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ON SMALL RING HOMOMORPHISMS

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The author studied the total quotient ring of a commutative ring R from the point of view of small R-submodules [2]. In this note, we shall extend those methods to a ring extension of R. Let R and R' be commutative rings and $f: R \to R'$ a ring homomorphism. If $f(R)$ is a small R-submodule of R', we say f being *small* or R being *small* in R'. In the first section, we shall give a criterion for R to be small in R' in terms of maximal ideals in R and R' and obtain fundamental properties of small homomorphisms. In the second section, we shall give a characterization of maximal ideals M by the multiplicative systems $R-M$ and small homomorphisms.

Throughout this note, we assume every ring R is a commutative ring with identity unless otherwise stated and very ring homomorphism is also unitary, i.e. $f(1)$ is the identity.

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1. Small homomorphisms

Let R be a (commutative) ring and let $M \supseteq N$ be R-modules. N is called a *small submodule in* M if it satisfies the following condition: the fact $M=N+T$ for some R-submodule T implies $T=M$. Let R' be commutative and $f: R \to R'$ a ring homomorphism. Then every R'-module may be regarded as an R-module via f. If $f(R)$ is a small R-submodule in R', we say that f is *small* or R is *small in* R'. Let A and A' be ideals in R and R', respectively. We put $f(A)R'=AR'$ and $f^{-1}(f(R)\cap A')=A'\cap R$. We shall denote the set of prime ideals by spec(R) and the set of maximal ideals by Spec(R). Then we have the induced map f_*: spec(R')$\to$spec(R).

The following lemma is well known and the proofs are trivial.

Lemma 0. 1) *Let $X \supseteq Y \supseteq Z$ be R-modules. If Z is a small R-submodule in Y, so is in X and if Y is small in X, so is Z.* 2) *Let W be an R-module and $f: X \to W$ an R-homomorphism. If Z is small in X, $f(Z)$ is small in W.* 3) *Furthermore, if U is a small submodule in W, $Z \oplus U$ is small in $X \oplus W$.*

Theorem 1. *Let R and R' be commutative rings and $f: R \to R'$ a ring homomorphism. Then the following conditions are equivalent.*

1) *f is a small homomorphism.*
2) *Every R-finitely generated submodule of R' is small in R'.*
3) $f_*(\mathrm{Spec}(R')) \cap \mathrm{Spec}(R) = \phi$.

Proof. We may assume $R = f(R) \subseteq R'$.

1)→2). Let $N = \sum_{i=1}^{l} n_i R$ be a finitely generated R-submodule in R'. We consider a standard exact sequence: $F = \sum_{i=1}^{l} \oplus u_i R' \xrightarrow{h} \sum_{i=1}^{l} n_i R' \to 0$. Since R is small in R', $\sum_{i=1}^{l} \oplus u_i R$ is small in F from Lemma 0. Hence, $N = h(\sum_{i=1}^{l} \oplus u_i R)$ is small in $\sum_{i=1}^{l} n_i R'$ and so in R' from Lemma 0.

2)→1). It is trivial.

1)→3). Let M' be a maximal ideal in R' and put $M = f_*(M') = R \cap M'$. If M is maximal, R/M is a subfield of R'/M'. Hence, there exists an R-submodule L in R' such that $L \supseteq M'$, $L \neq R'$ and $R' = R + L$, which is a contradiction.

3)→1). Let M be a maximal ideal in R. If $MR' \neq R'$, we can take a maximal ideal M' in R' containing MR'. Then $M = M' \cap R$. Hence, $MR' = R'$ for every $M \in \mathrm{Spec}(R)$. Now, we assume $R' = R + T$ for an R-submodule T in R'. Then $R'_M = R'MR_M = R_M M + T_M$. Since R'_M / T_M is a finitely generated R_M-module, $R'_M = T_M$ from Nakayama's Lemma. Hence, $R' = T$.

REMARKS. 1. The condition 3) is equivalent to 3') *$MR' = R'$ for $M \in \mathrm{Spec}(R)$.*

2. In case R' is a non-commutative ring but an R-algebra, Theorem 1 remains valid. We assume that R is a non-commutative ring with Jacobson radical J such that R/J is artinian. Then we obtain form the above proof that *$f: R \to R'$ is small as a right R-module if and only if $R'J = R'$.* Hence, if R is right perfect [1], then any ring extension f is never small. We note that the concept of small homomorphism as a right R-module is defferent from one as a left R-modules in case of non-commutative rings.

3. The following is also valid for non-commutative rings from Lemma 0, 2).

Let R, R' and R'' be rings and $f: R \to R'$, $g: R' \to R''$ ring homomorphisms. If f is small, then gf is small.

We shall give several fundamental properties of a small homomorphism as applications of Theorem 1.

Proposition 2. *Let K be a field and R a subring of K. Then R is small in K if and only if R is not a field.*

Let P be in spec(R). By μ_P we shall denote the natural homomorphism of R to R_P.

Proposition 3. *Let $f: R \to R'$ be a ring homomorphism.*

1) *f is small if and only if $f_M: R_M \to R'_M$ is small for every M in* Spec(R). 2) *For $P \in$ spec(R), f_P is small if and only if $P \notin \operatorname{Im} f_*$. In this case $f_P \mu_P$ is also small.* 3) *For $P' \in$ spec(R') and $P = f_*(P')$, $f_{P'}: R_P \to R'_{P'}$ is never small, but $f_{P'} \mu_P$ is small if and only if μ_P is small, namely $P \notin$* Spec(R).

Proof. 1) $MR' = R'$ for $M \in$ Spec(R) if and only if $(MR')_N = R'_N$ for every $N \in$ Spec(R). 2) It is clear from a commutative diagram

$$\begin{array}{ccc} \operatorname{spec}(R_P) & \xleftarrow{f_{P*}} & \operatorname{spec}(R'_P) \\ \big\downarrow \wr & & \big\downarrow \wr \\ A = \{Q \in \operatorname{spec}(R) \mid Q \subseteq P\} & \xleftarrow{f_*} & f_*^{-1}(A) \quad (\subseteq \operatorname{spec}(R')) . \end{array}$$

3) It is clear that $f_{P'*}(P'R'_P) = R_P P$. Hence, $f_{P'}$ is not small. Furthermore, from Theorem 1 $f_{P'}\mu_P$ is small if and only if P is not maximal.

Proposition 4. *Let f: $R \to R'$ be a ring homomorphism. Then the following are equivalent.*

1) *For any ring homomorphism g: $R' \to R''$, g is small if and only if gf is small.*
2) $f_*^{-1}(\text{Spec}(R)) = \text{Spec}(R')$.

Proof. 1)→2). Let M' be maximal in R'. Since μ'_M: $R' \to R'_{M'}$ is not small from Theorem 1, $\mu'_M f$ is not small Hence, $f_*(M') = (\mu_{M'} f)_*(M'R'_{M'})$ is maximal. Let P' be in spec(R') − Spec(R'). Then $\mu_{P'}$: $R' \to R'_{P'}$ is small from Theorem 1. Hence, $\mu_{P'} f$ is small. Therefore, $f_*(P') = (f \mu_{P'})_*(P'R'_{P'})$ is not maximal by Theorem 1.
2)→1). We assume g is small. Then $g_*(M'')$ is in spec(R')−Spec(R') for any maximal ideal M'' in R''. Hence, $(gf)_*(M'')$ is not maximal from 2). Therefore, gf is small from Theorem 1. Conversely, we assume gf is small. Then $(gf)_*(M'')$ is not maximal and so $g_*(M'')$ is not maximal from 2). Therefore, g is small.

If $R' = R_M$ for a maximal ideal M, $R' = R(x)$ or R' is integral over R, then they satisfy the above conditions [3].

Let A be an ideal in R. By ρ_A we denote the natural epimorphism of R to R/A.

Proposition 5. *Let R and R' be rings and f: $R \to R'$ a ring homomorphism. Then the following statements are equivalent.*

1) *f is small.*
2) *$\rho_{M'} f$ is small for every M' in Spec(R').*

3) *$\rho_{MR'} f$ is small for every M in Spec(R).*
4) *$\rho_{J'} f$ is small for the Jacobson radical J' of R'.*
5) *$\rho_{JR'} f$ is small for the Jacosbon radical J of R.*

Proof. 1)$\leftrightarrow$2) and 1)$\leftrightarrow$3) are clear from Remark 3, Proposition 2 and Theorem 1.
4)$\rightarrow$1). Let M' be a maximal ideal in R'. Then $\rho_{M'} f=\rho_{M'/J'}\rho_{J'} f$ is small from Proposition 1. Hence, f is small by 2).
5)$\rightarrow$1). We can prove it similarly to the above by using 3).

REMARK 4. If R' (resp. R) is local, we can replace 2) (resp. 3)) by $\rho_{A'} f$ (resp. $\rho_{AR'} f$) for some ideal A' (resp. A such that $AR'\neq R'$).

Proposition 6. *Let $R\xrightarrow{f}R'\xrightarrow{g}R''$ be rings and ring homomorphisms. We assume that R' is local and gf is small, then either f or g is small,* (see Example 2 below).

Proof. Let M' be the unique maximal ideal in R'. If $R\cap M'$ is maximal, $R''=R''(R\cap M')=R''M'$ from Theorem 1.

2. Quotient rings

Let S be a multiplicative system in R. If $\mu_S: R\rightarrow R_S$ is small, S is called *large*. If S satisfies the following two conditions, we call S *critical*.

1) *If $S\subsetneqq S'$, S' is large.*
2) *If $S\supseteq S'$, S' is not large,*

where S' is a multiplicative system in R.

We obtain immediately from Theorem 1

Proposition 7 ([2]). *Let S be a multiplicative system. Then the following are equivalent.*

1) *S is large.*
2) *$M\cap S\neq\phi$ for every M in* Spec(R).

Theorem 8. *Let R be a commutative ring. Then there exists a one-to-one mapping between* Spec(R) *and the set of critical multiplicative systems S in R as follows: $M=R-S$ and $S=R-M$, where $M\in$* Spec(R).

Proof. Let M be a maximal ideal and $S=R-M$. Then it is clear from Proposition 7 that S is critical. Conversely, let S be critical. Since S is not large, there exists a maximal ideal M' such that $M'\cap S=\phi$ from Proposition 7. Then we obtain again from Proposition 7 and the definition that $S=R-M'$.

Proposition 9. *R is never small for any non-zero ring homomorphism $f: R\rightarrow$*

R' if and only if MR_M is a nil ideal for every maximal ideal M in R.

Proof. "Only if" part. We may assume R is local from Proposition 3. If there exists m in M which is not nil, then $\{m_i\}_i$ is large from Proposition 7. Which is a contradiction. "If" part. If $R'M=R'$, $1=\sum_{i=1}^{t} r'_i m_i$; $r'_i \in R'$, $m_i \in M$. There exists s in $R-M$ such that $sm_i^n=0$ for all i and some n. Then $s=s(\sum_{i=1}^{t} r'_i m_i)^{tn}$ $=0$.

Proposition 10. *Let R be an integral domain and K the field of quotients. Then R is local if and only if R is small in any subring T is K such that $T \supset R$ and there exists an element $a^{-1} \in T-R$, $a \in R$.*

Proof. Let R be a local and T as above. Then $\{a^i\}_i$ is large from Proposition 7. Hence, R is small in T by Remark 3. Conversely, let M be maximal. Then R is not small in R_M. Hence, $R=R_M$ from the assumption.

Proposition 11. *Let R be a domain with K quotient field. Then the following are equivalent.*

1) *Let R' be an over ring of R. If R is small in R', $R'=K$.*
2) Krull dim $R=1$ i.e. *every non-zero prime is maximal in R.*

Proof. 1)→2). Let P be a non-zero prime ideal. Then $R_P=R$ or R is not small in R_P. Hence, P is maximal from Proposition 7.
2)→1). Let $K \supsetneqq R'$ be an over ring and R be small in R'. Then for every maximal ideal M', $M' \cap R \neq 0$ is not maximal, which is a contradiction.

Proposition 12. *Let R be a Dedekind domain and L an R-submodule in K containing R. Then*

1) *R is small in L if and only if $L \supset \sum_{P} P^{-1}$*

where P runs through the set $\boldsymbol{P}$ of non-zero primes in R.
If L is a subring, then

2) *R is small in L if and only if $K=L$, and L is small in K as an R-module if and only if $L=R$.*

Proof. Since $K/R=\sum_{P} \oplus(\sum_{n} P^{-n}/R)$ and every R-submodule in $\sum_{n} P^{-n}/R$ is of P^{-m}/R, $L=\sum_{P} P^{-n(P)}$; $n(P) \geqslant 0$. First, we shall show R is small in $\sum_{P} P^{-1}$ $=A$. If $A=R+T$, $A_P=P^{-1}R_P=R_P+T_P$. Let $R_P P=(p)$ and $R_P \cap T_P=(p^e)$. Then $p^{-1}=r+ts^{-1}$; $r \in R_P$, $t \in T$ and $s \in R-P$. Hence, $s(1-rp)=tp \in T_P \cap R_P$ and so $(1-rp) \in (p^e)$. Therefore, $e=1$ and $A_P=T_P$ for every P. Accordingly, $A=T$. Next, we consider a submodule $A(L)=\sum_{P \not\supseteq L} P^{-n(P)}$. Then we can show as above $A(L)=R+LA(L)$ and $A(L) \neq LA(L)$. Hence, R is not small in $A(L)$.

We have proved 1). 2) is clear from 1) and the structure of K/R.

EXAMPLES 1) Let K be a field and x an indeterminant. Then K is not small in $K(x)$, however $K[x]$ ($\supset K$) is small in $K(x)$ (cf. Lemma 0).

2) Let Z be the ring of integers with Q quotient field and p a prime. Then $Z_p[x]$ is not small in $Q[x]$, since $(px-1)$ is a maximal ideal in $Z_p[x]$ such that $Q[x](px-1)\neq Q[x]$. Hence, Proposition 6 is not true without the assumption "local."

3) Let $R=K[x,y]_{(x,y)}$. Then R is not small in $R[yx^{-1}]$ as an R-module and $R[yx^{-1}]$ does not contain any element a^{-1} as in Proposition 10.

4) $Z_p=Z_p/((xp-1)\cap Z_p)$ is small in $Q=Z_p[x]/(xp-1)$, but Z_p is not small in $Z_p[x]$ (cf. Proposition 5).

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