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TIGHT 4-DESIGNS

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The purpose of this paper is to fill a gap in [1]. The result is as follows:

Theorem. 1. *There exist only finitely many tight* 4-*designs.*

Actually our result is considerably more explicit. Furthermore we do not think that there exist tight 4-designs other than Witt designs.

[1] and [2] will be quoted as (T) and (C). We use the same notation as in (T).

1. $d_1=e-1$

Lemma 1. *Put* $w=v-2k$. *Then we have that*

$$w^2 = v^2-2(2ae+a+1)v+2(6ae+3a-1) = (v-2)^2-2(2ae+a-1)(v-3)\,. \tag{1}$$

Proof. By (C), (5) we have that

$$(a^2-1)k^2-\{4a^2e^2+4a(a-2)e+(3a-1)(a-1)\}k +2a^2e(e+1)(2ae+a-3) = 0\,. \tag{2}$$

Since k and $v-k$ are solutions of (2), we have that

$$(a^2-1)k(v-k) = 2a^2e(e+1)(2ae+a-3) \tag{3}$$

and

$$(a^2-1)v = 4a^2e^2+4a(a-2)e+(3a-1)(a-1)\,. \tag{4}$$

Since $w^2-v^2=-4k(v-k)$, (1) follows from (3) and (4).

The next lemma is crucial.

Lemma 2. *There exists a positive integer N such that*

$$(2ae+a-1)N = (a^2-1)^2\,. \tag{5}$$

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Proof. By (30) and (47) of (T) we have that

$$2(v-2)y = 2kx = (v-w)(v-w-2ae-a-1)\,. \tag{6}$$

By Lemma 1 the right-hand side of (6) equals

$$\begin{aligned}(7)\qquad & (v-w)^2-(v-w)(2ae+a+1)\\ &= v^2+w^2-2wv-(2ae+a+1)v+(2ae+a+1)w\\ &= 2v^2-(2w+6ae+3a+3)v+2(6ae+3a-1)+(2ae+a+1)w\\ &= (v-2)\{2(v-w)-6ae-3a+1\}+(2ae+a-3)w\,.\end{aligned}$$

Now by (C), (3) we have that

$$(a^2-1)(v-2) = (2ae+a-1)(2ae+a-3)\,. \tag{8}$$

Hence from (8) we have that

$$(2ae+a-3)w = \frac{(a^2-1)(v-2)w}{2ae+a-1}\,. \tag{9}$$

Furthermore, if a is odd, then the second factor of the first term of the right-hand side of (7) is even. Thus by (8), (9) and Lemma 1 we see that $2ae+a-1$ is a divisor of $(a^2-1)^2$.

From (5) we have that $N\equiv-1 \pmod{a}$. Hence we may put

$$N = Ma-1\,, \tag{10}$$

where M is a positive integer. Then from (5) and (10) we have that

$$2e(Ma-1) = (a-1)(a^2+a-1-M)\,. \tag{11}$$

Lemma 3. *Assume that* $d_1<e-1$. *Then we have that*

$$M<\tfrac{1}{4}a\,. \tag{12}$$

Proof. Assume that $M\geqq\frac{1}{4}a$. Then from (112) of (T) and (11) we have that

$$(4a+2)(\tfrac{1}{4}a^2-1)\leqq(a-1)(a^2+a-1-\tfrac{1}{4}a)\,,$$

which implies that $a\leqq 4$.

Lemma 4. $a\neq M^2+M-1$.

Proof. Assume that $a=M^2+M-1$. Then from (11) we have that $2e=M^3+2M^2-M-2$. By (C), (3) a^2-1 is a divisor of $4(e-a)(e-a+1)$. Now we have that $\dfrac{4(e-a)(e-a+1)}{a^2-1}=M^2-2M+1+\dfrac{4}{M+1}$. Thus $M=1$ or $M=4$. If $M=1$, then $a=1$. If $M=3$, then $a=11$ and $e=20$. Then from (C), (5) we have that

$$k^2-1682k+379456=0\,,$$

which has no rational integral solution.

From now on we assume that $d_1<e-1$, though some of the assertions may be independent from this assumption.

By Lemma 3 we may put

$$(13)\qquad a=zM+L\,,$$

where z is a positive integer and $0\leqq L<M$. Now from (11) we have that $-2e\equiv M+1 \pmod{a}$. So we may put

$$(14)\qquad 2e=Ka-M-1\,,$$

where K is a positive integer.

Lemma 5. $2e=za+a-M-1$.

Proof. By (11) and (14) we have that $K(Ma-1)=a^2-2+M^2$. By (13) this implies that

$$(15)\qquad K=z+\frac{La-2+M^2+z}{Ma-1}\,.$$

By Lemma 3 the last term of (15) must be equal to 1. So Lemma 5 follows from (11) and (14). Moreover, from (15) we have that

$$(16)\qquad Ma+1=La+M^2+z\,.$$

Lemma 6. *We have that*

$$(17)\qquad 2e<a^{3/2}+a\,.$$

Proof. If $z\geqq M$, then by (13) $M^2\leqq a$. So from (16) we have that $L=M-1$. Then from (13) and (16) we have that $a=M^2+z-1=zM+M-1$. Thus $M=z$ and $a=M^2+M-1$ contradicting Lemma 4. Thus $M>z$ and so by (13) we have that $a<M^2$. Then by (13) we have that

$$2e=za+a-M-1\leqq(a^2-La)/M+a-M-1<a^{3/2}+a\,.$$

Lemma 7. $2ae+a-1$ *divides* $(J+1)(2e-a+1)$. (See p. 519 of [1].)

Proof. Since $a(2e-a+1)=(2ae+a-1)-(a^2-1)$, the greatest common divisor of $2ae+a-1$ and a^2-1 equals that of $2e-a+1$ and a^2-1. Thus by Lemma 2 we see that $2ae+a-1$ divides $(2e-a+1)^2$. Since $(2e-a+1)^2+2(2ae+a-1)=(a^2-1)+(v-2)-(J+1)$, by (8) and Lemma 2 we see that $2ae+a-1$ divides $(a^2-1)(J+1)$ and hence $(2e-a+1)(J+1)$.

Lemma 8. *We have that*

(18) $$2e > a^{3/2} + a\,.$$

Proof. By Lemma 7 and (C), (2) we have that $J+1>a$. So by (C), (1) we have that

$$4(e-a)(e-a+1) > (a+3)(a^2-1)\,,$$

which implies that

$$2e > a^{3/2} + 2a - 2 > a^{3/2} + a\,.$$

(17) and (18) are obviously in contradiction. Thus we obtain that $d_1 = e-1$.

2. Diophantine equation

Two cases are remaining, namely Cases 2 and 3 of (C), § 1.

Lemma 9. *Case 2 cannot occur.*

Proof. By (iii) of (C), § 1, Case 2 and (C), (1) we have that $J=-2$. Since Lemma 7 holds without the assumption that $d_1 < e-1$, we see that $2ae+a-1$ divides $2e-a+1$. This implies that $a=1$.

Thus by (C), § 1 we finally have that

(19) $$v = 4e^2 + 4e + 2$$

and

(20) $$a^2 - 8ae + 4e^2 - 4a + 4e + 3 = 0\,.$$

By (19) and (20) we have that

(21) $$2(v-2) = (2e-a+1)(6e-a+3)$$

and

(22) $$2(2ea+a-1) = (2e-a+1)^2\,.$$

Then by (1), (21) and (22) we obtain that

(23) $$4f^2 = (6e-a+3)^2 - 4(4e^2+4e-1)\,,$$

where f is apositive integer.

Since $(4e+2-a)^2 = 3(2e+1)^2 - 2$ by (20), from (23) we have that $(2f^2-3)^2 = (2e+1)^2\{3(2e+1)^2-2\}$. Thus $2e+1$ and f are integral rational solutions of

(24) $$3X^4 - 4Y^4 - 2X^2 + 12Y^2 - 9 = 0\,.$$

Since (24) has only finitely many integral rational solutions ([3], p. 276), it follows that there exist only finitely many possibilities for a and e being parameters of tight 4-designs.

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