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# 5-FOLD TRANSITIVE PERMUTATION GROUPS IN WHICH THE STABILIZER OF FIVE POINTS HAS A NORMAL SYLOW 2-SUBGROUP

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## 1. Introduction

In this paper we shall prove the following theorem.

**Theorem.** *Let $G$ be a 5-fold transitive permutation group on a set $\Omega=\{1,2,\cdots,n\}$. Let $P$ be a Sylow 2-subgroup of $G_{12345}$. If $P$ is a nonidentity normal subgroup of $G_{12345}$, then $G$ is one of the following groups: $S_7$, $A_9$ or $M_{24}$.*

The idea of the proof of the theorem is derived from Oyama [7].

In order to prove the theorem, we shall use the following two lemmas, which will be proved in Sections 3 and 4.

**Lemma 1.** *Let $G$ be a permutation group on $\Omega=\{1,2,\cdots,n\}$ satisfying the following three conditions.*

(i) *For any five points $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and $\alpha_5$ in $\Omega$, the order of $G_{\alpha_1\cdots\alpha_5}$ is even.*

(ii) *For any five points $\alpha_1$, $\alpha_2$, $\alpha_3$, $\alpha_4$ and $\alpha_5$ in $\Omega$, a Sylow 2-subgroup of $G_{\alpha_1\cdots\alpha_5}$ is normal in $G_{\alpha_1\cdots\alpha_5}$.*

(iii) *Any involution in $G$ fixes at most seven points.*

*Then $G$ is $S_7$ or $A_9$.*

**Lemma 2.** *Let $G$ be a permutation group on $\Omega=\{1,2,\cdots,n\}$ satisfying the following four conditions.*

(i) *For any five points $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and $\alpha_5$ in $\Omega$, the order of $G_{\alpha_1\cdots\alpha_5}$ is even.*

(ii) *For any five points $\alpha_1$, $\alpha_2$, $\alpha_3$, $\alpha_4$ and $\alpha_5$ in $\Omega$, a Sylow 2-subgroup of $G_{\alpha_1\cdots\alpha_5}$ is normal in $G_{\alpha_1\cdots\alpha_5}$.*

(iii) *Any involution in $G$ fixes at most nine points.*

(iv) *For any 2-subgroup $X$ fixing exactly nine points, $N(X)^{I(X)}\leqq A_9$.*

*Then $G$ is $S_7$ or $A_9$.*

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We shall use the same notation as in [3].

## 2. Proof of the Theorem

Let $G$ be a group satisfying the assumption of the theorem.

Let $P$ be the unique Sylow 2-subgroup of $G_{12345}$. If $P$ is semiregular on $\Omega\text{-}I(P)$ or $|I(P)|>6$, then $G$ is $S_7$, $A_9$ or $M_{24}$by [2], [3], [4] and [5]. Hence from now on we assume that $P$ is not semiregular on $\Omega\text{-}I(P)$ and that $|I(P)|\leqq 6$, and we prove that this case does not arise. If $|I(P)|=6$, then $|I(G_{12345})|=6$, a contradiction to [1]. Hence $|I(P)|=5$.

Let $r$ be $\mathrm{Max}|I(a)|$, where $a$ ranges over all involutions in $G$. Since $P$ is not semiregular on $\Omega\text{-}I(P)$, we have $r\geqq 7$.

Suppose $r=7$. Let $t$ be a point of a minimal orbit of $P$ in $\Omega\text{-}I(P)$. It is easily seen that $N(P_t)^{I(P_t)}=S_7$. By [6], we have a contradiction.

Suppose $r=9$. Let $Q$ be a subgroup of $P$ such that the order of $Q$ is maximal among all subgroups of $P$ fixing exactly nine points. By Lemma 1, $N(Q)^{I(Q)}=A_9$. Again by [6], we have a contradiction. Thus we have $r\geqq 11$.

Let $Q$ be a subgroup of $P$ such that the order of $Q$ is maximal among all subgroups of $P$ fixing more than nine points. Set $N=N(Q)^{I(Q)}$. Then $N$ satisfies the following conditions.

(i) *$N$ is a permutation group on $I(Q)$, and its degree is not less than eleven.*

(ii) *For any five points $\alpha_1$, $\alpha_2$, $\alpha_3$, $\alpha_4$ and $\alpha_5$ in $I(Q)$, the order of $N_{\alpha_1\cdots\alpha_5}$ is even.*

(iii) *For any five points $\alpha_1$, $\alpha_2$, $\alpha_3$, $\alpha_4$ and $\alpha_5$ in $I(Q)$, a Sylow 2-subgroup of $N_{\alpha_1\cdots\alpha_5}$ is normal in $N_{\alpha_1\cdots\alpha_5}$.*

(iv) *Any imvolution fixes at most nine points.*

By Lemma 1, $N$ has an involution fixing exactly nine points. Let $X$ be any 2-subgroup of $N$ fixing exactly nine points. Set $\Delta=I(X)$. Let $S$ be the Sylow 2-subgroup of $G_\Delta$. Since $I(S)=\Delta$, we have $N_G(S)^{I(S)}=A_9$ by Lemma 1. Since $S$ is a characteristic subgroup of $G_\Delta$, $N$ satisfies the following condition.

(v) *For any 2-subgroup $X$ fixing exactly nine points, $N_N(X)^{I(X)}\leqq A_9$.*

Considering the permutation group $N$, we have a final contradiction by Lemma 2.

## 3. Proof of Lemma 1

Let $G$ be a permutation group satisfying the assumptions of Lemma 1. If $G$ has no involution fixing seven points, then $G$ is $S_7$ or $A_9$ by [8, Lemma 6] and [2]. Hence from now on we assume that $G$ has an involution fixing exactly seven points, and we prove Lemma 1 by way of contradiction. We may assume that $G$ has an involution $a$ fixing exactly $1,2,\cdots,7$ and

$$a=(1)(2)\cdots(7)(8\ 9)\cdots.$$

Set $T=C(a)_{89}$.

(1) *For any three points $i, j$ and $k$ in $I(a)$, there is an involution in $T_{ijk}$. Any involution in $T$ is not the identity on $I(a)$.*

Proof. Since $a$ normalizes $G_{89ijk}$ and $G_{89ijk}$ is of even order, $G_{89ijk}$ has an involution $x$ commuting with $a$. Then $x \in T_{ijk}$. Since $|I(a)|=7$ and $I(x) \supseteqq \{8, 9\}$, any involution in $T$ is not the identity on $I(a)$ by (iii).

(2) *For any three points $i, j$ and $k$ in $I(a)$, a Sylow 2-subgroup of $T_{ijk}$ is normal in $T_{ijk}$, and so a Sylow 2-subgroup of $T_{ijk}^{I(a)}$ is normal in $T_{ijk}^{I(a)}$.*

Proof. Let $S$ be a Sylow 2-subgroup of $T_{ijk}$. Since $S$ is a Sylow 2-subgroup of $C(a)_{89ijk}$, $S$ is a normal subgroup of $C(a)_{89ijk}$ by (ii).

We have the following property from (2).

(3) *If $x_1^{I(a)}$ and $x_2^{I(a)}$ are involutions in $T^{I(a)}$ with $|I(x_1^{I(a)}) \cap I(x_2^{I(a)})| \geqq 3$, then $x_1^{I(a)} x_2^{I(a)}$ is a 2-element of $T^{I(a)}$.*

(4) *Since $|I(a)|=7$, $T^{I(a)}$ is one of the following groups.*
(a) *$T^{I(a)}$ is intransitive and has an orbit of length one or two.*
(b) *$T^{I(a)}$ is intransitive and has an orbit of length three.*
(c) *$T^{I(a)}$ is primitive.*

(5) *The case* (a) *does not hold.*

Proof. Suppose $T^{I(a)}$ has an orbit of length one or two. We may assume that either $\{1\}$ or $\{1, 2\}$ is such an orbit. By (1), $T_{234}$ has an involution $x_1$. We may assume that

$$x_1 = (1)(2)(3)(4)(5\ 6)(7) \cdots .$$

Similarly $T_{235}$ has an involution $x_2$ of the form

$$x_2 = (1)(2)(3)(4)(5)(6\ 7) \cdots, (1)(2)(3)(5)(4\ 6)(7) \cdots \text{ or } (1)(2)(3)(5)(4\ 7)(6) \cdots .$$

If the first or the second alternative holds, then $|I(x_1^{I(a)}) \cap I(x_2^{I(a)})|=4$, and $x_1^{I(a)} x_2^{I(a)}$ is not a 2-element, a contradiction to (3). Thus $x_2=(1)(2)(3)(5)(4\ 7)(6) \cdots$. Again by (1), $T_{245}$ has an involution $x_3$ of the form

$$x_3 = (1)(2)(4)(5)(3)(6\ 7) \cdots, (1)(2)(4)(5)(3\ 6)(7) \cdots \text{ or } (1)(2)(4)(5)(3\ 7)(6) \cdots .$$

In every case, we get a contradiction to (3) by considering either $x_1^{I(a)} x_3^{I(a)}$ or $x_2^{I(a)} x_3^{I(a)}$.

(6) *The case (b) does not hold.*

Proof. Suppose $T^{I(a)}$ has an orbit of length three. We may assume that $\{1,2,3\}$ is such an orbit of length three. By (5), $\{4,5,6,7\}$ is a $T^{I(a)}$-orbit. By (1), $T_{456}$ has an involution $x_1$. We may assume that

$$x_1 = (1\ 2)(3)(4)(5)(6)(7)\cdots.$$

Since $\{1,2,3\}$ is a $T^{I(a)}$-orbit, $T$ has an element $y$ of the form

$$y = (1\ 2\ 3)\cdots.$$

Set $x_2 = x_1^y$, then $x_2 = (2\ 3)(1)(4)(5)(6)(7)\cdots$. So, $|I(x_1^{I(a)}) \cap I(x_2^{I(a)})| = 4$, and $x_1x_2 = (1\ 3\ 2)(4)(5)(6)(7)\cdots$, which is a contradiction. Hence $T^{I(a)}$ has no orbit of length three.

(7) *We show that the case (c) does not hold, and complete the proof of Lemma 1.*

Proof. Suppose $T^{I(a)}$ is primitive. By (1), we have $T^{I(a)} \geqq A_7$ (cf.e.g.[10]). Therefore for any involution $x$ in $G$ fixing exactly seven points, $C(x)^{I(x)} \geqq A_7$.

Let $\Gamma$ be any subset of $\Omega$ with $|\Gamma| = 5$. Set $\Gamma = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$. By (i), $G_{\alpha_1\cdots\alpha_5}$ has an involution. If $G_{\alpha_1\cdots\alpha_5}$ has an involution $x_1$ fixing exactly seven points, then $C(x_1)^{I(x_1)} \geqq A_7$. Hence $G_{\{\alpha_1,\cdots,\alpha_5\}}^{\{\alpha_1,\cdots,\alpha_5\}} = S_5$. Suppose that $G_{\alpha_1\cdots\alpha_5}$ has no involution fixing seven points. Let $x_2$ be an involution in $G_{\alpha_1\cdots\alpha_5}$. Let $x_2 = (\alpha_1)\cdots(\alpha_5)(\beta_1\ \beta_2)\cdots$. It is easily seen that $C(x_2)_{\beta_1\beta_2}^{I(x_2)} = S_5$. Hence $G_{\{\alpha_1,\cdots,\alpha_5\}}^{\{\alpha_1,\cdots,\alpha_5\}} = S_5$. Thus we have $G_{\{\alpha_1,\cdots,\alpha_5\}}^{\{\alpha_1,\cdots,\alpha_5\}} = S_5$ in either case. Therefore by [9, Lemma 3], $G$ is 4-fold transitive on $\Omega$.

Let $x$ be an involution in $G$ fixing seven points. Let $S$ be the Sylow 2-subgroup of $G_{I(x)}$. Since $C(x)^{I(x)} \geqq A_7$, we have $N(S)^{I(S)} \geqq A_7$. By [6], we get a contradiction.

Thus we complete the proof of Lemma 1.

## 4. Proof of Lemma 2

Let $G$ be a permutation group satisfying the assumptions of Lemma 2. If $G$ has no involution fixing nine points, then $G$ is $S_7$ or $A_9$ by Lemma 1. Hence from now on we assume that $G$ has an involution fixing exactly nine points, and we prove Lemma 2 by way of contradiction. We may assume that $G$ has an involution $a$ fixing $1, 2, \cdots, 9$ and

$$a = (1)(2)\cdots(9)(10\ 11)\cdots.$$

Set $T = C(a)_{10\,11}$.

(1) *For any three points $i, j$ and $k$ in $I(a)$, there is an involution in $T_{ijk}$. Any involution in $T$ is not the identity on $I(a)$.*

(2) *For any three points i,j and k in I(a), a Sylow 2-subgroup of* $T_{ijk}$ *is normal in* $T_{ijk}$, *and so a Sylow 2-subgroup of* $T_{ijk}^{I(a)}$ *is normal in* $T_{ijk}^{I(a)}$.

(3) *If* $x_1^{I(a)}$ *and* $x_2^{I(a)}$ *are involutions in* $T^{I(a)}$ *with* $|I(x_1^{I(a)})\cap I(x_2^{I(a)})|\geqq 3$, *then* $x_1^{I(a)}x_2^{I(a)}$ *is a 2-element of* $T^{I(a)}$.

The proofs of (1), (2) and (3) are similar to the proofs of (1), (2) and (3) in Section 3 respectively.

(4) *Since* $|I(a)|=9$, $T^{I(a)}$ *is one of the following groups.*
(a) $T^{I(a)}$ *is intransitive and has an orbit of length one or two.*
(b) $T^{I(a)}$ *is either an intransitive group with an orbit of length three, or a transitive but imprimitive group with three blocks of length three.*
(c) $T^{I(a)}$ *is intransitive and has an orbit of length four.*
(d) $T^{I(a)}$ *is primitive.*

(5) *The case* (a) *does not hold.*

Proof. Suppose $T^{I(a)}$ has an orbit of length one or two. We may assume that either {1} or {1,2} is such an orbit. By (1), $T_{234}$ has an involution $x_1$. By the assumption (iv), we may assume that

$$x_1=(1)(2)(3)(4)(5)(6\ 7)(8\ 9)\cdots.$$

Similarly $T_{236}$ has an involution $x_2$. We may assume without loss of generality that

$$\begin{aligned} x_2^{I(a)}=\ &(1)(2)(3)(6)(7)(4\ 5)(8\ 9)\cdots\ \alpha\ ,\\ &(1)(2)(3)(6)(7)(4\ 8)(5\ 9)\cdots\ \beta\ ,\\ &(1)(2)(3)(6)(8)(7\ 9)(4\ 5)\cdots\ \gamma\ ,\\ &(1)(2)(3)(6)(8)(7\ 4)(5\ 9)\cdots\ \delta\ ,\\ &(1)(2)(3)(6)(4)(5\ 7)(8\ 9)\cdots\ \varepsilon\quad\text{or}\\ &(1)(2)(3)(6)(4)(7\ 8)(5\ 9)\cdots\ \zeta\ .\end{aligned}$$

If $x_2^{I(a)}$ is of the form $\delta$, $\varepsilon$ or $\zeta$, then $|I(x_1^{I(a)})\cap I(x_2^{I(a)})|\geqq 3$, and $x_1^{I(a)}x_2^{I(a)}$ is not a 2-element, a contradiction to (3). Hence $x_2^{I(a)}$ is of the form $\alpha$, $\beta$ or $\gamma$. $T_{269}$ has an involution $x_3$. $x_3^{I(a)}$ is of the form

$$\begin{aligned} x_3^{I(a)}=\ &(1)(2)(6)(9)(3)(4\ 5)(7\ 8)\cdots ①\ ,\\ &(1)(2)(6)(9)(3)(4\ 7)(5\ 8)\cdots ②\ ,\\ &(1)(2)(6)(9)(3)(4\ 8)(5\ 7)\cdots ③\ ,\\ &(1)(2)(6)(9)(4)(3\ 5)(7\ 8)\cdots ④\ ,\\ &(1)(2)(6)(9)(4)(3\ 7)(5\ 8)\cdots ⑤\ ,\\ &(1)(2)(6)(9)(4)(3\ 8)(5\ 7)\cdots ⑥\ ,\end{aligned}$$

$$(1)(2)(6)(9)(5)(3\ 4)(7\ 8)\cdots ⑦,$$
$$(1)(2)(6)(9)(5)(3\ 7)(4\ 8)\cdots ⑧,$$
$$(1)(2)(6)(9)(5)(3\ 8)(4\ 7)\cdots ⑨,$$
$$(1)(2)(6)(9)(7)(3\ 4)(5\ 8)\cdots ⑩,$$
$$(1)(2)(6)(9)(7)(3\ 5)(4\ 8)\cdots ⑪.$$
$$(1)(2)(6)(9)(7)(3\ 8)(4\ 5)\cdots ⑫,$$
$$(1)(2)(6)(9)(8)(3\ 4)(5\ 7)\cdots ⑬,$$
$$(1)(2)(6)(9)(8)(3\ 5)(4\ 7)\cdots ⑭ \text{ or}$$
$$(1)(2)(6)(9)(8)(3\ 7)(4\ 5)\cdots ⑮.$$

If $x^{I(a)}$ is of the form ②, ③, ⑤, ⑥, ⑧ or ⑨, then $|I(x_1^{I(a)})\cap I(x_3^{I(a)})|=3$, and $x_1^{I(a)}x_3^{I(a)}$ is not a 2-element, which is a contradiction. Suppose $x_3^{I(a)}$ is of the form ⑩. Then $x_1x_3=(1)(2)(3\ 4)(5\ 8\ 9)(6\ 7)\cdots$, and $(x_1x_3)^2=(1)(2)(3)(4)(5\ 9\ 8)(6)(7)\cdots$. Set $y=(x_1x_3)^2$ and $x_4=x_1^y$. Then $x_4=(1)(2)(3)(4)(9)(6\ 7)(5\ 8)\cdots$. So, $|I(x_1^{I(a)})\cap I(x_4^{I(a)})|=4$, and $x_1x_4=(1)(2)(3)(4)(5\ 8\ 9)(6)(7)\cdots$, which is a contradiction. If $x_3^{I(a)}$ is of the form ⑪, ⑫, ⑬, ⑭ or ⑮, we have a contradiction by the same argument as in the case ⑩. Hence $x_3^{I(a)}$ is of the form ①, ④ or ⑦.

Suppose $x_2^{I(a)}$ is of the form $\alpha$ or $\gamma$. Since $x_3^{I(a)}$ is of the form ①, ④ or ⑦, we get a contradiction by considering $x_2^{I(a)}x_3^{I(a)}$.

Suppose $x_2^{I(a)}$ is of the form $\beta$. If $x_3^{I(a)}$ is of the form ① or ④, we get a contradiction by considering $x_2^{I(a)}x_3^{I(a)}$. Suppose $x_3^{I(a)}$ is of the form ⑦. Then $x_2x_3=(1)(2)(6)(5\ 9)(3\ 4\ 7\ 8)\cdots$. Set $x_5=(x_2x_3)^2$, then $x_5=(1)(2)(6)(5)(9)(3\ 7)(4\ 8)\cdots$. So, $|I(x_1^{I(a)})\cap I(x_5^{I(a)})|=3$, and $x_1x_5=(1)(2)(5)(3\ 7\ 6)(4\ 8\ 9)\cdots$, which is a contradiction. Thus $T^{I(a)}$ has neither orbit of length one nor orbit of length two.

(6) *The case* (*b*) *does not hold.*

Proof. Suppose $T^{I(a)}$ has an orbit of length three or three blocks of length three. We may assume that $\{1,2,3\}$ is such an orbit or a blcok.

Assume that $T^{I(a)}$ has three orbits of length three or three blocks of length three. We may assume that $\{1,2,3\}$, $\{4,5,6\}$ and $\{7,8,9\}$ are the orbits or the blocks. $T_{124}$ has an involution $x_1$. By the assumption (iv),

$$x_1=(1)(2)(3)(4)(5\ 6)\cdots.$$

Similarly $T_{125}$ has an involution $x_2$ of the form

$$x_2=(1)(2)(3)(5)(4\ 6)\cdots.$$

So, $|I(x_1^{I(a)})\cap I(x_2^{I(a)})|\geqq 3$, and $x_1x_2=(1)(2)(3)(4\ 6\ 5)\cdots$, which is a contradiction.

By (5) and the above, we have that $\{1, 2, 3\}$ and $\{4, 5, 6, 7, 8, 9\}$ are the

$T^{I(a)}$-orbits. Since $3 \,\Big|\, |\{4, 5, \cdots, 9\}|$, we may assume that $T$ has an element $y$ of the form

$$y=(4\ 5\ 6)\cdots .$$

$T_{789}$ has an involution $x_1$. We may assume that

$$x_1=(1\ 2)(3)(4\ 5)(6)(7)(8)(9)\cdots .$$

Set $x_2=x_1^y$, then $x_2=(5\ 6)(4)(7)(8)(9)\cdots$. So, $|I(x_1^{I(a)})\cap I(x_2^{I(a)})|\geqq 3$, and $x_1x_2=(4\ 6\ 5)(7)(8)(9)\cdots$, which is a contradiction. Thus $T^{I(a)}$ has neither orbit of length three nor block of length three.

(7) *The case* ($c$) *does not hold.*

Proof. Suppose $T^{I(a)}$ has an orbit of length four. We may assume that $\{1,2,3,4\}$ is a $T^{I(a)}$-orbit. By (5) and (6), $\{5,6,7,8,9\}$ is a $T^{I(a)}$-orbit. Since $5 \,\Big|\, |\{5, 6, 7, 8, 9\}|$, we may assume that $T$ has an element $y$ of the form

$$y=(1)(2)(3)(4)(5\ 6 7\ 8\ 9)\cdots .$$

$T_{123}$ has an involution $x_1$. We may assume that $x_1$ is of the form

$$\begin{aligned} x_1=&(1)(2)(3)(4)(5)(6\ 7)(8\ 9)\cdots ,\\ &(1)(2)(3)(4)(5)(6\ 8)(7\ 9)\cdots \text{ or}\\ &(1)(2)(3)(4)(5)(6\ 9)(7\ 8)\cdots .\end{aligned}$$

Set $x_2=x_1^y$. Then $x_2$ is of the following form respectively:

$$\begin{aligned} x_2=&(1)(2)(3)(4)(6)(7\ 8)(5\ 9)\cdots ,\\ &(1)(2)(3)(4)(6)(7\ 9)(5\ 8)\cdots \text{ or}\\ &(1)(2)(3)(4)(6)(5\ 7)(8\ 9)\cdots .\end{aligned}$$

In any case, we get a contradiction by considering $x_1^{I(a)}x_2^{I(a)}$.

(8) *We show that the case* ($d$) *does not hold, and complete the proof of Lemma* 2.

Proof. If $T^{I(a)}$ is primitive, then by (1) and the assumption (iv), we have $T^{I(a)}=A_9$(cf.e.g.[10]). But this contradicts (2). Thus $T^{I(a)}$ is not primitive.

Thus we complete the proof of Lemma 2.

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