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REMARKS ON MULTIPLY TRANSITIVE PERMUTATION GROUPS

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1. Introduction

In [5], T. Oyama determined all 4-fold transitive permutation groups in which the stabilizer of four points has an orbit of length two. On the other hand, in Yoshizawa [8], 5-fold transitive permutation groups in which the stabilizer of five points has a normal Sylow 2-subgroup have been determined. In this note we give some analogous version of these results for any odd prime p on $2p$ (or $2p+1$)-fold transitive permutation groups.

Theorem 1. *Let p be an odd prime $\geqslant 5$. Let G be a $2p$-fold transitive permutation group on $\Omega=\{1, 2, \cdots, n\}$. If $G_{1,2,\cdots,2p}$ has an orbit on $\Omega-\{1, 2, \cdots, 2p\}$ whose length is less than p, then G is one of $S_n(2p+1\leqslant n\leqslant 3p-1)$ and $A_n(2p+2\leqslant n\leqslant 3p-1)$.*

Corollary. *Let p be an odd prime $\geqslant 5$. Let D be a $2p$-$(v, k, 1)$ design with $2p<k<3p$. If an automorphism group G of D is $2p$-fold transitive on the set of points of D,then D is a $2p$-$(k, k, 1)$ design.*

Theorem 2. *Let p be an odd prime $\geqslant 5$. Let G be a $2p$-fold transitive permutation group on $\Omega=\{1, 2, \cdots, n\}$. Let P be a Sylow p-subgroup of $G_{1,2,\cdots,2p}$. If P is a normal subgroup of $G_{1,2,\cdots,2p}$, then G is one of $S_n(2p\leqslant n\leqslant 3p-1)$ and $A_n(2p+2\leqslant n\leqslant 3p-1)$.*

Theorem 3. *Let G be a 7-fold transitive permutation group on $\Omega=\{1,2,\cdots,n\}$. Let P be a Sylow 3-subgroup of $G_{1,2,\cdots,7}$. If P is a normal subgroup of $G_{1,2,\cdots,7}$, then G is S_7, S_8, S_9, S_{10}, A_9 or A_{10}.*

We shall use the same notation as in [4].

2. Proof of Theorem 1

Let G be a group satisfying the assumption of Theorem 1. By [4] and [5], if $G_{1,2,\cdots,2p}$ has an orbit on $\Omega-\{1, 2, \cdots, 2p\}$ whose length is one or two, then G is S_{2p+1}, S_{2p+2} or A_{2p+2}. Hence we may assume that $G_{1,2,\cdots,2p}$ has an orbit Δ

such that $3 \leqslant |\Delta| \leqslant p-1$.

Let P be a Sylow p-subgroup of $G_{1,2,\cdots,2p}$. If $P=1$, then G is one of S_n $(2p+3 \leqslant n \leqslant 3p-1)$ and $A_n(2p+3 \leqslant n \leqslant 3p-1)$ by [1]. From now on we assume that $P \neq 1$, and prove that this case does not occur. Since $3 \leqslant |\Delta| \leqslant p-1$, we have $I(P) \supseteq \Delta \cup \{1, 2, \cdots, 2p\}$ and $N_G(P)^{I(P)} = S_{2p+3}, \cdots, S_{3p-1}, A_{2p+3}, \cdots$ or A_{3p-1} by [1]. Therefore $N_G(P)_{1,2,\cdots,2p}^{I(P)-\{1,2,\cdots,2p\}} = S_3, \cdots, S_{p-1}, A_3, \cdots$ or A_{p-1}, and $I(P) = \Delta \cup \{1, 2, \cdots, 2p\}$. This shows that $I(P)$ is independent of the choice of Sylow p-subgroup P of $G_{1,2,\cdots,2p}$ and is uniquely determined by $G_{1,2,\cdots,2p}$.

Let Q be a subgroup of P such that the order of Q is maximal among all subgroups of P fixing more than $|I(P)|$ points. Set $N = N_G(Q)^{I(Q)}$, and $r = |\Delta|$. N has an element a of order p fixing $2p+r$ points. We may assume that

$$a = (1)(2)\cdots(2p+r)(2p+r+1, \cdots, 2p+r+p)\cdots .$$

Set $T = C_N(a)_{2p+r+1,\cdots,2p+r+p}^{I(a)}$ and $\Lambda = I(a)$. Then T satisfies the following two properties.

(i) T is a permutation group on Λ. $|\Lambda| = 2p+r$ and $3 \leqslant r \leqslant p-1$.

(ii) For any p points $\alpha_1, \alpha_2, \cdots, \alpha_p$ in Λ, a Sylow p-subgroup S of $T_{\alpha_1,\cdots,\alpha_p}$ is a cyclic group of order p generated by a p-cycle, and $|I(S)| = p+r$. Moreover $I(S)$ is independent of the choice of Sylow p-subgroup S of $T_{\alpha_1,\cdots,\alpha_p}$ and is uniquely determined by $T_{\alpha_1\cdots\alpha_p}$.

Suppose that T is primitive. Since $r \geqslant 3$ and T has a p-cycle, $T \geqslant A_{2p+r}$ by Theorem 13.9 in [7]. This contradicts (ii).

Suppose that T is imprimitive, and let the set $\{\Delta_1, \cdots, \Delta_s\}$ be a nontrivial complete block system. Assume $|\Delta_1| \leqslant p$. For each $i \in \{1, \cdots, s\}$, let δ_i be a point of Δ_i. By considering $T_{\delta_1,\cdots,\delta_p}(s \geqslant p)$ or $T_{\delta_1,\cdots,\delta_s}(s<p)$, we have a contradiction by (ii). Assume $|\Delta_1| > p$. Then $s=2$ and $\Delta_1 \cup \Delta_2 = \Lambda$ by (i). Let Γ_1 be a subset of Δ_1 with $|\Delta_1 - \Gamma_1| = p$, and let δ be a point of $\Delta_1 - \Gamma_1$. Since $|\Delta_1 - (\Gamma_1 \cup \{\delta\})| = p-1$, for every subset Γ_2 of Δ_2 with $|\Delta_2 - \Gamma_2| = p$, $T_{\Gamma_1 \cup \{\delta\} \cup \Gamma_2}$ has a p-cycle on $\Delta_2 - \Gamma_2$, contrary to (ii).

Therefore T is intransitive on Λ. Moreover by (ii), T has an orbit whose length is not less than p. If T has two orbits Δ_1 and Δ_2 such that $|\Delta_1| \geqslant p$ and $|\Delta_2| \geqslant p$, then we have a contradiction by the similar argument to the above. Hence T has a unique orbit Σ with $|\Sigma| \geqslant p$. By (ii), we have $2p \leqslant |\Sigma| < |\Lambda|$. Let Π be a subset of Σ with $|\Pi| + |\Lambda - \Sigma| = p$. Since $|\Lambda - \Sigma| < p$, for every subset Γ of $\Sigma - \Pi$ with $|\Gamma| = p - |\Pi|$, $T_{\Pi \cup \Gamma}$ has a p-cycle on $(\Sigma - \Pi) - \Gamma$, contrary to (ii).

Thus we complete the proof of Theorem 1.

3. Proof of Theorem 2

Let G be a group satisfying the assumption of Theorem 2. Let P be a

Sylow p-subgroup of $G_{1,2,\cdots,2p}$. If $P=1$, then G is one of S_n $(2p\leqslant n\leqslant 3p-1)$ and $A_n(2p+2\leqslant n\leqslant 3p-1)$ by [1]. From now on we assume that $P\neq 1$, and prove that this case does not occur. By [1] and Theorem 1, we have $N_G(P)^{I(P)}=S_{2p}$. By [2], we may assume that P is not semiregular on $\Omega-I(P)$.

Let Q be a subgroup of P such that the order of Q is maximal among all subgroups of P fixing more than $2p$ points. By [3, Lemma 6] and [2], $N_G(Q)^{I(Q)}\geqslant A^{I(Q)}=A_{3p}$. Since A_p is a simple group, we have a contradiction.

4. Proof of Theorem 3

Let G be a group satisfying the assumption of Theorem 3. Let P be a Sylow 3-subgroup of $G_{1,2\cdots,7}$. If $P=1$, then G is S_7, S_8, S_9, or A_9 by [1]. From now on we may assume that $P\neq 1$. Since $P\lhd G_{1\,2\,\cdots,\,7}$, we have $N_G(P)^{I(P)}=S_7$ by [1], [4] and [5]. If P is semiregular on $\Omega-I(P)$, then G is S_{10} or A_{10} by [2]. Hereafter we assume that P is not semiregular, and prove that this case does not occur.

Let Q be a subgroup of P such that the order of Q is maximal among all subgroups of P fixing more than ten points. Let $N=N_G(Q)^{I(Q)}$ and $\Gamma=I(Q)$. Then N is a permutation group on Γ, and $|\Gamma|\geqslant 13$ and $3\,\big|\,|\Gamma|-7$. If N has no element of order three fixing ten points, then N is S_{10} or A_{10} by [3, Lemma 6] and [2], which is a contradiction. Hence from now on we may assume that N has an element a of order three fixing exactly ten points. We may assume that

$$a=(1)(2)(3)(4)(5)(6)(7)(8)(9)(10)(11\ 12\ 13)\cdots .$$

Set $T=C_N(a)^{I(a)}_{11,12,13}$.

Suppose that T has an orbit of length one. Then we may assume that $\{1\}$ is a T-orbit. T_{2345} has an element x_1 of order three, and we may assume that $x_1=(1)(2)(3)(4)(5)(6)(7)(8\ 9\ 10)$. T_{2345} has an element x_2 of order three. Since a Sylow 3-subgroup of T_{1234} is normal in T_{1234}, x_1x_2 is a 3-element. Hence we may assume that $x_2=(1)(2)(3)(4)(8)(9)(10)(5\ 6\ 7)$. T_{2358} has an element x_3 of order three. Since a Sylow 3-subgroup of T_{1235} is normal in T_{1235}, x_1x_3 is a 3-element. Hence we may assume that $x_3=(1)(2)(3)(5)(8)(9)(10)(4\ 6\ 7)$, and so $x_2x_3=(1)(2)(3)(8)(9)(10)(4\ 6)(5\ 7)$. On the other hand, since x_2 and x_3 are 3-elements of T_{1238}, x_2x_3 is a 3-element. So, we have a contradiction.

By the same argument as the above, we have that G has no orbit of length two or three.

Suppose that T has an orbit of length four. Then we may assume that $\{1, 2, 3, 4\}$ is a T-orbit. Since T_{5678} has an element of order three, we may assume that T has an element of order three of the form $(1\ 2\ 3)(4)(5)(6)(7)(8)(9)(10)$. Since $T^{\{1234\}}$ is transitive, we have $T_{5,6,\cdots,10}^{\{1,2,3,4\}}\geqslant A_4$, which is a contradiction.

By the similar argument to the above, we have that T is neither an intransitive group with an orbit of length five nor an imprimitive group with two blocks of length five.

Finally, it is easily seen that T is neither an imprimitive group with five blocks of length two nor a primitive group (cf. [6]), and we complete the proof.

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