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Osaka J. Math.
16 (1979), 45-55

SOME DECOMPOSITIONS THEOREMS ON ABELIAN GROUPS AND THEIR GENERALISATIONS-II

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(Received December 1, 1977)

In [7] study of those modules M_R which satisfy the following two conditions was initiated:

(I) Every finitely generated submodule of every homomorphic image of M is a direct sum of uniserial modules.

(II) Given two uniserial submodules U and V of a homomorphic image of M, for any submodule W of U any non-zero homomorphism, $f: W \to V$ can be extended to a homomorphism $g: U \to V$ provided the composition length $d(U/W) \leq d(V/f(W))$.

It was shown that some of the well known decomposition theorems for torsion abelian groups, can be generalized to modules satisfying (I) and (II). Here we introduce another condition:

(III) For any finitely generated submodules N of M, $R/\text{ann}(N)$ is right artinian.

It can be easily seen that any torsion module over a bounded (*hnp*)-ring satisfies (I), (II) and (III). Let M be a module satisfying (I) and (II). The concept of h-pure submodules of M was introduced in [7]; if in addition M satisfies (III) it is shown in section one, that any submodule N of M is h-pure if and only if it is pure (Theorem (1.3)). Theorem (1.4) shows that any complement of $H_k(M)$ in M is a summand of M. In section 2, the concept of basic submodule is introduced. It is shown that any module M satisfying (I), (II) and (III) has a basic submodule and any two basic submodules of M are isomorphic (Theorem (2.7)). This result generalizes the corresponding well known result on basic subgroups of torsion abelian groups. In section 3, a decomposition theorem is proved; which states that given any module M satisfying (I) and (II), such that $M/\text{socle}(M)$ is decomposable then M is decomposable.

Preliminaries: Let M be a module satisfying (I) and (II). Let us recall some definitions from [6, 7]. An element x in M is said to be uniform if xR is a non-zero uniform (hence uniserial) submodule. For any uniform element x of M, its exponent $e(x)$ is defined to be equal to the composition length $d(xR)$;

This research was supported by UGC Research Grant No. 30-5(6562)/76(SR-II).

the height of x is the supremum of all $d(T/xR)$ where T is a uniserial submodule of M containing x. The height of x is denoted by $H_M(x)$ (or simply by $H(x)$). For any $k\geq 0$, $H_k(M)$ denotes the submodule of M generated by all those uniform elements x of M for which $H(x)\geq k$. A submodule N of M is said to be an h-pure submodule if $N\cap H_k(M)=H_k(N)$ for all k. M is said to be bounded if there exists a positive integer k such that $H(x)\leq k$ for all uniform elements x in M. M is said to be decomposable if it is a direct sum of uniserial modules. For definition and elementary properties of pure submodules we refer to Stenström [8]. For any ring R, $J(R)$ denotes the Jacobson radical of R.

Lemma 1.1. *Let M_R be a module satisfying (I), (II) and (III) and X be a uniserial submodule of M having*

$$X = X_0 > X_1 > X_2 \cdots > X_t = 0$$

as its unique composition series. If for $0\leq i\leq t-1$, $P_i=\text{ann}(X_i/X_{i+1})$ then $X_iP_i=X_{i+1}$.

Proof. Let $A=\text{ann}\,(X)$. Since $S=R/A$ is right artinian, $X_iJ(S)=X_{i+1}$ and $J(S)\subset P_i/A$, we have $X_iP_i=X_{i+1}$.

Lemma 1.2. *Let a module M_R satisfy (I) and (II). If for any finitely many uniform elements $x_1, x_2, \cdots, x_n$ in M*

$$\sum_{i=1}^{n} x_iR = \oplus \sum_{j=1}^{m} y_jR$$

where y_jR are uniserial, then $m\leq n$.

Proof. The result follows by induction on n.

The result that any submodule N of a torsion module over a bounded (hnp)-ring is pure if and only if it is h-pure was proved by M. Khan in [2]. The proof of the following is adapted from [2].

Theorem 1.3. *Let M_R be a module satisfying (I), (II) and (III) and N a submodule of M. Then N is h-pure if and only if it is a pure submodule.*

Proof. Let N be h-pure. Consider any finite system of linear equations

$$\sum_i x_i r_{ij} = s_j \in N$$

which admits a solution $\{x_i\}$ in M. Let $K=\sum x_iR+N$. Then K/N is a finitely generated module. So by condition (I).

$$K/N = \oplus \sum T_\alpha/N$$

where each T_α/N is uniserial. Then by [7, Lemma 2(i)], $T_\alpha=y_\alpha R\oplus N$. Hence

$$K = K_1 \oplus N$$

This gives that the above given system of equations are also solvable in N. Hence N is a pure submodule of M.

Let now N be a pure submodule. This immediately gives $MA \cap N = NA$ for all ideals A of R. Suppose for some k, $H_k(M) \cap N \neq H_k(N)$. We choose k smallest with $H_k(M) \cap N \neq H_k(N)$. We can find a uniform element x of smallest exponent such that $x \in H_k(M) \cap N$ but $x \notin H_k(N)$. Then $x \in H_{k-1}(N)$. By definition there exists a uniform element y in M such that $x \in yR$ and $d(yR/xR) = k$.

$x \in H_{k-1}(N)$ shows that there exist a uniform element $u \in N$ such that $x \in uR$ and $d(uR/xR) = k-1$. Let $zR = \text{socle}(xR)$ and $m = e(x)$. Then $d(uR/zR) = m + k - 2$, gives $H_N(z) \geq m+k-2$. Suppose $H_N(z) \geq m+k-1$. We can then find a uniform element $v \in N$ such that $z \in vR$ and $d(vR/zR) = m+k-1$. By condition (II), we get an isomorphism $\sigma: yR \to vR$ which is identity on zR. Then $x - \sigma(x)$ is a uniform element with $e(x-\sigma(x)) < e(x)$, $x - \sigma(x) \in N \cap H_k(M)$, but $x - \sigma(x) \notin H_k(N)$, since $\sigma(x) \in H_k(N)$. This contradicts the choice of x. Hence $H_N(z) = k+m-2 = d(uR/zR)$. So by [7, Lemma 1]

$$N = uR \oplus N_1$$

uR is also a pure submodule. Now $d(uR/zR) = d(yR/zR) - 1$. By (1.1) we can find prime ideals $P_1, P_2, \cdots, P_{m+k-1}$ such that R/P_i is simple artinian for all i and $yR > yP_1 > yP_1P_2 > \cdots > yP_1P_2\cdots P_{m+k-1} = 0$ with $zR = yP_1P_2\cdots P_{m+k-2}$. By condition (II) $uR \cong yP_1$ and hence $uP_2P_3\cdots P_{m+k-1} = 0$. However $yP_2P_3\cdots P_{m+k-1} \neq 0$. Thus $z \in MP_2P_3\cdots P_{m+k-1} \cap uR = uP_2P_3\cdots P_{m+k-1} = 0$. This is a contradiction. Hence N is an h-pure submodule of M.

The following theorem generalizes Erdelyi's theorem [1, Theorem (24.8)].

Theorem 1.4. *Let M be a module satisfying (I), (II) and (III) then for any $k \geq 1$, any complement of $H_k(M)$ is a summand of M.*

Proof. Let N be a complement of $H_k(M)$. Then N is bounded. If we show that N is a pure submodule, the result follows from [7, Theorem 3]. In view of (1.2) it is equivalent to showing $H_n(M) \cap N = H_n(N)$ for every n. Since $H_k(M) \cap N = 0 = H_k(N)$, the result holds for $n \geq k$. To apply induction we suppose that for some n with $0 \leq n < k$, $H_n(M) \cap N = H_n(N)$, we prove that same for $n+1$. Let the contrary hold. Then there exists a uniform element $x \in H_{n+1}(M) \cap N$ such that $x \notin H_{n+1}(N)$. Then $H_N(x) = n$. Now there exists a uniform element y in M such that $d(yR/xR) = n+1$. Let socle $(yR/xR) = x_1R/xR$. If $x_1 \in N$, we get $x_1 \in N \cap H_n(M) = H_n(N)$ and hence $x \in H_{n+1}(N)$. This is a contradiction. Consequently $x_1 \notin N$ and $(N + x_1R) \cap H_k(M) \neq 0$. Thus there exists a uniform element $z \in H_k(M)$ such that $z = u + x_1s$ for some $u \in N$ and $s \in R$. If $x_1sR \neq x_1R$, then $x_1sR \subset xR$ and $z \in N$; this is a contradiction to the fact that

$N\cap H_k(M)=0$. So $x_1sR=x_1R$ and $x_1=x_1ss'$, $s'\in R$. Then $zs'=us'+x_1\in(N+x_1R)\cap H_k(M)$ and $zs'\neq 0$. So we can suppose that $x_1s=x_1$. Thus $z=u+x_1$.

Let $P=\mathrm{ann}(x_1R/xR)$. By (1.1) $xR=x_1P$. So for any $r\in P$, $zr=0$ and $ur=-x_1r$. Now $H(z)\geq k>n$, $H(x_1)\geq n$, gives $u\in H_n(M)\cap N=H_n(N)$. For some $r_0\in P$, $x=x_1r_0=-ur_0$ If uR is uniform and $ur_0R<uR$, then $H_N(ur_0)\geq H_N(u)+1\geq n+1$ and hence $H_N(x)\geq n+1$; this is a contradiction. Hence the following two cases arise.

Case I: uR is uniform and $ur_0R=uR$. In this case $u=x_1b_0=xrb$ for some $b\in R$ and $z=x_1r_0b+x_1=x_1c$, $c\in R$. Thus $zR=x_1R$ and $x_1\in H_k(M)$. This shows that $H(x_1)\geq k$ and hence $H(x)\geq k+1$. This contradicts the fact that $N\cap H_{k+1}(N)=0$.

Case II: uR is not uniform. The fact that $u\in zR+x_1R$ and that zR, x_1R are uniform together with (1.2) yields $uR=u_1R\oplus u_2R$ with u_1, u_2 both uniform. Further we can take $u=u_1+u_2$. Then $xR=uP=u_1P\oplus u_2P$. So $u_1P=0$ or $u_2P=0$. To be definite let $u_2P=0$. Then u_2R is a simple R-module and $x_1R=u_1P$. Let $u_1P<u_1R$ then $xR=u_1P=v_1R$ for some $v_1\in u_1R$. Now $H(u_1)\geq\min(H(x_1),H(x))\geq n$. So by induction hypothesis $u_1\in H_n(N)$ and hence $H_N(v_1)\geq n+1$. Consequently $xR=v_1R$ gives $H_N(x)\geq n+1$. This is a contradiction. Thus $u_1R=u_1P=xR$. Hence $u_1=xa$, $a\in R$. Consequently $z=u_2+xa+x_1=u_2+x_1s$, $s\in R$. This reduces to case I and hence again gives a contradiction. Hence N is a pure submodule. This proves the theorem.

2. Basic submodules

DEFINITION 2.1. Let M be a module satisfying (I) and (II). A subset $\{x_\lambda:\lambda\in\Lambda\}$ of uniform elements of M is called h-pure independent if it is independent in the sense that $\sum x_\lambda R$ is direct, and $\sum x_\lambda R$ is an h-pure submodule of M.

The following Lemma generalizes [1, Lemma (29.1)].

Lemma 2.2. *Let a module M_R satisfy (I) and (II). An h-pure independent subset $\{x_\lambda:\lambda\in\Lambda\}$ is maximal if and only if M/L, where $L=\sum x_\lambda R$, is a direct sum of infinite length uniform submodules.*

Proof. The result follows from [7, Lemma 2 and Theorem 5].

This motivates the following:

DEFINITION 2.3. Let M be a module satisfying (I) and (II). A submodule B of M is called a basic submodule of M if it satisfies the following:

(i) B is an h-pure submodule.

(ii) B is a direct sum of uniserial modules.

(iii) M/B is a direct sum of uniform modules of infinite lengths.

[7, Lemma 2 and Theorem 5] and the fact that union of any chain of h-pure submodules is an h-pure submodule gives the following:

Lemma 2.4. *Any module satisfying* (*I*) and (*II*) *has a basic submodule.*

The main purpose of this section is to prove that any two basic submodules of a module M satisfying (I), (II) and (III) are isomorphic. The following theorem generalizes [1, Theorem (29.3)]. Since the proof is on similar lines it is omitted.

Theorem 2.5. *Let M be a module satisfying* (*I*), (*II*) *and* (*III*) *and B be a submodule of M such that $B=\oplus\sum_{n=1}^{\infty} B_n$, where each B_n is a direct sum of uniserial modules each of length n. Then B is a basic submodule of M, if and only if*

$$M=(B_1+\cdots+B_n)\oplus(B_n^*+H_n(M)) \quad \text{where} \quad B_n^*=\sum_{i>n} B_i\,.$$

The following theorem generalizes Szele's theorem [1, Theorem (29.4)].

Theorem 2.6. *Let M and B be as in* (2.5). *B is a basic submodule if and only if $B_1+\cdots+B_n$ is a summand of M and is maximal with respect to the property $(B_1+\cdots+B_n)\cap H_n(M)=0$.*

Proof. Let B be a basic submodule of M. From (2.5) $(B_1+\cdots+B_n)\cap H_n(M)=0$. Let N be a complement of $H_n(M)$ containing $B_1+\cdots+B_n$. N is a summand of M by (1.4). By [7, Corollary 1], N is a direct sum of uniserial modules. Suppose $N\neq B_1+\cdots+B_n$. Then we can find a uniform element $y\in N$ such that $B_1\oplus\cdots\oplus B_n\oplus yR$ is a summand of M. By using (2.5), we can suppose that $yR\subset B_n^*+H_n(M)$. Let $zR=\text{socle}(yR)$. Since $yR\cap H_n(M)=0$, and yR is a pure submodule, we get $H(z)\leq n-1$. Let

$$M'=B_n^*+H_n(M) \qquad \text{(i)}$$

If for every $i\geq n+1$,

$$C_i=\sum_{j=n+1}^{i} B_j \qquad \text{(ii)}$$

each C_i being pure and bounded, is a summand of M'. Further

$$M'=U_i(C_i+H_n(M))\,.$$

Consequently for some i,

$$z\in C_i+H_n(M)$$

Again

$$C_i = \oplus \sum_{\alpha} y_\alpha R \tag{iii}$$

where $y_\alpha R$ are uniserial. Also

$$M' = C_i \oplus D \tag{iv}$$

Now $z=c+x$, $c \in C_i$, $x \in H_n(M)$.
Using (iii) and (iv) we get

$$c = \sum_{\alpha} u_\alpha , \quad u_\alpha \in y_\alpha R$$
$$x = \sum_{\alpha} v_\alpha + d , \quad v_\alpha \in y_\alpha R , \quad d \in D .$$

Each $u_\alpha + v_\alpha$ being a homomorphic image of z must be either zero or be such that $(u_\alpha + v_\alpha)R$ is the minimal submodule of $y_\alpha R$. However as

$$C_i = \sum_{j=n+1}^{i} B_j ,$$

C_i is a direct sum of uniserial modules of lengths at least $n+1$. Consequently $H(u_\alpha + v_\alpha) \geq n$. Hence, as also $d \in H_n(M)$, we get $z \in H_n(M)$, by [6, Lemma 4]. This contradicts the fact that $H(z) \leq n-1$. This proves the necessity.

Conversely let B satisfy the given conditions. Then $B_1 + \cdots + B_n$ is a pure submodule of M, gives B is a pure submodule of M. If B is not a basic submodule in M, we can find a uniform element $u \in M$ such that $B \cap uR = 0$ and $B \oplus uR$ is a pure submodule (use [7, Lemma 2 and Theorem 5]). Let $d(uR)=n$. Then $(B_1 \oplus \cdots \oplus B_n \oplus uR) \cap H_n(M) = 0$. This contradicts the hypothesis. Hence the result follows.

Theorem 2.7. *Let a module M satisfy (I), (II) and (III). Then M has a basic submodule. Any two basic submodules of M are isomorphic.*

Proof. Existence follows from (2.4). Let B' and B be two basic submodules of M. We have

$$B = B_1 \oplus B_2 \oplus \cdots \oplus B_n \oplus \cdots \tag{i}$$
$$B' = B'_1 \oplus B'_2 \oplus \cdots \oplus B'_n \oplus \cdots \tag{ii}$$

where B_n, B'_n are direct sums of uniserial modules, each of length n. By (2.6)

$$M = (B_1 + \cdots + B_n) \oplus N_1 \tag{iii}$$
$$M = (B'_1 + \cdots + B'_n) \oplus N'_1 \tag{iv}$$

for some submodules N_1, N'_1 of M, containing $H_n(M)$ such that $H_n(M)$ is an essential submodule of each of them. Let $p\colon M \to B_1 + \cdots + B_n$ be projection given by (iii). By (2.6), $B'_n \cap N_1 = 0$ and hence $B'_n \cong p(B'_n)$. For each $i=1$, 2, ⋯, n, let

$$p_i\colon B_1+\cdots+B_n\to B_i$$

be natural projections. We claim that q, the restriction of $p_n p$ to B'_n is a monomorphism. Suppose $\ker q\neq 0$. As p restricted to B'_n is a monomorphism, we can find a minimal submodule xR of B'_n such that $p_n p(xR)=0$; clearly $p(xR)\neq 0$. Now $H(x)=n-1$. So there exists a uniform element $z\in B'_n$, such that $x\in zR$ and $d(zR)=n$. For some $i<n$, $p_i p(x)\neq 0$, since $p_n p(x)=0$. Then from socle $(zR)=xR\cong p_i p(xR)$, we get $zR\cong p_i p(zR)\subset B_1+\cdots+B_{n-1}$. But $d(zR)=n$, and $B_1+\cdots+B_{n-1}$ has no uniserial submodule of length exceeding $n-1$. Thus we get, a contradiction. Hence $q\colon B'_n\to B_n$ is a monomorphism. In particular we get a monomorphism;

$$\lambda\colon \text{socle }(B'_n)\to \text{socle }(B_n)$$

Similarly we get a monomorphism:

$$\mu\colon \text{socle }(B_n)\to \text{socle }(B'_n)$$

Consequently socle $(B_n)\cong$socle (B'_n)

Now $\qquad B_n=\oplus\sum_{i\in\Lambda}A_i$

and $\qquad B'_n=\oplus\sum_{j\in\Gamma}A_i$

where A_i and A'_j are uniserial modules, each of length n. Then

$$\text{socle }(B_n)=\oplus\textstyle\sum_i \text{socle }(A_i)$$
$$\text{socle }(B'_n)=\oplus\textstyle\sum_j \text{socle }(A'_j)\,,$$

we get a one-to-one mapping σ of Λ onto Γ such that socle (A_i)=socle $(A'_{\sigma(i)})$. By condition (II), $A_i\cong A'_{\sigma(i)}$. Hence $B_n\cong B'_n$. This in turn gives $B\cong B'$. This proves the theorem.

3. A decomposition theorem

Main purpose of this section is to prove the following:

Theorem 3.1. *If a module M satisfying (I) and (II), is such that for its socle S, M/S is decomposable, then M is also decomposble.*

We state the following without proof, since its proof is verbatim same as of Corollary 1 in [6].

Theorem 3.2. *Let M be a module satisfying (I) and (II), and P be its socle. M is a direct sum of uniserial modules if and only if P is a union of ascending sequence $P_n(n=1, 2, 3, \cdots)$ of submodules such that for each n, there exists a positive* *integer k_n with the property that $H(x) \leq k_n$ for every uniform element x of P_n.*

Lemma 3.3. *If a module M satisfying (I), (II) is such that for some $k \geq 0$, $H_k(M)$ is decomposable then M is decomposable.*

Proof. Let N be an h-pure submodule of M, maximal with respect to the property that $N \cap H_k(M) = 0$. N is bounded and decomposable. Further by [7, Theorem 3].

$$M = N \oplus K .$$

Let T be a complement of $H_k(M)$ containing N. If $T \neq N$, we can find a uniform element $z \in \text{socle}(T)$ such that $z \in K$. Now $H(z) = t \leq k-1$. If u is a uniform element in K with $z \in uR$ and $d(uR/zR) = t-1$, we get from [7, Lemma 1], that $K = uR \oplus K_1$. Then

$$M = N \oplus uR \oplus K_1$$

and $N + uR$ is an h-pure submodule of M containing N properly, having zero intersection with $H_k(M)$. This contradicts the choice of N. Hence N is a complement of $H_k(M)$. Further

$$H_k(M) = H_k(N) + H_k(K) = H_k(K)$$

Thus $H_k(M) \subset' K$. Hence to prove that M is decomposable we only need to show that K is decomposable. So without loss of generality we may suppose that $H_k(M) \subset' M$. So $S = \text{socle}(M) = \text{socle}(H_k(M))$. By hypothesis $H_k(M)$ is decomposable. So by (3.1), $S = \bigcup_n S_n$, where S_n $(n = 1, 2, \cdots)$ is an ascending sequence of submodules, such that for each n, we have a positive integer l_n such that the height of any uniform element x of S_n taken in $H_k(M)$ does not exceed l_n. Then the height of any uniform element x in S_n taken in M does not exceed $l_n + k$. So by (3.2) M is decomposable.

Proof of (3.1). In view of (3.3) it is enough to prove that $H_1(M)$ is decomposable. Now by hypothesis

$$\bar{M} = M/S = \oplus \sum_\alpha \bar{y}_\alpha R$$

where $\bar{y}_\alpha R$ are uniserial.

As seen in the proof of (3.3), without loss of generality we can suppose that $H_1(M) \subset' M$. In view of the condition (I) we take each y_α to be uniform in M. Now $d(y_\alpha R) \geq 2$. Let $x_\alpha R < y_\alpha R$ with $d(y_\alpha R / x_\alpha R) = 1$. Then $x_\alpha \in H_1(M)$. We claim,

$$H_1(M) = \oplus \sum_\alpha x_\alpha R$$

and this will prove the result. Suppose

$$x_\alpha R \cap (\sum_{i=1}^{n} x_i R) \neq 0$$

with $x_\alpha R \neq x_i R$ for $1 \leq i \leq n$. Then

$$z_\alpha R = x_\alpha R \cap (\sum_{i=1}^{n} x_i R) = \text{socle}\,(x_\alpha R) = y_\alpha R \cap (\sum_{i=1}^{n} y_i R)$$

Now $\sum_{i=1}^{n} y_i R = \oplus \sum_{j=1}^{m} u_j R$

where $u_j R$ are uniserial and by (1.2) $m \leq n$. If for some j, $d(u_j R) = 1$, we have

$$\oplus \sum_{i=1}^{n} \bar{y}_i R = \oplus \sum_{j=1}^{m} \bar{u}_j R \pmod{S}$$

and the right hand side has less than n terms. This is a contradiction. Therefore $d(u_j R) \geq 2$ for every j and $m = n$. We write

$$z_\alpha = v_1 + v_2 + \cdots + v_n\,, \quad v_i \in u_i R$$

We can find $t_\alpha \in y_\alpha R$ such that $d(t_\alpha R / z_\alpha R) = 1$. By condition (II), we get homomorphisms

$$\sigma_j:\ t_\alpha R \rightarrow u_j R$$

such that $\sigma_j(z_\alpha) = v_j$. Define

$$\sigma:\ t_\alpha R \rightarrow \oplus \sum_{j=1}^{n} u_j R$$

by $\sigma(y) = \sum_j \sigma_j(y)$, $y \in t_\alpha R$

Then σ is identity on $z_\alpha R$. Let

$$A = \{r \in R:\ t_\alpha r \in z_\alpha R\}$$

Then A is a maximal right ideal of R with $z_\alpha R = t_\alpha A$. So for $r \in A$, $t_\alpha r = \sigma(t_\alpha r) = \sum_{j=1}^{n} \sigma_j(t_\alpha) r$. Consequently $t_\alpha - \sigma(t_\alpha)$ is a uniform element such that

$$(t_\alpha - \sigma(t_\alpha))R \cong R/A$$

This gives $t_\alpha - \sigma(t_\alpha) \in S$. Hence,

$$t_\alpha \equiv \sigma(t_\alpha) \pmod{S}$$

Consequently $\bar{t}_\alpha \in \bar{y}_\alpha R \cap (\sum_{j=1}^{n} \bar{y}_j R) = \bar{0}$. Hence $t_\alpha \in S$. This is a contradiction. Therefore

$$\sum_\alpha x_\alpha R = \oplus \sum_\alpha x_\alpha R$$

This also yields $\sum_\alpha y_\alpha R = \oplus \sum_\alpha y_\alpha R$, since each $y_\alpha R$ is an essential extension of $x_\alpha R$. Consider any uniform element $x \in H_1(M)$ such that $x \notin S$. We can find a uniform element $y \in M$ such that, $x \in yR$ and $d(yR/xR)=1$. If

$$A = \{r \in R;\ yr \in xR\}$$

then A is a maximal right ideal of R and $xR = yA$. Now

$$\bar{y} = y + S = \bar{\xi}_1 + \bar{\xi}_2 + \cdots + \bar{\xi}_n\,, \quad \bar{\xi}_i \in \bar{y}_i R$$

for some $\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_n$ among $\bar{y}_\alpha$'s, $\alpha \in \Lambda$. We take $\xi_i \in y_i R$. If for any i, $\bar{\xi}_i \neq 0$, then the natural homomorphism η_i: $\bar{y}R \to \bar{\xi}_i R$ is non-zero; since $\bar{y}R$ is uniserial, it follows that $\operatorname{Ker} \eta_i \subset \bar{x}R = \bar{y}A$ and so $\xi_i R/\xi_i A \cong \bar{y}R/\bar{x}R \neq 0$. Thus $\bar{\xi}_i \neq 0$ implies $\bar{\xi}_i A \subset x_i R$. Consequently $\bar{x} \in \sum \bar{x}_i R$ and hence

$$x \in \sum_\alpha x_\alpha R + S\,.$$

$$H_1(M) = \sum x_\alpha R + S$$

This proves:

We claim: $S \subset \sum x_\alpha R$. If not we can find a uniform element $x \in S$ such that $x \notin \sum x_\alpha R$. Then $xR \cap \sum_\alpha x_\alpha R = 0$. We can find a uniform element $y \in M$ such that $x \in yR$ and $d(yR/xR)=1$. Now let $N' = yR + \sum y_\alpha R = yR \oplus (\sum y_\alpha R)$. Then

$$M/S = (\sum y_\alpha R + S)/S = (N' + S)/S \cong N'/\text{socle}\,(N')$$
$$\cong yR/xR \oplus \sum (y_\alpha R)/\text{socle}\,(y_\alpha R)$$

Therefore

$$\oplus \sum_\alpha y_\alpha R/\text{socle}\,(y_\alpha R) \cong (yR/xR) \oplus \sum_\alpha y_\alpha R/\text{socle}\,(y_\alpha R)$$

This isomorphism is natural. Hence $yR/xR = 0$. This is a contradiction. Hence $S \subset \sum_\alpha x_\alpha R$. This yields

$$H_1(M) = \oplus \sum_\alpha x_\alpha R$$

Hence the result follows.

We end this paper with a few remarks.

(1) Any module M over a commutative ring R satisfying (I) and (II) must satisfy (III). However, a simple faithful module over a nonartinian primitive ring trivially satisfies (I) and (II), but not (III).

(2) If a module M satisfies (I) and (II), then (II) gives that any uniserial submodule xR of M is quasi-injective. The example on page 362 in [3] is of a uniserial module which is not quasi-injective. This shows that although a uniserial module always satisfies (I), but it need not satisfy (II).

(3) If a commutative ring R, admits a faithful finitely generated module satisfying (I) and (II), then R is a principal ideal ring with d.c.c. It will be interesting to investigate the structure of noncommutative rings admitting faithful, finitely

generated modules satisfying conditions (I), (II) and (III).
(4) Consider a local ring R, with maximal ideal W, such that $W^2=0$, $Q=R/W$, a field with the property that $\dim_Q W=1$, and $\dim W_Q=2$. R is not a right principal ideal ring. However for any $x\neq 0$ in W, R/xR is a uniserial, injective, faithful, right R-module of length two, so it satisfies (I), (II) and (III). (See V. Dlab, and C. M. Ringel, Math. Ann. 195, (1972) Proposition 2)

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