Ōshima, Y.
Osaka J. Math.
15 (1978), 283-310

ON THE EQUILIBRIUM MEASURE OF RECURRENT MARKOV PROCESSES

Yoichi ŌSHIMA

(Received February 21, 1977)

0. Introduction

In the case of planar Brownian motion, if we denote $h(x, y)=-\frac{1}{\pi}log|x-y|$, the following results are well known (see [13], [16]). (i) If F is a non-polar compact set, then there exists a probability measure ξ_F on F such that $\int h(x, y)\xi_F(dy)$ equals a constant $R(F)$ on F except on a polar set. The measure ξ_F and the constant $R(F)$ are respectively called the quilibruim measure and Robin's constant of F. (ii) A compact set F is non-polar if and only if there exists a non-zero finite measure ξ on F such that $\int h(x, y)\xi(dy)$ is locally bounded.

In this paper we shall be concerned with the similar problem for recurrent Hunt processes with strong Feller resolvent. In our case, in place of $h(x, y)$, we shall use a density $g(x, y)$ of a potential kernel $G(x, dy)$ of X relative to the invariant measure $\mu(dy)$. Unfortunately, our density $g(x, y)$ is not equal to $h(x, y)$ in the case of planar Brownian motion but equal to $h(x, y)+f(x)+g(y)$ with some locally bounded functions f and g (see §4).

Now we shall outline the contents of this paper. Let X be a recurrent Hunt process with strong Feller resolvent and μ an invariant measure of X. If we are given a certain finite non-negative continuous additive functional A of X then we can construct a potential kernel G of X by means of time change and killing based upon A ([4], [12]). In this paper we shall suppose, for simplicity, that $A_t=\int_0^t I_C(X_s)ds$ for an arbitrary fixed non-null compact set C but the similar argument can be applicable for a large class of functionals A.

In section 1, some preliminary results are established. Among others, a potential kernel K_A and an invariant measure ν_A of the time changed process by A are described. In section 2, for any other finite non-negative continuous additive functional B, a potential kernel K_B and an invariant measure ν_B of the time changed process by B are constructed by making use of K_A and ν_A. In section 3, let us introduce the duality hypothesis that there exists a dual process $\hat{X}$ (of X relative to μ) satisfying those regularity conditions like X.

We shall then construct a kernel function $g(x, y)$ such that $g(\cdot, y)$ [resp. $g(x, \cdot)$] is finely [resp. confinely] continuous, finite except on a polar set and $K_B(x, dy) = g(x, y)\nu_B(dy)$ [resp. $\hat{K}_B(dx, y) = g(x, y)\nu_B(dx)$] for all continuous additive functionals B, where $\hat{K}_B$ is the potential kernel of the time changed process of $\hat{X}$ by the dual functional $\hat{B}$ of B. In particular, when $B_t = t$, we have $G(x, dy) = g(x, y)\mu(dy)$ and $\hat{G}(dx, y) = g(x, y)\mu(dx)$, where $\hat{G}$ is the potential kernel of $\hat{X}$ associated with $\hat{A}_t = \int_0^t I_C(\hat{X}_s)\,ds$ in the sense of section 2. In this sense, our function $g(x, y)$ may be called the potential kernel function associated with $(G, \hat{G})$. In section 4, we introduce the notion of potential kernel function $h(x, y)$ in a more general sense and then establish a relation between $h(x, y)$ and $g(x, y)$. In section 5, we shall show the equilibrium principle. This means that, if $F = supp\,(B)$, then there is a probability measure ξ_F on $supp\,(\hat{B})$ such that $\int g(x, y)\xi_F(dy) = R(F)$ on F. In our case, the equilibrium measure ξ_F and Robin's constant $R(F)$ have intuitive probablistic meanings. If X and $\hat{X}$ are equivalent, the results of section 5 have simpler forms and the analogous potential principles to classical potential theory hold. This case is treated in section 6. There a characterization of the equilibrium measure by means of energy is also given.

1. Notations and preliminary results

Let E be a locally compact Hausdorff space with countable base, $\mathcal{E}$ the Borel σ-field on E and $\mathcal{E}^*$ the σ-field obtained by the universal completion of $\mathcal{E}$. If $\mathcal{A}$ is a σ-field of subsets of E then the classes of all bounded $\mathcal{A}$-measurable functions, all bounded non-negative $\mathcal{A}$-measurable functions and all bounded $\mathcal{A}$-measurable functions with compact support are denoted by $b\mathcal{A}$, $b\mathcal{A}_+$ and $b\mathcal{A}_c$, respectively.

Throughout in this paper, let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ be a recurrent Hunt process on E with strong Feller resolvent, that is, a Hunt process satisfying

(i) (Recurrence); *For all* $f \in b\mathcal{E}_+$, $G^0f(x) = E^x[\int_0^\infty f(X_t)\,dt] \equiv 0$ *or* $\equiv \infty$ *on* E.

(ii) (Strong Feller property of resolvent); *For all* $p > 0$ *and* $f \in b\mathcal{E}$, $G^pf(x) = E^x[\int_0^\infty e^{-pt}f(X_t)\,dt]$ *is bounded continuous.*

In this case, it is well known that there exists a unique (except a constant multiple) invariant Radon measure μ of X, which is positive on every open sets (see [1], [2]). Let Φ be the family of all non-negative continuous additive functionals (abbreviated CAF) $A = (A_t)_{t \geq 0}$ of X such that $A_t < \infty$ *a.s.* for all $t < \infty$ and let Φ^+ be the subfamily of functionals $A \in \Phi$ which are not equivalent to the zero functional. If $A \in \Phi^+$ then $P^x(A_\infty = \infty) = 1$ for all x ([1]). For $A \in \Phi^+$ and $p \geqq 0$ we define a kernel K_A^p by

$$(1.1)\qquad K_A^p f(x) = E^x\Big[\int_0^\infty e^{-pA_t} f(X_t)\, dA_t\Big].$$

Note that $(K_A^p)_{p>0}$ is the resolvent of the time changed process of X by A.

Moreover, for $A, B \in \Phi^+$ and $p, q \geqq 0$ we define two auxiliary kernels $U_{A,B}^{p,q}$ and $V_{A,B}^{p,q}$ as follows:

$$(1.2)\qquad U_{A,B}^{p,q} f(x) = E^x\Big[\int_0^\infty e^{-pA_t - qB_t} f(X_t)\, dA_t\Big],$$

$$(1.3)\qquad V_{A,B}^{p,q} f(x) = E^x\Big[\int_0^\infty e^{-pA_t - qB_t} f(X_t)\, dB_t\Big].$$

Obviously, $U_{A,B}^{p,q} = V_{B,A}^{q,p}$. The family $(U_{A,B}^{p,q})_{p>0}$ is the resolvent of the time changed process by A_t of the e^{-qB_t}-subprocess of X. If $B_t = t$, we shall write $K_A^{p,q}$ for $U_{A,\{t\}}^{p,q}$ and $G_A^{p,q}$ for $V_{A,\{t\}}^{p,q}$, i.e.,

$$(1.4)\qquad K_A^{p,q} f(x) = E^x\Big[\int_0^\infty e^{-pA_t - qt} f(X_t)\, dA_t\Big],$$

$$(1.5)\qquad G_A^{p,q} f(x) = E^x\Big[\int_0^\infty e^{-pA_t - qt} f(X_t)\, dt\Big].$$

Note that $U_{A,B}^{p,0} = K_A^{p,0} = K_A^p$, $V_{A,B}^{0,q} = K_B^q$ and $G_A^{0,q} = G^q$ (the resolvent of X).

In the sequel, if there is no danger of confusion, the suffices A, B will be often omitted.

Lemma 1.1 (Nagasawa-Sato [10; theorem 2.1 and 2.2]). *Write $U^{p,q}$ and $V^{p,q}$ for $U_{A,B}^{p,q}$ and $V_{A,B}^{p,q}$. For all $p>0$, $p'>0$ $q\geqq 0$, $q'\geqq 0$ and $f \in b\mathcal{E}^*$,*

$$(1.6)\qquad U^{p,q}f - U^{p',q'}f + (p-p')\, U^{p,q}U^{p',q'}f + (q-q')\, V^{p,q}U^{p',q'}f = 0,$$

$$(1.7)\qquad V^{q,p}f - V^{q',p'}f + (p-p')\, V^{q,p}V^{q',p'}f + (q-q')\, U^{q,p}V^{q',p'}f = 0.$$

If, in particular, $U^{0,q_0}|f|$ [resp. $V^{q_0,0}|f|$] is bounded for some $q_0 \geqq 0$ then $U^{0,q}|f|$ [resp. $V^{q,0}|f|$] is bounded for all $q>0$ and (1.6) *[resp.* (1.7)*] holds for all $p, p', q, q' \geqq 0$ satifysing $p+q>0$ and $p'+q'>0$.*

Lemma 1.2 ([12; lemma 2.2]). *There exists an increasing sequence $\{E_n\}_{n\geqq 1}$ [resp. $\{F_n\}_{n\geqq 1}$] of subsets in $\mathcal{E}^*$ such that $\bigcup_{n\geqq 1} E_n = E$ [resp. $\bigcup_{n\geqq 1} F_n = E$] and $U^{0,1}(\cdot, E_n)$ [resp. $V^{1,0}(\cdot, F_n)$] is bounded for all $n \geqq 1$.*

Lemma 1.3 (Blumenthal-Getoor [3; III, section 5]). *If $A \in \Phi^+$ then $G_A^{p,0}(\cdot, F)$ is bounded for all compact set F and $p>0$.*

A set C is said to be null if it is a set of potential zero relative to $(G^p)_{p>0}$. Let C be an arbitrary (but fixed) non-null compact subset of E and let us assume that μ is normalized on C as $\mu(C) = 1$.

In the remainder of this paper, unless otherwise stated, the CAF $A=(A_t)_{t\geq 0}\in\Phi^+$ always represents the CAF defined by

$$(1.8)\qquad A_t=\int_0^t I_C(X_s)\,ds\,.$$

Then for every $B\in\Phi^+$,

$$U^{0,1}_{A,B}1(x)=E^x\Big[\int_0^\infty e^{-B_t}I_C(X_t)\,dt\Big]=G^{1,0}_B(x,C)$$

is bounded by lemma 1.3. Moreover we have

Lemma 1.4. *For any p, $q>0$ and $f\in b\mathcal{E}^*$, the functions K^p_Af and $G^{p,q}_Af$ are bounded continuous. In case $f\in b\mathcal{E}^*_c$, $G^{p,0}_Af$ is bounded continuous for all $p>0$.*

Proof. Drop the suffix A in the related kernels. For any $p>0$ and $f\in b\mathcal{E}^*$ we have, from (1.7),

$$G^pf-G^{p,p}f-pK^{0,p}G^{p,p}f=0\,.$$

Since $K^{0,p}g=G^p(I_Cg)$ for any $g\in b\mathcal{E}^*$ the function

$$\begin{aligned}G^{p,p}f&=G^pf-pG^p(I_CG^{p,p}f)\\&=G^p(f-pI_CG^{p,p}f)\end{aligned}$$

is bounded continuous by the strong Feller property of G^p. Therefore,

$$G^{p,q}f=G^{p,p}(f+(q-p)\,G^{p,p}G^{p,q}f)$$

is bounded continuous. If $f\in b\mathcal{E}^*_c$ then $G^{p,0}f$ is bounded by lemma 1.3, so that the above equality for $q=0$ shows that $G^{p,0}f$ is bounded continuous.

Since $(K^p_A)_{p>0}$ is a strong Feller resolvent by lemma 1.4, the mapping $x\to K^1_A(x,\cdot)$ of the compact set C into the space of measures over C is strongly continuous by a theorem of Mokobodzki (see Meyer [9]). Since, in addition, $K^1_A(x,\cdot)$ are equivalent for all $x\in E$, we have

$$\sup_{x,y\in C}\frac{1}{2}\|K^1_A(x,\cdot)-K^1_A(y,\cdot)\|\equiv a<1\,.$$

Thus there exists a unique invariant probability measure ν_A of K^1 such that

$$(1.9)\qquad \sup_{x\in E}\|(K^1_A)^{n+1}(x,\cdot)-\nu_A(\cdot)\|\leqq 2a^n$$

for all $n\geqq 0$ ([7; lemma 1.3]). Therefore the kernel

$$(1.10)\qquad K_A(x,F)=\sum_{n=1}^\infty[(K^1_A)^n(x,F)-\nu_A(F)]$$

is well defined and satisfies

$$(I-K_A^1)K_A f = K_A^1 f - \langle \nu_A, f\rangle$$

for all $f\in b\mathcal{E}^*$.

Lemma 1.5. *The kernel K_A defined by* (1.10) *satisfies*

$$(1.11)\qquad \lim_{p\to 0}\sup_{x\in E} \|K_A^p(x,\cdot)-\frac{\nu_A(\cdot)}{p}-K_A(x,\cdot)\| = 0\,,$$

in particular,

$$(1.12)\qquad \lim_{p\to 0}\sup_{x\in E} \|pK_A^p(x,\cdot)-\nu_A(\cdot)\| = 0\,.$$

Proof. From the resolvent equation for (K_A^p) we have

$$\begin{aligned} K_A^p(x,\cdot) &= K_A^1 \sum_{n=0}^{\infty}(1-p)^n(K_A^1)^n(x,\cdot) \\ &= \sum_{n=1}^{\infty}(1-p)^{n-1}\{(K_A^1)^n(x,\cdot)-\nu_A(\cdot)\}+\frac{\nu_A(\cdot)}{p} \end{aligned}$$

for all $x\in E$ and $0<p<1$. Thus it follows that

$$\begin{aligned} &\|K_A^p(x,\cdot)-\frac{\nu_A(\cdot)}{p}-K_A(x,\cdot)\| \\ &\leqq \sum_{n=1}^{\infty}\{1-(1-p)^{n-1}\}\|(K_A^1)^n(x,\cdot)-\nu_A(\cdot)\| \\ &\leqq \sum_{n=1}^{\infty}\{1-(1-p)^{n-1}\}\, 2a^{n-1} = 2\left\{\frac{1}{1-a}-\frac{1}{1-a(1-p)}\right\}. \end{aligned}$$

Therefore the lemma follows.

2. An invariant measure and a potential kernel of (K_B^p)

Similarly to [4] and [12], for any $B\in\Phi^+$, an invariant measure ν_B and a potential kernel K_B of $(K_B^p)_{p>0}$ can be constructed by making use of ν_A and K_A defined in section 1. In [12], we have treated only the case of $B_t=t$ but the same arguments are valid for all $B\in\Phi^+$. We shall outline it in the form of our present use.

For any $B\in\Phi^+$ define the measure ν_B by

$$(2.1)\qquad \nu_B=\nu_A V_{A,B}^{1,0}$$

Then ν_B charges no semipolar set and satisfies the following properties.

Lemma 2.1. *The measure ν_B is a σ-finite invariant measure of $(K_B^p)_{p>0}$. In particular, $\nu_{\{t\}}=\mu$.*

Proof. (cf. [12; theorem 2.7]) Since $V_{A,B}^{1,0}(\cdot, F_n)$ is bounded for all n by lemma 1.2, ν_B is σ-finite. Integrating the equality

$$V_{A,B}^{1,0}-K_B^p+K_A^1K_B^p-pV_{A,B}^{1,0}K_B^p=0$$

by ν_A we have

$$\nu_B=p\nu_BK_B^p\,,$$

that is, ν_B is an invariant measure of (K_B^p).

By the uniqueness of the invariant measure of X, $\nu_{\{t\}}$ is a constant multiple of μ, say,

$$\nu_{\{t\}}=\nu_AG_A^{1,0}=b\mu$$

for some constant b. Since $\mu(C)=1$ we have

$$b=\nu_AG_A^{1,0}(C)=\nu_AK^1(C)=\nu_A(C)=1\,,$$

Hence $\nu_{\{t\}}=\mu$.

Lemma 2.2 (cf. [4; proposition 2]). *For any B, $B'\in\Phi^+$,*

(2.2) $\quad \nu_{B'}=p\nu_BV_{B,B'}^{p,0}\,.$

In particular, ν_B is the measure associated with B in the sense of Revuz ([14]). *Moreover it holds that $\nu_A=\mu|_C$, where $\mu|_C$ is the restriction of μ to C.*

Proof. Similarly to lemma 1.1, we can prove easily that

(2.3) $\quad V_{A,B'}^{q,0}-V_{B,B'}^{p,0}+qK_A^qV_{B,B'}^{p,0}-pV_{A,B}^{q,0}V_{B,B'}^{p,0}=0$

for sufficiently many $f\in b\mathcal{E}^*$. Letting $q=1$ and integrating by ν_A, (2.2) follows. Set $B_t=t$ at (2.2) then $\nu_{B'}=p\nu_{\{t\}}V_{\{t\},B'}^{p,0}=p\mu K_{B'}^{0,p}$ by lemma 2.1. Hence $\nu_{B'}$ is the measure associated with B'. In particular, when $B'=A$, it follows that $\langle\nu_A,f\rangle=p\langle\mu,K_A^{0,p}f\rangle=p\langle\mu,G^pI_Cf\rangle=p\langle\mu G^p,I_Cf\rangle=\langle\mu,I_Cf\rangle=\langle\mu|_C,f\rangle$.

Define a kernel K_B by

(2.4) $\quad K_B(x,\cdot)=K_AV_{A,B}^{1,0}(x,\cdot)+V_{A,B}^{1,0}(x,\cdot)-\nu_B(\cdot)\,.$

In case $B_t=t$ we shall denote K_B by G, which is the kernel we have constructed in [12]. Obviously, $K_B(x,\cdot)$ is a σ-finite signed measure on E and, for any $n\geqq 1$, the total variation of $K_B(x,\cdot)$ on F_n are uniformly bounded for all $x\in E$ by (1.9) ana lemma 1.2. Similarly, for any compact set F, the total variation of $G(x,\cdot)$ on F is uniformly bounded for all $x\in E$ by lemma 1.3. If we denote the total variation of a measure on F by $\|\cdot\|_F$, then the following theorem holds.

Theorem 2.3. *For all $n\geqq 1$,*

(2.5) $\quad \limsup_{p\to 0}\sup_{x\in E}\|V_{A,B}^{p,0}(x,\cdot)-\frac{\nu_B(\cdot)}{p}-K_B(x,\cdot)\|_{F_n}=0\,,$

and in partciular,

(2.6) $$\lim_{p\to 0} \sup_{x\in E} \|pV^{p,0}_{A,B}(x,\ \cdot)-\nu_B(\cdot)\|_{F_n}=0\,.$$

If $B_t=t$ then we can take arbitrary compact set in place of F_n.

Proof. Write $V^{p,0}$ for $V^{p,0}_{A,B}$. For any Borel subset D of F_n

$$V^{p,0}(x,\ D)-V^{1,0}(x,\ D)+pK^p_A V^{1,0}(x,\ D)-K^p_A V^{1,0}(x,\ D)=0$$

from (1.7). This can be written, by noting (2.1),

$$\{V^{p,0}(x,\ D)-\frac{\nu_B(D)}{p}\}-V^{1,0}(x,\ D)+pK^p_A V^{1,0}(x,\ D)$$
$$-(K^p_A-\frac{1}{p}\nu_A)V^{1,0}(x,\ D)=0\,.$$

Thus we have

$$\begin{aligned}&\|V^{p,0}(x,\ \cdot)-\frac{\nu_B(\cdot)}{p}-K_B(x,\ \cdot)\|_{F_n}\\ &\quad\leqq\|\{pK^p_A(x,\ \cdot)-\nu_A(\cdot)\}V^{1,0}\|_{F_n}\\ &\qquad+\|\{K^p_A(x,\ \cdot)-\frac{\nu_A(\cdot)}{p}-K_A(x,\ \cdot)\}V^{1,0}\|_{F_n}\,.\end{aligned}$$

This proves the theorem from lemma 1.5.

Corollary 1. *If $f\in b\mathcal{E}^*$ vanishes outside of some F_n, then*

(2.7) $$(I-pK^p_B)K_Bf=K^p_Bf-U^{0,p}_{A,B}1\langle\nu_B,f\rangle$$

for all $p>0$. If $V^{1,0}_{A,B}1$ is bounded, then

(2.8) $$K_B(I-pK^p_B)f=K^p_Bf-\langle\nu_A,\ K^p_Bf\rangle$$

for all $p>0$ and $f\in b\mathcal{E}^*$.

Proof. Suppose that $V^{1,0}1$ is bounded, then obviously (2.5) holds for E in place of F_n, so we have

$$\begin{aligned}K_B(I-pK^p_B)f(x)&=\lim_{q\to 0}(V^{q,0}-\frac{1}{q}\nu_B)(I-pK^p_B)f(x)\\ &=\lim_{q\to 0}V^{q,0}(I-pK^p_B)f(x)=\lim_{q\to 0}(K^p_Bf-qK^p_AK^q_Bf)(x)\\ &=K^p_Bf(x)-\langle\nu_A,\ K^p_Bf\rangle\,,\end{aligned}$$

from (1.13). The proof of (2.7) is similar.

Let us denote

(2.9) $$N_B=\{f;f\in b\mathcal{E}^*,\ =0\ \textit{outside of some}\ F_n\ \textit{and}\ \langle\nu_B,f\rangle=0\}\,,$$

(2.10) $\quad N=\{f; f\in b\mathcal{E}^*, \langle\mu, f\rangle=0\}$.

Definition. If a kernel H on E satisfies the condition that (i) for any $f\in N_B$ [resp. $f\in N$], $Hf\in b\mathcal{E}^*$ and that (ii) for any $f\in N_B$ [resp. $f\in N$], $(I-pK_B^p)\, Hf=K_B^p f$ [resp. $(I-pG^p)Hf=G^p f$] for all $p>0$, then we shall say that *H is a potential kernel of $(K_B^p)_{p>0}$ [resp. X].*

Corollary 2. *The kernels K_B and G are the potential kernels of $(K_B^p)_{p>0}$ and X, respectively.*

Corollary 3. *For every compact set F, the function $G(\cdot, F)$ is finely continuous.*

Proof. Set $B_t=t$ at (2.7)

$$G(x, F)=pG^pG(x, F)+G^p(x, F)-(K_A^{0,p}1(x))\,\mu(F)\,.$$

Since $K_A^{0,p}1$ is p-excessive, the result is obvious.

3. Hypothesis of duality and the kernel function $g(x, y)$

In this section we shall assume that there exists a Hunt process $\hat{X}$ with strong Feller resolvent $\hat{G}^p$ such that X and $\hat{X}$ are in duality relative to μ. It follows that $\hat{X}$ is also recurrent and μ is the invariant measure of $\hat{X}$.

Let $\hat{\Phi}^+$ be the family of all non-zero non-negative finite continuous additive functionals of $\hat{X}$. For any $\hat{A}$, $\hat{B}\in\hat{\Phi}^+$, we define $\hat{U}_{A,B}^{p,q}$ etc. by

$$f\hat{U}_{A,B}^{p,q}(x)=\hat{E}^x\Big[\int_0^\infty e^{-p\hat{A}_t-q\hat{B}_t}f(\hat{X}_t)\,d\hat{A}_t\Big]$$

etc. (in general, a kernel with respect to the dual process $\hat{X}$ is written such as $\hat{K}(D, x)$, so that $\hat{K}$ operates to function from the right side and to measure from the left).

By Revuz [14; theorem VII. 1], for any $B\in\Phi^+$, there exists a polar set P_B and a CAF $\hat{B}\in\hat{\Phi}^+$ of $\hat{X}$ restricted to $E-P_B$ such that $\nu_B=\hat{K}_B^{0,1}\,\mu$. Also, by [14; theorem VII. 2], there exists a jointly measurable kernel function $g_B^{p,q}(x, y)$ satisfying

(i) *$g_B^{p,q}(\cdot, y)$ [resp. $g_B^{p,q}(x, \cdot)$] is finely [resp. cofinely] continuous and q-excessive [resp. q-coexcessive] relative to the resolvent $(G_B^{p,q})_{q>0}$ [resp. $(\hat{G}_B^{p,q})_{q>0}$] for all $p>0$ and $y\in E-P_B$ [resp. $x\in E$],*

(ii) *For all $p, q>0$ and $x\in E$, $K_B^{p,q}(x, dy)=g_B^{p,q}(x, y)\nu_B(dy)$, $G_B^{p,q}(x, dy)=g_B^{p,q}(x, y)\mu(dy)$ and for all $p, q>0$ and $y\in E-P_B$, $\hat{K}_B^{p,q}(dx, y)=g_B^{p,q}(x, y)\,\nu_B(dx)$, $\hat{G}_B^{p,q}(dx, y)=g_B^{p,q}(x, y)\,\mu(dx)$.*

As before, the set C with $\mu(C)=1$ is fixed and A is given by (1.8). If $B=A$, P_B may be supposed to be empty and the dual CAF of A is given exactly by

(3.1) $\hat{A}_t = \int_0^t I_C(\hat{X}_s)ds$.

In the following, unless otherwise stated, $\hat{A}$ always represents this CAF and we shall drop the suffix A in $g_A^{p,q}$. Further, we shall denote $g^q(x, y)$ for $g_A^{0,q}(x, y)$, which is Kunita-Watanabe's potential kernel function. Note that $g_B^{0,q}(x, y)=g^q(x, y)$ for all $B\in\Phi^+$. Form the resolvent equation (1.6), for any $q>0$,

$$g^{1,q}(x, y) = g^q(x, y)-K_A^{1,q}g^q(x, y) \tag{3.2}$$
$$= g^q(x, y)-g^q\hat{K}_A^{1,q}(x, y)$$

on $\{(x, y);\ g^q(x, y)<\infty\}$. Hence for any $y\in E$, $K_A^{1,q}g^q(x, y)=g^q\hat{K}_A^{1,q}(x, y)$ *a.a.* $x(\mu)$. Since both sides of the equality are q-excessive, it holds for all $x, y\in E$ (cf. Getoor [5; theorem 2.5]).

Lemma 3.1. *For all* $x\in E$ *and* $B\in\Phi^+$,

$$V_{A,B}^{1,0}(x, dy) = g^{1,0}(x, y)\nu_B(dy) . \tag{3.3}$$

Proof. Set $A_t'=A_t+qt$. Replacing A', $\{t\}$, B for A, B, B' in (2.3) we have

$$V_{A',B}^{1,0}f = V_{\{t\},B}^{q,0}f-K_{A'}^1V_{\{t\},B}^{q,0}f+qV_{A',\{t\}}^{1,0}V_{\{t\},B}^{q,0}f ,$$

for sufficiently many functions f. Noting that $V_{\{t\},B}^{q,0}=K_B^{0,q}$, $K_{A'}^1=K_A^{1,q}+qG_A^{1,q}$ and $V_{A',\{t\}}^{1,0}=G_A^{1,q}$, it follows that

$$\begin{aligned}
E^x[\int_0^\infty e^{-A_t-qt}f(X_t)\,dB_t] &= V_{A',B}^{1,0}f(x)\\
&= K_B^{0,q}f(x)-K_A^{1,q}K_B^{0,q}f(x)\\
&= \int\{g^q(x, y)-K_A^{1,q}g^q(x, y)\}f(y)\nu_B(dy)\\
&= \int\{g^q(x, y)-K_A^{1,q}g^q(x, y)\}f(y)\nu_B(dy)\\
&= \int\{g^{1,q}(x, y)f(y)\}\,\nu_B(dy)
\end{aligned}$$

The last equality follows from (3.2) since ν_B has no mass on the polar set $\{y;\ g^q(x, y)=\infty\}$. Letting $q\to 0$ we have the result.

Dually, if $\hat{B}$ is the dual CAF of B then

$$\hat{V}_{A,B}^{1,0}(dx, y) = g^{1,0}(x, y)\nu_B(dx) \qquad \textit{for all } y\in P_B . \tag{3.4}$$

Hence we have

Corollary. *For all* $f, g\in b(\mathcal{E}^*)^+$,

$$(3.5)\qquad \int f(x)V^{1,0}_{A,B}g(x)\nu_A(dx)=\int f\hat{U}^{1,0}_{A,B}(y)g(y)\nu_B(dy)$$

$$(3.6)\qquad \int f\hat{V}^{1,0}_{A,B}(y)g(y)\nu_A(dy)=\int f(x)U^{1,0}_{A,B}g(x)\nu_B(dx)\,.$$

Since

$$K_A(x,dy)=K^1_B(x,dy)-\nu_A(dy)+K_AK^1_A(x,dy)\,,$$

it is easy to show that, for each x, $K_A(x,\cdot)$ is absolutely continuous relative to ν_A and its density is given by

$$g^{1,0}(x,y)-1+K_Ag^{1,0}(x,y)$$

up to a set of ν_A-measure 0.

However, in order to solve the problem proposed in the introduction, we have to choose a more elaborated density $g(x,y)$. To do this, we need one more preliminary observation.

For all $x, y\in E$ and $n\geqq 1$, set

$$(3.7)\qquad f_n(x,y)=(\hat{K}^1_A)^{n-1}g^{1,0}(x,y)-1=g^{1,0}(K^1_A)^{n-1}(x,y)-1\,,$$

then

$$(3.8)\qquad \begin{aligned} f_n(x,y)\nu_A(dy)&=(K^1_A)^n(x,dy)-\nu_A(dy)\\ f_n(x,y)\nu_A(dx)&=(\hat{K}^1_A)^n(dx,y)-\nu_A(dx)\,. \end{aligned}$$

Since

$$\int\sum_{n=1}^{\infty}|f_n(x,y)|\nu_A(dy)=\sum_{n=1}^{\infty}\|(K^1_A)^n(x,\cdot)-\nu_A(\cdot)\|<\infty$$

for all $x\in E$ from (1.9) and (3.8), the series $\sum_{n=1}^{\infty}f_n(x,y)$ converges absolutely for $a.a.x(\nu_A)$. Similarly for all $y\in E$, $\sum_{n=1}^{\infty}f_n(x,y)$ converges absolutely for $a.a.x(\nu_A)$. Also

$$\begin{aligned}\int_D\sum_{n=1}^{\infty}f_n(x,y)\nu_A(dy)&=\sum_{n=1}^{\infty}\int_D f_n(x,y)\nu_A(dy)\\ &=\sum_{n=1}^{\infty}\{(K^1_A)^n(x,D)-\nu_A(D)\}=K_A(x,D)\end{aligned}$$

for all $D\in\mathcal{E}$, that is, $\sum_{n=1}^{\infty}f_n(x,\cdot)$ is a density of $K_A(x,\cdot)$ relative to ν_A. Dually, $\sum_{n=1}^{\infty}f_n(\cdot,y)$ is a density of

$$(3.9)\qquad \hat{K}_A(\cdot,y)=\sum_{n=1}^{\infty}\{(\hat{K}^1_A)^n(\cdot,y)-\nu_A(\cdot)\}$$

relative to ν_A. Here the proof of the strong convergence of (3.9) is similar to (1.9).

Lemma 3.2. *There exists a Borel subset Γ of $E\times E$ satisfying the following conditions.*

(i) *Set $\Gamma_y=\{x; (x, y)\in\Gamma\}$ and $\hat{\Gamma}_x=\{y; (x, y)\in\Gamma\}$, then Γ_y^c and $\hat{\Gamma}_x^c$ are polar for all $x, y\in E$.*

(ii) *For all $(x, y)\in\Gamma$, $\sum_{n=1}^{\infty} f_n(x, y)$ converges absolutely, $|K_A|g^{1,0}(x, y)<\infty$ and $g^{1,0}|\hat{K}_A|(x, y)<\infty$, where $|K_A|(x, \cdot)$ is the total variation measure of $K_A(x, \cdot)$.*

(iii) *For all $(x, y)\in\Gamma$,*

$$\sum_{n=1}^{\infty} f_n(x, y) = g^{1,0}(x, y)-1+K_A g^{1,0}(x, y) \tag{3.10}$$
$$= g^{1,0}(x, y)-1+g^{1,0}\hat{K}_A(x, y)\,.$$

We define the kernel function $g(x, y)$ by

$$g(x, y) = (3.10) \quad \text{if} \quad (x, y)\in\Gamma \tag{3.11}$$
$$= \infty \quad \text{if} \quad (x, y)\notin\Gamma\,.$$

By the lemma, it is easy to see that the function $g(\cdot, y)$ [resp. $g(x, \cdot)$] is finely [resp. cofinely] continuous on the fine [resp. cofine] open set Γ_y [resp. $\hat{\Gamma}_x$] for all y [resp. all x]$\in E$.

Proof. Noting that,

$$\begin{aligned}
|f_{n+1}(x, y)| &= |K_A^1\{(K_A^1)^{n-1}g^{1,0}-1\}(x, y)| \\
&= \Big|\int K_A^1(x, dz)\Big\{\iint (K_A^1)^{n-2}(z, du)g^{1,0}(u, v)g^{1,0}(v, y)\nu_A(dv) \\
&\qquad -\int g^{1,0}(v, y)\nu_A(dv)\Big\}\Big| \\
&= \Big|\int K_A^1(x, dz)\int \{(K_A^1)^{n-2}g^{1,0}(z, v)-1\}g^{1,0}(v, y)\nu_A(dv)\Big| \\
&= |K_A^1 f_{n-1}\hat{K}_A^1(x, y)| \leqq K_A^1|f_{n-1}|\hat{K}_A^1(x, y)
\end{aligned}$$

for $n\geqq 2$ and

$$\begin{aligned}
&\int |(K_A^1)^n(x, dz)-\nu_A(dz)|g^{1,0}(z, y) \\
&= \int |(K_A^1)^{n-1}g^{1,0}(x, z)-1|g^{1,0}(z, y)\nu_A(dz) \\
&= \int |K_A^1\{(K_A^1)^{n-2}g^{1,0}-1\}(x, z)|g^{1,0}(z, y)\nu_A(dz) \\
&\leqq K_A^1|f_{n-1}|\hat{K}_A^1(x, y)
\end{aligned}$$

for $n\geqq 2$, let us define the set Γ by

$$\Gamma = \{(x, y);\ |f_1|(x, y)+|f_2|(x, y)+\sum_{n=1}^{\infty} K_A^1|f_n|\hat{K}_A^1(x, y)<\infty\}\ .$$

Then the proofs of (ii) and (iii) are obvious. For the proof of (i) set

$$\xi_x(dy) = \delta_x(dy)+K_A^1(x, dy)+\sum_{n=1}^{\infty}(K_A^1|f_n|)(x, y)\nu_A(dy)\,.$$

Then

$$\xi_x(E) = 2+\sum_{n=1}^{\infty}\int K_A^1(x, dy)|(K_A^1)^n(y, \cdot)-\nu_A(\cdot)|<\infty\ .$$

Moreover, it is easy to see that,

$$\Gamma = \{(x, y);\ \int\xi_x(dz)g^{1,0}(z, y)<\infty\}\ .$$

Hence $\hat{\Gamma}_x^c$ is polar if and only if $\int\xi_x(dz)g^{1,0}(z, y)<\infty$ except on a polar set. Since $g^{1,0}$ is the potential kernel function of the e^{-A_t}-subprocess (which is a transient Hunt process on E (13; III.3.16])) of X, the potential $\int\xi_x(dz)g^{1,0}(z, y)$ of the bounded measure ξ_x is finite except on a polar set if it is finite for $a.a.y(\mu)$ ([3; VI.2.3]). Since, for all $f\in b\mathcal{E}_c^+$,

$$\begin{aligned}&\iint\xi_x(dz)g^{1,0}(z, y)f(y)\,\mu(dy)\\ &\quad=\int\xi_x(dz)\,G_A^{1,0}f(z)\leqq||G_A^{1,0}f||\,\xi_x(E)<\infty\ ,\end{aligned}$$

by lemma 1.3, $\int\xi_x(dz)g^{1,0}(z, y)<\infty a.a.y(\mu)$. Therefore $\hat{\Gamma}_x^c$ is polar. Similarly Γ_y^c is ploar.

Suppose we are given a CAF $B\in\Phi^+$ and let $\hat{B}$ be its dual CAF. Just as (2.4), define a kernel $\hat{K}_B$ by

$$\hat{K}_B(dx, y) = \hat{V}_{A,B}^{1,0}\hat{K}_A(dx, y)+\hat{V}_{A,B}^{1,0}(dx, y)-\nu_B(dx)\,, \tag{3.12}$$

for $y\notin P_B$, where $\hat{K}_A$ is the kernel defined by (3.9). In the case $B_t=t$ denote $\hat{G}_B$ by $\hat{G}$. For these kernels, the dual results of section 2 are valid.

Theorem 3.3. *For all $x\in E$, $y\in E$ and $z\in E-P_B$,*

$$\begin{aligned}&K_A(x, dy) = g(x, y)\nu_A(dy),\ \hat{K}_A(dx, y) = g(x, y)\nu_A(dx)\\ &K_B(x, dy) = g(x, y)\nu_B(dy)\quad\text{and}\quad\hat{K}_B(dx, z) = g(x, z)\nu_B(dx)\,.\end{aligned} \tag{3.13}$$

Proof. The first two equalities have been already proved. For the proof of the third equality, take a function $f\in b\mathcal{E}^+$ such that $V_{A,B}^{1,0}f$ is bounded. Then, since ν_B charges no polar set, it follows from lemma 3.1 and 3.2 that

$$\iint |g(x, y)| f(y) \nu_B(dy) \leqq V_{A,B}^{1;0} f(x) + \langle \nu_B, f \rangle + |K_A| V_{A,B}^{1;0} f(x) < \infty .$$

Hence

$$\int g(x, y) f(y) \nu_B(dy) = V_{A,B}^{1;0} f(x) - \langle \nu_B, f \rangle + K_A V_{A,B}^{1;0} f(x) = K_B f(x) .$$

The last equality follows similarly.

Corollary. *For all $x \in E$ and $y \in E$,*

$$G(x, dy) = g(x, y) \mu(dy) \quad \textit{and} \quad \hat{G}(dx, y) = g(x, y) \mu(dx) . \tag{3.14}$$

For a measure ξ on E, let us denote

$$G^{1,0}\xi(x) = \int g^{1,0}(x, y) \xi(dy) , \tag{3.15}$$

$$G\xi(x) = \int g(x, y) \xi(dy) , \tag{3.16}$$

if they are well defined.

Let X_A and $\hat{X}_A$ be the subprocesses of X and $\hat{X}$ by the multiplicative functionals $M_t = e^{-A_t}$ and $\hat{M}_t = e^{-\hat{A}_t}$, respectively. Then a set is polar if and only if it is polar relative to X_A or $\hat{X}_A$. Moreover, as we have seen at lemma 1.4, the resolvents $(G_A^{1,p})_{p>0}$ and $(\hat{G}_A^{1,p})_{p>0}$ of the processes X_A and $\hat{X}_A$ are strong Feller, so that, it is well known that a compact set F is non-polar if and only if $G^{1,0}\xi$ is locally bounded for some non-zero finite measure ξ on F. Also, it is well known that if $G^{1,0}\xi$ is locally bounded then ξ charges no polar set (see [3; p. 285]). Hence we have the following theorem.

Theorem 3.4. *If F is a compact subset of E, then F is non-polar if and only if there exists a non-zero finite measure ξ on F such that $\int |g(x, y)| \xi(dy)$ is locally bounded.*

Proof. It is enough to prove that $G^{1,0}\xi$ is locally bounded if and only if $\int |g(x, y)| \xi(dy)$ is locally bounded.

If $G^{1,0}\xi$ is locally bounded for some non-zero finite measure ξ then ξ charges no polar set and hence, in particular, $\xi(\hat{\Gamma}_x^c) = 0$ for all $x \in E$. So, it follows that,

$$\int | g(x, y) | \xi(dy) \leqq G^{1,0}\xi(x) + \xi(E) + |K_A| G^{1,0}\xi(x) .$$

In the right side of the inequality, since $G^{1,0}\xi$ is bounded on the compact set C, the last two terms are bounded. Therefore, $\int |g(x, y)| \xi(dy)$ is locally bounded.

Conversely, if $\int|g(x, y)|\xi(dy)$ is locally bounded then $\xi(\hat{\Gamma}_x^c)=0$ from the definition of $g(x, y)$. Therefore, for any $x\in E$,

$$g(x, y) = g^{1,0}(x, y)-1+\int g^{1,0}(x, z)g(z, y)\nu_A(dz)$$

$a.a.y(\xi)$. Thus

$$\begin{aligned} G^{1,0}\xi(x) &\leqq \int|g(x, y)|\xi(dy)+\xi(E) \\ &\quad +\int g^{1,0}(x, z)\{\int|g(z, y)|\xi(dy)\}\nu_A(dz)\,. \end{aligned}$$

Since $G^{1,0}\nu_A(x)=K_A^1 1(x)=1$,

$$\begin{aligned} &\int g^{1,0}(x, z)\{\int|g(z, y)|\xi(dy)\}\,\nu_A(dz) \\ &\leqq \sup_{z\in C}\int|g(z, y)|\xi(dy)\,. \end{aligned}$$

Therefore the theorem is proved.

4. Potential kernel functions

By the corollary of theorem 3.3, we shall say that $g(x, y)$ is *the potential kernel function associated with* $(G, \hat{G})$. Moreover the kernel function $g(x, y)$ satisfies several regularity conditions (corollaries 2 and 3 of theorem 2.3, lemma 3.2).

We now extend the notion of potential kernel functions.

Definition. An $\mathcal{E}^*\times\mathcal{E}^*$-measurable kernel function $h(x, y)$ is said to be a *potential kernel function* if the following conditions are satisfied.

(i) Set $H(x, dy)=h(x, y)\mu(dy)$ and $\hat{H}(dx, y)=h(x, y)\mu(dx)$. Then H and $\hat{H}$ are the potential kernels of X and $\hat{X}$ such that Hf and $f\hat{H}$ are well defined and locally bounded for all $f\in b\mathcal{E}_c^*$. Moreover, the functions $H(\cdot, F)$ and $\hat{H}(F, \cdot)$ are finely and cofinely continuous for any compact set F, respectively.

(ii) The sections $(\Gamma_h)_y^c$ and $(\hat{\Gamma}_h)_x^c$ (see §3) of the set $\Gamma_h^c=\{(x, y);\ |h(x, y)|=\infty\}$ are polar sets and the functions $h(\cdot, y)$ and $h(x, \cdot)$ are finely and cofinely continuous on the fine and cofine open sets $(\Gamma_h)_y$ and $(\hat{\Gamma}_h)_x$ for all $x, y\in E$, respectively.

We shall show how any potential kernel function $h(x, y)$ is related to $g(x, y)$. Recall that $\Gamma^c=\{(x, y);\ |g(x, y)|=\infty\}$.

Theorem 4.1. *If $h(x, y)$ is a potential kernel function of X, then*

$$(4.1) \qquad g(x, y) = h(x, y) - H(x, C) - \hat{H}(C, y) + H(C, C),$$

for all $(x, y) \in \Gamma \cap \Gamma_h$, *where* $H(C, C) = \int_C H(x, C)\mu(dx)$.

Proof. If $f \in N$ then, by (i), $Gf - Hf$ is bounded and satisfies $(I - pG^p)(Gf - Hf) = 0$, so that, $Gf - Hf$ equals a constant on E. Particularly, set $f = I_F - \mu(F)I_C \in N$ for a relatively compact set $F \in \mathcal{E}^*$ then, since $G(\cdot, C) = 0$,

$$(4.2) \qquad G(x, F) - H(x, F) + H(x, C)\mu(F) = a$$

for some constant a. Integrating both sides of (4.2) by $\nu_A = \mu|_C$ and noting that $\nu_A G = 0$, we have

$$-H(C, F) + H(C, C)\mu(F) = a.$$

Thus,

$$G(x, F) = H(x, F) - H(x, C)\mu(F) - H(C, F) + H(C, C)\mu(F).$$

Therefore, for all $x \in E$, (4.1) holds for $a.a. y(\mu)$. Since μ is equivalent to $\hat{G}^p(\cdot, y)$ for all $p > 0$ and $y \in E([1])$, μ charges all cofine open sets. Hence, for all $x \in E$, (4.1) holds for cofinely dense $y \in E$. Since both sides of (4.1) are cofinely continuous relative to y on the cofine open set $\hat{\Gamma}_x \cap (\hat{\Gamma}_h)_x$, (4.1) holds for all $y \in \hat{\Gamma}_x \cap (\hat{\Gamma}_h)_x$.

If $B \in \Phi^+$ then, since the associated measure ν_B of B has no mass on any semipolar set, we have

Corollary 1. *If $h(x, y)$ is a potential kernel function of X, then the kernels $H_B(x, dy) = h(x, y)\nu_B(dy)$ and $\hat{H}_B(dx, y) = h(x, y)\nu_B(dx)$ are potential kernels of (K_B^p) and $(\hat{K}_B^p)$, respectively.*

Corollary 2. *Let $h(x, y)$ be a potential kernel function such that $\Gamma_h \subset \Gamma$, then a compact subset F of E is non-polar if and only if $\int |h(x, y)| \xi(dy)$ is locally bounded for some non-zero finite measure ξ on F. In particular, if X and $\hat{X}$ are equivalent, then F is non-polar iff $\int |h(x, y)| \xi(dy)$ is bounded on F for some ξ as above.*

Proof. It is enough to show that $\int |g(x, y)| \xi(dy)$ is locally bounded if and only if $\int |h(x, y)| \xi(dy)$ is locally bounded.

If $\int |g(x, y)| \xi(dy)$ is locally bounded, then ξ charges no polar set by theorem 3.4 and, in particular, $\xi(\{\hat{\Gamma}_x \cap (\hat{\Gamma}_h)_x\}^c) = 0$. Hence it follows from (4.1) that $\int |h(x, y)| \xi(dy)$ is locally bounded.

Conversely, if $\int |h(x, y)| \xi(dy)$ is locally bounded, then ξ has no mass on $(\hat{\Gamma}_h)_x^c$ for all $x \in E$, so that (4.1) holds $a.a. y(\xi)$ for all $x \in E$. Hence $\int |g(x, y)| \xi(dy)$ is locally bounded. If X and $\hat{X}$ are equivalent, then all semipolar sets are polar.

Hence $\sup_{x\in E} G^{1,0}\xi(x)=\sup_{x\in F} G^{1,0}\xi(x)$ ([3]), so that the last part of the corollary is obvious from the proof of theorem 3.4.

Remark. If $h(x, y)$ is a potential kernel function of X, then the kernel function $h'(x, y)$ defined by

$$(4.3)\qquad \begin{aligned} h'(x, y) &= h(x, y) \qquad &&\text{if}\quad (x, y)\in\Gamma\cap\Gamma_h \\ &= \infty &&\text{if}\quad (x, y)\notin\Gamma\cap\Gamma_h \end{aligned}$$

is a potential kernel function of X. For this kernel function, the hypothesis $\Gamma_{h'}\subset\Gamma$ of the corollary 2 holds obviously.

Remark. So far we have fixed a compact set C and assumed that $\mu(C)=1$. If we delete such normalization condition, the only minor change is necessary; ν_A equals $[\mu(C)]^{-1}\mu|_C$ for $\mu|_C$ and $\nu_{\{t\}}$ equals $[\mu(C)]^{-1}\mu$ for μ. It then follows that $G(x, dy)=g(x, y)[\mu(C)]^{-1}\mu(dy)$.

For two compact sets C_1 and C_2, let G_1, G_2 and g_1, g_2 be their associated potential kernels and kernel functions. Let μ be an arbitrary invariant measure (not necessarily normalized either on C_1 or C_2). By an argument similar to the proof of theorem 4.1, we have $G_1(x, F)-\dfrac{G_1(C_2, F)}{\mu(C_2)}=G_2(x, F)-\dfrac{\mu(F)}{\mu(C_1)}G_2(x, C_1)$. By the preceding remark, we obtain the following relation:

$$(4.4)\qquad \frac{g_1(x, y)}{\mu(C_1)}-\frac{1}{\mu(C_2)}G_1(C_2, y) = \frac{g_2(x, y)}{\mu(C_2)}-\frac{1}{\mu(C_1)}G_2(x, C_1)$$

on $\Gamma g_1\cap\Gamma g_2$.

5. Equilibrium measure

Let $\boldsymbol{F}$ be the family of all non-empty relatively compact sets F which is the fine support of some CAF $B\in\Phi^+$. In this section we shall fix a set $F\in\boldsymbol{F}$ and the corresponding CAF B. Let $\{\hat{F}_n\}_{n\geq 1}$ be an increasing sequence satisfying that $\cup\,\hat{F}_n=E$ and $\hat{V}^{1,0}_{A,B}(\hat{F}_n, \cdot)$ are bounded for all n. The existence of such a sequence is the same as in lemma 1.2. Define the continuous additive functionals $B^n\in\Phi$ by $B^n_t=\int_0^t I_{F_n\cap\hat{F}_n}(X_s)dB_s$. Then the fine support of each B^n is relatively compact. The kernels defined by A and B^n are denoted by $U^{p,q}_n$ and $V^{p,q}_n$. By the definition of B^n, $V^{1,0}_n|f|$ and $|f|\,\hat{V}^{1,0}_n$ are bounded for all $f\in b\mathcal{E}^*$. Let ν_B be the measure assoicated with B as before and set $\nu_n(\cdot)=\nu_B(\cdot\cap F_n\cap\hat{F}_n)$. The fine support of ν_B is equal to F (see [14; remark II.2]) and ν_n is the measure associated with B^n. Write K_n for K_{B^n}. It follows that K_nf is well defined and bounded for all $f\in b\mathcal{E}^*$.

Lemma 5.1. *If $B^n\neq 0$ then, for all $p>0$,* (5.1) $\quad pK_n(U_n^{0,p}1)+U_n^{0,p}1\equiv R_n(p)$

is a finite constant on E.

Proof. If $B^n\neq 0$ then $B^n\in\Phi^+$, so that, by theorem 2.3, formulas (1.6) and (2.2), and lemma 1.5,

$$\begin{aligned}pK_n(U_n^{0,p}1)(x) &= p\lim_{q\to 0}\{V_n^{q,0}(x,\cdot)-\frac{1}{q}\nu_n\}\,U_n^{0,p}1\\ &=\lim_{q\to 0}\{pV_n^{q,0}U_n^{0,p}1(x)-\frac{1}{q}\}\\ &=\lim_{q\to 0}\{qK_A^qU_n^{0,p}1(x)-U_n^{0,p}1(x)\}\\ &=\nu_AU_n^{0,p}1-U_n^{0,p}1(x)\,.\end{aligned}$$

Therefore,

(5.2) $\quad pK_n(U_p^{0,p}1)(x)+U_n^{0,p}1(x)=\nu_AU_n^{0,p}1=R_n(p)$

is a constant.

Let T_F be the hitting time of the set F, $\tau=inf\,\{t;\,B_t>0\}$ and $\tau^n=inf\,\{t;\,B_t^n>0\}$, where $inf\,\phi=\infty$. Then, $T_F=\tau$ *a.s.* (see [3; proposition V.3.5]) and $\tau^n\downarrow\tau$ *a,.s.* as $n\uparrow\infty$. Since $R_n(p)=E^{\nu_A}[\int_0^\infty e^{-pB_t^n}I_C(X_t)\,dt]$ decreases when n or p increases, the limit

(5.3) $\quad R(F)=\lim_{n\to\infty}\lim_{p\to\infty}R_n(p)=\lim_{p\to\infty}\lim_{n\to\infty}R_n(p)$

exists and it is finite since $B\neq 0$ (see the proof of lemma 5.2 below).

DEFINITION. We shall call the constant $R(F)$ as Robin's constant of F (relative to the potential kernel function $g(x, y)$).

Lemma 5.2. $R(F)=E^{\nu_A}[\int_0^{T_F}I_C(X_t)\,dt]$.

Proof. Since $B\neq 0$, $U_n^{0,1}1$ is bounded for all large n. Hence, for all $p\geqq 1$ and large n,

$$R_n(p)\leqq R_n(1)=\nu_AU_n^{0,1}1<\infty\,,$$

Therefore, by the Lebesque theorem,

$$\begin{aligned}\lim_{p\to\infty}R_n(p) &=\lim_{p\to\infty}E^{\nu_A}[\int_0^\infty e^{-pB_t^n}I_C(X_t)\,dt]\\ &=E^{\nu_A}[\int_0^\infty(\lim_{p\to\infty}e^{-B_t^n})I_C(X_t)\,dt]\\ &=E^{\nu_A}[\int_0^{\tau_n}I_C(X_t)\,dt]\,.\end{aligned}$$

Letting $n\to\infty$ we have the result.

REMARK. From the lemma 5.2, Robin's constant $R(F)$ of F does not depend on the choice of B.

Lemma 5.3. *If $F\in F$ then there exists a probability measure ξ_F on $\bar{F}$ such that*

$$\lim_{n\to\infty}\lim_{p\to\infty} V_n^{1,0}(pU_n^{0,p}1)(x) = G^{1,0}\xi_F(x) \tag{5.4}$$

for a.a.$x(\mu)$. Moreover, $V_n^{1,0}(pU_n^{0,p}1)(x)$ are uniformly bounded for all $x\in E$, $p\geqq 1$ and large n.

Proof. From (1.6), for all $p>0$ and $n\geqq 1$ such that $B^n\neq 0$,

$$V_n^{1,0}(pU_n^{0,p}1)(x) = 1-U_n^{0,p}1(x)+K_A^1 U_n^{0,p}1(x)\,.$$

As in the proof of lemma 5.2, $U_n^{0,p}1(x)$ are uniformly bounded for all $x\in E$, $p\geqq 1$ and $n\geqq 1$ such that $B^n\neq 0$, and

$$\lim_{n\to\infty}\lim_{p\to\infty} U_n^{0,p}1(x) = U^{0,\infty}1(x) \equiv E^x[\int_0^T I_C(X_t)\,dt]\,.$$

Hence,

$$\lim_{n\to\infty}\lim_{p\to\infty} V_n^{1,0}(pU_n^{0,p}1)(x) = 1-U^{0,\infty}1(x)+K_A^1 U^{0,\infty}1(x)\,,$$

boundedly. Define a measure $\xi_{p,n}$ on the compact set $\bar{F}$ by $\xi_{p,n}(dy)=pU_n^{0,p}1(y)\nu_n(dy)$ for $p>0$ and $n\geqq 1$ such that $B^n\neq 0$, then

$$\xi_{p,n}(E) = \langle \nu_n,\, pU_n^{0,p}1\rangle = \langle \nu_A,\, 1\rangle = 1\,.$$

Thus there exists a sequence $p_k\to\infty$ such that $\{\xi_{p_k,n}\}_{k\geqq 1}$ converges weakly to a probability measure ξ_n on $\bar{F}$ as $k\to\infty$, for all n. Therefore, we can choose a subsequence $\{\xi_{n_m}\}$ of $\{\xi_n\}$ which converges weakly to a probability measure ξ_F on $\bar{F}$ as $m\to\infty$. Taking an arbitrary function $f\in b\mathcal{E}_c^*$, we have

$$\begin{aligned}
&\int f(x)\{1-U^{0,\infty}1(x)+K_A^1 U^{0,\infty}1(x)\}\,\mu(dx)\\
&= \lim_{m\to\infty}\lim_{k\to\infty}\int f(x)V_{n_m}^{1,0}(p_k U_{n_m}^{0,p_k}1)(x)\,\mu(dx)\\
&= \lim_{m\to\infty}\lim_{k\to\infty}\int f(x)\,G^{1,0}\xi_{p_k,n_m}(x)\,\mu(dx)\\
&= \lim_{m\to\infty}\lim_{k\to\infty}\int f\hat{G}_A^{1,0}(y)\,\xi_{p_k,n_m}(dy)\\
&= \int f\hat{G}_A^{1,0}(y)\,\xi_F(dy) = \int f(x)\,G^{1,0}\xi_F(x)\,\mu(dx)\,,
\end{aligned}$$

where we used the boundedness and continuity of $f\hat{G}_A^{1,0}$, which follows from the dual facts of lemmas 1.3 and 1.4. Therefore,

$$(5.5)\qquad 1-U^{0,\infty}1(x)+K_A^1U^{0,\infty}1(x)=G^{1,0}\xi_F(x)\,,\qquad \text{for } a.a.x(\mu)\,.$$

Let $\hat{B}$ be the dual CAF of B as in section 3 and let $\hat{F}$ be the cofine support of B. As before, $\hat{F}$ is the cofine support of ν_B. Set $\hat{\tau}=inf\,\{t;\,\hat{B}_t>0\}$, then $\hat{\tau}=\hat{T}_F$ *a.s.* $\hat{P}^x$ for all $x\in E-P_B$, where $\hat{T}_F$ is the hitting time of $\hat{F}$ relative to $\hat{X}$.

Lemma 5.4. *For all* $f\in b\mathcal{E}^*$,

$$(5.6)\qquad \int f(y)\,\xi_F(dy)=\hat{E}^{\nu_A}[f(\hat{X}_{\hat{\tau}})]\,.$$

In particular, ξ_F *is a probability measure on* $\hat{F}$ *which attains no mass on any polar set.*

Proof. It is enough to show the equality (5.6) for $f\in C_c$. If $f\in C_c$, then by the corollary of lemma 3.1,

$$\begin{aligned}\int f(y)\,\xi_F(dy)&=\lim_{m\to\infty}\lim_{k\to\infty}\int f(y)\,p_kU_{n_m}^{0,p_k}1(y)\,\nu_{n_m}(dy)\\&=\lim_{m\to\infty}\lim_{k\to\infty}\int p_kf\hat{V}_{n_m}^{0,p_k}(y)\,\nu_A(dy)\\&=\lim_{m\to\infty}\lim_{k\to\infty}\hat{E}^{\nu_A}[p_k\int_0^\infty e^{-p_k(\hat{B}^{n_m})_t}f(\hat{X}_t)\,d(\hat{B}^{n_m})_t]\\&=\lim_{m\to\infty}\lim_{k\to\infty}\hat{E}^{\nu_A}[\int_0^\infty e^{-u}f(\hat{X}_{\hat{\tau}_{n_m}(u/p_k)})\,du]\,,\end{aligned}$$

where $(\hat{B}^n)_t=\int_0^t I_{F_n\cap F_n}(\hat{X}_u)\,d\hat{B}_u$ is the dual CAF of B^n and $\hat{\tau}_n(s)=inf\,\{u;\,(\hat{B}^n)_u>s\}$. Since, for all $n\geqq 1$, $\hat{\tau}_n(s)\to\hat{\tau}_n\equiv\hat{\tau}_n(0)$ *a.s. as* $s\to 0$,

$$\begin{aligned}\int f(y)\,\xi_F(dy)&=\lim_{m\to\infty}\hat{E}^{\nu_A}[\int_0^\infty e^{-u}f(\hat{X}_{\hat{\tau}_{n_m}})\,du]\\&=\lim_{m\to\infty}\hat{E}^{\nu_A}[f(\hat{X}_{\hat{\tau}_{n_m}})]\,.\end{aligned}$$

Also, since $\hat{\tau}_{n_m}\to\hat{\tau}$ *a.s.* when $m\to\infty$, the lemma follows.

Theorem 5.5 (Equilibrium principle). *Let* $F\in\mathcal{E}^*$ *be a relatively compact subset of* E *and suppose that there exists a CAF* $B\in\Phi^+$ *with fine support* F. *Then there exists a unique probability measure* ξ_F *on* $\hat{F}$ *such that*

$$(5.7)\qquad G\xi_F(x)=a\ constant\ on\ F.$$

Here, $\hat{F}$ *is the cofine support of the dual CAF* $\hat{B}$ *of* B *and the constant is equal to Robin's constant* $R(F)$ *of* F. *The measure* ξ_F *is given by* (5.6) *and called the equilibrium measure of* F.

Proof. Let us show that the measure ξ_F in lemma 5.3 satisfies

(5.8) $\qquad G\xi_F(x)+U^{0,\infty}1(x)=R(F)\qquad$ everywhere on E.

This proves (5.7) since $U^{0,\infty}1(x)=E^x[\int_0^\tau I_C(X_t)\,dt]=0$ on F. From (2.2), (2.4) and lemma 5.1,

$$\begin{aligned}R_n(p) &= K_n(pU_n^{0,p}1)(x)+U_n^{0,p}1(x)\\ &= K_AV_n^{1,0}(pU_n^{1,p}1)(x)+V_n^{1,0}(pU_n^{0,p}1)(x)\\ &\quad -p\nu_nU_n^{0,p}1+U_n^{0,p}1(x)\\ &= K_AV_n^{1,0}(pU_n^{0,p}1)(x)+V_n^{1,0}(pU_n^{0,p}1)(x)\\ &\quad -1+U_n^{0,p}1(x)\,,\end{aligned}$$

since $p\nu_nU_n^{0,p}1=\nu_A(C)=1$. Let $p\to\infty$ and $n\to\infty$, then, as we have seen in lemmas 5.2 and 5.3, $R_n(p)\to R(F)$ and $V_n^{1,0}(pU_n^{0,p}1)(x)\to G^{1,0}\xi_F(x)$ *a.a.* $x(\mu)$, boundedly,. Since $K_A(x,\cdot)$ is a bounded signed measure and which is absolutely continuous relative to μ, we have, for *a.a.*$x(\mu)$,

$$\begin{aligned}(5.9)\qquad R(F) &= K_AG^{1,0}\xi_F(x)+G^{1,0}\xi_F(x)-1+U^{0,\infty}1(x)\\ &= \int\{K_Ag^{1,0}(x,y)+g^{1,0}(x,y)-1\}\,\xi_F(dy)+U^{0,\infty}1(x)\\ &= G\xi_F(x)+U^{0,\infty}1(x)\,,\end{aligned}$$

from lemmas 3.2 and 5.4. Denote

$$\xi(dy)=\xi_F(dy)+\int\sum_{n=1}^{\infty}\{(\hat{K}_A^1)^n-\nu_A\}\,g^{1,0}(dy,z)\,\xi_F(dz)\,,$$

then ξ is a bounded signed measure on $\bar{F}$ and

$$G\xi_F(x)\doteq G^{1,0}\xi(x)-1\,.$$

Since $G^{1,0}\xi_F(x)$ is bounded, $G^{1,0}\xi(x)$ is the difference of two bounded excessive functions relative to $(G_A^{1,p})_{p>0}$. Therefore,

$$\lim_{p\to\infty} pG_A^{1,p}G^{1,0}\xi(x)=G^{1,0}\xi(x)\qquad \text{for all } x\in E\,.$$

Moreover,

$$pG_A^{1,p}1(x)=E^x[\int_0^\infty exp\,(-\int_0^{t/p}I_C(X_s)\,ds-t)\,dt]\to 1\,,$$

as $p\to\infty$ for all $x\in E$ and $G_A^{1,p}(x,\cdot)$ is absolutely continuous relative to μ, for all $x\in E$ and $p>0$. Thus, operating $pG_A^{1,p}$ to both sides of (5.9) and letting $p\to\infty$, we have

$$R(F)=G\xi_F(x)+\lim_{p\to\infty} pG_A^{1,p}U^{0,\infty}1(x)\qquad \text{for all } x\in E\,.$$

Therefore, it is enough to show that

$$(5.10)\qquad \lim_{p} pG_A^{1,p}U^{0,\infty}1(x) = U^{0,\infty}1(x) \qquad \text{for all } x\in E.$$

Let $p>1$ then,

$$\begin{aligned}
&pG_A^{1,p}U^{0,\infty}1(x)\\
&= E^x\Big[\int_0^\infty p\, exp\,\{-\int_0^t I_C(X_s)\,ds-pt\}E^{X_t}\Big[\int_0^\tau I_C(X_u)\,du\Big]\,dt\Big]\\
&\leqq E^x\Big[\int_0^\infty pe^{-pt}\{\int_t^{t+\tau\circ\theta_t} I_C(X_u)\,du\}\,dt\Big]\\
&\leqq E^x\Big[\int_0^\infty e^{-t}\{\int_{t/p}^{(t/p)+\tau\circ\theta_{t/p}} I_C(X_u)\,du\}\,dt\Big]\\
&\leqq E^x\Big[\int_0^\infty e^{-t}\{\int_t^{t+\tau\circ\theta_t} I_C(X_u)\,du\}\,dt\Big]+1\\
&\leqq ||U^{0,\infty}1||+1\,.
\end{aligned}$$

Thus, noting that $\lim_{t\to 0}(t+\tau\circ\theta_t)=\tau$ (see [3; p. 214]), by the Lebesgue theorem,

$$\begin{aligned}
&\lim_{p\to\infty} pG_A^{1,p}\,U^{0,\infty}1(x)\\
&= \lim_{p\to\infty} E^x\Big[\int_0^\infty exp\,\{-\int_0^{t/p} I_C(X_s)\,ds-t\}\,dt\int_{t/p}^{(t/p)+\tau\circ\theta_{t/p}} I_C(X_u)\,du\Big]\\
&= \int_0^\infty e^{-t}E^x\Big[\int_0^\tau I_C(X_u)\,du\Big]\,dt = U^{0,\infty}1(x)\,.
\end{aligned}$$

Now, it remains only the proof of the uniqueness. Let ξ be a bounded signed measure on $\hat{F}$ satisfying $\xi(E)=0$ and $G\xi(x)=a$, for some constant a, on F. For the proof of uniqueness we claim that $\xi=0$. Integrating both sides of $G\xi(x)=a$ $(x\in F)$ by $f(x)\nu_n(dx)$, we have

$$(5.11)\qquad \int f\hat{K}_n(y)\,\xi(dy) = a\langle \nu_n, f\rangle$$

for all $f\in b\mathcal{E}^*$ and $n\geqq 1$, where $\hat{K}_n(dx, y)=g(x, y)\nu_n(dx)$ as before. Set $f=g(I-p\hat{K}^p_{B^n})$ for a bounded continuous function g. It follows, from the dual result of (2.8), that

$$f\hat{K}_n(y) = g(I-p\hat{K}^p_{B^n})\hat{K}_n(y) = g\hat{K}^p_{B^n}(y)-\langle g\hat{K}^p_{B^n}, \nu_A\rangle\,,$$

for all $y\notin P_{B^n}$, $n\geqq 1$ and $p>0$. Substituting this function into (5.11), we have

$$\int g\hat{K}^p_{B^n}(y)\,\xi(dy) = 0 \qquad \text{for all } n\geqq 1 \text{ and } p>0\,,$$

because $\xi(E)=0$, $\langle g(I-p\hat{K}^p_{B^n}), \nu_n\rangle=0$ and ξ vanishes outside of $\hat{F}\subset E-P_B\subset E-P_{B^n}$. Therefore, similarly to the proof of lemma 5.4, we have

$$\int \hat{E}^y[g(\hat{X}_{\hat{\tau}})]\,\xi(dy) = 0\,.$$

This implies that $\int g(y)\,\xi(dy)=0$, since $\hat{P}^y[\hat{\tau}=0]=1$ for all $y\in\hat{P}$.

6. Symmetric case

In this section we shall assume, in addition, that $g^p(x, y)=g^p(y, x)$ for all $p>0$ and $x, y\in E$, that is, X and $\hat{X}$ are equivalent. In this case, as is well known (see [3; proposition VI. 4. 10]), any semipolar set is polar. Hence, for every compact set F, the set $F-F^r$ is polar, where F^r is the set of all regular points of F (see [3; II. 3.3]). Therefore F is a projective set (see [3; V. 4.5]). Hence, by considering the projection of CAF $\{t\}$, there exists a CAF B such that

$$E^x[e^{-T_F}] = E^x[\int_0^\infty e^{-t}dB_t] \tag{6.1}$$

and $supp\,(B)=F^r$ ([3; V. 4.6 and 4.7]), where $supp\,(B)$ is the fine support of B. Obviously, F is a polar set if and only if the corresponding CAF B is zero. Let $T=inf\,\{t\ ;\ B_t=\infty\}$. We have

$$1 \geqq E^x[e^{-T_F}] \geqq E^x[\int_0^T e^{-t}\,dB_t] \geqq E^x[e^{-T}B_T]\,.$$

This implies that $T=\infty$ *a.s.* P^x for all $x\in E$, that is, $B\in\Phi$. Let $\hat{B}$ be the dual CAF of B then, under our present hypothesis, the corresponding polar set P_B may be supposed empty (see the proof of [14 ; VII. 1]) and the cofine support $\hat{F}$ of $\hat{B}$ is equal to ${}^rF=F^r$, since the fine and cofine topologies conicide, where ${}^rF$ is the set of all coregular points of F. Therefore, by theorem 5.5 we have

Theorem 6.1. *If F is a non-polar compact subset of E, then there exists a unique probability measure ξ_F on F^r such that*

$$G\xi_F(x) = R(F) \quad on\ F^r\,. \tag{6.2}$$

Here, the measure ξ_F and the constant $R(F)$ are given by

$$\xi_F(dy) = \hat{P}^{\nu_A}[\hat{X}_{\hat{T}_F}\in dy] \quad and \tag{6.3}$$

$$R(F) = E^{\nu_A}[\int_0^{T_F} I_C(X_t)\,dt]\,, \tag{6.4}$$

respectively. The measure ξ_F is called the equilibrium measure of F (relative to the potential kernel function $g(x, y)$).

Corollary. *Under the hypothesis of theorem* 6.1 *there exists a unique probability measure ξ_F on F such that $G\xi_F$ is bounded on F and satisfies* (6.2).

Proof. It is enough to prove the uniqueness. Suppose that a measure ξ on F satisfies the conditions of the corollary. Then since $G\xi$ is bounded on F, ξ charges no polar set (see the proofs of theorem 3.4 and corollary 2 of theorem 4.1). Hence ξ is a measure on F^r, so that the corollary follows from theorem 6.1.

REMARK. By the proof of the corollary, the result of the corollary may be replaced by the following result. "There exists a unique probability measure on F which attains no mass on any polar set and satisfies (6.2)".

By using the relation (4.1) of $g(x, y)$ and an arbitrary potential kernel function $h(x, y)$, we would like to investigate the equilibrium principle relative to $h(x, y)$. At present, however, we can get only a partial result on this problem; we have to impose very strong conditions on $h(x, y)$ and we do not know even if the logarithmic potential kernel function of planar Brownian motion satisfies these conditions. Our conditions are the following.

(H1) For every compact set D,

$$\lim_{p\to 0} \sup_{x,y\in D} |g^p(x, y)-\phi(p)-h(x, y)| = 0 \tag{6.5}$$

for some function ϕ and a potential kernel function h.

(H2) For all $p>0$ and bounded continuous function f, $K_B^p f$ is continuous, where B is a CAF with fine support F^r, as before.

To find the equilibrium measure ξ relative to h, we shall attempt a formal calculation. Suppose that a probability measure ξ on F satisfies $H\xi(x)=\int h(x, y)\xi(dy)=a$ *on* F^r for some constant a. Then, from (4.1), for all $x\in F^r$

$$\begin{aligned} G\xi(x) &= H\xi(x)-H(x, C)-\int \hat{H}(C, y)\xi(dy)+H(C, C) \\ &= -H(x, C)+a-\int \hat{H}(C, y)\xi(dy)+H(C, C)\,. \end{aligned}$$

Operating $I-pK_B^p$ and integrating by $f d\nu_B$, it follows that

$$\langle f, (I-pK_B^p)G\xi\rangle_{\nu_B} = -\langle f, (I-pK_B^p)H(\cdot, C)\rangle_{\nu_B}\,. \tag{6.6}$$

The left side of (6.6) becomes

$$\begin{aligned} \langle f, (I-pK_B^p)G\xi\rangle_{\nu_B} &= \langle f(I-p\hat{K}_B^p)\hat{K}_B, \xi\rangle \\ &= \langle f\hat{K}_B^p, \xi\rangle-\langle f\hat{K}_B^p, \nu_A\rangle \end{aligned}$$

from the dual formula of (2.8).

On the other hand, the right side of (6.6) becomes

$$-\langle f, (I-pK_B^p)H(\cdot, C)\rangle_{\nu_B} = -\lim_{q\to 0}\langle f, (I-pK_B^p)(G^q(\cdot, C)-\phi(q))\rangle_{\nu_B}$$
$$= -\lim_{q\to 0}\langle f, (I-pK_B^p)G^q(\cdot, C)\rangle_{\nu_B} = \lim_{q\to 0}\langle f, -G_B^{p,0}(\cdot, C)+qG_B^{p,0}G^q(\cdot, C)\rangle_{\nu_B}$$
$$= -\langle f\hat{K}_B^p, \nu_A\rangle+\lim_{q\to 0}\langle f, qG_B^{p,0}G^q(\cdot, C)\rangle_{\nu_B}.$$

Hence (6.6) becomes

$$\langle f\hat{K}_B^p, \xi\rangle = \lim_{q\to 0}\langle f, qG_B^{p,0}G^q(\cdot, C)\rangle_{\nu_B}.$$

Multiplying p and letting $p\to\infty$ we have

$$\text{(6.7)}\qquad \langle f, \xi\rangle = \lim_{p\to\infty}\lim_{q\to 0} pq\langle f, G_B^{p,0}G^q(\cdot, C)\rangle_{\nu_B}.$$

Theorem 6.2. *Let F be a non-polar compact subset of E. Under the hypothesis* (H1) *and* (H2), *there exists a unique probability measure ξ on F^r such that*

$$\text{(6.8)}\qquad H\xi = \text{a constant on } F^r.$$

Proof. Since

$$\langle 1, pqG_B^{p,0}G^q(\cdot, C)\rangle_{\nu_B} = pq\langle 1\hat{K}_B^p\hat{G}^q, \nu_A\rangle = 1,$$

the measure $pqG_B^{p,0}G^q(x, C)\nu_B(dx)$ is a probability measure on F for any $p, q>0$. Hence, for all $p>0$, we can choose a sequence $q_n\to 0$ and a probability measure ξ_p on F such that $pq_nG_B^{p,0}G^{q_n}(x, C)\nu_B(dx)\to\xi_p(dx)$, weakly. Similarly, there exists a sequence $p_m\to\infty$ and a probability measure ξ on F such that $\xi_{p_m}\to\xi$, weakly. From the hypothesis (H2), for all bounded continuous function f,

$$\langle p_kf\hat{K}_B^{p_k},\xi\rangle = \lim_m\lim_n\langle p_kf\hat{K}_B^{p_k}, p_mq_nG_B^{p_m,0}G^{q_n}(\cdot, C)\rangle_{\nu_B}$$
$$= \lim_m\lim_n\langle p_kp_mq_nf\hat{K}_B^{p_k}\hat{K}_B^{p_m}\hat{G}^{q_n}, \nu_A\rangle$$
$$= \lim_m\lim_n\frac{1}{p_m-p_k}\langle p_kp_mq_nf(\hat{K}_B^{p_k}-\hat{K}_B^{p_m})\hat{G}^{q_n},\nu_A\rangle$$
$$= \lim_m\lim_n\frac{1}{p_m-p_k}p_kp_mq_n\langle f\hat{K}_B^{p_k}\hat{G}^{q_n}, \nu_A\rangle$$
$$= \lim_n p_kq_n\langle f\hat{K}_B^{p_k}\hat{G}^{q_n}, \nu_A\rangle = \langle f, \xi_k\rangle.$$

Hence, letting $k\to\infty$, it follows that

$$\lim_k\langle p_kf\hat{K}_B^{p_k}, \xi\rangle = \langle f, \xi\rangle,$$

that is, $\xi=\hat{E}^\xi[f(\hat{X}_{\hat{T}})]=E^\xi[f(X_{T_F})]$. So that ξ is a measure on F^r.

To prove (6.8), let f be a bounded continous function with compact support. By restricting the CAT B as in section 5, we may suppose that $V_{A,B}^{1,0}1$ is bounded. Then, since $f\hat{G}$ is bounded and continuous from (2.7), we have

$$\begin{aligned}\langle f, G\xi\rangle_\mu &= \langle f\hat{G}, \xi\rangle = \lim_m \lim_n \langle f\hat{G}, p_m q_n G_B^{p_m,0} G^{q_n}(\cdot, C)\rangle_{\nu_B} \\ &= \lim_m \lim_n p_m \langle f\hat{G}, G_B^{p_m,0}(\cdot, C) - G^{q_n}(\cdot, C) + p_m K_B^{p_m} G^{q_n}(\cdot, C)\rangle_{\nu_B} \\ &= \lim_m \lim_n p_m \langle f, K_B G_B^{p_m,0}(\cdot, C) - K_B(I - p_m K_B^{p_m}) G^{q_n}(\cdot, C)\rangle_\mu \\ &= \lim_m \lim_n p_m \langle f, K_B G_B^{p_m,0}(\cdot, C) - K_B^{p_m} G^{q_n}(\cdot, C) + \nu_A K_B^{p_m} G^{q_n}(C)\rangle_\mu\end{aligned}$$

from (2.8). By the definition of K_B and H,

$$\begin{aligned}\lim_m p_m K_B G_B^{p_m,0}(x, C) &= \lim_m \lim_{q\to 0} p_m \left\{V_{A,B}^{q,0} - \frac{\nu_B}{q}\right\} U_{A,B}^{0,p_m} 1(x) \\ &= \lim_m \lim_{q\to 0} \left\{p_m V_{A,B}^{q,0} U_{A,B}^{0,p_m} 1(x) - \frac{1}{q}\right\} \\ &= \lim_m \lim_{q\to 0} \left\{K_A^q 1 - U_{A,B}^{0,p_m} 1 + q K_A^q U_{A,B}^{0,p_m} 1 - \frac{1}{q}\right\}(x) \\ &= \lim_m \{-U_{A,B}^{0,p_m} 1 + \nu_A U_{A,B}^{0,p_m} 1\}(x) \\ &= -E^x\left[\int_0^{T_F} I_C(X_s)\, ds\right] + E^{\nu_A}\left[\int_0^{T_F} I_C(X_s)\, ds\right],\end{aligned}$$

$$\begin{aligned}&\lim_m \lim_n p_m \{K_B^{p_m} G^{q_n}(x, C) - \nu_A K_B^{p_m} G^{q_n}(C)\} \\ &\quad = \lim_m \lim_n p_m [K_B^{p_m}\{G^{q_n}(x, C) - \phi(q_n)\} - \nu_A K_B^{p_m}\{G^{q_n}(\cdot, C) - \phi(q_n)\}] \\ &\quad = \lim_m p_m \{K_B^{p_m} H(x, C) - \nu_A K_B^{p_m} H(C)\} \\ &\quad = E^x[H(X_{T_F}, C)] - E^{\nu_A}[H(X_{T_F}, C)].\end{aligned}$$

Hence

$$\begin{aligned}\langle f, G\xi\rangle_\mu = \langle f, -E^\cdot\left[\int_0^{T_F} I_C(X_s)\, ds\right] - E^\cdot[E(X_{T_F}, C]\rangle_\mu \\ + \langle f, 1\rangle_\mu E^{\nu_A}\left[\int_0^{T_F} I_C(X_s)\, ds + H(X_{T_B}, C)\right].\end{aligned}$$

Therefore

$$G\xi(x) = -E^x\left[\int_0^{T_F} I_C(X_s)\, ds + H(X_{T_F}, C)\right] + E^{\nu_A}\left[\int_0^{T_F} I_C(X_s)\, ds + H(X_{T_F}, C)\right]$$

for *a.a.x* (μ). In particular,

$$G\xi(x) = -H(x, C) + E^{\nu_A}\left[\int_0^{T_F} I_C(X_s)\, ds + H(X_{T_F}, C)\right]$$

for *a.a.x*$\in F^r$ (μ), and hence for all $x \in F^r$. Hence, by (4.1), (6.8) holds. If ξ_1 and ξ_2 are measures on F^r satisfying (6.8), then $G(\xi_1 - \xi_2)$ equals to a constant on F^r. Hence $\xi_1 = \xi_2$ by the proof of theorem 5.5.

In the classical case, the equilibrium measure is characterized as the measure which minimize the energy. In our case, the analogous result holds. Denote $\mathfrak{M}$ the family of all bounded signed measures ξ on E with compact support such that $\int |g(x, y)| |\xi|(dy)$ is bounded, $\mathcal{M}^+ = \{\xi \geqq 0; \xi \in \mathcal{M}\}$ and $\mathcal{M}^0 = \{\xi \in \mathcal{M}; \xi(E) = 0\}$. For $\xi, \zeta \in \mathcal{M}$, define the mutual energy of ξ and ζ by

$$(6.9) \qquad (\xi, \zeta) = \iint g(x, y) \xi(dx) \zeta(dy) .$$

Denote (ξ, ξ) by $I(\xi)$ and call it the energy of ξ.

Lemma 6.3. *If $\xi \in \mathcal{M}^0$, then $I(\xi)$ is non-negative. Moreover, $I(\xi) = 0$ if and only if $\xi = 0$.*

Proof. Suppose that $\xi \in \mathcal{M}^0$. Since $G^{1,0}|\xi|(x)$ is bounded,

$$\int \sum_{n=1}^{\infty} |(K_A^1)^n - \nu_A|(x, dz) G^{1,0}|\xi|(z)$$
$$\leqq \|G^{1,0}|\xi|\| \sum_{n=1}^{\infty} \|(K_A^1)(x, \cdot) - \nu_A\|$$

converges uniformly in x. Hence for any $\varepsilon > 0$ there exists a number N such that

$$\left| \sum_{n=N}^{\infty} \int \{(K_A^1)^n - \nu_A\} (x, dz) G^{1,0}\xi(z) \right| < \varepsilon$$

for all $x \in E$. From our definition of $g(x, y)$, for $(x, y) \in \Gamma$,

$$g(x, y) = g^{1,0}(x, y) - 1 + \sum_{n=1}^{N-1} \{(K_A^1)^n - \nu_A\} g^{1,0}(x, y)$$
$$+ \varepsilon(x, y, N) ,$$

where $\varepsilon(x, y, N) = \sum_{n=N}^{\infty} \{(K_A^1)^n - \nu_A\} g^{1,0}(x, y)$. Since $\xi(\hat{\Gamma}_x) = 0$, for all $x \in E$,

$$I(\xi) = \iint g^{1,0}(x, y) \xi(dx) \xi(dy) + \sum_{n=1}^{N-1} \iint (K_A^1)^n g^{1,0}(x, y) \xi(dx)(dy)$$
$$+ \iint \varepsilon(x, y, N) \xi(dx) \xi(dy) .$$

From the resolvent equation (1.7), we have

$$g^{2,0}(x, y) - g^{1,0}(x, y) + K_A^2 g^{1,0}(x, y) = 0 .$$

This combined with $g^{1,0}(x, y) \geqq g^{2,0}(x, y)$, we have

$$K_A^2 g^{2,0}(x, y) \leqq K_A^2 g^{1,0}(x, y) \leqq g^{1,0}(x, y) ,$$

so that

$$\int g^{2,0}(x, z) g^{2,0}(z, y) \nu_A(dz) \leqq g^{1,0}(x, y) .$$

Hence we have, from the symmetry of $g^2(x, y)$

$$\iint g^{1,0}(x, y) \xi(dx) \xi(dy) \geqq \int \{ \int g^{2,0}(x, y) \xi(dy) \}^2 \nu_A(dx) \geqq 0 .$$

Similarly it follows that

$$\sum_{n=1}^{N-1}\iint (K_A^1)^n g^{1,0}(x,y)\xi(dx)\xi(dy)\geqq 0\,.$$

Therefore $I(\xi)\geqq -\varepsilon$ and hence $I(\xi)\geqq 0$.

Suppose that $I(\xi)=0$. By a routine argument, we have $|(\xi,\zeta)|^2\leqq I(\xi)I(\zeta)$ for all $\zeta\in\mathcal{M}^0$. Hence $(\xi,\zeta)=0$ for all $\zeta\in\mathcal{M}^0$. This implies that $G\xi$ equlas to a constant on E. Integrating by ν_A, we can see that the constant is equals to 0. Hence $\xi=0$.

Theorem 6.4. *The equilibrium measure ξ_F of a compact set F is the unique measure which attains the*

(6.10) $\quad$ *min* $\{I(\xi);\ \xi\in\mathcal{M}^+,\ \xi(E)=1,$ *support of* $\xi\subseteqq F\}$,

and Robin's *constant $R(F)$ equals the minimum value of* (6.10).

Proof. The proof is similar to the classical case [16]. If a measure ξ satisfies the conditions of (6.10), then, since $G\xi_F=R(F)$ on F except a polar subset of F and ξ charges no polar set,

$$\begin{aligned} I(\xi) &= I(\xi-\xi_F)-I(\xi_F)+2(\xi,\xi_F)\\ &= I(\xi-\xi_F)+R(F)\,. \end{aligned}$$

Since $\xi-\xi_F\in\mathcal{M}^0$, this implies the result by lemma 6.2.

KUMAMOTO UNIVERSITY

References

[1] J. Azema, M. Kaplan Duflo and D. Revuz: *Mesure invariante sur les classes recurrentes des processus de Markov,* Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **8** (1967), 157–181.

[2] ————: *Note sur la mesure invariante des processus de markov recurrents,* Ann. Inst. Henri Poincare Sect. B, **3** (1967), 397–402.

[3] R.M. Blumenthal and R.K. Getoor: Markov processes and potential theory, Academic Press, New York, 1968.

[4] M. Brancovan: *Fonctionneless additives speciales d'un processus de Harris,* C.R. Acad Sc. Paris Serie A, **283** (1976), 57–59.

[5] R.K. Getoor: *Multiplicative functional of dual processes,* Ann. Inst. Fourier, Grenoble **21** (1971), 43–83.

[6] R. Kondo: *On weak potential operators for recurenet markov processes,* J. Math. Kyoto Univ. **11** (1971), 11–44.

[7] R. Kondo and Y. Oshima: *A characterization of weak potential kernels for strong Feller recurrent Markov chains,* Proc. of 2nd Japan-USSR symp. on prob. theory, Lecture Notes in Math., 330, Springer, Berlin, (1973).

[8] H. Kunita and T. Watanabe: *Markov processes and Martin boundaries part* 1, Illinois J. Math. **9** (1965), 485–526.

[9] P.A. Meyer: *Les resolvantes fortement Felleriennes d'après Mokobodzki,* Sem. de probabilités II, Lecture Notes in Math. **51**, Springer, Berlin, (1968), 171–174.

[10] M. Nagasawa and K. Sato: *Some theorems on time change and killing of Markov processes,* Kodai Math. Sem. Rep. **15** (1963), 195–219.

[11] J. Neveu: *Potentiel Markovien recurrent des chaines de Harris,* Ann. Inst. Fourier, Grenoble **22** (1972), 85–130.

[12] Y. Oshima: *On a construction of a recurrent potential kernel by means of time change and killing,* J. Math. Soc. Japan. **29** (1977), 151–159.

[13] S.C. Port and C.J. Stone: *Logarithmic potentials and planar Brownian motion,* Proc. 6th Berkeley Symposium of probability and Math. Stat., U. of California Press, III (1972), 177–192.

[14] D. Revuz: *Mesures associees aux fonctionelles additives de Markov,* Trans. Amer. Math. Soc. **148** (1970), 501–531.

[15] K. Takasu: *On the eigenvalues of recurrent potential kernels,* Hiroshima Math. J. **2** (1972), 19–31.

[16] J. Takeuchi, S. Watanabe and T. Yamada: *Stable processes,* Seminar on prob. **13** (1962), (Japanese).