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ON CHARACTERISTIC CLASSES OF RIEMANNIAN FOLIATIONS

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0. Introduction

In [6], Lazarov and Pasternack defined characteristic classes for Riemannian foliations and investigated their properties very closely. Their theory is a special case of the theory of characteristic classes for foliated bundles due to Kamber and Tondeur [4]. From this point of view, the characteristic classes are defined by looking at the unique Riemannian connection on the orthonormal frame bundle of the foliation, whose structure group is the orthogonal group $O(n)$ (n is the codimension of the foliation). However if we enlarge the structure group to $E(n)$, the group of Euclidean motions on $\boldsymbol{R}^n$, and if we look at a system of differential forms defined by considering the Cartan connection, then we obtain more charateristic classes than those defined by Lazarov and Pasternack. The purpose of this note is to clarify this point. Thus this note could be considered as an addendum to [6].

In § 1 we give the main construction of the characteristic classes and in § 2, the concept of "*p-th* scalar curvature" is defined for every Riemannian manifold. In § 3 the cohomology of a truncated Weil algebra of $\mathfrak{e}(n)$, the Lie algebra of $E(n)$, is determined and § 4 is devoted to the study of continuous variation of the new characteristic classes.

1. Construction of the characteristic classes

Let F be a Riemannian foliation on a smooth manifold M defined by a maximal family of submersions

$$f_\alpha\colon U_\alpha \to (\boldsymbol{R}^n_\alpha, g_\alpha)$$

from open sets U_α in M to a Riemannian manifold $(\boldsymbol{R}^n_\alpha, g_\alpha)$ (g_α is a Riemannian metric on $\boldsymbol{R}^n$) such that for every $x \in U_\alpha \cap U_\beta$ there exists a local isometry $\gamma_{\beta\alpha}$: neighborhood of $f_\alpha(x) \to$ neighborhood of $f_\beta(x)$ with $f_\beta = \gamma_{\beta\alpha} f_\alpha$ near x. Now let $O(\boldsymbol{R}^n_\alpha)$ be the orthonormal frame bundle of $\boldsymbol{R}^n_\alpha$. Since $O(\boldsymbol{R}^n_\beta) | \gamma_{\beta\alpha}(U) =$

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$\gamma_{\beta\alpha}^*(O(\boldsymbol{R}_\alpha^n)\,|\,U)$ where U is a small neighborhood of $f_\alpha(x)$, we can define a principal bundle $O(F)$ over M such that $O(F)\,|\,U_\alpha = f_\alpha^*(O(\boldsymbol{R}_\alpha^n)\,|\,f_\alpha(U_\alpha))$. We call $O(F)$ the orthonormal frame bundle of the foliation F. Now since the canonical form and the Riemannian connection form of Riemannian manifolds are preserved by isometries, we can define $\boldsymbol{R}^n$ and $\mathfrak{so}(n)$ valued one forms θ_0 and θ_1 on $O(F)$ such that $\theta_0\,|\,U_\alpha = f_\alpha^*(\theta_0^\alpha)$ and $\theta_1\,|\,U_\alpha = f_\alpha^*(\theta_1^\alpha)$, where θ_0^α and θ_1^α are the canonical form and the Riemannian connection form of $\boldsymbol{R}_\alpha^n$, respectively. We call θ_0 and θ_1 the canonical form and the Riemannian connection form of F. We may also consider the pair (θ_0, θ_1) as an $\mathfrak{e}(n)$-valued one form on $O(F)$ whose restriction to U_α is the pull back under f_α^* of the $\mathfrak{e}(n)$-valued one form $(\theta_0^\alpha, \theta_1^\alpha)$ on $O(\boldsymbol{R}_\alpha^n)$, which may be considered as the unique torsionfree Cartan connection form of $\boldsymbol{R}_\alpha^n$. With respect to the usual basis of $\mathfrak{e}(n) = \boldsymbol{R}^n \oplus \mathfrak{so}(n)$, we can represent θ_0 and θ_1 by n forms $\theta^1, \theta^2, \cdots, \theta^n$ and a skew symmetric matrix of differential forms θ_j^i. Now if we denote $W(\mathfrak{e}(n))$ for the Weil algebra of $\mathfrak{e}(n)$, then θ_0 and θ_1 define a *d.g.a.* map

$$\varphi\colon\ W(\mathfrak{e}(n)) \to \Omega^*(O(F))$$

where $\Omega^*(O(F))$ is the de Rham complex of $O(F)$. Let $\omega^i, \omega_j^i, \Omega^i, \Omega_j^i \in W(\mathfrak{e}(n))$ be the universal connection and curvature forms corresponding to the usual basis of $\mathfrak{e}(n)$. Then φ satisfies $\varphi(\omega^i) = \theta^i$ and $\varphi(\omega_j^i) = \theta_j^i$. Now we know the following conditions (cf. [5]).

$$\text{(1.1)}\qquad \begin{aligned} &\text{(i)} && d\theta^i = -\sum_j \theta_j^i \wedge \theta^j \qquad \text{(torsionfree-ness)},\\ &\text{(ii)} && d\theta_j^i = -\sum_k \theta_k^i \wedge \theta_j^k + \Theta_j^i,\\ & && \text{where } \Theta_j^i = \frac{1}{2}\sum_{k,l} R_{jkl}^i \theta^k \wedge \theta^l,\\ &\text{(iii)} && \sum_j \Theta_j^i \wedge \theta^j = 0 \qquad \text{(the first Bianchi's identity)}. \end{aligned}$$

In view of these conditions, we define an ideal I of $W(\mathfrak{e}(n))$ as the one generated by the following elements.

(1.2)
- (i)′ Ω^i,
- (ii)′ elements whose "length" l is greater than n, where l is defined by the conditions: $l(\omega_j^i) = l(\Omega^i) = 0$, $l(\omega^i) = 1$ and $l(\Omega_j^i) = 2$.
- (iii)′ $\sum_j \Omega_j^i \wedge \omega^j$.

Then it is easy to see that I is a subcomplex of $W(\mathfrak{e}(n))$. The condition (1.1) shows that $\varphi(I) = 0$. Therefore, if we denote $\tilde{W}(\mathfrak{e}(n)) = W(\mathfrak{e}(n))/I$, then φ induces a d.g.a. map

$$\varphi\colon\ \tilde{W}(\mathfrak{e}(n)) \to \Omega^*(O(F))\,.$$

Now suppose that the normal bundle of F is trivialized by a cross section $s: M \to O(F)$, then we obtain

$$H^*(\tilde{W}(\mathfrak{e}((n))) \to H^*_{DR}(O(F)) \xrightarrow{s^*} H^*_{DR}(M).$$

Since this construction is functorial, we finally obtain

$$H^*(\tilde{W}(\mathfrak{e}(n))) \to H^*(BR\bar{\Gamma}_n; \boldsymbol{R}) \tag{1.3}$$

where $BR\bar{\Gamma}_n$ is the classifying space for codimension n Riemannian Haefliger structures with trivial normal bundles (cf. [6]). In the general case we have

$$\begin{aligned} &H^*(\tilde{W}(\mathfrak{e}(n))_{O(n)}) \to H^*(RR\Gamma_n; \boldsymbol{R}) \\ &H^*(\tilde{W}(\mathfrak{e}(n))_{SO(n)}) \to H^*(BR\Gamma_n^+; \boldsymbol{R}), \end{aligned} \tag{1.4}$$

where the left hand sides are the cohomology of subcomplexes $\tilde{W}(\mathfrak{e}(n))_{O(n)}$ (resp. $\tilde{W}(\mathfrak{e}(n))_{SO(n)}$) of $O(n)$ (resp. $SO(n)$) basic elements of $\tilde{W}(\mathfrak{e}(n))$ and $BR\Gamma_n$ (resp. $BR\Gamma_n^+$) are the classifying space for the Riemannian (resp. oriented Riemannian) Haefliger structures. This is our construction of the characteristic classes for Riemannian foliations. Now if we ignore the canonical form θ_0, then we obtain

$$H^*(\tilde{W}(\mathfrak{so}(n))) \to H^*(BR\bar{\Gamma}_n; \boldsymbol{R})$$

where $\tilde{W}(\mathfrak{so}(n)) = W(\mathfrak{so}(n))$ modulo the ideal $I \cap W(\mathfrak{so}(n))$. This is nothing but the characteristic classes defined by Kamber and Tondeur [4] and is the same as those defined by Lazarov and Pasternack [6].

2. Scalar curvatures

In this section, we define the notion of "*p-th* scalar curvature" for every Riemannian manifold M of dimension n, where p is an even integer $\leqq n$. First of all we recall the concept of *p-th* sectional curvature γ_p defined by Allendoerfer and studied by Thorpe [8]. Let $G_p(M)$ be the Grassman bundle of tangent p-planes of M. For every p-plane $(x, P) \in G_p(M)$, $\gamma_p(x, P)$ is defined to be the Lipschitz-Killing curvature at $x \in M$ of the p-dimensional submanifold of M geodesic at x and tangent to P at x. Thus γ_p is a smooth function on $G_p(M)$. In terms of the curvature tensor R of M, γ_p is expressed by

$$\begin{aligned} \gamma_p(x, P) = \frac{(-1)^{p/2}}{2^{p/2}p!} \sum_{\sigma,\tau} &\operatorname{sgn}(\sigma)\operatorname{sgn}(\tau) g(R(u_{\sigma(1)}, u_{\sigma(2)})u_{\tau(1)}, u_{\tau(2)}) \\ &\cdots g(R(u_{\sigma(p-1)}, u_{\sigma(p)})u_{\tau(p-1)}, u_{\tau(p)}) \end{aligned} \tag{2.1}$$

where g is the metric tensor of M, $u_1, \cdots, u_p$ is an orthonormal basis for P and σ, τ range over the *p-th* symmetric group S_p. Now if we average this γ_p over each fibre of $G_p(M)\to M$, then we obtain a real valued smooth function R_p on M. Let us call this function the "*p-th* scalar curvature" of M. In terms of the curvature tensor R, R_p is expressed by the formula,

$$R_p(x) = \frac{(-1)^{p/2}}{2^{p/2}p!\binom{n}{p}} \sum_i \gamma_p(x, P_i)\,. \tag{2.2}$$

Here the sum ranges over all p-tuples $i=(i(1), \cdots, i(p))$ with $1\leqq i(1)<\cdots<i(p)\leqq n$ and P_i is the p-plane at $x\in M$ spanned by $u_{i(1)}, \cdots, u_{i(p)}$, where $u_1, \cdots, u_n$ is any orthonormal frame at x. R_2 is the usual scalar curvature of M (up to a non-zero constant) and R_n is the Lipschitz-Killing curvature if n is even. Now as in § 1, let θ^i, $i=1, \cdots, n$ and θ^i_j be the canonical form and the Riemannian connection form of M. Thus they are one forms defined on $O(M)$, the orthonormal frame bundle of M. Let us define a smooth function $\tilde{R}^i_{jkl}$ on $O(M)$ by

$$\begin{aligned} d\theta^i_j &= -\sum_k \theta^i_k\wedge\theta^k_j+\Theta^i_j\,, \\ \Theta^i_j &= \frac{1}{2}\sum_{k,l}\tilde{R}^i_{jkl}\,\theta^k\wedge\theta^l\,. \end{aligned} \tag{2.3}$$

For every even integer $p\leqq n$, we define a smooth function $\tilde{R}_p$ on $O(M)$ as follows. We consider the n-form:

$$\det((n-p)\theta, (p/2)\Theta) = \sum_\sigma \operatorname{sgn}(\sigma)\theta^{\sigma(1)}\wedge\cdots\wedge\theta^{\sigma(n-p)}\wedge\Theta^{\sigma(n-p+1)}_{\sigma(n-p+2)}\wedge\cdots\wedge\Theta^{\sigma(n-1)}_{\sigma(n)},$$

where σ ranges over the *n-th* symmetric group S_n. Then $\tilde{R}_p$ is defined to be a function satisfying the equality.

$$\det((n-p)\theta, (p/2)\Theta) = \tilde{R}_p\theta^1\wedge\cdots\wedge\theta^n\,. \tag{2.4}$$

Then it is easy to see that the function $\tilde{R}_p$ is constant on each fibre of the bundle $\pi\colon O(M)\to M$. In fact we have

Proposition 2.1. $\tilde{R}_p=(-1)^{p/2}n!\pi^*R_p$.

Proof. Let $x\in M$ and let $u_1, \cdots, u_n$ be an orthonormal frame at x. We choose a coordinate around x such that $\frac{\partial}{\partial x_i}=u_i$ for $i=1, \cdots, n$. If R^i_{jkl} is the component of R with respect to this coordinate, then from (2,2) we have

$$R_p(x) = \frac{(-1)^{p/2}}{2^{p/2}p!\binom{n}{p}} \sum_{\sigma,\tau,i} \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)R^{i(\tau(1))}_{i(\tau(2))i(\sigma(1))i(\sigma(2))}\cdots R^{i(\tau(p-1))}_{i(\tau(p))i(\sigma(p-1))i(\sigma(p))} \tag{2.5}$$

where $i=(i(1), \cdots, i(p))$ ranges over all p-tuples with $1\leqq i(1)<\cdots<i(p)\leqq n$. On the other hand we have

$$(2.6)\qquad \tilde{R}_p(x, u) = \frac{1}{2^{p/2}} \sum_{\sigma, j} \operatorname{sgn}(\sigma) \operatorname{sgn}\begin{pmatrix} 1 & \cdots & n-p & & \cdots & n \\ \sigma(1) & \cdots & \sigma(n-p) & j(1) & \cdots & j(p) \end{pmatrix} \tilde{R}^{\sigma(n-p+1)}_{\sigma(n-p+2)j(1)j(2)} \cdots \tilde{R}^{\sigma(n-1)}_{\sigma(n)j(p-1)j(p)}$$

where $j=(j(1), \cdots, j(p))$ ranges over every permutation of $(n-p+1, \cdots, n)$. We have

$$(2.7)\qquad \tilde{R}_p(x, u) = \frac{1}{2^{p/2}} \sum_{\sigma, j} \operatorname{sgn}\begin{pmatrix} \sigma(n-p+1) & \cdots & \sigma(n) \\ j(1) & \cdots & j(p) \end{pmatrix} \tilde{R}^{\sigma(n-p+1)}_{\sigma(n-p+2)j(1)j(2)} \cdots \tilde{R}^{\sigma(n-1)}_{\sigma(n)j(p-1)j(p)}$$
$$= \frac{(n-p)!}{2^{p/2}} \sum_{i,j} \operatorname{sgn}\begin{pmatrix} i(1) & \cdots & i(p) \\ j(1) & \cdots & j(p) \end{pmatrix} \tilde{R}^{i(1)}_{i(2)j(1)j(2)} \cdots \tilde{R}^{i(p-1)}_{i(p)j(p-1)j(p)},$$

where $i=(i(1), \cdots, i(p))$ ranges over all p-tuples with $1 \leq i(k) \leq n$, $i(k) \neq i(l)$ for $k \neq l$ and $j=(j(1), \cdots, j(p))$ ranges over all permutations of i. Now by the choice of the coordinate $x_1, \cdots, x_n$, clearly

$$R^i_{jkl}(x) = \tilde{R}^i_{jkl}(x, u)$$

for all i, j, k, l. Therefore by comparing (2.5) and (2.7) we obtain

$$\tilde{R}_p(x, u) = (-1)^{p/2} n! R_p(x).$$

Since it is easy to see that $\tilde{R}_p(x, u)$ does not depend on the choice of the frame u, this completes the proof. q.e.d.

Now let $f: M \to N$ be an isometry of Riemannian manifolds M, N. Then from the definition of R_p, it is clear that $R_p(M) = f^* R_p(N)$. Next we investigate how the p-th scalar curvature behaves under the scale change $g \to k^2 g$. Let $\bar{R}_p$ be the p-th scalar curvature of the Riemannian manifold $(M, k^2 g)$. Then we have

Proposition 2.2. $\bar{R}_p = k^{-p} R_p$.

Proof. This follows from an elementary calculation.

3. Cohomology of $\tilde{W}(\mathfrak{e}(n))$

In this section we compute the cohomology of the truncated Weil algebra $\tilde{W}(\mathfrak{e}(n))$. For each even integer $p \leq n$, let r_p be an element of $\tilde{W}(\mathfrak{e}(n))$ defined by

$$r_p = \det((n-p)\omega, (p/2)\Omega)$$
$$= \sum_{\sigma} \operatorname{sgn}(\sigma) \omega^{\sigma(1)} \cdots \omega^{\sigma(n-p)} \Omega^{\sigma(n-p+1)}_{\sigma(n-p+2)} \cdots \Omega^{\sigma(n-1)}_{\sigma(n)}.$$

Then it is easy to see that r_p is closed so that it defines a cohomology class in $H^*(\tilde{W}(\mathfrak{e}(n)))$. Next we define $h_i \in \tilde{W}(\mathfrak{e}(n))$ $(i=2, 4, \cdots, n-1)$ for n odd and $h_i, h_\chi \in \tilde{W}(\mathfrak{e}(n))$ $(i=2, 4, \cdots, n-2)$ for n even by

$$h_i = \widetilde{TP_i}\,, \quad h_\chi = \widetilde{T\chi}\,,$$

where P_i, χ are the *i-th* Pontrjagin form and the Euler form respectively, T denotes Chern-Simons' transgression form [2] (at the Weil algebra level) and $\sim$ is the projection modulo the ideal I. Let

$$\begin{aligned} E_n &= E(h_2, h_4, \cdots, h_{n-1}) && n \text{ odd}\,, \\ &= E(h_2, h_4, \cdots, h_{n-2}, h_\chi) && n \text{ even}\,, \end{aligned}$$

be the exterior algebra generated by $h_2, \cdots$, and let r_pE_n be the vector space over $\boldsymbol{R}$ with basis $\{r_px_i\}$ where $\{x_i\}$ is a basis of E_n. Then by the truncation, every element r_px_i is closed. We have the following.

Theorem 3.1. $H^*(\hat{W}(\mathfrak{e}(n))) = H^*(\hat{W}(\mathfrak{so}(n))) \oplus \sum_{\substack{0 \leq p < n \\ p:\text{ even}}} r_pE_n\,.$

Here $H^*(W(\mathfrak{so}(n)))$ has been determined by Kamber and Tondeur and is isomorphic to $H^*(RW_n)$, where RW_n is the differential complex defined by Lazarov and Pasternack [6]. Before proving the Theorem we describe the geometric meaning of the second term of Theorem 3.1. Thus let F be a codimension n Riemannian foliation on a smooth manifold M defined by sumbersions $f_\alpha\colon U_\alpha \to \boldsymbol{R}^n_\alpha$ (see § 1). We define a smooth function R_p on M, for every even integer p with $0 \leq p < n$, as follows. R_0 is the identity function of M and for $p \geq 2$, $R_p | U_\alpha = f^*_\alpha(R^\alpha_p)$, where R^α_p is the p-th scalar curvature of $\boldsymbol{R}^n_\alpha$ defined in § 2. This is well-defined because R_p is invariant under isometries. Also we have an n-form v on M such that $v | U_\alpha = f^*_\alpha$(volume form of $\boldsymbol{R}^n_\alpha$). With these understood, r_p of the foliation F is represented by the n-form $(-1)^{p/2} n! R_p v$ (cf. (2.4) and Proposition 2.1.). Now assume that we have a cross section $s\colon M \to O(F)$ and let $TP_i(F)$, $T\chi(F)$ be the Chern-Simons' transgression forms corresponding to the Riemannian connection on $O(F)$. Then h_i (resp. h_χ) of the foliation is represented by the form $s^*TP_i(F)$ (resp. $s^*T\chi(F)$). Now we prove our Theorem 3.1.

Proof of Theorem 3.1. Let us define the "weight" function w on the elements of $\hat{W}(\mathfrak{e}(n))$ by $w(\omega^i) = 1$, $w(\omega^i_j) = w(\Omega^i_j) = 0$, and define $J_0 = \{x \in \hat{W}(\mathfrak{e}(n));\, w(x) = 0\}$, $J_+ = \{x \in \hat{W}(\mathfrak{e}(n));\, w(x) > 0\}$. Then it is easy to see that both J_0 and J_+ are subcomplexes of $\hat{W}(\mathfrak{e}(n))$. Moreover we have $\hat{W}(\mathfrak{e}(n)) = J_0 \oplus J_+$. Therefore

$$H^*(\hat{W}(\mathfrak{e}(n))) = H^*(J_0) \oplus H^*(J_+)\,. \tag{3.1}$$

Now let us define a decreasing filtration F^p on J_+ by

$$F^p = \{x \in J_+;\, l(x) \geq p\}$$

where l is the "length" on $\hat{W}(\mathfrak{e}(n))$ induced from that on $W(\mathfrak{e}(n))$. Let

$\{E_r^{p,q}, d_r\}$ be the spectral sequence associated with this filtration. Define $M_p=$ sub-vector space of J_+ spanned by $\omega^{i(1)}\cdots\omega^{i(p-2k)}\Omega_{j(2)}^{j(1)}\cdots\Omega_{j(2k)}^{j(2k-1)}$ for all i, j, k. Then $\mathfrak{so}(n)$ acts on M_p by the Lie derivation. Thus M_p is an $\mathfrak{so}(n)$-module. Let $C^r(\mathfrak{so}(n); M_p)$ be the set of r-cochains on $\mathfrak{so}(n)$ with coefficient in M_p. Then it is easy to see that

$$E_0^{p,q} \cong C^q(\mathfrak{so}(n); M_p)\,. \tag{3.2}$$

Moreover the following diagram is commutative up to sign.

$$\begin{array}{ccc} E_0^{p,q} & \cong & C^q(\mathfrak{so}(n); M_p) \\ \downarrow{\scriptstyle d_0} & & \downarrow{\scriptstyle d} \\ E_0^{p,q+1} & \cong & C^{q+1}(\mathfrak{so}(n); M_p) \end{array} \tag{3.3}$$

where d is the differential of the complex $C^*(\mathfrak{so}(n); M_p)$. Therefore we obtain

$$E_1^{p,q} \cong H^q(\mathfrak{so}(n); M_p)\,. \tag{3.4}$$

Now since $\mathfrak{so}(n)$ is simple, by a theorem in [3], we have

$$H^q(\mathfrak{so}(n), M_p) = H^q(\mathfrak{so}(n)) \otimes M_p^{\mathfrak{so}(n)} \tag{3.5}$$

where $M_p^{\mathfrak{so}(n)}$ is the $\mathfrak{so}(n)$-invariant subspace of M_p. Now by the form of the action of $\mathfrak{so}(n)$ on M_p, we can apply a theorem of Weyl [9] on the $\mathfrak{so}(n)$-invariants to obtain

$$\begin{aligned} M_p^{\mathfrak{so}(n)} &= 0 & 1 \leqq p < n \\ &= \text{vector space with basis } r_k\text{: } k \text{ even, } 0 \leqq k < n \qquad & p = n\,. \end{aligned} \tag{3.6}$$

Now since r_k is closed and has length n, it is easy to see that $d_1 = d_2 = \cdots = 0$. Hence we have $E_1^{p,q} = \cdots = E_\infty^{p,q}$ and this implies

$$\begin{aligned} H^*(J_+) &\cong \sum_{\substack{p:\,\text{even} \\ 0 \leqq p < n}} r_p H^*(\mathfrak{so}(n)) \\ &= \sum_p r_p E_n\,. \end{aligned} \tag{3.7}$$

On the other hand, clearly we have $J_0 = \tilde{W}(\mathfrak{so}(n))$ and a similar argument as above shows that

$$H^*(J_0) \cong H^*(RW_n) \tag{3.8}$$

where RW_n is now considered as a subcomplex of $\tilde{W}(\mathfrak{so}(n))$. (3.7) and (3.8) prove our Theorem. q.e.d.

Let $I(SO(n))$ $(I(O(n)))$ be the ring of invariant polynomials of $SO(n)$ $(O(n))$. Then we can consider that $I(SO(n))$ and $I(O(n))$ are subcomplexes of $W(\mathfrak{e}(n))$. Let $\tilde{I}(SO(n))=I(SO(n))/I\cap I(SO(n))$ and $\tilde{I}(O(n))=I(O(n))/I\cap I(O(n))$. Then, by similar arguments we obtain

Theorem 3.2. $$H^*(\tilde{W}(\mathfrak{e}(n))_{SO(n)})=\tilde{I}(SO(n))\oplus\sum_{\substack{p:\text{even}\\ 0\leq p<n}} r_p R\,,$$
$$H^*(\tilde{W}(\mathfrak{e}(n))_{O(n)})=\tilde{I}(O(n))\,.$$

REMARK 3.3. One may hope that one can obtain more characteristic classes for smooth foliations than those defined by Bott and Haefliger [1] by considering the Cartan connection. However this is false because we have an isomorphism $H^*(\tilde{W}(\mathfrak{gl}(n;\boldsymbol{R})))=H^*(\tilde{W}(\mathfrak{a}(n;\boldsymbol{R})))$ where $\mathfrak{a}(n;\boldsymbol{R})$ is the Lie algebra of the *n-th* affine group and $\tilde{W}$ denotes Weil algebras modulo certain ideals which are constructed by a similar argument as in the Riemannian case.

4. Continuous variation

In this section we prove that the new characteristic classes $\sum_p r_p E_n$ defined in § 3 vary continuously and independently under deformations of Riemannian foliations. Precisely we prove

Theorem 4.1. *Let* $\dim H^k(SO(n))=d$. *Then there is a surjective homomorphism*

$$H_{n+k}(B R\bar{\Gamma}_n;Z)\to \boldsymbol{R}^{d[(n+1)/2]}\to 0\,.$$

As before, let M be an oriented Riemannian manifold and let $\pi:SO(M)\to M$ be the oriented orthonormal frame bundle of M. We consider the codimension n Riemannian foliation F on $SO(M)$ induced from the given Riemannian structure on M by the projection π. The oriented orthonormal frame bundle of this foliation, $SO(F)$, is the pull back of the principal bundle π: $SO(M)\to M$ by the map π. Thus we have $SO(F)=\{(x;u,v);x\in M,\ u,v\in\pi^{-1}(x)\}$ and there is a commutative diagram

$$\begin{array}{ccc} SO(F) & \xrightarrow{f} & SO(M) \\ \downarrow{\scriptstyle\bar{\pi}} & & \downarrow{\scriptstyle\pi} \\ SO(M) & \xrightarrow{\pi} & M \end{array} \tag{4.1}$$

where $f(x;u,v)=(x,v)$ and $\bar{\pi}(x;u,v)=(x,u)$. Now we define a cross section s: $SO(M)\to SO(F)$ of the bundle $\bar{\pi}$ by $s(x,u)=(x;u,u)$. Then clearly the composition map $f\circ s$: $SO(M)\to SO(F)\to SO(M)$ is the identity. Henceforth we denote $F(M)$ for the foliation on $SO(M)$ described above with the trivialization s of the normal bundle. Now assume that M satisfies the following conditions.

(4.2)
(i) $P_i(M)$, $\chi(M)\equiv 0$ where $P_i(M)$ and $\chi(M)$ are the Pontrjagin and the Euler forms of M, respectively.
(ii) M is parallelizable so that there is given a bundle isomorphism i: $M\times SO(n)\sim SO(M)$.

Let $\tau(P_i)$, $\tau(\chi)\in H^*(SO(n); \boldsymbol{R})$ be the transgression images of the Pontrjagin class P_i and the Euler class χ, and let $TP_i(M)$, $T\chi(M)\in\Omega^*(SO(M))$ be the Chern-Simons' transgression forms of M corresponding to the Riemannian connection on $SO(M)$. Then h_i and h_χ of the foliation $F(M)$ are represented by $s^*f^*TP_i(M)$ and $s^*f^*T\chi(M)$. But since $f\circ s=id.$, we obtain

$$(4.3)\qquad \begin{aligned} h_i(F(M)) &= TP_i(M) \\ h_\chi(F(M)) &= T\chi(M)\,. \end{aligned}$$

By the assumption (4.2)-(i), both $TP_i(M)$ and $T\chi(M)$ are closed forms and define cohomology classes in $H^*(SO(M))$ which is isomorphic to $H^*(M)\otimes H^*(SO(n))$ under the homomorphism i^*. (Hereafter we identify $H^*(SO(M))$ with $H^*(M)\otimes H^*(SO(n))$ by i^*.) Since the forms $TP_i(M)$, $T\chi(M)$ restricted to each fibre are closed and represent the cohomology classes $\tau(P_i)$ and $\tau(\chi)$ (cf. [2]), we have

$$(4.4)\qquad \begin{aligned} [h_i(F(M))] &\equiv 1\times\tau(P_i) \qquad \text{modulo } I\,, \\ [h_\chi(F(M))] &\equiv 1\times\tau(\chi) \qquad \text{modulo } I\,, \end{aligned}$$

where [ ] denotes the cohomology class and I is the ideal $\tilde{H}^*(M)\otimes H^*(SO(n))$ of $H^*(M)\otimes H^*(SO(n))\simeq H^*(SO(M))$. Now for each even integer p with $0\leqq p<n$, we have the p-th scalar curvature $R_p(M)$ of M. (R_0 is defined to be the identity function of M.) Then clearly $R_p(F(M))=\pi^*R_p(M)$ and the characteristic class r_p of $F(M)$ is represented by

$$(4.5)\qquad r_p(F(M)) = (-1)^{p/2}n!\int_M R_p(M)v(M)\cdot\pi^*[M]\,,$$

where $v(M)$ is the volume form of M and $[M]$ is the fundamental cohomology class. From (4.4) and (4.5) we obtain

Proposition 4.2. *Let $F(M)$ be as above and saume that M satisfies the condition* (4.2). *Moreover assume $\int_M R_p(M)v(M)\neq 0$ for an even integer p. Then the cohomology classes $r_ph_{i_1}\cdots h_{i_l}$, $r_ph_{i_1}\cdots h_{i_l}h_\chi$ of $F(M)$ are represented by*

$$r_ph_{i_1}\cdots h_{i_l}(F(M)) = (-1)^{p/2}n!\int_M R_p(M)v(M)\cdot[M]\times\tau(P_{i_1})\cdots\tau(P_{i_l})\,,$$

$$r_ph_{i_1}\cdots h_{i_l}h_\chi(F(M)) = (-1)^{p/2}n!\int_M R_p(M)v(M)\cdot[M]\times\tau(P_{i_1})\cdots\tau(P_{i_l})\tau(\chi)\,.$$

Now we consider the special case when M is the Riemannian product $S^1\times S^{n-1}$ of unit spheres. For a unit sphere S^i, clearly we have $R_p(S^i)=1$ for every p (cf. [8]), and from the definition of R_p, it is easy to see that

$$R_p(S^1\times S^{n-1}) = \frac{\binom{n-1}{p}}{\binom{n}{p}} = \frac{n}{n-p}.$$

Since $S^1\times S^{n-1}$ satisfies the condition (4.2), from Proposition 4.2 we obtain

Proposition 4.3. *The characteristic classes of $F(S^1\times S^{n-1})$ are given by*

$$r_p h_{i_1}\cdots h_{i_l}(F(S^1\times S^{n-1})) = (-1)^{p/2} n!\frac{n}{n-p} v_1 v_{n-1}[S^1\times S^{n-1}]\times \tau(P_{i_1})\cdots\tau(P_{i_l}),$$

$$r_p h_{i_1}\cdots h_{i_l} h_\chi(F(S^1\times S^{n-1})) = (-1)^{p/2} n!\frac{n}{n-p} v_1 v_{n-1}[S^1\times S^{n-1}]\times \tau(P_{i_1})\cdots\tau(P_{i_l})\tau(\chi),$$

where v_i is the volume of the unit sphere S^i.

Next we consider the Riemannian manifold $(S^1\times S^{n-1})_k$ which is obtained from $S^1\times S^{n-1}$ by the scale change $g\to k^2 g$. Since the Chern-Simons' TP form is invariant under the scale change, from Proposition 2.2 and Proposition 4.3 we have

Proposition 4.4. *The characteristic classes of $F((S^1\times S^{n-1})_k)$ are given by*

$$r_p h_{i_1}\cdots h_{i_l}(F((S^1\times S^{n-1})_k)) = (-1)^{p/2} n!\frac{n}{n-p} k^{n-p} v_1 v_{n-1}[S^1\times S^{n-1}]\times \tau(P_{i_1})\cdots\tau(P_{i_l}),$$

$$r_p h_{i_1}\cdots h_{i_l} h_\chi(F((S^1\times S^{n-1})_k)) = (-1)^{p/2} n!\frac{n}{n-p} k^{n-p} v_1 v_{n-1}[S^1\times S^{n-1}]\times \tau(P_{i_1})\cdots\tau(P_{i_l})\tau(\chi).$$

Now we are in a position to prove Theorem 4.1. In view of Proposition 4.4, it is enough to prove that the homomorphism

$$\psi\colon H_n(B\bar{\Gamma}_n; Z)\to \boldsymbol{R}^{[(n+1)/2]}$$

defined by the characteristic classes $\{r_p\}_{0\leq p<n\ p\,\mathrm{even}}$ is a surjection. Now the foliation $F((S^1\times S^{n-1})_k)$ on $(S^1\times S^{n-1})_k$ defines a homology class $\alpha_k\in H_n(B\bar{\Gamma}_n; \boldsymbol{Z})$

and by Proposition 4.4 its characteristic numbers are given by

$$r_p(\alpha_k) = c_p k^{n-p}$$

for a non-zero c_p ($0 \leqq p < n$, p even). We consider the homology class $\alpha(k) = \alpha_{k(1)} + \alpha_{k(2)} + \cdots + \alpha_{k([(n+1)/2])}$ where $k = \left(k(1), \cdots, k\left(\left[\frac{n+1}{2}\right]\right)\right)$ is an $\boldsymbol{R}^{[(n+1)/2]}$-valued variable. The characteristic numbers of $\alpha(k)$ are given by

$$r_p(\alpha(k)) = \sum_i c_p k_i^{n-p},$$

where the sum ranges over $i = 1, \cdots, \left[\frac{n+1}{2}\right]$. Now let $f\colon \boldsymbol{R}^{[(n+1)/2]} \to \boldsymbol{R}^{[(n+1)/2]}$ be the map defined by

$$\begin{aligned} f\left(k(1), \cdots, k\left(\left[\frac{n+1}{2}\right]\right)\right) &= (r_0(\alpha(k)), \cdots, r_{2[(n-1)/2]}(\alpha(k))) \\ &= \left(\sum_i c_0 k_i^n, \cdots, \sum_i c_{2[(n-1)/2]} k_i^{n-2[(n-1)/2]}\right) \end{aligned}$$

Then f is smooth and it is easy to see that the determinant of the Jacobian matrix of f is not constantly zero. Therefore we conclude that $\operatorname{Im} f$ contains an inner point. Hence $\operatorname{Im} \psi$ contains also an inner point. Since $\operatorname{Im} \psi$ is a subgroup of $\boldsymbol{R}^{[(n+1)/2]}$, it follows that ψ is surjective. This completes the proof. q.e.d.

REMARK 4.5 Lazarov and Pasternack [7] proved that certain characteristic classes for Riemannian foliations defined by them vary continuously by using the residue formula for zero-points of a Killing vector field.

If we use the sphere S^n instead of $S^1 \times S^{n-1}$, then we can prove the following Theorems, which are refinements of Theorem 4.1.

Theorem 4.6. *The characteristic classes $\{r_p\}_{0 \leqq p < n\ p\,\text{even}}$ define a surjective homomorphism*

$$\pi_n(BR\Gamma_n^+) \to \boldsymbol{R}^{[(n+1)/2]} \to 0.$$

Theorem 4.7. *If n is even, then the characteristic classes $\{r_p h_\chi\}_{0 \leqq p < n, p\,\text{even}}$ define a surjective homomorphism*

$$\pi_{2n-1}(BR\bar{\Gamma}_n) \to \boldsymbol{R}^{[(n+1)/2]} \to 0.$$

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