(S^{a,0}\times X)$, where $\delta_a^*\colon h^{p,q}(S^{a,0}\times X)\to\tilde{h}^{p,q+1}(\Sigma^{a,0}\wedge X_{+})$ is the connecting morphism.

Identify $B^{r,0}=B^{1,0}\times\cdots\times B^{1,0}$, $B^{2r,0}=B^{r,0}\times B^{r,0}$ and $S^{2r,0}=\partial B^{2r,0}$, and let $\tilde{d}_r\colon S^{1,0}\times(B^{r,0},S^{r,0})\to S^{1,0}\times(S^{2r,0},S^{r,0}\times S^{r,0})$ be the τ-map of τ-pairs defined by

$$\begin{aligned} \tilde{d}_r(1,t)&=(1,t+|t|-1,t-|t|+1)\,,\\ \tilde{d}_r(-1,t)&=(-1,t-|t|+1,t+|t|-1)\,, \end{aligned}$$

where $t=(t_1,\cdots,t_r)\in B^{r,0}$, $|t|=(|t_1|,\cdots,|t_r|)$, and ± 1 denotes $(\pm 1,\cdots,\pm 1)$ in $B^{r,0}$. The induced map of quotient spaces

$$S_{+}^{1,0}\wedge\Sigma^{r,0}\to S_{+}^{1,0}\wedge(S_{+}^{2r,0}/S_{+}^{r,0}\wedge S_{+}^{r,0})$$

is also denoted by the same notation $\tilde{d}_r$.

Let

$$h_\theta\colon S^{1,0}\times S^{r,0}\to S^{1,0}\times S^{r,0}\times S^{r,0}\,,$$

$0\leqq\theta\leqq 1$, be the τ-map defined by

$$h_\theta(1, t) = (1, t+\theta|t|-\theta, t-\theta|t|+\theta),$$
$$h_\theta(-1, t) = (-1, t-\theta|t|+\theta, t+\theta|t|-\theta)$$

for $t=(t_1, \cdots, t_r)\in S^{r,0}=\partial B^{r,0}$, where $\pm\theta$ denotes $(\pm\theta, \cdots, \pm\theta)$ in $B^{r,0}$. Then h_θ, $0\leqq\theta\leqq 1$, gives a τ-homotopy such that

$$h_1 = \tilde{d}_r | S^{1,0}\times S^{r,0} \text{ and } h_0 = 1\times d_r,$$

where d_r: $S^{r,0}\to S^{r,0}\times S^{r,0}$ is the diagonal map. Hence

$$((\tilde{d}_r | S^{1,0}\times S^{r,0})\times 1)^* = (1\times d_r\times 1)^* . \tag{4.24}$$

Let

$$\delta_r^*: \tilde{h}^{c,d}(S_+^{r,0}\wedge W_+) \to \tilde{h}^{c,d+1}(\Sigma^{r,0}\wedge W_+),$$
$$\bar{\delta}_r^*: \tilde{h}^{c,d}(S_+^{1,0}\wedge S_+^{r,0}\wedge W_+) \to \tilde{h}^{c,d+1}(S_+^{1,0}\wedge\Sigma^{r,0}\wedge W_+)$$

and

$$\bar{\Delta}_{r,r}: \tilde{h}^{c,d}(S_+^{1,0}\wedge S_+^{r,0}\wedge S_+^{r,0}\wedge W_+) \to \tilde{h}^{c,d+1}(S_+^{1,0}\wedge(S_+^{2r,0}/S_+^{r,0}\wedge S_+^{r,0})\wedge W_+)$$

be connecting morphisms of their corresponding τ-cofibrations. By naturality of connecting morphisms we get the commutativities

$$(\pi\wedge 1\wedge 1)^*\circ\delta_r^* = \bar{\delta}_r^*\circ(\pi\wedge 1\wedge 1)^*,$$
$$(\pi\wedge 1\wedge 1)^*\circ\Delta_{r,r} = \bar{\Delta}_{r,r}\circ(\pi\wedge 1\wedge 1)^*,$$

where π: $S_+^{1,0}\to\Sigma^{0,0}$ is the τ-map to collapse $S^{1,0}$ to 0. (4.24) implies the following commutativity

$$(\tilde{d}_r\wedge 1)^*\circ\bar{\Delta}_{r,r} = \bar{\delta}_r^*\circ(1\wedge d_r\wedge 1)^* .$$

Thus we get

$$(\pi\wedge 1\wedge 1)^*\circ\delta_r^*\circ(d_r\wedge 1)^* = (\tilde{d}_r\wedge 1)^*\circ(\pi\wedge 1\wedge 1)^*\circ\Delta_{r,r}$$

on $\tilde{h}^{c,d}(S_+^{r,0}\wedge S_+^{r,0}\wedge W_+)$. Combining this equality with (4.22) we get

$$\begin{aligned}&(\pi\wedge 1\wedge 1)^*\circ\delta_r^*\circ(d_r\wedge 1)^* \\ &= ((\pi_r'\circ(\pi\wedge 1)\circ\tilde{d}_r)\wedge 1)^*\circ\delta_r'+((\pi_r''\circ(\pi\wedge 1)\circ\tilde{d}_r)\wedge 1)^*\circ\delta_r'' .\end{aligned} \tag{4.25}$$

Lemma 4.26. $\pi_r' \circ(\pi\wedge 1)\circ\tilde{d}_r \simeq_\tau \eta_{1,r}\wedge 1$,
$\pi_r''\circ(\pi\wedge 1)\circ\tilde{d}_r \simeq_\tau(\eta_{1,r}\circ\tau)\wedge 1$

after identifying $S_+^{r,0}\wedge\Sigma^{r,0}=\Sigma^{r,0}\wedge S_+^{r,0}$ *by switching factors, where* τ *is the involution of* $S^{1,0}$.

Proof. $\pi_r'\circ(\pi\wedge 1)\circ\tilde{d}_r$: $S_+^{1,0}\wedge\Sigma^{r,0}\to\Sigma^{r,0}\wedge S_+^{r,0}$ is the map:

$$(1,\ t) \mapsto \begin{cases} * & \text{if } t_i \leqq 0 \text{ for some } i,\ 1 \leqq i \leqq r, \\ (2t-1,\ 1,\ \cdots,\ 1) & \text{otherwise}, \end{cases}$$

$$(-1,\ t) \mapsto \begin{cases} * & \text{if } t_i \geqq 0 \text{ for some } i,\ 1 \leqq i \leqq r, \\ (2t+1,\ -1,\ \cdots,\ -1) & \text{otherwise}, \end{cases}$$

where $t=(t_1, \cdots, t_r) \in B^{r,0}$. Define a τ-homotopy φ'_s, $0 \leqq s \leqq 1$, by

$$\varphi'_s(1,\ t) = \begin{cases} * & \text{if } t_i \leqq (s-1)/(s+1) \text{ for some } i,\ 1 \leqq i \leqq r, \\ (t+s(t-1),\ 1,\ \cdots,\ 1) & \text{otherwise}, \end{cases}$$

$$\varphi'_s(-1,\ t) = \begin{cases} * & \text{if } t_i \geqq (1-s)/(s+1) \text{ for some } i,\ 1 \leqq i \leqq r, \\ (t+s(t+1),\ -1,\ \cdots,\ -1) & \text{othererwise}. \end{cases}$$

Then $\varphi'_1=\pi'_r \circ (\pi \wedge 1) \circ \tilde{d}_r$ and $\varphi'_0=\eta'_{1,r} \wedge 1$, where $\eta'_{1,r}(1)=(1, \cdots, 1)$ and $\eta'_{1,r}(-1)=(-1, \cdots, -1)$, and $\eta'_{1,r} \simeq_\tau \eta_{1,r}$ since $(-1, \cdots, -1)$ and $(1, \cdots, 1)$ are antipodes of $S^{r,0}$.

$$\pi''_r \circ (\pi \wedge 1) \circ \tilde{d}_r \colon S^{1,0}_+ \wedge \Sigma^{r,0} \to S^{r,0}_+ \wedge \Sigma^{r,0} \text{ is the map:}$$

$$(1,\ t) \mapsto \begin{cases} * & \text{if } t_i \geqq 0 \text{ for some } i,\ 1 \leqq i \leqq r, \\ (-1,\ \cdots,\ -1,\ 2t+1) & \text{otherwise}, \end{cases}$$

$$(-1,\ t) \mapsto \begin{cases} * & \text{if } t_i \leqq 0 \text{ for some } i,\ 1 \leqq i \leqq r, \\ (1,\ \cdots,\ 1,\ 2t-1) & \text{otherwise}. \end{cases}$$

Define a τ-homotopy φ''_s, $0 \leqq s \leqq 1$, by

$$\varphi''_s(1,\ t) = \begin{cases} * & \text{if } t_i \geqq (1-s)/(s+1) \text{ for some } i,\ 1 \leqq i \leqq r, \\ (-1,\ \cdots,\ -1,\ t+s(t+1)) & \text{otherwise}, \end{cases}$$

$$\varphi''_s(-1,\ t) = \begin{cases} * & \text{if } t_i \leqq (s-1)/(s+1) \text{ for some } i,\ 1 \leqq i \leqq r, \\ (1,\ \cdots,\ 1,\ t+s(t-1)) & \text{otherwise}. \end{cases}$$

Then, $\varphi''_0=(\eta'_{1,r} \circ \tau) \wedge 1$ and $\varphi''_1=\pi''_r \circ (\pi \wedge 1) \circ \tilde{d}_r$. q.e.d.

Proof of Proposition 4.19. By (2.11) there exist elements $u \in h^{p,q}(S^{r,0} \times X)$ and $v \in h^{p',q'}(S^{r,0} \times Y)$ such that $\eta^*_{1,r}u=x$ and $\eta^*_{1,r}v=y$. Then

$$(*) \qquad \begin{aligned} \mu_1(x \otimes y) &= (d \times 1)^* \circ \mu(x \otimes y) \\ &= (d \times 1)^* \circ (\eta_{1,r} \times \eta_{1,r} \times 1)^* \circ \mu(u \otimes v) \\ &= \eta^*_{1,r} \circ (d_r \times 1)^* \circ \mu(u \otimes v) \end{aligned}$$

by naturality of μ and the commutativity

$$(\eta_{1,r} \times \eta_{1,r}) \circ d = d_r \circ \eta_{1,r}\,.$$

Put

$$a=\delta_r u,\ b=\delta_r v \ \text{ and } \ c=\delta_r\circ(d_r\times 1)^*\circ\mu(u\otimes v)\,.$$

Then

$$k_1 x=i_1^{r-1}(a),\ k_1 y=i_1^{r-1}(b) \ \text{ and } \ k_1\circ\mu_1(x\otimes y)=i_1^{r-1}(c)$$

by (2.10) and (*). Now

$$\begin{aligned} j_1(c) &= (\pi\times 1\times 1)^*\circ\delta_r\circ(d_r\times 1)^*\circ\mu(u\otimes v) \\ &= \sigma^{-r,0}\circ(\pi\times 1\times 1)^*\circ\delta_r^*\circ(d_r\times 1)^*\circ\mu(u\otimes v) \\ &= \sigma^{-r,0}(1\times\eta_{1,r}\times 1)^*\circ\delta_r'\circ\mu(u\otimes v) \\ &\quad +\sigma^{-r,0}\circ((\eta_{1,r}\circ\tau)\times 1\times 1)^*\circ\delta_r''\circ\mu(u\otimes v) \end{aligned}$$

by (4.25) and (4.26). Here

$$\begin{aligned} &(1\times\eta_{1,r}\times 1)^*\circ\delta_r'\circ\mu(u\otimes v)=(1\times\eta_{1,r}\times 1)^*\circ\mu(\delta_r^* u\otimes v) \\ &=\mu''(\delta_r^* u\otimes\eta_{1,r}^* v)=\mu''(\delta_r^* u\otimes y) \end{aligned}$$

by (4.23) and naturality of μ. Similarly

$$((\eta_{1,r}\circ\tau)\times 1\times 1)^*\circ\delta_r''\circ\mu(u\otimes v)=(-1)^{p+q}\mu'(\bar{x}\otimes\delta_r^* v)\,.$$

Thus

$$\begin{aligned} j_1(c) &= \mu''(\sigma^{-r,0}\circ\delta_r^* u\otimes y)+(-1)^{(p+q)(r-1)}\mu'(\bar{x}\otimes\sigma^{-r,0}\circ\delta_r^* v) \\ &= \mu''(\delta_r u\otimes y)+(-1)^{(p+q)(r-1)}\mu'(\bar{x}\otimes\delta_r v)\,, \end{aligned}$$

which concludes the proposition.

Proposition 4.19 implies easily the derivation property of d_r, whereby the signs become unnecessary as will be seen in what follows.

Lemma 4.27. $\rho=-1$ *on* Im k_1.

Proof. k_1 is a $\wedge$-module map, and $\rho=-1$ on $E_1^{p,q}$ by [1], Proposition 3.6. Thus

$$\rho\cdot k_1=k_1\cdot\rho=k_1\circ(-1)=-k_1\,. \qquad \text{q.e.d.}$$

Proposition 4.28. *For* $r>1$, Im k_r *and* Im d_r *consists only of elements of order* 2.

Proof. Let $r>1$ and $x\in Z_r^{p,q}$. Then $k_1 x\in \operatorname{Im} i_1$. By the above lemma and [1], Proposition 4.2, we see that

$$-k_1 x=\rho\cdot k_1 x=k_1 x\,,$$

i.e., $k_1 x$ is an element of order 2, which concludes the proposition since k_r is induced by $k_1|Z_r^{p,q}$ and $d_r=j_r\circ k_r$. q.e.d.

Propositions 4.19 and 4.28 imply

Theorem 4.29. *Let $h^{*,*}$ be a multiplicative τ-cohomology theory. In its associated forgetful spectral sequences, natural multiplications*

$$\mu_r\colon E_r^{p,q}(X)\otimes E_r^{p',q'}(Y)\rightarrow E_r^{p+p',q+q'}(X\times Y)$$

are induced for $1\leqq r\leqq\infty$, and there hold derivation formulas

$$d_r\circ\mu_r(x\otimes y)=\mu_r(d_r x\otimes y)+\mu_r(\bar{x}\otimes d_r y)$$

for $1\leqq r<\infty$.

Remark. The derivation property of d_1 can be rewritten also as

$$d_1\circ\mu_1(x\otimes y)=\mu_1(d_1 x\otimes\bar{y})+\mu_1(x\otimes d_1 y)\,. \tag{4.30}$$

To see this, first we remark that the involution in the E_1-term is multiplicative, and $\overline{d_1 x}=d_1 x$ by (4.15). Then, making use of (4.16) we can deform the derivation formula of Theorem 4.29 into the form (4.30).

Let $1\in h^{0,0}(pt)$ be the unity for the multiplication in $h^{*,*}$. By (4.4) $j_1 1\in E_1^{0,0}(pt)$ plays as the unity for μ_1, which is denoted also by 1. Then, $k_1 1=0$, whence 1 remains in all $E_r^{0,0}(pt)$ and is the unity for all μ_r, $1\leqq r\leqq\infty$, since μ_r is induced by μ_1.

Put

$$u_r=\varpi_r(1)\in E_r^{-a_r,a_r}(pt)\,, \tag{4.31}$$

where ϖ_r is the periodicity isomorphism of Theorem 3.10. Then, for any $x\in E_r^{p,q}(X)$

$$\begin{aligned}\varpi_r(x)&=\varpi_r\circ\mu_r(1\otimes x)\\&=(\omega_{r,a_r}\times 1)^*\circ\mu_r(1\otimes x)\\&=\mu_r(u_r\otimes x)\,,\end{aligned}$$

i.e., the periodicity isomorphism ϖ_r is obtained by the multiplication with u_r. We call u_r the *r-th periodicity element.* It is a standard matter to see that all periodicity isomorphism of type (r, n) of E_r-terms can be obtained by a multiplication with a power of u_r (up to signs) and u_r is invertible. In fact,

$$u_r^{-1}=\varpi_r^{-1}(1)\,. \tag{4.32}$$

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