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NON-DEFORMABILITY OF EINSTEIN METRICS

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Introduction

Let M be a compact connected C^∞-manifold and g be an Einstein metric on M. By an Einstein deformation of g we mean a 1-parameter family $g(t)$ of Einstein metrics on M such that $g(0)=g$ and the volume of $g(t)$ is constant for t. If for each Einstein deformation $g(t)$ of g there exists a 1-parameter family $\gamma(t)$ of diffeomorphisms such that $g(t)=\gamma(t)^*g$ (resp. $g'(0)=\frac{d}{dt}|_0\gamma(t)^*g$) then g is said to be non-deformable (resp. infinitesimally non-deformable). M. Berger and D. Ebin [1, Lemma 7.4] show that the Einstein structure of the standard sphere is infinitesimally non-deformable, by using the fact that the operator L associated to the curvature tensor of the standard sphere is positive definite. In this paper, the main theorem (Theorem 3.3) gives a criterion for an Einstein structure to be non-deformable, improving their method of estimating eigenvalues of the operator L. As an application we see, for example, that the Einstein structure of a compact irreducible locally symmetric space M of non-compact type with dim $M>2$ is non-deformable. (Corollary 3.5).

To prove the main theorem we have to relate infinitesimal non-deformability to non-deformability. For this purpose we need a smooth slice theorem. The slice theorem (Theorem 2.1) in the H^s-situation (D. Ebin [5, Theorem 7.1], [6, Theorem 8.20]) being in continuous category, we shall improve this continuous slice theorem to a smooth slice theorem (Theorem 2.2) in the ILH-situation. Owing to this we get a theorem (Theorem 2.11) which relates infinitesimal non-deformability to non-deformability.

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1. Preliminaries

First, we introduce notation which will be used throughout this paper. Let M be an n-dimensional, connected and compact C^∞-manifold without boundary, and we always assume $n>2$. For a riemannian manifold (M,g), we consider the riemannian connection and use the following notation;

S^2; the symmetric covariant 2-tensor bundle over M,

$C^\infty(T)$; the vector space of all C^∞-sections of a tensor bundle T over M,

S_0^2; the space of all symmetric covariant 2-tensors whose trace is zero,

$(\, , \,)$; the inner product in fibers of a tensor bundle defined by the riemannian structure,

$\langle \, , \, \rangle$; the global inner product for sections of a tensor bundle over M, i.e., $\langle \, , \, \rangle = \int_M (\, , \,) v_g$, v_g being the volume element defined by g,

R; the curvature tensor,

ρ ; the Ricci tensor,

τ ; the scalar curvature,

∇; the covariant derivation on $C^\infty(T)$,

δ ; the formal adjoint of ∇ with respect to $\langle \, , \, \rangle$,

δ^*; the formal adjoint of $\delta | C^\infty(S^2)$,

$\Delta = \delta d$; the Laplacian operating on the space $C^\infty(M)$ of C^∞-functions on M,

$\bar{\Delta} = \delta \nabla$; the rough Laplacian operating on $C^\infty(T)$,

Hess$=\nabla d$; the Hessian on $C^\infty(M)$.

We shall use the Einstein's convention, although we use $\sum$ if necessary. We shall apply the following formulae throughout the paper.

$$\begin{aligned}
&R^k_{i\,jl}\xi^l = \nabla_i\nabla_j\xi^k - \nabla_j\nabla_i\xi^k, \; R_{ijkl} = R^m_{i\,jk}g_{ml}\,, \\
&\rho_{ij} = -R^l_{i\,lj}, \; \tau = \rho^l_l\,, \\
&(\delta S)^{i_1\cdots i_r}_{j_2\cdots j_s} = -\nabla^l S^{i_1\cdots i_r}_{lj_2\cdots j_s}\,, \; (\delta^*\xi)_{ij} = \frac{1}{2}(\nabla_i\xi_j + \nabla_j\xi_i)\,, \\
&\Delta f = -\nabla^l d_l f, \; (\bar{\Delta} S)^{i_1\cdots i_r}_{j_1\cdots j_s} = -\nabla^l\nabla_l S^{i_1\cdots i_r}_{j_1\cdots j_s}\,.
\end{aligned}$$

(For the standard sphere, $R_{1212}<0$, $\rho_{11}>0$ and $\tau>0$, with respect to orthonormal frame.)

Let (M,g) be an Einstein manifold. If tr $h=0$ then

$$g^{ij}R_i{}^k{}_j{}^l h_{kl} = -\rho^{kl}h_{kl} = 0\,.$$

Hence we can define *the operator* L: $S_0^2 \rightarrow S_0^2$ by

$$(Lh)_{ij} = R_i{}^k{}_j{}^l h_{kl}\,.$$

Next, we recall the following concepts defined by H. Omori [12, pp. 168–169]. A topological vector space E is called an *ILH-space*, if E is an inverse limit of Hilbert spaces $\{E_i\}_{i=1,2,\cdots}$ such that if $j \geqq i$ $E_i \supset E_j$ and the inclusion is a bounded linear operator. We denote $E = \varprojlim E_i$.

A topological space X is called a *C^k-ILH-manifold modeled on E,* if X has the following properties C1 and C2.

C1) X is an inverse limit of C^k-Hilbert manifolds $\{X_i\}_{i=1,2,\cdots}$ such that each X_i is modeled on E_i and $X_i \supset X_j$ if $j \geqq i$.

2) Let x be any point of X. For each i there are an open neighbourhood $U_i(x)$ of x in X_i and a homeomorphism ψ_i from $U_i(x)$ onto an open subset V_i in E_i which gives a C^k-coordiante around x in X_i and satisfies $U_i(x) \supset U_j(x)$ if $j \geqq i$ and $\psi_{i+1}(y) = \psi_i(y)$ for every $y \in U_{i+1}(x)$.

Let X be a C^k-ILH-manifold ($k \geqq 1$). Let TX_i be the tangent bundle of X_i. Then the inverse limit $TX = \varprojlim TX_i$ is called the *ILH-tangent bundle* of X.

Let X, Y be C^k-ILH-manifolds. A mapping $\phi: X \to Y$ is said to be C^l-*ILH-differentiable* ($l \leqq k$), if ϕ is an inverse limit of C^l-defferentiable mappings, that is, for every i, there are a positive integer $j(i)$ and a C^l-mapping $\phi_i: X_{j(i)} \to Y_i$ such that $\phi_i(x) = \phi_{i+1}(x)$ for every $x \in X_{j(i+1)}$ and $\phi = \varprojlim \phi_i$.

If X is a C^k-ILH-manifold for all $k \geqq 0$, we call X an *ILH-manifold*. For ILH-manifolds X, Y, if ϕ is C^k-ILH-defferentiable for all $k \geqq 0$, we say that ϕ is *ILH-differentiable*. We denote by $T_x X_i$ the tangent space of X_i at x and put $T_x X = \varprojlim T_x X_i$. Also we denote by

$$T^r \phi_i(x) \colon \prod_{l=1}^r T_x X_{j(i)} \to T_{\phi_x} Y_i$$

the r-th derivative of ϕ_i at $x \in X$. Then, it is easy to check that $\{T^r \phi_i(x)\}_{i=1,2,\dots}$ has an inverse limit

$$\varprojlim T^r \phi_i(x) \colon \prod_{l=1}^r T_x X \to T_{\phi_x} Y .$$

We call this inverse limit the *r-th derivative* of ϕ and denote it by $T^r \phi(x)$.

A topological group is called an *ILH-Lie group*, if it is an ILH-manifold and the group operations are ILH-mappings.

We can easily see that the space $\mathcal{M}$ of all smooth riemannian metrics on M is an ILH-manifold. (See D. Ebin [5, p. 15], [6, Proposition 5.8] and H. Omori [12, p.170].) We know that the group $\mathcal{D}$ of all diffeomorphisms of M is an ILH-Lie group, and the natural action $A: \mathcal{D} \times \mathcal{M} \to \mathcal{M}$ is ILH-differentiable. (See [12, Lemma 2.5].)

Let $g \in \mathcal{M}$. By a deformation of g we mean a C^∞-curve $g(t): I \to \mathcal{M}$ such that $g(0) = g$, where I is an open interval containning 0 in $\boldsymbol{R}$. Since $\mathcal{M}$ is a positive cone in the vector space of all symmetric covariant 2-tensors on M, we may identify the differential $g'(0)$ of a deformation $g(t)$ with a symmetric covariant 2-tensor field on M. We call such a tensor field an *infinitesimal deformatoin*, or simply an *i-deformation*.

When we consider a deformation $g(t)$ of g, the covariant derivation, the curvature tensor or the Ricci tensor with respect to each $g(t)$ will be *denoted by* ∇_t, $R(t)$ or $\rho_{g(t)}$. Also, we always raise or lower indices of tensors with respect to $g(t)$, and we *denote by* ' the differentiation with respect to t. It is clear that the differential at $t=0$ of the tensors R, ρ, τ etc. depend only on the i-deformation that $g(t)$ defines.

2. Deformations and infinitesimal deformations

Let M be a compact connected C^∞-manifold. We *denote by* $\mathcal{M}^s$ the space of all H^s-metrics on M and by $\mathcal{D}^s$ the space of all H^s-diffeomorphisms of M, where H^s means an object which has partial derivatives defined almost everywhere up to order s and such that each partial derivative is square integrable. We know that the space $\mathcal{M}^s$ and the space $\mathcal{D}^s$ are Hilbert manifolds if s is sufficiently large. Moreover, the usual action A: $\mathcal{D}\times\mathcal{M}\to\mathcal{M}$ extends to a continuous mapping A^s: $\mathcal{D}^{s+1}\times\mathcal{M}^s\to\mathcal{M}^s$. (See D. Ebin [5,p.18], [6, Proposition 4.24].)

D. Ebin gave the following

Theorem 2.1 (D. Ebin [5, Theorem 7.1], [6, Theorem 8.20]). *For each $g\in\mathcal{M}$, there is a submanifold S^s_g of $\mathcal{M}^s$ with the following properties.*

(1) *If $\gamma\in I_g$, then $\gamma^*(S^s_g)=S^s_g$.*

(2) *Let $\gamma\in\mathcal{D}^{s+1}$. If $\gamma^*(S^s_g)\cap S^s_g\neq\phi$, then $\gamma\in I_g$.*

(3) *There are a neighbourhood U^{s+1} of the coset I_g in the right coset space $\mathcal{D}^{s+1}/I_g$ and a local cross section χ^{s+1}: $\mathcal{D}^{s+1}/I_g\to\mathcal{D}^{s+1}$ defined on U^{s+1} such that if the mapping F^s: $U^{s+1}\times S^s_g\to\mathcal{M}^s$ is defined by $F^s(u,s)=\chi^{s+1}(u)^*s$, then F^s is a homeomorphism onto a neighbourhood of g.*

Outline of the proof. Canonically we can construct a riemannian metric on $\mathcal{M}^s$ which is invariant under the action A of $\mathcal{D}$. At any point $g\in\mathcal{M}^s$, ψ^s_g: $\mathcal{D}^{s+1}\to\mathcal{M}^s$ is defined by $\psi^s_g(\eta)=A^s(\eta,g)$ for $\eta\in\mathcal{D}^{s+1}$. If $g\in\mathcal{M}$, ψ^s_g is smooth. Also for $\eta\in\mathcal{D}^{s+1}$, we identify the tangent spaces $T_\eta(\mathcal{D}^{s+1})$ or $T_{\psi^s_g(\eta)}(\mathcal{M}^s)$ with the space of H^s-sections of some vector bundle over M. Then, $T_\eta\psi^s_g$ becomes a first order linear differential operator. It turns out that this operator has an injective symbol, and so its range is closed in $T_{\psi^s_g(\eta)}(\mathcal{M}^s)$.

The right coset space $\mathcal{D}^{s+1}/I_g$ has an induced manifold structure and admits a smooth local cross section χ^{s+1}: $U^{s+1}\to\mathcal{D}^{s+1}$. ψ^s_g induces a mapping ϕ^s_g: $\mathcal{D}^{s+1}/I_g\to\mathcal{M}^s$. ϕ^s_g is an injective immersion and we see directly that it is a diffeomorphism onto the closed orbit O^s_g.

Using the riemannian metric on $\mathcal{M}^s$, we obtain a smooth normal bundle π^s: $\nu^s\to O^s_g$. Moreover, the exponential mapping $\exp^s$ on $\mathcal{M}^s$ is defined on a neighbourhood W^s of the zero-section of ν^s and it is a diffeomorphism. We put $S^s_g=\exp^s W^s_g$, where W^s_g is the fibre on g.

Also, we know that for any $\eta\in\mathcal{D}^{s+1}$ a smooth mapping η^*: $\mathcal{M}^s\to\mathcal{M}^s$ is defined by $\eta^*(g)=A(\eta,g)$ and η^* is an isometry. Therefore, if $\exp^s$ is defined for a vector V in $T(\mathcal{M}^s)$, $\exp^s$ is defined for $T\eta^*(V)$ and we have $\eta^*\exp^s V=\exp^s T\eta^* V$.

Combining these informations, we can prove the slice theorem in the H^s-situation. Moreover, if we define the mapping F^s: $U^{s+1}\times S^s_g\to\mathcal{M}^s$ by $F^s(u,s)=A^s(\chi^{s+1}(u), s)$, for $z\in\exp^s W^s$ we have

$$(F^s)^{-1}(z) = ((\phi_g^s)^{-1}\circ\pi^s\circ(\exp^s)^{-1}(z), A((\chi^{s+1}\circ(\phi_g^s)^{-1}\circ\pi^s\circ(\exp^s)^{-1}(z))^{-1},z)).$$

We shall need the following slice theorem which improve Theorem 2.1 to the C^∞-situation.

Theorem 2.2. *We denote by $\mathcal{M}$ the ILH-manifold formed by all riemannian metrics on M, and by $\mathcal{D}$ the ILH-Lie group of all diffeomrophisms on M. The group $\mathcal{D}$ acts on $\mathcal{M}$ in a canonical way. For each $g\in\mathcal{M}$, there is an ILH-submanifold S_g of $\mathcal{M}$ with the following properties. Let I_g be the group of all isometries of the riemannian manifold (M,g).*

(S1) *If γ belongs to I_g, then $\gamma^*(S_g)=S_g$.*

(S2) *Let $\gamma\in\mathcal{D}$. If $\gamma^*(S_g)\cap S_g\neq\phi$, then $\gamma\in I_g$.*

(S3) *There are a neighbourhood U of the point I_g in the right coset space $\mathcal{D}/I_g$ and a local cross section χ: $\mathcal{D}/I_g\to\mathcal{D}$ defined on U such that if the mapping F: $U\times S_g\to\mathcal{M}$ is defined by $F(u,s)=\chi(u)^*s$, then F is an ILH-diffeomorphism onto a neighbourhood of g.*

We need the following lemmas.

Lemma 2.3. *$\mathcal{D}/I_g$ is an ILH-manifold.*

Lemma 2.4. *Put $U=U^s\cap\mathcal{D}/I_g$. Then $\chi^s(U)$ is contained in $\mathcal{D}$ and the mapping $\chi=\chi^s|U$ is ILH-differentiable.*

Lemma 2.5. *Put $W=W^s\cap T\mathcal{M}$. Then $\exp^s(W)$ is contained in $\mathcal{M}$ and the mapping $\exp=\exp^s|W$ is an ILH-diffeomorphism. Hence $S_g=S_g^s\cap\mathcal{M}$ is an ILH-submanifold of $\mathcal{M}$.*

These lemmas will be proved in below.

Lemma 2.6 [12, Lemma 2.5]. *$A^s(\mathcal{D}\times\mathcal{M})$ is contained in $\mathcal{M}$ and the mapping $A=A^s|\mathcal{D}\times\mathcal{M}$ is ILH-differentiable.*

Lemma 2.7 [12, Lemma 1.14]. *If the mapping $\tilde{i}$: $\mathcal{D}\to\mathcal{D}$ is defined by $\tilde{i}(\eta)=\eta^{-1}$ for $\eta\in\mathcal{D}$, then $\tilde{i}$ is ILH-differentiable.*

Proof of Theorem 2.2. Combining these lemmas and the proof of Theorem 2.1, the mappings $F=F^s|U\times S_g$ and $F^{-1}=(F^s)^{-1}|\exp W$ are compositions of ILH-mappings, and so F is an ILH-diffeomorphism, which proves Theorem 2.2.

Proof of Lemma 2.3. We know that $\mathcal{D}^s/I_g$ is a Hilbert manifold. We shall prove that the inclusion $\tilde{i}^s$: $\mathcal{D}^{s+1}/I_g\to\mathcal{D}^sI_g$ is smooth. By [5, Corollary 5.11] or [6, Corollary 7.16], $\tilde{i}^s$ is smooth if and only if $\tilde{i}^s\circ p^{s+1}$: $\mathcal{D}^{s+1}\to\mathcal{D}^s/I_g$ is smooth, where p^{s+1}: $\mathcal{D}^{s+1}\to\mathcal{D}^{s+1}/I_g$ is the natural projection. We can easily see $\tilde{i}^s\circ p^{s+1}=$

$p^s \circ i^s$, where $i^s: \mathcal{D}^{s+1} \to \mathcal{D}^s$ is the inclusion. Since i^s and p^s are smooth, $\tilde{i}^s$ is smooth.

Proof of Lemma 2.4. By [5, Proposition 5.10] or [6, Proposition 7.15], $\mathcal{D}^s/I_g$ admits a smooth local cross section around any coset. We denote by χ^s_x the local cross section around $x \in \mathcal{D}^s/I_g$ and put $\chi^s = \chi^s_{I_g}$. Let U^s be the domain of χ^s and set $U^r = U^s \cap \mathcal{D}^r/I_g$ and $\chi^r = \chi^s | U^r$ for $r \geqq s$. If $u \in U^r$, there is an element $a \in D^r$ such that $u = I_g a$ and $\chi^r(u) \in I_g a \subset \mathcal{D}^r$. Hence we have $\chi^r(U^r) \subset \mathcal{D}^r$. To prove that χ^r is smooth, we shall show that if we define a mapping $\nu: (p^s)^{-1}(U^s) \to I_g$ by $\nu(\eta) = \eta(\chi^s \circ p^s \eta)^{-1}$, then ν is smooth. By [5, Lemma 5.5] or [6, Corollary 7.7], the composition: $I_g \times \mathcal{D}^s \to \mathcal{D}^s$ is smooth. Hence, if we define a mapping $\psi: I_g \times U^s \to \mathcal{D}^s$ by $\psi(\xi, x) = \xi \chi^s(x)$, then ψ is smooth. On the other hand, we have $\psi^{-1}(\eta) = (\nu(\eta), p^s(\eta))$ and p^s is smooth. For ν, we fix a positive integer i such that the composition: $\mathcal{D}^s \times \mathcal{D}^i \to \mathcal{D}^i$ and the inverse: $\mathcal{D}^s \to \mathcal{D}^i$ are C^1-mappings. ([12, Lemma 1.13 and Lemma 1.14]. Suppose that s is sufficientry large.) Then, we see directry that ν is a C^1-mapping into $\mathcal{D}^i$. But I_g contains the image of ν and I_g is a submanifold of $\mathcal{D}^i$ (see [5, Corollary 5.4] or [6, Theorem 7.1]). Hence, ν is a C^1-mapping into I_g. Therefore, we know that ψ is smooth and ψ^{-1} is a C^1-mapping. By the inverse function theorem, ψ^{-1} is smooth and so ν is smooth.

Now, we shall prove the smoothness of χ^r around any $x \in U^r$. There is a smooth local cross section χ^r_x on a neighbourhood V of x. Therefore the mapping $\nu \circ$"inclusion"$\circ \chi^r_x: V \to I_g$ is smooth and we have $\nu \circ$"inclusion"$\circ \chi^r_x(y)$ $= \chi^r_x(y)(\chi^s(y))^{-1} = \chi^r_x(y)(\chi^r(y))^{-1}$. Since we know that $\chi^r(y) = ((\chi^r_x(y))(\chi^r(y))^{-1})^{-1}\chi^r_x(y)$ and the inverse: $I_g \to I_g$ and the composition: $I_g \times \mathcal{D}^r \to \mathcal{D}^r$ are smooth, the mapping $\chi^r: V \to \mathcal{D}^r$ is smooth.

Proof of Lemma 2.5. Let $\bar{W}^s$ be an open subset of $T\mathcal{M}^s$ such that $W^s = \nu^s \cap \bar{W}^s$. Set $\bar{W}^r = \bar{W}^s \cap T\mathcal{M}^r$, $W^r = \bar{W}^s \cap \nu^r$, $\exp^r = \exp^s | \bar{W}^r$ and $(\exp^{-1})^r = (\exp^s | W^s)^{-1} | \exp^s(W^s) \cap \mathcal{M}^r$. The mappings $\exp^s: \bar{W}^s \to \mathcal{M}^s$ and $(\exp^{-1})^s$: $\exp^s(W^s) \to T\mathcal{M}^s$ are smooth and commute with the action of $\mathcal{D}$. Hence, by the following Lemma 2.8, $\exp^r(\bar{W}^r)$ and $(\exp^{-1})^r(\exp(W^s) \cap \mathcal{M}^r)$ are contained in $\mathcal{M}^r$ and $T\mathcal{M}^r$ respectively, and the mappings $\exp^r: \bar{W}^r \to \mathcal{M}^r$ and $(\exp^{-1})^r$: $\exp^s(W^s) \cap \mathcal{M}^r \to T\mathcal{M}^r$ are smooth for $r \geqq s$. But W^r is a submanifold of $\bar{W}^r$ and $(\exp^{-1})^r(\exp^s(W^s) \cap \mathcal{M}^r)$ is contained in W^r, which implies that $\exp^r: W^r \to \exp^s(W^s) \cap \mathcal{M}^r$ is a diffeomorphism. Thus we see that exp is an ILH-diffeomorphism onto $\exp^s(W^s) \cap \mathcal{M}$.

Lemma 2.8. *Let E and F be vector bundles over M associated with the frame bundle (e.g., T, T^*, S^2, $T \times T^*$, the k-th jet bundle $J^k(T)$ etc.).* Any $\eta \in \mathcal{D}$ defines a *natural linear mapping* $\eta^*: H^0(E) \to H^0(E)$. *Let A be an open subset of $H^s(E)$ and let $f: A \to H^s(F)$ be a smooth mapping which commutes with the action of*

$\mathcal{D}$. Put $A^r = A \cap H^r(E)$ for $r \geqq s$. Then $f(A^r)$ is contained in $H^r(F)$ and $f \mid A^r \to H^r(F)$ is smooth.

Proof of Lemma 2.8. We shall prove that if this lemma holds for $r=i$, then the same is true for $r=i+1$. The induction will then complete the proof. First, by induction, we shall prove that $\eta^* \circ T^k f(a) = T^k f(\eta^* a) \circ \eta^*$ for all positive integer k. If $\eta^* \circ T^l f(a) = T^l f(\eta^* a) \circ \eta^*$, then we have

$$\begin{aligned} \eta^* \circ T^{l+1} f(a)(v, v_1, \cdots, v_l) &= \eta^* \frac{d}{dt}\Big|_0 T^l f(a+tv)(v_1, \cdots, v_l) \\ &= \frac{d}{dt}\Big|_0 T^l f(\eta^* a + t\eta^* v)(\eta^* v_1, \cdots, \eta^* v_l) \\ &= T^{l+1} f(\eta^* a)(\eta^* v, \eta^* v_1, \cdots, \eta^* v_l)\,. \end{aligned}$$

Let V be a vector field on M and let η_t be the 1-parameter subgroup of diffeomorphisms generated by V. For sufficiently small t, $\eta_t^* a \in A^i$ if $a \in A^i$. Hence we get

$$\begin{aligned} \mathcal{L}_v T^k f(a)(v_1, \cdots, v_k) &= \frac{d}{dt}\Big|_0 \eta_t^* T^k f(a)(v_1, \cdots, v_k) \\ &= \frac{d}{dt}\Big|_0 T^k f(\eta_t^* a)(\eta_t^* v_1, \cdots, \eta_t^* v_k) \\ &= T^{k+1} f(a)(\mathcal{L}_v a, v_1, \cdots, v_k) + T^k f(a)(\mathcal{L}_v v_1, v_2, \cdots, v_k) \\ &\quad + \cdots + T^k f(a)(v_1, \cdots, v_{k-1}, \mathcal{L}_v v_k)\,. \end{aligned}$$

Next, we shall prove that $f(A^{i+1}) \subset H^{i+1}(F)$, and $f \mid A^{i+1}: A^{i+1} \to H^{i+1}(F)$ is continuous and that if $f \mid A^{i+1}$ is a C^k-mapping and $T^k(f \mid A^{i+1}) = T^k f \mid A^{i+1} \times H^{i+1}(E) \times \cdots \times H^{i+1}(E)$, then $f \mid A^{i+1}$ is a C^{k+1}-mapping and $T^{k+1}(f \mid A^{i+1}) = T^{k+1} f \mid A^{i+1} \times H^{i+1}(E) \times \cdots \times H^{i+1}(E)$. Then, by the hypothesis of the induction, $f \mid A^{i+1}$ is smooth.

If $a \in A^{i+1}$, then $\mathcal{L}_v a \in H^i(E)$ for all $V \in C^\infty(T)$. Hence $\mathcal{L}_v f(a) = Tf(a)(\mathcal{L}_v a) \in H^i(F)$, which implies that $f(A^{i+1}) \subset H^{i+1}(F)$. If a sequence $\{a_n\}$ converges to a in A^{i+1}, then $\{\mathcal{L}_v a_n\}$ converges to $\mathcal{L}_v a$ in $H^i(E)$ for all $V \in C^\infty(T)$. Hence $\{\mathcal{L}_v f(a_n) = Tf(a_n)(\mathcal{L}_v a_n)\}$ converges to $T_f(a)(\mathcal{L}_v a) = \mathcal{L}_v f(a)$ in $H^i(F)$, which implies that $f \mid A^{i+1}$ is continuous. By the same calculation, we check easily that $T^j f(A^{i+1} \times H^{i+1}(E) \times \cdots \times H^{i+1}(E)) \subset H^{i+1}(F)$ and $T^j f \mid A^{i+1} \times H^{i+1}(E) \times \cdots \times H^{i+1}(E)$ is continuous. We assume that $f \mid A^{i+1}$ is a C^k-mapping and $T^k(f \mid A^{i+1}) = T^k f \mid A^{i+1} \times H^{i+1}(E) \times \cdots \times H^{i+1}(E)$. Define a mapping

$$\nu: A^{i+1} \times H^{i+1}(E) \times \underbrace{\{H^{i+1}(E) \times \cdots \times H^{i+1}(E)\}}_{k\text{-terms}} \to H^{i+1}(F)$$

by
$$\nu(a, v, \boldsymbol{v}) = T^k(f \mid A^{i+1})(a+v)(\boldsymbol{v}) - T^k(f \mid A^{i+1})(a)(\boldsymbol{v}) - T^{k+1} f(a)(v, \boldsymbol{v})\,.$$

Then, by the assumption,

$$\nu(a,v,\boldsymbol{v}) = T^k f(a+v)(\boldsymbol{v}) - T^k f(a)(\boldsymbol{v}) - T^{k+1}f(a)(v,\boldsymbol{v})$$

and $$\nu(a,tv,\boldsymbol{v}) = T^k f(a+tv,\boldsymbol{v}) - T^k f(a)(\boldsymbol{v}) - tT^{k+1}f(a)(v,\boldsymbol{v}) .$$

By differentiation with respect to the H^i-topology, we get

$$\nu(a,v,\boldsymbol{v}) = \int_0^1\int_0^t T^{k+2}f(a+uv)(v,v,\boldsymbol{v})dudt .$$

Since $|T^{k+2}f|$ is continuous with respect to the H^{i+1}-topology, we have $|\nu(a,v,\boldsymbol{v})|/|v| \leqq \max_{|b-a|<\varepsilon} |T^{k+2}f(b)||v||\boldsymbol{v}|$, where ε is sufficiently small and $|\ |$ is the H^{i+1}-norm. Therefore, $T^k(f|A^{i+1})$ is differentiable and $T^{k+1}(f|A^{i+1})$ coincides with the continuous mapping $T^{k+1}f|A^{i+1}\times H^{i+1}(E)\times\cdots\times H^{i+1}(E)$. Q.E.D.

A deformation $g(t)$ contained in a $\mathcal{D}$-orbit O_g of g is called *trivial,* since each $(M,g(t))$ is isometric to (M,g). On the other hand, a deformation contained in S_g is said to be *essential with respect to g*. According to M. Berger and D. Ebin [1, §3, (3.1)], we can identify the tangent spaces $T_g(O_g)$ and $T_g(S_g)$ at g with Im δ^* and Ker δ. We call therefore an element of Im δ^* a *trivial i-deformation* and an element of Ker δ an *essential i-deformation.*

Let $g(t)$ and $\tilde{g}(t)$ be deformations of g. If there is a 1-parameter family of diffeomorphisms $\gamma(t)$ satisfying $g(t)=\gamma(t)^*\tilde{g}(t)$, then $g(t)$ is said to be *equivalent to* $\tilde{g}(t)$. Theorem 2.2 implies that every deformation is equivalent to an essential deformation (by restricting the range of t to some open interval containing 0).

Lemma 2.9. *If $g'(t)$ is trivial with respect to $g(t)$ (i.e., $g'(t)\in$ Im $\delta^*_{g(t)}$) for each t, then $g(t)$ is a trivial deformation.*

Proof. D. Ebin [5, Theorem 8.1 or 6, Propsition 8.30] shows that for given $g\in\mathcal{M}$ and any neighbourhood V of the identity in $\mathcal{D}$, there is a neighbourhood H of g in $\mathcal{M}$ such that if $\psi\in H$ there is $\gamma\in V$ satisfying $\gamma^{-1}I_\psi\gamma\subset I_g$. So, we find dim $I_{g(t)}$ is upper semi-continuous. Let W be a connected component of the set of all t such that dim $I_{g(t)}$ is minimum. Then W is open in I. Fixing $t_0\in W$, we shall apply Theorem 2.2 for $g(t_0)$.

Let $\tilde{g}(t)$ be a deformation equivalent to $g(t)$ contained in $S_{g(t_0)}$. First we prove $\tilde{g}'(t_1)=0$ for all t_1 for which $\tilde{g}(t_1)$ is defined. If $\gamma\in I_{\tilde{g}(t_1)}$, then $\gamma^*\tilde{g}(t_1)=\tilde{g}(t_1)\in S_{g(t_0)}$ and so $\gamma\in I_{g(t_0)}$, because of the property (S2) in Theorem 2.2. This implies $I_{\tilde{g}(t_1)}\subset I_{g(t_0)}$. Since $t_0\in W$, it follows that any Killing vector field with respect to $g(t_0)$ is a Killing vector field with respect to $\tilde{g}(t_1)$. Now, because $\tilde{g}'(t_1)$ is trivial with respect to $\tilde{g}(t_1)$, there is $\xi\in T_{\mathrm{Id}}(\mathcal{D})$ such that $\tilde{g}'(t_1)=TA_{(\mathrm{Id},\tilde{g}(t_1))}(\xi,0)$, where A is the map $\mathcal{D}\times\mathcal{M}\to\mathcal{M}$ defined by the action of

$\mathcal{D}$ on $\mathcal{M}$ and TA is the differential of A. Denote by π the natural projection from $\mathcal{D}$ to $\mathcal{D}/I_{g(t_0)}$ and let χ be as in Theorem 2.2. Put $\bar{\xi}=T\chi\circ T\pi(\xi)$. Then $\xi-\bar{\xi}$ is a Killing vector field with respect to $g(t_0)$, and so with respect to $\tilde{g}(t_1)$ also. Therefore $TA_{(\mathrm{Id},\tilde{g}(t_1))}(\xi-\bar{\xi},0)=0$, $\tilde{g}(t_1)$ being fixed under the action of $I_{\tilde{g}(t_1)}$.

Now, set $F^{-1}=p\times q$ where $p\colon \mathcal{M}\to\mathcal{D}/I_{g(t_0)}$ and $q\colon \mathcal{M}\to S_{g(t_0)}$. Since $\tilde{g}'(t_1)$ is tangent to $S_{g(t_0)}$, $Tp(\tilde{g}'(t_1))=0$. On the other hand,

$$\begin{aligned}\tilde{g}'(t_1) &= TA_{(\mathrm{Id},\tilde{g}(t_1))}(\xi,0)\\ &= TA_{(\mathrm{Id},\tilde{g}(t_1))}(\xi-\bar{\xi},0)+TA_{(\mathrm{Id},\tilde{g}(t_1))}(\bar{\xi},0)\\ &= TA_{(\mathrm{Id},\tilde{g}(t_1))}(T\chi\circ T\pi(\xi),0)\\ &= TF_{(I_{g(t_0)},\tilde{g}(t_1))}(T\pi(\xi),0)\,,\end{aligned}$$

hence $Tq(\tilde{g}'(t_1))=0$. But $Tp\times Tq$ is an isomorphism, and therefore $\tilde{g}'(t_1)=0$. We have thus proved that $\tilde{g}(t)$ is constant on W, and so $g(t)$ is trivial on W. By [5, Proposition 6.13 or 6, Theorem 8.10], a $\mathcal{D}$-orbit is closed in M. Let a be an end point of W. Since W is open, $a\notin W$. If $a\in I$, then $g(a)\in O_{g(t_0)}$, and so $g(a)$ is isometric to $g(t_0)$, which contradicts $a\notin W$. Hence $W=I$. Q.E.D.

Let $\mathcal{P}$ be a subset of $\mathcal{M}$ invariant under the action of $\mathcal{D}$. For $g\in\mathcal{P}$, we *denote by* $\mathcal{P}_g$ the vector space which is spanned by all i-deformations $g'(0)$ defined by deformations $g(t)$ contained in $\mathcal{P}$.

DEFINITION 2.10. If all deformations of g contained in $\mathcal{P}$ are trivial then g is said to be *non-deformable* (in the sence of $\mathcal{P}$). If $\mathcal{P}_g\subset\mathrm{Im}\ \delta_g^*$ then g is said to be *infinitesimally non-deformable* (in the sence of $\mathcal{P}$).

Theorem 2.11. *Let $\mathcal{P}$ be a $\mathcal{D}$-invariant subset of $\mathcal{M}$. If there is a $\mathcal{D}$-invariant open set W of $\mathcal{P}$ such that all metrics in W are infinitesimally non-deformable, then every $g\in W$ is non-deformable.*

Proof. Let $g(t)\colon I\to\mathcal{P}$ be any deformation of $g\in W$ contained in $\mathcal{P}$. Let J be the subset of I of all t such that $g(t)\in W$, and J_1 be the connected component of J containing 0. Then $g(t)$ is infinitesimally non-deformable for each $t\in J_1$, and so, by Lemma 2.9, $g(t)\,|\,J_1$ is trivial. If J_1 does not coincide with I, then there is an end point t_0 of J_1 in I. Since $\mathcal{D}$-orbits in $\mathcal{M}$ are closed, $g(t_0)$ is isometric to g, which contradicts $g(t_0)\notin W$. Thus $J_1=I$. Q.E.D.

3. Einstein deformations

DEFINITION 3.1 We *denote by* $\mathcal{E}$ the space of all Einstein metrics on M whose volume is some constant c. A deformation contained in $\mathcal{E}$ is called an *Einstein deformation.* If all Enistein deformations of $g\in\mathcal{E}$ are trivial, then g is said to be *non-deformable.* (cf. Definition 2.10)

Lemma 3.2. *Let $g(t)$ be an Einstein deformation of g. Then the essential* *component h of the i-deformation* $g'(0)$ *(i.e.,* $g'(0)=h+\delta^*\xi$ *and* $\delta h=0$*) satisfies the following equalities:*

$$\bar{\Delta}h+2Lh=0, \quad \mathrm{tr}\ h=0,$$

where the operator $L: S_0^2 \to S_0^2$ *is defined in* **1**; $(Lh)_{ij}=R_i{}^k{}_j{}^l h_{kl}$.

Proof. See M. Berger and D. Ebin [1, Lemma 7.1, (7.1)].

Theorem 3.3. *Let* (M,g) *be a compact Einstein manifold with* $\rho=\varepsilon g$, ρ *being the Ricci tensor. Denote by* α_0 *the minimum eigenvalue on M of the operator L. If* $\alpha_0>\min\left\{\varepsilon, -\frac{1}{2}\varepsilon\right\}$, *then* (M,g) *is non-deformable.*

Proof. Owing to Theorem 2.11 and Lemma 3.2, it is sufficient to prove that if h is an i-deformation of g such that $\delta h=0$, $\bar{\Delta}h+2Lh=0$ and tr $h=0$ then $h=0$. First we define the operators $\mathfrak{S}\nabla: C^\infty(S^2)\to C^\infty(T_3^0)$ and $S\nabla: C^\infty(S^2)\to C^\infty(T_3^0)$ by

$$(\mathfrak{S}\nabla h)(X,Y,Z)=\alpha(\nabla_X h)(Y,Z)+\beta(\nabla_Y h)(Z,X)+\gamma(\nabla_Z h)(X,Y)$$
$$(S\nabla h)(X,Y,Z)=(\nabla_Y h)(Z,X)$$

where, $\alpha, \beta, \gamma\in \boldsymbol{R}$, $\alpha^2+\beta^2+\gamma^2=1$. Set $u=\alpha\beta+\beta\gamma+\gamma\alpha$. Then the minimum and the maximum of u are $-\frac{1}{2}$ and 1 respectively. By simple computations, we have

$$\langle\mathfrak{S}\nabla h, \mathfrak{S}\nabla h\rangle = \langle\nabla h, \nabla h\rangle+2u\langle S\nabla h, \nabla h\rangle$$
$$= \langle\bar{\Delta}h, h\rangle+2u\langle\delta S\nabla h, h\rangle.$$

Now,

$$(\delta S\nabla h)_{ij} = -\nabla^k(S\nabla h)_{kij} = -\nabla^k\nabla_i h_{jk}$$
$$= g^{km}R^l_{mij}h_{lk}+g^{km}R^l_{mik}h_{jl}-\nabla_i\nabla^k h_{jk}$$
$$= -(Lh)_{ij}-\rho_i^l h_{jl}+(\nabla\delta h)_{ij}.$$

Therefore, we get

$$\langle\bar{\Delta}h-2uLh-2u\varepsilon h+2u\nabla\delta h, h\rangle\geqq 0.$$

Here, we set $\delta h=0$ and $\bar{\Delta}h=-2Lh$. Then

$$u\varepsilon\langle h, h\rangle \leqq -(1+u)\langle Lh, h\rangle.$$

Thus, if $h\neq 0$ then we have $\alpha_0\leqq\varepsilon$ and $\alpha_0\leqq-\frac{1}{2}\varepsilon$, by setting $u=-\frac{1}{2}$, 1, respectively. Q.E.D.

Let N be a riemannian manifold and $O_p=X_i$ be an orthonormal frame at $p\in N$. Then $\sigma_{ij}=-R_{ijij}$ is the sectional curvautre if $i\neq j$, and is zero if $i=j$. We count the number of j such that $\sigma_{i_0 j}=0$ for an index i_0, and call the maximum of such numbers the *flat dimension* fd(N) of N when p, O_p, i_0 run over respective sets. For example, if N has negative curvature, then fd$(N)=1$.

Proposition 3.4. *If an Einstein manifold (M,g) has non-positive sectional curvature, and if its universal riemannian covering $(\tilde{M},\tilde{g})$ is the product of the riemannian manifolds $\tilde{M}_a (1\leqq a\leqq k)$ satisfying $2\mathrm{fd}(\tilde{M}_a)<\dim \tilde{M}_a$, then g is non-deformable. Especially, an Einstein manifold (M,g) is non-deformable, if all irreducible component of $(\tilde{M},\tilde{g})$ have negative sectional curvature and are of dimension>2.*

Proof. (I) First, we consider the case that $\tilde{M}$ itself is such that $2\mathrm{fd}(\tilde{M})<\dim\tilde{M}$. Put $r=\mathrm{fd}(\tilde{M})$. Fix a point m in $\tilde{M}$ and let $Lh=\alpha h$ for a non-zero symmetric bilinear form h whose trace is zero. Using an orthnormal frame $\{X_i\}$ at m, we diagonalize h with respect to $\tilde{g}$, and set $h^{ii}=x^i$. Then, $\sum x^i=0$ and

$$R_{ijkl}h^{ik}h^{jl}=\sum_{i,j} R_{ijij}x^ix^j=-\sum_{i,j}\sigma_{ij}x^ix^j\,.$$

Now, let (y_i) be an eigenvector of the matrix (σ_{ij}) belonging to an eigenvalue λ. By changing order of coordinates if necessary, we can assume that $y_r=\max_j|y_j|$ and $\sigma_{ri}<0$ for all $i>r$. Then,

$$-\lambda y_r=-\sum_i \sigma_{ir}y_i\geqq\sum_i \sigma_{ir}y_r=\mathcal{E}y_r\,.$$

So $-\lambda\geqq\mathcal{E}$ and, if the equality holds, then we have $y_i=-y_r$ for all $i>r$, which implies

$$\sum_i y_i=\sum_{i>r}y_i+\sum_{i\leqq r}y_i\leqq-(n-r)y_r+ry_r=-(n-2r)y_r<0\,.$$

Therefore, for (x_i) such that $\sum_i x^i=0$, we have

$$-\sum_{i,j}\sigma_{ij}x^ix^j>\mathcal{E}\sum_i (x^i)^2\,.$$

Hence, $\alpha(h,h)=-\sum_{i,j}\sigma_{ij}x^ix^j>\mathcal{E}\sum_i (x^i)^2=\mathcal{E}(h,h)$.

Thus we get $\alpha>\mathcal{E}$. Our assertion follows then from Theorem 3.3.

(II) Now we consider the general case. Corresponding to the decomposition $(\tilde{M},\tilde{g})=\prod_a(\tilde{M}_a,\tilde{g}_a)$, the curvature tensor decomposes. Hence, the Ricci tensor $\tilde{\rho}$ of $\tilde{M}$ has the decompostion $\tilde{\rho}=\sum_a\tilde{\rho}_a$ where $\tilde{\rho}_a$ is the Ricci tensor of $\tilde{M}_a$. Therefore $\tilde{\rho}_a=\mathcal{E}\tilde{g}_a$. Moreover, $S^2_0(\tilde{M})$ and the operator $\tilde{L}$ on $S^2_0(\tilde{M})$ decomposes as follows;

$$S^2_0(\tilde{M})=(\bigoplus_a S^2_0(\tilde{M}_a))\oplus((\bigoplus_a \boldsymbol{R}\tilde{g}_a)\cap S^2_0(\tilde{M}))\oplus\sum_{a\neq b}S^2(\tilde{M}_a,\tilde{M}_b)\,,$$
$$\tilde{L}\,|\,S^2_0(\tilde{M}_a)=\tilde{L}_a\,,$$
$$\tilde{L}\,|\,(\oplus(\boldsymbol{R}\tilde{g}_a))\cap S^2_0(\tilde{M})=-\mathcal{E}\,,$$
$$\tilde{L}\,|\,S^2(\tilde{M}_a,\tilde{M}_b)=0\quad\text{for}\quad a\neq b\,,$$

where $\tilde{L}_a$ is the operator of $\tilde{M}_a$ and

$$S^2(\tilde{M}_a,\tilde{M}_b)=\{h\in S^2(\tilde{M}_a\times\tilde{M}_b);\ h(T\tilde{M}_c,T\tilde{M}_c)=0\ \text{ for }\ c=a,b\}\,.$$

Since the curvature of $(M,g)\leqq 0$, ε is negative. Then, combined with what we have proved in (I), we get $\alpha_0>\varepsilon$ and our assertion follows from Theorem 3.3 Q.E.D.

Corollary 3.5. *Let (M,g) be a compact Einstein manifold. If M is a locally symmetric space of non-compact type, and the dimension of every irreducible component of the universal covering $(\tilde{M},\tilde{g})$ of (M,g) is greater than 2, then (M,g) is non-deformable.*

Proof. Let G/K be a symmetric space which is the universal covering of (M,g). Since the dimension of every irreducible component of G/K is greater than 2, we may assume that G has no simple factor of dimension 3. On the other hand A. Weil [13, §10] shows that if G has no simple factor of dimension 3, then $\alpha_0>\varepsilon$. Thus the proof reduces to Theorem 3.3.

Remark 3.6. Theorem 24.1′ in G.D. Mostow [10] implies that if (M,g_1) and (M,g_2) are locally symmetric spaces of non-compact type without 2-dimensional factors locally, then g_1 and g_2 are isometric up to normalizing constants. (cf. E. Calabi [3, Theorem 1], A. Weil [13, Theorem 1])

Corollary 3.7. *If the sectional curvature of a compact Einstien manifold (M,g) ranges in the interval $\left(\frac{n-2}{2n-1},1\right]$, then (M,g) is non-deformable.*

Proof. We easily see that $\varepsilon=\frac{1}{n}\sum_{i\neq j}\sigma_{ij}$, hence the condition implies $\varepsilon>(n-2)(n-1)/(2n-1)$. By virtue of Theorem 3.3, it is sufficient to prove $\alpha_0+\frac{1}{2}\varepsilon>0$. In the same way as for the proof I of Proposition 3.4, we may set $h^{ii}=x^i$ with $\sum x^i=0$. We can assume that there is an integer c such that $y^i=x^i\geqq 0$ for any $i\leqq c$, and $z^i=-x^i>0$ for any $i>c$. Set $\sum_{i\leqq c}y^i=\sum_{i>c}z^i=A$. Then, since $\sum x^i=0$,

$$\begin{aligned}(Lh,h)+\frac{1}{2}(\varepsilon h,h) &= -\sum_{i,j}\sigma_{ij}x^ix^j+\frac{1}{2}\varepsilon\sum_i(x^i)^2+\sum_i x^i\sum_j x^j\\
&= \left(1+\frac{1}{2}\varepsilon\right)\{\sum_{i\leqq c}(y^i)^2+\sum_{i>c}(z^i)^2+\sum_{i\neq j,\, i,j\leqq c}(1-\sigma_{ij})y^iy^j\\
&\quad+\sum_{i\neq j,\, i,j>c}(1-\sigma_{ij})z^iz^j-2\sum_{i\leqq c,\, j>c}(1-\sigma_{ij})y^iy^j\\
&> \frac{n(n+1)}{2(2n-1)}\{\sum_{i\leqq c}(y^i)^2+\sum_{i>c}(z^i)^2\}-2\frac{n+1}{2n-1}A^2\\
&\geqq \frac{n(n+1)}{2(2n-1)}\left(\frac{1}{c}A^2+\frac{1}{n-c}A^2\right)-2\frac{n+1}{2n-1}A^2\\
&\geqq \frac{n(n+1)}{2(2n-1)}\frac{4}{n}A^2-2\frac{n+1}{2n-1}A^2=0\,.\end{aligned}$$

REMARK 3.8. Y. Muto [11, Theorem] shows that every Einstein metric near a metric with positive constant sectiona curvature is of positive constant sectional curvature.

REMARK 3.9. Even if (M,g) is a non-deformable Einstein metric, M may have an Einstein metric $\tilde{g}$ which is not isometric to g. In fact, G.R. Jensen [8, pp. 612–613] constructs a non-standard Einstein metric $\tilde{g}$ on S^{4p+3}. The author does not know whether $\tilde{g}$ is non-deformable or not.

Finally, by a direct computation, we may apply Theorem 3.3 to the manifold M whose universal covering $\tilde{M}$ is an irreducible symmetric space G/K of compact type.

I. the case where $\tilde{M}$ is hermitian symmetric

In this case, the eigenvalue of the generalized operator $\tilde{L}$: $S^2 \to S^2$ are calculated by E. Calabi and E. Vesentini [4, p. 502, Table 2] and A. Borel [2, Corollary 4.6, 4.7]. See Table 1. Here we omit 0 and $-\varepsilon$, which are always eigenvalues of $\tilde{L}$. The eigenspace corresponding to this eigenvalue $-\varepsilon$ is generated by g. Hence, this is not an eigenvalue of our operator L on S_0^2. We conclude that the following three classes are non-deformable.

AIII $(p=1)$, $(q=1)$
DIII $(p \geqq 6)$
EVII

Table 1

type	$\dim_C M$	G/M	$\alpha\varepsilon^{-1}$/multiplicity			
AIII	pq	$SU(p+q)$ $S(U_p \times U_q)$	$2(p+q)^{-1}$ $2\binom{p+1}{2}\binom{q+1}{2}$	$-2(p+q)^{-1}$ $2\binom{p}{2}\binom{q}{2}$	$-p(p+q)^{-1}$ q^2-1	$-q(p+q)^{-1}$ p^2-1
DIII	$\binom{p}{2}$	$SO(2p)$ $U(p)$	$(p-1)^{-1}$ $\frac{1}{6}p^2(p^2-1)$	$-2(p-1)^{-1}$ $2\binom{p}{4}$	$-\frac{1}{2}(p-2)(p-1)^{-1}$ p^2-1	
CI	$\binom{p+1}{2}$	$Sp(p)$ $U(p)$	$2(p+1)^{-1}$ $2\binom{p+3}{4}$	$-(p+1)^{-1}$ $\frac{1}{6}p^2(p^2-1)$	$-\frac{1}{2}(p+2)(p+1)^{-1}$ p^2-1	
BDI	p	$SO(p+2)$ $SO(p)\times T^1$	$2p^{-1}$ $(p-1)(p+2)$	$-(p-2)p^{-1}$ 2	$-2p^{-1}$ $\binom{p}{2}$	
EIII	16	E_6 Spin (10)$\cdot T^1$	$\frac{1}{6}$ 252	$-\frac{1}{2}$ 20	$-\frac{1}{3}$ 45	
EVII	27	E_7 $E_6 \times T^1$	$\frac{1}{9}$ 702	$-\frac{4}{9}$ 54	$-\frac{1}{3}$ 78	

II) Other cases

By easy but complicated computations we can compute α_0. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{m}$ be the orthogonal decomposition with respect to the Killing form on $\mathfrak{g}$, where $\mathfrak{k}$ is the Lie algebra of K. Then the tangent space $T_{eK}(\tilde{M})$ at the identity coset is canonically identified with $\mathfrak{m}$, and we know that $R(X,Y)Z=-[[X,Y],Z]$ for $X,Y,Z\in\mathfrak{m}$. (See S. Kobayashi and K. Nomizu [9, p. 231 Theorem 3.2].) We can compute the eigenvalue of the curvature operator L which is identified with the linear endomorphism on $S_0^2(\mathfrak{m})$, and we get Table 2 for the type BDI and CII. Hence the following symmetric spaces are non-deformable, where we assume $p\geqq q$;

$$BDI \quad (p\geqq 3,\ q=1),\ (q\geqq p-1,\ p+q\geqq 7)$$
$$CII \quad (p=q=1),\ (p\geqq 3,\ q=1).$$

Table 2

type	n	G/K	(*)	$\alpha\varepsilon^{-1}$
BDI	q	$\frac{SO(p+q)}{SO(p)\times SO(q)}$	$p>q=1$	$(p-1)^{-1}$
			$p\geqq q\geqq 2$	$\pm 2(p+q-2)^{-1},\ (2-p)(p+q-2)^{-1},\ (2-q)(p+q-2)^{-1}$
CII	$4pq$	$\frac{Sp(p+q)}{Sp(p)\times Sp(q)}$	$p=q=1$	$\frac{1}{3}$
			$p>q=1$	$-(p+2)^{-1},\ (p+2)^{-1}$
			$p\geqq q>1$	$\pm(p+q+1)^{-1},\ -(p+1)(p+q+1)^{-1},\ -(q+1)(p+q+1)^{-1}$

(*) condition

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