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EXTENSION OF VALUATIONS ON DISTRIBUTIVE LATTICES

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0. Introduction

The process of extending a measure on a ring to a measure on the generated σ-ring has been discussed by many authors. The primitive extension theorem which is stated in terms of a non-negative real-valued set function can be generalized in two directions. One generalization is concerned with the range space of the measure. Extension theorems in this direction are given in [9], [3], [11], [16], [8] (the range space is a Banach space), [7] (a locally convex space), [22], [21], [18] (an abelian topological group). Some authors ([9], [3], [16], [7]), in the vector-valued cases, are based on the extendability principle of uniformly continuous maps, while the Carathéodory's method is adopted by others ([11], [22], [21], [18]) and Zorn's lemma in [8]. Among them, M. Sion [21] proved that the monotone-convergence condition (Fox's condition) was necessary and sufficient for a group valued measure on a field to be extended to a measure on the generated σ-field. The other type of generalization is the abstraction of the domain of the measure. In this direction, discussions on real-valued modulr functions (valuations) on certain types of lattices are seen in [4] Ch. XI, [1], [12], [20]. Here we note that extension theorems of Daniell integrals of real-valued functions taking values in a Banach space [15] and in a topological group [19] have been obtained. In reference to integrals, group valued or some abstract valued integration theories rae seen in [24], [25], [26], [2].

Under these circumstances, the main purpose of this paper is to establish a general process of extending a group valued *valuation* [10] on a sublattice of a distributive lattice to a valuation on the generated δ-sublattice (Theorem 1). If the lattice is relatively complemented, then the process yields a generalization of the extension theorems in [22] and [21]. In case the lattice is an l-group (lattice-ordered group [4]) an extension theorem of a group valued Daniell integral of l-group valued functions is obtained (the valuation is defined on a subgroup of the l-group of all functions taking values in an l-group). To accomplish the process we introduce the notion of a *relative inverse* of an element of the lattice by some axioms which unify the relatively complemented lattice theory and the l-group theory. The relative inverse is interpreted as the relative complement in the former case and is defined in terms of the difference operation of the group in the latter case. Here the extension process is based on the Carathéodory's method and the monotone-convergence condition considered in [8] and [21] plays an essential role. Finally, the maximal extension theorem in [23] is generalized in Theorem 2 and a general completion theory is given in section 3. The notion of completion is generalized in some extent from that in [13] and [20].

1. δ-lattices and the first extension theorem

A lattice M will be called a δ^+-*lattice* if, for any $x_i \in M$, $i=1, 2, \cdots$, with upper bounds,

1) there exists $\bigcup_{i=1}^{\infty} x_i = \sup\{x_i \mid i \in \boldsymbol{N}\}$,

2) $x \cap (\bigcup_{i=1}^{\infty} x_i) = \bigcup_{i=1}^{\infty} (x \cap x_i)$ for any $x \in M$.

Dually, a lattice M is a δ^--*lattice* if, for any $x_i \in M$, $i=1, 2, \cdots$, with lower bounds,

1*) there exists $\bigcap_{i=1}^{\infty} x_i = \inf\{x_i \mid i \in \boldsymbol{N}\}$,

2*) $x \cup (\bigcap_{i=1}^{\infty} x_i) = \bigcap_{i=1}^{\infty} (x \cup x_i)$ for any $x \in M$.

If a δ^+-lattice M is at the same time a δ^--lattice, then we say that M is a δ-*lattice*. A δ-lattice is necessarily a distributive lattice.

A sublattice R of a δ^+-lattice M is called a δ^+[respectively, σ^+]-*sublattice* if $\bigcup_{i=1}^{\infty} x_i \in R$ for any $x_i \in R$, $i \in \boldsymbol{N}$, with upper bounds in R [respectively, M]. Dually, a sublattice R of a δ^--lattice M is a $\delta^-[\sigma^-]$-*sublattice* if $\bigcap_{i=1}^{\infty} x_i \in R$ for any $x_i \in R$, $i \in \boldsymbol{N}$, with lower bounds in $R[M]$. If a $\delta^+[\sigma^+]$-sublattice R of a δ-lattice M is also a $\delta^-[\sigma^-]$-sublattice, then R is called a $\delta[\sigma]$-*sublattice* of M. Obviously, any $\sigma[\sigma^+, \sigma^-]$-sublattice of a $\delta[\delta^+, \delta^-]$-lattice M is a $\delta[\delta^+, \delta^-]$-sublattice.

Suppose that M is a $\delta[\delta^+, \delta^-]$-lattice and R is a subset of M. Then any intersection of $\delta[\delta^+, \delta^-]$-sublattices of M containing R is a $\delta[\delta^+, \delta^-]$-sublattice of M containing R. Hence there exists the smallest $\delta[\delta^+, \delta^-]$-sublattice of M containing R. This sublattice is called the $\delta[\delta^+, \delta^-]$-sublattice of M *generated* by R and is denoted by $R^{\delta}[R^{\delta^+}, R^{\delta^-}]$. Similarly there exists the smallest $\sigma[\sigma^+, \sigma^-]$-sublattice of M containing R, which is called the $\sigma[\sigma^+, \sigma^-]$-sublattice of M *generated* by R and is denoted by $R^{\sigma}[R^{\sigma^+}, R^{\sigma^-}]$.

Throughout this paper, we assume the following:

Assumption I. *M is a lattice and G is a topological additive (abelian) group.*

Let R be a subset of M and μ a map of R into G.

Then we shall say that μ is $\delta^+[\delta^-]$-*convergent* if $\mu(x_i) \to \mu(x)$ $(i \to \infty)$ for any

$x_i \in R$, $i \in \boldsymbol{N}$, and for any $x \in R$ such that $x_i \uparrow x$ $(i \to \infty)$[1] $[x_i \downarrow x$ $(i \to \infty)]$. μ is *δ-convergent* if μ is δ^+-convergent and δ^--convergent.

The map μ is called to be $\delta^+[\delta^-]$-*fundamental* if the sequence $\mu(x_i)$, $i \in \boldsymbol{N}$, is fundamental (Cauchy) for any increasing [decreasing] sequence $x_i \in R$, $i \in \boldsymbol{N}$, with upper [lower] bounds in R. μ is *δ-fundamental* if μ is δ^+-fundamental and δ^--fundamental.

Assume that R is a sublattice of M. Then the map μ is called a *valuation* on R if

$$\mu(x)+\mu(y) = \mu(x \cup y)+\mu(x \cap y)$$

for any $x, y \in R$.

EXAMPLE 1.1. Let $\boldsymbol{m}$ be a set and M the set of all subsets of $\boldsymbol{m}$. Then M is a complemented δ-lattice when we define the relation $x \leqq y$ by $x \subset y$ for each $x, y \in M$. A ring of subsets of $\boldsymbol{m}$ is defined to be a relatively complemented sublattice of M containing the smallest element ϕ of M. Let R be such a ring and μ a map of R into G. Then the following conditions are equivalent:

1) $\mu(x \cup y) = \mu(x)+\mu(y)$ for any $x, y \in R$ such that $x \cap y = \phi$.

2) μ is a valuation such that $\mu(\phi)=0$.

If these conditions are satisfied and if G is separated (Hausdorff), then the δ-convergence of μ is equivalent to the countable additivity of μ.

EXAMPLE 1.2. In the above example, suppose that $\boldsymbol{m}$ is an infinite set and that R is the ring of all subsets x of $\boldsymbol{m}$ such that the set x or the complement x^c of x is finite. Let us assume that G is the topological group $\boldsymbol{R}$ of all real numbers. Then

1) Let us put $\mu(x) = \mathrm{Card}\,(x)$ (the number of elements in x) and $\mu(x^c) = -\mathrm{Card}\,(x)$ for each finite subset $x \subset \boldsymbol{m}$. If the set $\boldsymbol{m}$ is uncountable, then μ is a δ-convergent valuation on R such that $\mu(\phi)=0$. But μ is not δ-fundamental.

2) Let us put $\mu(x)=0$ and $\mu(x^c)=1$ for each finite subset $x \subset \boldsymbol{m}$. Then μ is a δ-fundamental valuation on R such that $\mu(\phi)=0$. But μ is not δ-convergent if the set $\boldsymbol{m}$ is countable.

EXAMPLE 1.3. Let E be a set and K the set of all real numbers (or generally a complete totally ordered additive group or a complete l-group). Then the set M of all K-valued functions defined on E is a δ-lattice under the usual order relation. Further M is naturally considered to be an l-group and any group homomorphism μ of an l-subgroup R of M into G is a valuation such that $\mu(0)=0$.

It is easy to see the following

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1) This means that $x_i \leqq x_{i+1}$ for any $i \in \boldsymbol{N}$ and x is the supremum $\bigcup_{i=1}^{\infty} x_i$ of the subset $\{x_i \mid i \in \boldsymbol{N}\}$ of M.

Proposition 1.1. *Let R be a sublattice of M and μ a map of R into G. Assume that M is a $\delta[\delta^+, \delta^-]$-lattice and consider the conditions:*

1) *R is a $\delta[\delta^+, \delta^-]$-sublattice and μ is $\delta[\delta^+, \delta^-]$-convergent.*

2) *There exists a $\delta[\delta^+, \delta^-]$-sublattice $\tilde{R}$ of M containing R and μ is extended to a $\delta[\delta^+, \delta^-]$-convergent map $\tilde{\mu}$ of $\tilde{R}$ into G.*

3) *μ is $\delta[\delta^+, \delta^-]$-fundamental.*

Then it holds that 1)$\Rightarrow$2)$\Rightarrow$3).

The following theorem asserts under some assumptions that the condition 3) in the above proposition implies the condition 2).

Theorem 1. *Suppose that M is a δ-lattice and that G is a separated and complete topological additive group. Let R be a sublattice of M and μ a G-valued δ-convergent and δ-fundamental valuation on R. Then each of the following two conditions is sufficient for μ to be extended to a G-valued δ-convergent valuation μ^δ on R^δ:*

1) *M is a relatively complemented lattice and R is a relatively complemented sublattice of M.*

2) *M is an l-group, R is a subgroup of M, and μ is a group homomorphism.*

If μ^δ exists, then it is unique. Moreover, R^δ is relatively complemented in case 1), *and R^δ is a subgroup and μ^δ is a group homomorphism in case* 2).

A proof of the theorem will be stated in section 9. In this section we shall give somemore notations and some lemmas.

We denote by $\sum[\sum^*]$ the set of all maps ξ of $\boldsymbol{N}$ into M satisfying the condition: there exists an $x \in M$ such that $\xi(i) \leqq \xi(i+1) \leqq x[x \leqq \xi(i+1) \leqq \xi(i)]$ for any $i \in \boldsymbol{N}$. For each $x \in M$ and each $\xi, \eta \in \sum[\sum^*]$ we define maps $x \cup \xi$, $x \cap \xi$, $\xi \cup \eta$, and $\xi \cap \eta$ of $\boldsymbol{N}$ into M by $(x \cup \xi)(i) = x \cup \xi(i)$, $(x \cap \xi)(i) = x \cap \xi(i)$, $(\xi \cup \eta)(i) = \xi(i) \cup \eta(i)$, and $(\xi \cap \eta)(i) = \xi(i) \cap \eta(i)$, respectively, for any $i \in \boldsymbol{N}$.

For a subset A of M, we denote by $\sum(A)[\sum^*(A)]$ the set of all $\xi \in \sum[\sum^*]$ such that $\xi(i) \in A$ for any $i \in \boldsymbol{N}$, and by $\sum_0(A)$ $[\sum_0^*(A)]$ the set of all $\xi \in \sum(A)$ $[\sum^*(A)]$ such that there exists an $x \in A$ with $\xi(i) \leqq x[x \leqq \xi(i)]$ for any $i \in \boldsymbol{N}$.

Then immediately we have

Lemma 1.1. *If $x \in M$ and if $\xi, \eta \in \sum[\sum^*]$, then $x \cup \xi$, $x \cap \xi$, $\xi \cup \eta$, and $\xi \cap \eta$ are elements of $\sum[\sum^*]$.*

Corollary. *If R is a sublattice of M and if $\sum'$ denotes one of the sets $\sum(R)$, $\sum^*(R)$, $\sum_0(R)$, and $\sum_0^*(R)$, then $x \cup \xi$, $x \cap \xi$, $\xi \cup \eta$, and $\xi \cap \eta$ are elements of $\sum'$ for any $x \in R$ and any $\xi, \eta \in \sum'$.*

Under the assumption that M is a $\delta^+[\delta^-]$-lattice, we shall write $\bar{\xi} = \bigcup_{i=1}^{\infty} \xi(i)$ $[\bar{\xi} = \bigcap_{i=1}^{\infty} \xi(i)]$ for each $\xi \in \sum[\sum^*]$ and $\bar{\Theta} = \{\bar{\xi} \mid \xi \in \Theta\}$ for each $\Theta \subset \sum[\sum^*]$.

Then the definition of a $\delta^+[\delta^-]$-lattice implies

Lemma 1.2. *If M is a $\delta^+[\delta^-]$-lattice, then* $\overline{x\cup\xi}=x\cup\bar{\xi}$, $\overline{x\cap\xi}=x\cap\bar{\xi}$, $\overline{\xi\cup\eta}=\bar{\xi}\cup\bar{\eta}$, *and* $\overline{\xi\cap\eta}=\bar{\xi}\cap\bar{\eta}$ *for any* $x\in M$ *and* $\xi,\eta\in\sum[\sum^*]$.

In this section we assume the following

Assumption 1.1. *M is a δ^+-lattice and R is a sublattice of M.*

Then we have

Lemma 1.3. *Let α be an element of $\sum(\overline{\sum(R)})$. Then there exists a $\xi\in\sum(R)$ such that*

1) $\xi(i)\leqq\alpha(i)$ *for any* $i\in \boldsymbol{N}$,
2) $\bar{\xi}=\bar{\alpha}$.

If, for each $i\in\boldsymbol{N}$, $x_i\leqq\alpha(i)$ for some $x_i\in R$, then the condition 1) *can be replaced by*

1′) $x_i\leqq\xi(i)\leqq\alpha(i)$ *for any* $i\in\boldsymbol{N}$.

Proof. Since $\alpha(i)\in\overline{\sum(R)}$, for each $i\in\boldsymbol{N}$, there exists a $\xi_i\in\sum(R)$ such that $\alpha(i)=\bar{\xi}_i$. In case $x_i\leqq\alpha(i)$ for some $x_i\in R$, we can consider $x_i\cup\xi_i$ in place of ξ_i and hence it may be assumed that $\xi_i(j)\geqq x_i$ for any $j\in\boldsymbol{N}$. Putting $\xi(i)=\bigcup_{k=1}^{i}\xi_k(i)$ for each $i\in\boldsymbol{N}$, we have a map ξ of $\boldsymbol{N}$ into R such that $\xi(i)\leqq\bigcup_{k=1}^{i}\xi_k(i+1)\leqq\xi(i+1)$ and $\xi(i)\leqq\bigcup_{k=1}^{i}\bar{\xi}_k=\bigcup_{k=1}^{i}\alpha(k)=\alpha(i)\leqq\bar{\alpha}$. Hence ξ is an element of $\sum(R)$ satisfying the condition 1) (and 1′) under the additional assumption) and the condition $\bar{\xi}\leqq\bar{\alpha}$. Let us show that $\bar{\alpha}\leqq\bar{\xi}$. For each $i,j\in\boldsymbol{N}$, putting $l=\max\{i,j\}$ we have $\xi_i(j)\leqq\xi_i(l)\leqq\bigcup_{k=1}^{l}\xi_k(l)=\xi(l)\leqq\bar{\xi}$. This implies $\alpha(i)=\bar{\xi}_i\leqq\bar{\xi}$ for each $i\in\boldsymbol{N}$ and hence $\bar{\alpha}\leqq\bar{\xi}$.

Lemma 1.4. $\overline{\sum(R)}=R^{\sigma+}$ *and* $\overline{\sum_0(R)}=R^{\delta+}$.

Proof. Corollary to Lemma 1.1. and Lemma 1.2 imply that $\overline{\sum(R)}[\overline{\sum_0(R)}]$ is a sublattice of M. Suppose that a sequence $a_i\in\overline{\sum(R)}[\overline{\sum_0(R)}]$, $i\in\boldsymbol{N}$, has an upper bound in $M[\overline{\sum_0(R)}]$. Putting $\alpha(i)=\bigcup_{k=1}^{i}a_k$ we have an element α of $\sum(\overline{\sum(R)})[\sum_0(\overline{\sum_0(R)})]$. Hence Lemma 1.3 implies the existence of a $\xi\in\sum(R)$ $[\sum_0(R)]$ such that $\bar{\xi}=\bar{\alpha}$ and therefore $\bigcup_{i=1}^{\infty}a_i=\bigcup_{i=1}^{\infty}\alpha(i)=\bar{\alpha}=\bar{\xi}\in\overline{\sum(R)}[\overline{\sum_0(R)}]$. This implies that $\overline{\sum(R)}[\overline{\sum_0(R)}]$ is a $\sigma^+[\delta^+]$-sublattice of M containing R so that $R^{\sigma+}\subset\overline{\sum(R)}[R^{\delta+}\subset\overline{\sum_0(R)}]$. The reverse inclusion is obvious.

Corollary 1. *The following conditions are equivalent*:

1) *R is a σ^+-sublattice of M.*

2) $\overline{\sum(R)} \subset R$.

3) $\overline{\sum(R)} = R$.

Corollary 2. *The following conditions are equivalent*:

1) *R is a δ^+-sublattice of M.*

2) $\overline{\sum_0(R)} \subset R$.

3) $\overline{\sum_0(R)} = R$.

Assumption 1.2. *M is a δ-lattice.*

Lemma 1.5. *For any $x \in R^\delta$ there exist a, $b \in R$ such that $a \leqq x \leqq b$.*

Proof. This follows from the fact that the set $\{x \mid x \in M$ and $a \leqq x \leqq b$ for some a, $b \in R\}$ is a δ-sublattice of M containing R.

Lemma 1.6. *If a subset A of M contains R, $\overline{\sum_0(A)}$, and $\overline{\sum_0^*(A)}$, then it contains R^δ.*

Proof. Let us put $\mathcal{S} = \{S \mid S \subset R^\delta$ and $\overline{\sum_0(S)} \cup \overline{\sum_0^*(S)} \subset S\}$. Considering $A \cap R^\delta$ in place of A, we can assume that $R \subset A \in \mathcal{S}$. For the intersection S_0 of all S's such that $R \subset S \in \mathcal{S}$ it is easy to see that $R \subset S_0 \in \mathcal{S}$. For any $x \in R^\delta$ let us show that the set $S(x) = \{y \mid y \in R^\delta$, $x \cup y \in S_0$, and $x \cap y \in S_0\}$ is an element of $\mathcal{S}$. For any $\eta \in \sum_0(S(x))$ it follows from $S(x) \subset R^\delta$ that $\bar{\eta} \in R^\delta$. Since $\bar{\eta} \leqq y_0$ for some $y_0 \in S(x)$ we have $x \cup \eta$, $x \cap \eta \in \sum_0(S_0)$. Hence $x \cup \bar{\eta} = \overline{x \cup \eta} \in \overline{\sum_0(S_0)} \subset S_0$ and $x \cap \bar{\eta} \in S_0$, implying that $\bar{\eta} \in S(x)$. Thus we have $\overline{\sum_0(S(x))} \subset S(x)$ and dually $\overline{\sum_0^*(S(x))} \subset S(x)$, which prove that $S(x) \in \mathcal{S}$. If $t \in R$, then we have $R \subset S(t)$ so that $S_0 \subset S(t)$. Hence any $x \in S_0$ satisfies $x \in S(t)$ and therefore $t \in S(x)$. Thus, for any $x \in S_0$, we have $R \subset S(x)$ so that $S_0 \subset S(x)$. This implies that S_0 is a sublattice of M. Since $S_0 \in \mathcal{S}$ implies that S_0 is a δ-sublattice containing R we have $R^\delta \subset S_0 \subset A$, which proves the lemma.

Corollary. *Assume that G is separated. If δ-convergent maps μ and ν of R^δ into G are such that $\mu(x) = \nu(x)$ for any $x \in R$, then it holds that $\mu = \nu$.*

Proof. It is easily verified that the subset $A = \{x \mid x \in R^\delta$ and $\mu(x) = \nu(x)\}$ of M satisfies the condition in the lemma. Hence $R^\delta \subset A$ and this implies $\mu = \nu$.

Note that the above corollary implies the uniqueness of the valuation μ^δ in Theorem 1.

2. r.i. lattices and the second extension theorem

Suppose that for each a, x, $b \in M$ with $a \leqq x \leqq b$ an element ${}^a x^b$ of M is defined subject to the conditions:

1) $a \leqq x \leqq y \leqq b$ implies $a \leqq {}^a y^b \leqq {}^a x^b \leqq b$,

2) $a \leqq x \leqq y \leqq b$ implies ${}^a x^y = {}^a({}^x y^b)^b$ and ${}^x y^b = {}^a({}^a x^y)^b$,

3) ${}^{x \cap y} x^{x \cup y} = y$,

for each a, x, y, $b \in M$. Then we say that M is a *relatively inversible* lattice or an *r.i.* lattice and that ${}^a x^b$ is the *relative inverse* of x in the interval $[a, b]$.

If M is an r.i. lattice, then a sublattice R of M is called to be *relatively inversible* or *r.i.* if ${}^a x^b \in R$ for any a, x, $b \in R$ with $a \leqq x \leqq b$.

If M is an r.i. lattice and if R is an r.i. sublattice of M, then a map μ of R into G is called an *r.i. valuation* if

$$\mu(a) + \mu(b) = \mu(x) + \mu({}^a x^b)$$

for any a, x, $b \in R$ with $a \leqq x \leqq b$. The above condition 3) implies that any r.i. valuation is a valuation.

EXAMPLE 2.1. Suppose that M is an *r.c.* (an abbreviation for *relatively complemented*) distributive lattice. Let us denote by ${}^a x^b$ the relative complement of x in the interval $[a, b]$ for each a, x, $b \in M$ with $a \leqq x \leqq b$. Then M is an r.i. lattice. In fact, the condition 2) is verified as follows: putting $p = {}^a x^y$ and $q = {}^x y^b$ we have $p \cap q = (p \cap y) \cap q = p \cap (y \cap q) = p \cap x = a$ and dually $p \cup q = b$ so that $p = {}^a q^b$ and $q = {}^a p^b$. A sublattice R of M is r.i., by definition, if and only if r.c.. If R is an r.i. sublattice of M, then a map μ of R into G is an r.i.valuation if and only if μ is a valuation.

We note that the condition 3) in the definition of an r.i. lattice implies the uniqueness of the manner in which an r.c. distributive lattice is considered to be an r.i. lattice.

EXAMPLE 2.2. Suppose that M is an l-group and put ${}^a x^b = a + b - x$ for each a, x, $b \in M$ with $a \leqq x \leqq b$. Then M is an r.i. lattice as is verified by simple computations. Let R be an l-subgroup of M. Then R is an r.i. sublattice of M and any group homomorphism μ of R into G is an r.i. valuation.

If M is an r.c. distributive lattice or an l-group, then we shall consider M to be an r.i. lattice in the manners in Examples 2.1 and 2.2, respectively.

Then we have

Theorem 2. *Suppose that M is an r.i. δ-lattice and that G is a separated and complete topological additive group. Let R be an r.i. δ-sublattice of M and μ a G-valued δ-convergent r.i. valuation on R. Then there exists a δ-sublattice $\bar{R}$ of M such that $R \subset \bar{R} \subset R^\sigma$ and μ is extended to a G-valued δ-convergent valuation $\bar{\mu}$ on $\bar{R}$ satisfying the condition: if P is a δ-sublattice of M such that $R \subset P \subset R^\sigma$ and if a G-valued δ-convergent valuation ν on P is an extension of μ, then P is contained in $\bar{R}$ and ν is the restriction of $\bar{\mu}$. Moreover, if M is an r.c. lattice, then $\bar{R}$ is an r.c. sublattice of M. If M is an l-group, if R is a subgroup of M, and if $\mu(0)=0$, then $\bar{R}$ is a subgroup of M and $\bar{\mu}$ is a group homomorphism.*

A proof of the theorem will be given in section 9. In this section we shall state some properties of r.i. lattices.

Assumption 2.1. *M is an r.i. lattice.*

Proposition 2.1. *For any $a, b \in M$, the map $x \to {}^{a}x^{b}$ is an involutive dual automorphism of the interval $[a, b]$.*

Proof. It suffices to show that ${}^{a}({}^{a}x^{b})^{b}=x$ for each $x \in [a, b]$ and this follows from $x \leqq {}^{x}b^{b}={}^{a}({}^{a}x^{b})^{b}={}^{a}a^{x} \leqq x$.

Corollary. *For any $a, x, b \in M$ with $a \leqq x \leqq b$, the map $s \to {}^{a}x^{s}$ is an isomorphism of $[x, b]$ onto $[a, {}^{a}x^{b}]$ and the map $t \to {}^{t}x^{b}$ is an isomorphism of $[a, x]$ onto $[{}^{a}x^{b}, b]$.*

Proof. Since the map $s \to {}^{a}x^{s}$ is the composite map of two dual isomorphisms $s \to p={}^{x}s^{b}$ and $p \to {}^{a}p^{b}$, this map is an isomorphism. Dually, so is the map $t \to {}^{t}x^{b}$.

Let us write $(x-y)_{e}={}^{e \cap y}y^{y \cup x}$ for each $e, x, y \in M$. Then,

Proposition 2.2. *For any $e, x, y \in M$.*
1) $(x-y)_{e}=(x \cup y-y)_{y \cap e}$,
2) $(x-y \cup e)_{e}=(x \cup e-y)_{e}=(x-y)_{e} \cup e$,
2*) $(x-x \cap y)_{e}=(x-y)_{x \cap e}=x \cap (x-y)_{e}$.

Proof. 1) follows immediately from the definition of $(x-y)_{e}$. 2) follows from $(x-y \cup e)_{e}={}^{e \cap (y \cup e)}(y \cup e)^{(y \cup e) \cup x}={}^{e}(y \cup e)^{y \cup x \cup e}={}^{e \cap y}({}^{e \cap y}e^{y \cup e})^{y \cup x \cup e}={}^{e \cap y}y^{y \cup (x \cup e)}$ $=(x \cup e-y)_{e}={}^{e \cap y}y^{(y \cup x) \cup (y \cup e)}={}^{e \cap y}y^{y \cup x} \cup {}^{e \cap y}y^{y \cup e}=(x-y)_{e} \cup e$. 2*) is the dual of 2).

Note that the condition 1) in the proposition is written in an equivalent form:

1′) $(x-y)_{e}=(x \cup y-y)_{e}=(x-y)_{y \cap e}$,

or generally

1″) $(x-y)_{e}=(x'-y)_{e'}$ for any $e', x' \in M$ such that $y \cap e \leqq e' \leqq e$ and $x \leqq x' \leqq x \cup y$.

Corollary 1. *If $e, x, y \in M$ are such that $e \leqq x$, then*

$$(x-y)_{e} = (x-x \cap (y \cup e))_{e} = (x-(x \cap y) \cup e)_{e}\,.$$

Proof. $e \leqq x$ implies $(x-y)_{e}=(x \cup e-y)_{e}=(x-y \cup e)_{e}=(x-y \cup e)_{x \cap e}=(x-x \cap (y \cup e))_{e}$ and dually we have $(x-y)_{e}=(x-(x \cap y) \cup e)_{e}$.

Corollary 2. *For any $e, x, y \in M$, it holds that*

$$(x \cup y) \cap e \leqq (x-y)_{e} \leqq x \cup (y \cap e)\,.$$

Proof. Putting $f=y \cap e$ and $z=x \cup f$ we have $(x-y)_{e}=(x-y)_{f}=(x-y \cup f)_{f}=(z-y)_{f}=(z-y)_{z \cap f}=z \cap (z-y)_{f} \leqq z=x \cup (y \cap e)$ and dually $(x \cup y) \cap e \leqq (x-y)_{e}$.

The above corollary gives an inequality $(x \cup y) \cap z \leqq x \cup (y \cap z)$ for $x, y, z \in M$. If $x \leqq z$ the reverse inequality immediately follows and this shows that M is a modular lattice. Further we have (see [4], p. 36, Exercise 3)

Proposition 2.3. *M is a distributive lattice.*

Proof. For $x, y, z \in M$ it holds that $(x \cup y) \cap z \leqq \{x \cup (y \cap z)\} \cap \{y \cup (x \cap z)\} \leqq (y \cap z) \cup [x \cap \{y \cup (x \cap z)\}] \leqq (y \cap z) \cup \{(x \cap y) \cup (x \cap z)\}$. This implies $a \leqq b$ for $a = \{(x \cup y) \cap z\} \cup (x \cap y)$ and $b = (y \cap z) \cup (z \cap x) \cup (x \cap y)$. Since the reverse is easily seen we have $a = b$, and dually $a^* = b^*$ for $a^* = (x \cup y) \cap \{z \cup (x \cap y)\}$ and $b^* = (y \cup z) \cap (z \cup x) \cap (x \cup y)$. The modularity implies $a = a^*$ and hence $b = b^*$, which proves that M is distributive ([4], p. 32, Theorem 8).

Proposition 2.4. *Let $e, x \in M$ be such that $e \leqq x$. Then the map $y \to (x - y)_e$ is a lattice (and hence order) dual homomorphism*[1] *of M into itself.*

Proof. Corollary 1 to Proposition 2.2 implies that the map is the composite map of the homomorphism $y \to z = x \cap (y \cup e)$ and the dual isomorphism $z \to (x - z)_e = {}^e z^x$.

Proposition 2.5.

1) *For any $e, y \in M$, the map $x \to (x - y)_e$ is a lattice endomorphism of M.*
2) *For any $x, y \in M$, the map $e \to (x - y)_e$ is a lattice endomorphism of M.*

Proof. 1) The map is the composite of the homomorphism $x \to s = y \cup x$ and the isomorphism $s \to {}^{e \cap y} y^s$. 2) is the dual of 1).

Obviously, a sublattice R of M is r.i. if and only if $(x - y)_e \in R$ for any $e, x, y \in R$.

Here we state two propositions concerning with r.i. valuations.

Proposition 2.6. *Suppose that μ is an r.i. valuation on an r.i. sublattice R of M. Then*

1) *μ is δ-convergent if μ is δ^+-convergent or δ^--convergent.*
2) *μ is δ-fundamental if μ is δ^+-fundamental or δ^--fundamental.*

Proof. Let $x_i \in R$, $i \in \boldsymbol{N}$, be any decreasing sequence with a lower bound $a \in R$ and let us put $b = x_1$. Since $a \leqq x_i \leqq b$, for $y_i = {}^a x_i{}^b$ we have an increasing sequence $y_i \in R$, $i \in \boldsymbol{N}$, with an upper bound $b \in R$ such that $\mu(a) + \mu(b) = \mu(x_i) + \mu(y_i)$ for any $i \in \boldsymbol{N}$. 2) If μ is δ^+-fundamental, then the sequence $\mu(y_i)$, $i \in \boldsymbol{N}$, is fundamental and so is the sequence $\mu(x_i) = \mu(a) + \mu(b) - \mu(y_i)$, $i \in \boldsymbol{N}$, which implies that μ is δ^--fundamental. 1) Suppose that μ is δ^+-convergent and $x_i \downarrow x$ $(i \to \infty)$ for some $x \in R$. Since $a \leqq x \leqq b$, Proposition 2.1 implies that

1) This means that $(x - y_1 \cup y_2)_e = (x - y_1)_e \cap (x - y_2)_e$ and $(x - y_1 \cap y_2)_e = (x - y_1)_e \cup (x - y_2)_e$ for any $y_1, y_2 \in M$.

$y_i \uparrow y$ $(i \to \infty)$ for $y = {}^a x^b \in R$. Hence the sequence $\mu(y_i)$, $i \in \boldsymbol{N}$, converges to $\mu(y)$ so that the sequence $\mu(x_i)$, $i \in \boldsymbol{N}$, to $\mu(a)+\mu(b)-\mu(y)=\mu(x)$, which implies that μ is δ^--convergent. Thus the duality implies the proposition.

Proposition 2.7. *Assume that M is an l-group and that an r.i. sublattice R of M is a subsemigroup containing* 0. *Then, a map μ of R into G is a homomorphism if and only if μ is an r.i. valuation such that $\mu(0)=0$.*

Proof. Let us assume that μ is an r.i. valuation such that $\mu(0)=0$. Then $\mu(a)+\mu(b)=\mu(t)+\mu({}^a t^b)$ for any a, t, $b \in R$ such that $a \leqq t \leqq b$. Considering one of a, t, and b to be 0, we have the following lemma: if p, $q \in R$ satisfy one of the three conditions $p \leqq 0 \leqq q$, $0 \leqq p \cap q$, and $p \cup q \leqq 0$, then $\mu(p+q)=\mu(p)+\mu(q)$. (In case, for example, $0 \leqq p \cap q$, put $a=0$, $t=p$, and $b=p+q$.) For each $x \in R$, putting $x_+ = x \cup 0$ and $x_- = x \cap 0$ we have x_+, $x_- \in R$ and the lemma implies $\mu(x)=\mu(x_+)+\mu(x_-)$.

Let x and y be any elements of R and put $z=x+y$. Then we are to prove that $\mu(z)=\mu(x)+\mu(y)$. Since x_+, x_-, y_+, $y_- \in R$, for $u=x_+ + y_-$ and $v=y_+ + x_-$ we have u, $v \in R$ and the above lemma implies $\mu(u)=\mu(x_+)+\mu(y_-)$ and $\mu(v)=\mu(y_+)+\mu(x_-)$. Since $u+v=x+y=z$, since $u=x \cup 0 + y_- = (x+y_-) \cup y_- \leqq (x+y) \cup 0 = z_+$, and since $v \leqq z_+$, it follows that $u_+ + v_+ = u \cup 0 + v \cup 0 = (u+v \cup 0) \cup (v \cup 0) = \{(u+v) \cup u\} \cup (v \cup 0) = (u+v)_+ \cup (u \cup v) = z_+ \cup (u \cup v) = z_+$. Hence the above lemma implies $\mu(z_+)=\mu(u_+)+\mu(v_+)$. Further $u_- + v_- = (u+v)-(u_+ + v_+) = z - z_+ = z_-$ implies $\mu(z_-)=\mu(u_-)+\mu(v_-)$. Thus we have $\mu(z)=\mu(z_+)+\mu(z_-) = \mu(u_+)+\mu(v_+)+\mu(u_-)+\mu(v_-)=\mu(u)+\mu(v)=\mu(x_+)+\mu(y_-)+\mu(y_+)+\mu(x_-)=\mu(x)+\mu(y)$, proving that μ is a homomorphism. The converse is obvious and hence the proposition holds.

Proposition 2.8. *If M is a δ^+-lattice or a δ^--lattice, then M is a δ-lattice.*

Proof. Let, for example, M be a δ^+-lattice. Suppose that a sequence $x_i \in M$, $i \in \boldsymbol{N}$, has a lower bound $a_0 \in M$ and let $x \in M$ be any. Putting $a = x \cap a_0$ and $b = x \cup x_1$ we have $a \leqq x \leqq b$ and $a \leqq x_i$ for each $i \in \boldsymbol{N}$. For each $t \in M$ such that $t \leqq x$ it follows from $a \leqq b \cap (t \cup x_i) \leqq b$ that $a \leqq {}^a(b \cap (t \cup x_i))^b \leqq b$. Since M is a δ^+-lattice, we have an element $p(t) = \bigcup_{i=1}^{\infty} {}^a(b \cap (t \cup x_i))^b$ of M such that $a \leqq p(t) \leqq b$. Proposition 2.1 and the inequality $t \leqq x$ imply ${}^a p(t)^b = \bigcap_{i=1}^{\infty} (b \cap (t \cup x_i)) = \bigcap_{i=1}^{\infty} (t \cup x_i)$. This shows the existence of $\bigcap_{i=1}^{\infty} x_i = {}^a p(a)^b \in M$. Further $x \leqq b$ implies $b \cap (x \cup x_i) = x \cup (b \cap x_i)$ and similarly $b \cap (a \cup x_i) = a \cup (b \cap x_i) = b \cap x_i$ so that $p(x) = \bigcup_{i=1}^{\infty} {}^a(x \cup (b \cap x_i))^b = \bigcup_{i=1}^{\infty} ({}^a x^b \cap {}^a(b \cap x_i)^b) = {}^a x^b \cap p(a)$. Thus we have $\bigcap_{i=1}^{\infty} (x \cup x_i) = {}^a p(x)^b = {}^a({}^a x^b \cap p(a))^b = x \cup {}^a p(a)^b = x \cup (\bigcap_{i=1}^{\infty} x_i)$. This proves that M is a δ^--lattice and hence a δ-lattice.

Assumption 2.2. *M is a δ-lattice.*

Lemma 2.1. *For any e, x, $y \in M$ with $e \leqq x$ and for any $y_i \in M$, $i \in \boldsymbol{N}$,*

1) $y = \bigcup_{i=1}^{\infty} y_i$ *implies* $(x-y)_e = \bigcap_{i=1}^{\infty} (x-y_i)_e$,

2) $y = \bigcap_{i=1}^{\infty} y_i$ *implies* $(x-y)_e = \bigcup_{i=1}^{\infty} (x-y_i)_e$.

Proof. Put $p(t) = x \cap (t \cup e)$ for each $t \in M$. Then Assumption 2.2 implies $p(\bigcup_{i=1}^{\infty} y_i) = \bigcup_{i=1}^{\infty} p(y_i)$ and $p(\bigcap_{i=1}^{\infty} y_i) = \bigcap_{i=1}^{\infty} p(y_i)$. Hence the argument in the proof of Proposition 2.4 proves the lemma.

Lemma 2.2. *For any e, x, y, $t \in M$ and for any $t_i \in M$, $i \in \boldsymbol{N}$,*

1) $t = \bigcup_{i=1}^{\infty} t_i$ *implies* $(t-y)_e = \bigcup_{i=1}^{\infty} (t_i-y)_e$ *and* $(x-y)_t = \bigcup_{i=1}^{\infty} (x-y)_{t_i}$,

2) $t = \bigcap_{i=1}^{\infty} t_i$ *implies* $(t-y)_e = \bigcap_{i=1}^{\infty} (t_i-y)_e$ *and* $(x-y)_t = \bigcap_{i=1}^{\infty} (x-y)_{t_i}$.

Proof. 2) Since we can write $t = \bigcap_{i=1}^{\infty} s_i$ for a bounded sequence $s_i = t_1 \cap t_i$, $i \in \boldsymbol{N}$, it follows from the equality $y \cup t = \bigcap_{i=1}^{\infty} (y \cup s_i)$ and Corollary to Proposition 2.1 that $(t-y)_e = \bigcap_{i=1}^{\infty} (s_i-y)_e$. Since Proposition 2.5 implies $(s_i-y)_e = (t_1-y)_e \cap (t_i-y)_e$ we have $(t-y)_e = \bigcap_{i=1}^{\infty} (t_i-y)_e$. The latter equality $(x-y)_t = \bigcap_{i=1}^{\infty} (x-y)_{t_i}$ is easily seen. 1) follows dually.

Proposition 2.9. *An r.i. sublattice R of M is a δ-sublattice if R is a δ^+-sublattice or a δ^--sublattice.*

Proof. We may assume that R is a δ^--sublattice. Then it suffices to show that $\bar{\xi} \in R$ for any $\xi \in \sum_0(R)$ (Corollary 2 to Lemma 1.4). The sequence $\xi(i)$, $i \in \boldsymbol{N}$, has a lower bound $a = \xi(1) \in R$ and an upper bound $b \in R$. Since ${}^a\xi(i)^b \in R$, our assumption implies the existence of $x = \bigcap_{i=1}^{\infty} {}^a\xi(i)^b \in R$. Thus we have $\bar{\xi} = \bigcup_{i=1}^{\infty} \xi(i) = {}^a x^b \in R$.

Proposition 2.10. *If R is an r.i. sublattice of M, then R^δ is an r.i. sublattice.*

Proof. 1) For the subset $A = \{y \mid y \in R^\delta$ and $(x-y)_e \in R^\delta$ for any $x, e \in R\}$ of M, let us show that $R^\delta \subset A$. Since $R \subset A$ follows immediately, it suffices, by Lemma 1.6, to prove that $\bar{\eta} \in A$ for any $\eta \in \sum_0(A) \cup \sum_0^*(A)$. The duality implies that we may assume $\eta \in \sum_0(A)$. The relation $\bar{\eta} \in R^\delta$ follows from $A \subset R^\delta$. For any $x, e \in R$, putting $x' = x \cup e$ we have $e \leqq x' \in R$. Since the sequence $(x'-\eta(i))_e \in R^\delta$, $i \in \boldsymbol{N}$, has a lower bound $\bar{\eta} \cap e \in R^\delta$, Lemma 2.1 implies $(x'-\bar{\eta})_e = \bigcap_{i=1}^{\infty} (x'-\eta(i))_e \in R^\delta$. Hence Proposition 2.2 and its Corollary 2 imply $(x-\bar{\eta})_e =$

$(x-\bar{\eta})_e \cup \{e \cap (x \cup \bar{\eta})\} = \{(x-\bar{\eta})_e \cup e\} \cap (x \cup \bar{\eta}) = (x'-\bar{\eta})_e \cap (x \cup \bar{\eta}) \in R^\delta$, proving that $\bar{\eta} \in A$. 2) For each y, $e \in R^\delta$ and for $B = \{x \mid x \in R^\delta$ and $(x-y)_e \in R^\delta\}$ it holds that $\overline{\sum_0(B)} \cup \overline{\sum_0^*(B)} \subset B$. In fact, for any $\xi \in \sum_0(B)[\sum_0^*(B)]$ it is obvious that $\bar{\xi} \in R^\delta$. Since the sequence $(\xi(i)-y)_e \in R^\delta$, $i \in \boldsymbol{N}$, has an upper bound $\bar{\xi} \cup y \in R^\delta$[a lower bound $y \cap e \in R^\delta$], Lemma 2.2 implies $(\bar{\xi}-y)_e = \bigcup_{i=1}^{\infty} (\xi(i)-y)_e$ $[= \bigcap_{i=1}^{\infty} (\xi(i)-y)_e] \in R^\delta$, proving that $\bar{\xi} \in B$. Dually we have: 2*) For each $x, y \in R^\delta$ and for $C = \{e \mid e \in R^\delta$ and $(x-y)_e \in R^\delta\}$ it holds that $\overline{\sum_0(C)} \cup \overline{\sum_0^*(C)} \subset C$. 3) If $y \in R^\delta$ and if $e \in R$, then 1) implies that the set B in 2) contains R and hence 2) implies that $R^\delta \subset B$. Hence, for each $x, y \in R^\delta$, the set C in 2*) contains R so that $R^\delta \subset C$. This implies that $(x-y)_e \in R^\delta$ for any x, y, $e \in R^\delta$ or that R^δ is an r.i. sublattice.

Proposition 2.11. *If M is an l-group and if R is an l-subgroup of M, then R^δ is a subgroup of M.*

Proof. 1) For any $x \in R$ and for $A = \{y \mid y \in R^\delta$ and $x-y \in R^\delta\}$ let us show that $R^\delta \subset A$. For any $\eta \in \sum_0(A)$ we have $\bar{\eta} \in R^\delta$ and $\bar{\eta} \leqq y_0$ for some $y_0 \in R$. Since $x-y_0 \in R \subset R^\delta$ is a lower bound of the sequence $x-\eta(i) \in R^\delta$, $i \in \boldsymbol{N}$, it follows that $x-\bar{\eta} = \bigcap_{i=1}^{\infty} (x-\eta(i)) \in R^\delta$. Thus we have $\bar{\eta} \in A$ and dually $\overline{\eta'} \in A$ for any $\eta' \in \sum_0^*(A)$. It is obvious that $R \subset A$ and hence Lemma 1.6 implies that $R^\delta \subset A$. 2) For any $y \in R^\delta$ it follows from 1) that R is contained in $B = \{x \mid x \in R^\delta$ and $x-y \in R^\delta\}$. Further we have $\overline{\sum_0(B)} \cup \overline{\sum_0^*(B)} \subset B$ so that $R^\delta \subset B$ and this implies the proposition.

Lemma 2.3. *If R is an r.i. sublattice of M and if a δ^+-sublattice $\tilde{R}$ of M is such that $R^{\delta^-} \subset \tilde{R} \subset R^{\sigma+}$, then $\tilde{R}$ is an r.i. δ-sublattice of M.*

Proof. It is sufficient to show that $z = (x-y)_e$ belongs to $\tilde{R}$ for any $e, x, y \in \tilde{R}$ with $e \leqq x$. Since $\tilde{R} \subset R^{\sigma+} = \overline{\sum(R)}$ there exist ξ' η, $\varepsilon \in \sum(R)$ such that $\bar{\xi}' = x$, $\bar{\eta} = y$, and $\bar{\varepsilon} = e$. For $\xi = \xi' \cup \varepsilon$ we have a $\xi \in \sum(R)$ such that $\bar{\xi} = \bar{\xi}' \cup \bar{\varepsilon} = x \cup e = x$ and such that $\varepsilon(k) \leqq \xi(k)$ for any $k \in \boldsymbol{N}$. For the sequence $z_k = (\xi(k)-\eta(k))_{\varepsilon(k)} \in R$, $k \in \boldsymbol{N}$, Proposition 2.5 and Corollary 2 to Proposition 2.2 imply $z_k \geqq (\xi(1)-\eta(k))_{\varepsilon(1)} \geqq \xi(1) \cap \varepsilon(1) \in R$. Let us put $\zeta(j) = \bigcap_{k=j}^{\infty} z_k$ for each $j \in \boldsymbol{N}$. Then $\zeta(j) \in R^{\delta^-} \subset \tilde{R}$ and $\zeta(j-1) \leqq \zeta(j) \leqq z_j \leqq \xi(j) \cup \varepsilon(j) = \xi(j) \leqq x \in \tilde{R}$, which imply $\zeta \in \sum_0(\tilde{R})$. For $i \leqq i$ we have $(\xi(i)-y)_{\varepsilon(j)} \leqq (\xi(j)-y)_{\varepsilon(j)} = \bigcap_{k=j}^{\infty} (\xi(k)-y)_{\varepsilon(k)} \leqq \bigcap_{k=j}^{\infty} z_k = \zeta(j) \leqq \bigcap_{k=j}^{\infty} (x-\eta(k))_e = (x-\bigcup_{k=j}^{\infty} \eta(k))_e = z$. Hence $(\xi(i)-y)_e = \bigcup_{j=i}^{\infty} (\xi(i)-y)_{\varepsilon(j)} \leqq \bigcup_{j=i}^{\infty} \zeta(j) = \bar{\zeta} \leqq z$ so that $z = \bigcup_{i=1}^{\infty} (\xi(i)-y)_e \leqq \bar{\zeta} \leqq z$. Thus we have $z = \bar{\zeta} \in \overline{\sum_0(\tilde{R})} \subset \tilde{R}$ and this proves the lemma.

3. Completion of valuations

Throughout this paper we denote by $\mathcal{U}$ the set of all symmetric neighbourhoods of $0 \in G$.

Assumption 3.1. *R is a sublattice of M and μ is a valuation on R.*

Let us denote by $\hat{R}$ the set of all $a \in M$ satisfying the condition: for any $U \in \mathcal{U}$ there exist $x, y \in R$ with $x \leqq a \leqq y$ such that $\mu(s)-\mu(t) \in U$ for any $s, t \in R$ with $x \leqq s \leqq y$ and $x \leqq t \leqq y$.

It is easily seen that the part "$\mu(s)-\mu(t) \in U$ for any $s, t \in R$ with $x \leqq s \leqq y$ and $x \leqq t \leqq y$" in the above condition can be replaced by each of "$\mu(s)-\mu(x) \in U$ for any $s \in R$ with $x \leqq s \leqq y$" and "$\mu(s)-\mu(y) \in U$ for any $s \in R$ with $x \leqq s \leqq y$".

Proposition 3.1. *Let $\bar{R}$ be the set of all $a \in M$ satisfying the condition: there exist $x, y \in R$ with $x \leqq a \leqq y$ such that $\mu(x)=\mu(s)=\mu(y)$ for any $s \in R$ with $x \leqq s \leqq y$. Then it holds that $R \subset \bar{R} \subset \hat{R}$. If G is separated and satisfies the first condition of countability, if M is a δ-lattice and if R is a δ-sublattice of M, then it holds that $\bar{R}=\hat{R}$.*

Proof. The relation $R \subset \bar{R} \subset \hat{R}$ being obvious, it suffices to show under the additional conditions that $a \in \bar{R}$ for any $a \in \hat{R}$. Let $\{U_i \mid i \in \boldsymbol{N}\}$ be a countable base of the system of neighbourhoods of $0 \in G$. Then for each $i \in \boldsymbol{N}$ there exist $x_i, y_i \in R$ with $x_i \leqq a \leqq y_i$ such that $\mu(s)-\mu(t) \in U_i$ for any $s, t \in R$ with $x_i \leqq s \leqq y_i$ and $x_i \leqq t \leqq y_i$. Since R is a δ-sublattice we have $x=\bigcup_{i=1}^{\infty} x_i \in R$ and $y=\bigcap_{i=1}^{\infty} y_i \in R$ with $x \leqq a \leqq y$. Then for any $s \in R$ with $x \leqq s \leqq y$ we have $\mu(s)=\mu(x)$, which implies that $a \in \bar{R}$.

The following two examples show that the relation $\bar{R}=\hat{R}$ in the above proposition not necessarily holds if R is not a δ-sublattice or if G fails to satisfy the first condition of countability.

EXAMPLE 3.1. The l-group M of all real-valued functions on a fixed interval $E=[a, b]$ in the real line $\boldsymbol{R}$ is a δ-lattice (Example 1.3) and the set R of all continuous functions in M is an l-subgroup of M. Putting $G=\boldsymbol{R}$ and denoting by $\mu(f)$ the integral (in the usual sense) of $f \in R$ over E we have a δ-convergent valuation μ on R. Then $\bar{R}$ coincides with R while $\hat{R}$ is the set of all Riemann-integrable functions on E.

EXAMPLE 3.2. Suppose that the set $\boldsymbol{m}$ in Example 1.1 is uncountable and let R be the ring of all subsets x of $\boldsymbol{m}$ such that x or x^c is countable. Then R is a δ-sublattice of M. Let G be the topological group of all real-valued functions on $\boldsymbol{m}$ with the weak topology and let us denote by $\mu(x)$ the characteristic function of $x \in R$. Then μ is a δ-convergent valuation on R and it is easy to see that $\bar{R}=R \neq M=\hat{R}$.

REMARK. If we replace the topology of G with the discrete one, then the set $\bar{R}$ in Proposition 3.1 coincides with $\hat{R}$. Hence some properties of $\bar{R}$ are derived from those of $\hat{R}$. For example the following proposition implies that $\bar{R}$ is a sublattice of M.

Proposition 3.2. *$\hat{R}$ is a sublattice of M containing R.*

Proof. Let $a_1, a_2 \in \hat{R}$ and $U \in \mathcal{U}$ be any elements. Then, for $i=1, 2$, there exist $x_i, y_i \in R$ with $x_i \leqq a_i \leqq y_i$ such that $\mu(s)-\mu(t) \in U$ for any $s, t \in R$ with $x_i \leqq s \leqq y_i$ and $x_i \leqq t \leqq y_i$. For $x=x_1 \cap x_2$ and $y=y_1 \cap y_2$, let us prove that $\mu(u)-\mu(x) \in 2U$ for any $u \in R$ with $x \leqq u \leqq y$. It follows from $u \cup x_1 \in R$ and $x_1 \leqq u \cup x_1 \leqq y_1$ that $\mu(u)-\mu(u \cap x_1)=\mu(u \cup x_1)-\mu(x_1) \in U$. Since $u \cap x_1 \in R$ is such that $x \leqq u \cap x_1 \leqq y$, similarly we have $\mu(u \cap x_1)-\mu((u \cap x_1) \cap x_2) \in U$. Thus it is proved that $\mu(u)-\mu(x)=\mu(u)-\mu((u \cap x_1) \cap x_2) \in 2U$. Since the elements $x, y \in R$ satisfy $x \leqq a_1 \cap a_2 \leqq y$, that which is proved above implies $a_1 \cap a_2 \in \hat{R}$. Dually we have $a_1 \cup a_2 \in \hat{R}$ for any $a_1, a_2 \in \hat{R}$ and this proves the proposition.

We say that the valuation μ is *complete* if $\hat{R}=R$.

Lemma 3.1. *Assume that M is a δ^+-lattice and that R and $\tilde{R}$ are δ^+-sublattices of M such that $R \subset \tilde{R} \subset R^{\sigma+}$. If μ is complete and extended to a valuation $\tilde{\mu}$ on $\tilde{R}$, then $\tilde{\mu}$ is complete.*

Proof. We are to prove that $a \in \tilde{R}$ for any $a \in \hat{\tilde{R}}$. First let us show that $x \cap a \in R$ for any $x \in R$. For any $U \in \mathcal{U}$ the assumption $a \in \hat{\tilde{R}}$ implies the existence of $u, v \in \tilde{R}$ with $u \leqq a \leqq v$ such that $\mu(s)-\mu(t) \in U$ for any $s, t \in \tilde{R}$ with $u \leqq s \leqq v$ and $u \leqq t \leqq v$. We can write $u=\bar{\alpha}$ for some $\alpha \in \sum(R)$ and hence it follows from $x \cap \alpha \in \sum_0(R)$ that $x \cap u=\overline{x \cap \alpha} \in R$. Likewise $x \cap v \in R$ and further we have $x \cap u \leqq x \cap a \leqq x \cap v$. Let $s_i \in R$ be such that $x \cap u \leqq s_i \leqq x \cap v$ for $i=1, 2$. Putting $t_i=s_i \cup u$ we have $t_i \in \tilde{R}$ with $u \leqq t_i \leqq v$ so that $\tilde{\mu}(t_1)-\tilde{\mu}(t_2) \in U$. Hence it follows from $\mu(s_i)+\tilde{\mu}(u)=\tilde{\mu}(s_i \cup u)+\tilde{\mu}(s_i \cap u)=\tilde{\mu}(t_i)+\tilde{\mu}(x \cap u)$ that $\mu(s_1)-\mu(s_2)=\tilde{\mu}(t_1)-\tilde{\mu}(t_2) \in U$. Thus it is proved that $x \cap a \in \hat{R}=R$ for any $x \in R$. Now the relation $a \in \tilde{R}$ is proved as follows. There exists a $v \in \tilde{R}$ such that $a \leqq v$ and we can write $v \in \bar{\xi}$ for some $\xi \in \sum(R)$. Since $\tilde{R} \supset R \ni \xi(i) \cap a \leqq \bar{\xi}=v \in \tilde{R}$ implies $a \cap \xi \in \sum_0(\tilde{R})$ we have $a=\overline{a \cap \xi} \in \tilde{R}$.

A map $\hat{\mu}$ of $\hat{R}$ into G is called a *completion* of μ if: for any $a \in \hat{R}$ and for any $U \in \mathcal{U}$ there exist $x, y \in R$ with $x \leqq a \leqq y$ such that $\mu(s)-\hat{\mu}(a) \in U$ for any $s \in R$ with $x \leqq s \leqq y$.

Assumption 3.2. *G is separated and complete.*

Proposition 3.3. *There uniquely exists a completion $\hat{\mu}$ of μ. Moreover $\hat{\mu}$ is a valuation and is an extension of μ.*

Proof. Suppose that a completion $\hat{\mu}$ of μ exists. The uniqueness of $\hat{\mu}$ and the fact that $\hat{\mu}$ is an extension of μ immediately follow from the assumption that G is separated. Let us show that $\hat{\mu}$ is a valuation. Let $a_1, a_2 \in \hat{R}$ and $U \in \mathcal{U}$ be any elements. Then, for $i=1, 2$, there exist $x_i, y_i \in R$ with $x_i \leqq a_i \leqq y_i$ such that $\mu(s)-\mu(t) \in U$ for any $s, t \in R$ with $x_i \leqq s \leqq y_i$ and $x_i \leqq t \leqq y_i$. Here we assert that $\mu(u)-\hat{\mu}(a_1 \cap a_2) \in 5U$ for any $u \in R$ with $x_1 \cap x_2 \leqq u \leqq y_1 \cap y_2$. In fact, we have $\mu(u)-\mu(x_1 \cap x_2) \in 2U$ as is seen in the proof of Proposition 3.2. Since $x_1 \cap x_2 \leqq a_1 \cap a_2 \leqq y_1 \cap y_2$ the definition of $\hat{\mu}$ implies the existence of a $u_0 \in R$ such that $x_1 \cap x_2 \leqq u_0 \leqq y_1 \cap y_2$ and $\mu(u_0)-\hat{\mu}(a_1 \cap a_2) \in U$. Since $\mu(u_0)-\mu(x_1 \cap x_2) \in 2U$ we have $\mu(u)-\hat{\mu}(a_1 \cap a_2) \in 5U$, which implies that our assertion is true. Dually we have $\mu(v)-\hat{\mu}(a_1 \cup a_2) \in 5U$ for any $v \in R$ with $x_1 \cup x_2 \leqq v \leqq y_1 \cup y_2$. Hence it follows that $\hat{\mu}(a_1 \cap a_2)+\hat{\mu}(a_1 \cup a_2) \in \mu(x_1 \cap x_2)+\mu(x_1 \cup x_2)+10U=\mu(x_1)+\mu(x_2)+10U$. Since, for $i=1, 2$, $\mu(s_i)-\hat{\mu}(a_i) \in U$ for some $s_i \in R$ with $x_i \leqq s_i \leqq y_i$ and since $\mu(s_i)-\mu(x_i) \in U$ we have $\mu(x_i)-\hat{\mu}(a_i) \in 2U$. Thus it is proved that $\hat{\mu}(a_1 \cap a_2)+\hat{\mu}(a_1 \cup a_2) \in \hat{\mu}(a_1)+\hat{\mu}(a_2)+14U$ for any $a_1, a_2 \in \hat{R}$ and any $U \in \mathcal{U}$, which proves that $\hat{\mu}$ is a valuation.

Now it suffices to show the existence of $\hat{\mu}$. For each $a \in \hat{R}$ we have a directed set $\Lambda(a)=\{x \mid x \in R \text{ and } x \leqq a\}$ and hence a directed sequence $\mu(x)$, $x \in \Lambda(a)$, in G. This sequence is easily seen to be fundamental and hence converges to an element $\hat{\mu}(a)$ of G. Thus we can define a map $\hat{\mu}$ of $\hat{R}$ into G. Let $a \in \hat{R}$ and $U \in \mathcal{U}$ be any elements. Then for a $V \in \mathcal{U}$ such that $2V \subset U$ there exist $x_1, y \in R$ with $x_1 \leqq a \leqq y$ such that $\mu(s)-\mu(t) \in V$ for any $s, t \in R$ with $x_1 \leqq s \leqq y$ and $x_1 \leqq t \leqq y$. Further there exists $x_2 \in R$ with $x_2 \leqq a$ such that $\mu(x')-\hat{\mu}(a) \in V$ for any $x' \in R$ with $x_2 \leqq x' \leqq a$. Putting $x=x_1 \cup x_2$ we have $x, y \in R$ with $x \leqq a \leqq y$ and, for any $s \in R$ with $x \leqq s \leqq y$, $\mu(s)-\hat{\mu}(a)=\{\mu(s)-\mu(x)\}+\{\mu(x)-\hat{\mu}(a)\} \in 2V \subset U$. Hence $\hat{\mu}$ is a completion of μ and this completes the proof.

The completion of μ will be denoted by $\hat{\mu}$.

EXAMPLE 3.3. 1) Proposition 3.1 implies that Lebesgue measure (restricted on the ring of the sets of measure finite) is the completion (in our sense) of Borel measure.

2) If μ is the valuation in Example 3.1, then $\hat{\mu}(f)$ means the Riemann integral of $f \in \hat{R}$ over E. In general, n-dimensional Riemann integral is obtained as the completion of the integral of continuous functions.

3) For the valuation μ in Example 3.2, $\hat{\mu}(x)$ means the characteristic function of $x \in \hat{R}=M$.

Lemma 3.2. *For any $a \in \hat{R}$ and for any $U \in \mathcal{U}$ there exist $x, y \in R$ with $x \leqq a \leqq y$ such that $\hat{\mu}(u)-\hat{\mu}(v) \in U$ for any $u, v \in \hat{R}$ with $x \leqq u \leqq y$ and $x \leqq v \leqq y$.*

Proof, For a $V \in \mathcal{U}$ such that $3V \subset U$, there are $x, y \in R$ with $x \leqq a \leqq y$ such that $\mu(s)-\mu(t) \in V$ for any $s, t \in R$ with $x \leqq s \leqq y$ and $x \leqq t \leqq y$. Then for any $u_i \in \hat{R}$ with $x \leqq u_i \leqq y$, $i=1, 2$, there exists an $s_i \in R$ such that $x \leqq s_i \leqq y$ and

$\mu(s_i)-\hat{\mu}(u_i)\in V$. Thus we have $\hat{\mu}(u_1)-\hat{\mu}(u_2)=-\{\mu(s_1)-\hat{\mu}(u_1)\}+\{\mu(s_2)-\hat{\mu}(u_2)\}+\{\mu(s_1)-\mu(s_2)\}\in 3V\subset U$, proving the lemma.

Proposition 3.4. *$\hat{\mu}$ is complete.*

Proof. It suffices to prove that $a\in\hat{R}$ for any $a\in\hat{\hat{R}}$. For any $U\in\mathcal{U}$, let $V\in\mathcal{U}$ be such that $4V\subset U$. Then there exist $u_1, u_2\in\hat{R}$ with $u_1\leqq a\leqq u_2$ such that $\hat{\mu}(p)-\hat{\mu}(u_1)\in V$ for any $p\in\hat{R}$ with $u_1\leqq p\leqq u_2$. For $i=1, 2$, Lemma 3.2 implies the existence of $x_i, y_i\in R$ with $x_i\leqq u_i\leqq y_i$ such that $\hat{\mu}(q)-\hat{\mu}(u_i)\in V$ for any $q\in\hat{R}$ with $x_i\leqq q\leqq y_i$. Thus we have $x_1, y_2\in R$ with $x_1\leqq u_1\leqq a\leqq u_2\leqq y_2$. Let $s\in R$ be such that $x_1\leqq s\leqq y_2$. For $v=s\cup u_1\in\hat{R}$ it follows from $x_1\leqq s\cap u_1\leqq y_1$ that $\mu(s)-\hat{\mu}(v)=\hat{\mu}(s\cap u_1)-\hat{\mu}(u_1)\in V$. This implies $\mu(x_1)-\hat{\mu}(u_1)\in V$ as a special case. Further $x_2\leqq v\cup u_2\leqq y_2$ implies $\hat{\mu}(v)-\hat{\mu}(v\cap u_2)=\hat{\mu}(v\cup u_2)-\hat{\mu}(u_2)\in V$. Finally $u_1\leqq v\cap u_2\leqq u_2$ implies $\hat{\mu}(v\cap u_2)-\hat{\mu}(u_1)\in V$ and hence we have $\mu(s)-\mu(x_1)\in 4V\subset U$. Thus it is proved that $a\in\hat{R}$.

Proposition 3.5. *If M is a δ^+-lattice, if R is a δ^+-sublattice, and if μ is δ^+-convergent, then $\hat{R}$ is a δ^+-sublattice and $\hat{\mu}$ is δ^+-convergent.*

Proof. For any $\alpha\in\sum_0(\hat{R})$, it suffices to prove 1) $\bar{\alpha}\in\hat{R}$ and 2) $\hat{\mu}(\alpha(i))\to\hat{\mu}(\bar{\alpha})$ $(i\to\infty)$. For any $U\in\mathcal{U}$ let $U_0\in\mathcal{U}$ be such that $5U_0\subset U$ and $U_i\in\mathcal{U}$ such that $2U_i\subset U_{i-1}$ for each $i\in\boldsymbol{N}$. Since $\bar{\alpha}\leqq a$ for some $a\in\hat{R}$, we have a $z\in R$ with $a\leqq z$ such that $\mu(t)-\hat{\mu}(a)\in U_0$ for any $t\in R$ with $a\leqq t\leqq z$. For each $i\in\boldsymbol{N}$, Lemma 3.2 implies the existence of $x_i, y_i\in R$ with $x_i\leqq\alpha(i)\leqq y_i\leqq z$ such that $\hat{\mu}(u)-\hat{\mu}(v)\in U_i$ for any $u, v\in\hat{R}$ with $x_i\leqq u\leqq y_i$ and $x_i\leqq v\leqq y_i$. We may assume $x_i\leqq x_{i+1}$ for each $i\in\boldsymbol{N}$. For $x=\bigcup_{i=1}^{\infty}x_i$ and $y=\bigcup_{i=1}^{\infty}y_i$ we have $x, y\in R$ with $x\leqq\bar{\alpha}\leqq y\leqq z$. Let $s_0\in R$ be such that $x\leqq s_0\leqq y$. Putting $s_i=s_0\cup y_1\cup y_2\cup\cdots\cup y_i$ we have $s_i\in R$ for $i=0, 1, 2,\cdots$. Since $s_i\uparrow y$ $(i\to\infty)$ implies $\mu(s_i)\to\mu(y)$ $(i\to\infty)$, there exists an $m\in\boldsymbol{N}$ such that $\mu(s_m)-\mu(y)\in U_0$. Since $x_i\leqq s_{i-1}\cap y_i\leqq y_i$ implies $\mu(s_{i-1})-\mu(s_i)=\mu(s_{i-1})-\mu(s_{i-1}\cup y_i)=\mu(s_{i-1}\cap y_i)-\mu(y_i)\in U_i$ it follows that $\mu(s_0)-\mu(y)=\sum_{i=1}^{m}\{\mu(s_{i-1})-\mu(s_i)\}+\{\mu(s_m)-\mu(y)\}\in\sum_{i=1}^{m}U_i+U_0\subset 2U_0\subset U$. Hence it is proved that 1) $\bar{\alpha}\in\hat{R}$. To prove 2), we may assume that $a=\bar{\alpha}$. Then $\bar{\alpha}\leqq y\leqq z$ implies $\mu(y)-\hat{\mu}(\bar{\alpha})\in U_0$. If we put $s_0=x$ the above argument implies $\mu(x)-\mu(y)\in 2U_0$. Since $x_i\uparrow x$ $(i\to\infty)$ there exists an $n\in\boldsymbol{N}$ such that $\mu(x_i)-\mu(x)\in U_0$ for any $i\geqq n$. Then, for any $i\geqq n$, $\hat{\mu}(\alpha(i))-\mu(x_i)\in U_i\subset U_0$ implies $\hat{\mu}(\alpha(i))-\hat{\mu}(\bar{\alpha})\in 5U_0\subset U$, proving 2).

Corollary. *If M is a δ-lattice, if R is a δ-sublattice, and if μ is δ-convergent, then $\hat{R}$ is a δ-sublattice and $\hat{\mu}$ is δ-convergent.*

Proposition 3.6. *If M is an r.i. lattice, if R is an r.i. sublattice, and if μ is an r.i. valuation, then $\hat{R}$ is an r.i. sublattice and $\hat{\mu}$ is an r.i. valuation.*

Proof. Let a_0, a_1, and a_2 be elements of $\hat{R}$ such that $a_0 \leqq a_1 \leqq a_2$ and let us put $a = {}^{a_0}a_1{}^{a_2}$. For any $U \in \mathcal{U}$, let $V \in \mathcal{U}$ be such that $7V \subset U$. Then, for $i=0$, 1, and 2, there exist x_i, $y_i \in R$ with $x_i \leqq a_i \leqq y_i$ such that $\hat{\mu}(u) - \hat{\mu}(v) \in V$ for any u, $v \in \hat{R}$ with $x_i \leqq u \leqq y_i$ and $x_i \leqq v \leqq y_i$. Since $a_0 \leqq a_1 \leqq a_2$ we may assume that $x_0 \leqq x_1$ and $y_1 \leqq y_2$. Putting $x = {}^{x_0}y_1{}^{y_1 \cup x_2}$ and $y = {}^{y_0 \cap x_1}x_1{}^{y_2}$ we have $x, y \in R$ with $x \leqq {}^{a_0}y_1{}^{y_1 \cup a_2} = {}^{a_0}({}^{y_1 \cap a_2}a_2{}^{y_1 \cup a_2})^{y_1 \cup a_2} = {}^{a_0}(y_1 \cap a_2)^{a_2} \leqq {}^{a_0}a_1{}^{a_2} = a$, and hence the duality implies $x \leqq a \leqq y$. Suppose that s, $t \in R$ are such that $x \leqq s \leqq y$ and $x \leqq t \leqq y$. For $s' = {}^{x_0}s^{y_2} \in R$ we have ${}^{y_1}(y_1 \cup x_2)^{y_2} = {}^{x_0}({}^{x_0}y_1{}^{y_1 \cup x_2})^{y_2} \geqq s'$. Dually $s' \geqq {}^{x_0}(y_0 \cap x_1)^{x_1}$ and hence $s' \cap y_1 \geqq {}^{x_0}(y_0 \cap x_1)^{x_1}$. Now, for any $r \in R$ with ${}^{y_1}(y_1 \cup x_2)^{y_2} \geqq r$, it follows from $y_2 \geqq {}^{y_1}(y_1 \cup x_2)^{y_2} \geqq r \cup y_1 \geqq y_1$ that $x_2 \leqq y_1 \cup x_2 \leqq {}^{y_1}(r \cup y_1)^{y_2} \leqq y_2$ and hence $\mu(r \cap y_1) - \mu(r) = \mu(y_1) - \mu(r \cup y_1) = \mu({}^{y_1}(r \cup y_1)^{y_2}) - \mu(y_2) \in V$. Dually we have $\mu(x_1 \cup r') - \mu(r') \in V$ for any $r' \in R$ with $r' \geqq {}^{x_0}(y_0 \cap x_1)^{x_1}$. Considering r and r' to be s' and $s' \cap y_1$, respectively, for $u = x_1 \cup (s' \cap y_1) \in R$ we have $\mu(u) - \mu(s') = \{\mu(x_1 \cup r') - \mu(r')\} + \{\mu(r \cap y_1) - \mu(r)\} \in 2V$. This relation and the one $\mu(u) - \mu(x_1) \in V$, which follows from $x_1 \leqq u \leqq y_1$, imply $\mu(x_0) + \mu(y_2) - \mu(s) = \mu(s') \in \mu(x_1) + 3V$. Likewise it follows that $\mu(x_0) + \mu(y_2) - \mu(t) \in \mu(x_1) + 3V$ and hence $\mu(s) - \mu(t) \in 6V \subset U$. This proves that $a \in \hat{R}$, implying that $\hat{R}$ is an r.i. sublattice.

In the above argument, to prove that $\hat{\mu}$ is an r.i. valuation, we may assume that the element $s \in R$ satisfies the condition $\mu(s) - \hat{\mu}(a) \in V$. Then we have $\hat{\mu}(a_0) + \hat{\mu}(a_2) - \hat{\mu}(a) - \hat{\mu}(a_1) \in \mu(x_0) + \mu(y_2) - \mu(s) - \mu(x_1) + 4V \subset 7V \subset U$, which implies that $\hat{\mu}$ is an r.i. valuation and this completes the proof.

Proposition 3.7. *If M is an l-group, if R is a subgroup, and if μ is a homomorphism, then $\hat{R}$ is a subgroup and $\hat{\mu}$ is a homomorphism.*

Proof. Let $a_1, a_2 \in \hat{R}$ and $U \in \mathcal{U}$ be any elements. Then, for $i=1, 2$, there are x_i, $y_i \in R$ with $x_i \leqq a_i \leqq y_i$ such that $\mu(s) - \mu(t) \in U$ for any s, $t \in R$ with $x_i \leqq s \leqq y_i$ and $x_i \leqq t \leqq y_i$. For $x = x_1 - y_2$ and $y = y_1 - x_2$ we have $x, y \in R$ with $x \leqq a_1 - a_2 \leqq y$. Let $s \in R$ be such that $x \leqq s \leqq y$ and put $s_1 = x_1 \cup (s + x_2)$ and $s_2 = s_1 - s$. Then it is easily verified that $s_i \in R$ and $x_i \leqq s_i \leqq y_i$ for $i=1, 2$. Hence we have $\mu(s) - \mu(x) = \{\mu(s_1) - \mu(x_1)\} - \{\mu(s_2) - \mu(y_2)\} \in 2U$ and this proves that $a_1 - a_2 \in \hat{R}$ or that $\hat{R}$ is a subgroup of M. Further Propositions 2.7 and 3.6 imply that $\hat{\mu}$ is a homomorphism.

4. A common extension of two valuations

In this section we shall give some lemmas to prove Theorem 2.

Assumption 4.1. *M is a distributive lattice and R is a subset of M. Further $\underset{\sim}{R}$ and $\tilde{R}$ are sublattices of M.*

We denote by $\bar{R}$ the set of all $a \in M$ such that $x \cap a \in \underset{\sim}{R}$ and $x \cup a \in \tilde{R}$ for any $x \in R$. Then

Proposition 4.1. *$\bar{R}$ is a sublattice of M.*

Proof. If $a, b \in \bar{R}$, then $x \in R$ implies $x \cap (a \cap b) = (x \cap a) \cap (x \cap b) \in \underset{\sim}{R}$ and $x \cup (a \cap b) = (x \cup a) \cap (x \cup b) \in \tilde{R}$ so that $a \cap b \in \bar{R}$. Hence the duality implies that $\bar{R}$ is a sublattice.

Proposition 4.2. *If M is a $\delta^+[\delta^-, \delta]$-lattice and if $\underset{\sim}{R}$ and $\tilde{R}$ are $\delta^+[\delta^-, \delta]$-sublattices, then $\bar{R}$ is a $\delta^+[\delta^-, \delta]$-sublattice.*

Proof. Let α be any element of $\sum_0(\bar{R})$. Then for any $x \in R$ we have $x \cap \alpha \in \sum_0(\underset{\sim}{R})$ and $x \cup \alpha \in \sum_0(\tilde{R})$. Hence $x \cap \bar{\alpha} = \overline{x \cap \alpha} \in \underset{\sim}{R}$ and $x \cup \bar{\alpha} = \overline{x \cup \alpha} \in \tilde{R}$ so that $\bar{\alpha} \in \bar{R}$. This proves that $\bar{R}$ is a δ^+-sublattice.

Lemma 4.1. *If M is a δ^+-lattice, if $\tilde{R}$ is a σ^+-sublattice, and if $\underset{\sim}{R}$ is a δ^+-sublattice containing R, then $\bar{R}$ is a σ^+-sublattice.*

Proof. Let $\alpha \in \sum(\bar{R})$ be any element. Then, for any $x \in R$, $x \cap \alpha(i) \leqq x \in R \subset \underset{\sim}{R}$ and $x \cup \alpha(i) \leqq x \cup \bar{\alpha} \in M$ imply $x \cap \alpha \in \sum_0(\underset{\sim}{R})$ and $x \cup \alpha \in \sum(\tilde{R})$ so that $x \cap \bar{\alpha} = \overline{x \cap \alpha} \in \underset{\sim}{R}$ and $x \cup \bar{\alpha} \in \tilde{R}$. This implies $\bar{\alpha} \in \bar{R}$, proving the lemma.

Corollary. *Assume that M is an r.i. δ-lattice and that R is a non-empty r.i. δ-sublattice. If $\underset{\sim}{R} = R^{\sigma-}$ and if $\tilde{R} = R^{\sigma+}$, then $\bar{R} = R^\sigma$.*

Proof. Lemma 2.3 implies that $\tilde{R}$ and $\underset{\sim}{R}$ are δ-sublattices and hence our lemma that $\bar{R}$ is a σ-sublattice. Obviously $R \subset \bar{R}$ and therefore $R^\sigma \subset \bar{R}$. For the converse, let $a \in \bar{R}$ be any element. Since $x \cup a \in \tilde{R} = \overline{\sum(R)}$ for some $x \in R$, we have a $\xi \in \sum(R)$ such that $x \cup a = \bar{\xi}$. Since $a \cap \xi(i) \in \underset{\sim}{R} \subset R^\sigma$ for each $i \in N$ we have $a \cap \xi \in \sum(R^\sigma)$ so that $a = a \cap \bar{\xi} = \overline{a \cap \xi} \in R^\sigma$. This implies $\bar{R} \subset R^\sigma$ and hence $\bar{R} = R^\sigma$.

Proposition 4.3. *If M is an r.c. lattice and if $\underset{\sim}{R}$ and $\tilde{R}$ are r.c. sublattices, then $\bar{R}$ is an r.c. sublattice.*

Proof. Suppose that $a \in \bar{R}$ and $b \in M$ are such that $a \cap b \in \bar{R}$ and $a \cup b \in \bar{R}$. Then for any $x \in R$ we have $(x \cap a) \cap (x \cap b) = x \cap (a \cap b) \in \underset{\sim}{R}$ and $(x \cap a) \cup (x \cap b) = x \cap (a \cup b) \in \underset{\sim}{R}$. Since $\underset{\sim}{R}$ is r.c., $x \cap a \in \underset{\sim}{R}$ implies $x \cap b \in \underset{\sim}{R}$. Dually $x \cup b \in \tilde{R}$ for any $x \in R$, and hence $b \in \bar{R}$.

Assumption 4.2. *R is non-empty and contained in $\underset{\sim}{R} \cap \tilde{R}$. If $x \in R$, then $x \cup a \in \tilde{R}$ for any $a \in \underset{\sim}{R}$ and $s \cap b \in \underset{\sim}{R}$ for any $b \in \tilde{R}$. $\underset{\sim}{\mu}$ and $\tilde{\mu}$ are valuations on $\underset{\sim}{R}$ and $\tilde{R}$, respectively, such that $\underset{\sim}{\mu}(u) = \tilde{\mu}(u)$ for any $u \in \underset{\sim}{R} \cap \tilde{R}$.*

Proposition 4.4. *It holds that $\underset{\sim}{R} \cup \tilde{R} \subset \bar{R}$. Further there uniquely exists a valuation $\bar{\mu}$ on $\bar{R}$ such that $\bar{\mu}$ is a common extension of $\underset{\sim}{\mu}$ and $\tilde{\mu}$.*

Proof. The first assertion is easily verified. Let x be a fixed element of R

and let us put $\bar{\mu}(a)=\underset{\sim}{\mu}(x\cap a)+\tilde{\mu}(x\cup a)-\underset{\sim}{\mu}(x)$ for each $a\in\bar{R}$. Then we have a map $\bar{\mu}$ of $\bar{R}$ into G. If $a, b\in\bar{R}$, then $\bar{\mu}(a\cup b)+\underset{\sim}{\mu}(x)=\underset{\sim}{\mu}(x\cap(a\cup b))+\tilde{\mu}(x\cup(a\cup b))=\underset{\sim}{\mu}((x\cap a)\cup(x\cap b))+\tilde{\mu}((x\cup a)\cup(x\cup b))$ and $\bar{\mu}(a\cap b)+\underset{\sim}{\mu}(x)=\underset{\sim}{\mu}((x\cap a)\cap(x\cap b))+\tilde{\mu}((x\cup a)\cap(x\cup b))$ so that $\bar{\mu}(a\cup b)+\bar{\mu}(a\cap b)+2\underset{\sim}{\mu}(x)=\{\underset{\sim}{\mu}(x\cap a)+\underset{\sim}{\mu}(x\cap b)\}+\{\tilde{\mu}(x\cup a)+\tilde{\mu}(x\cup b)\}=\{\bar{\mu}(a)+\underset{\sim}{\mu}(x)\}+\{\bar{\mu}(b)+\underset{\sim}{\mu}(x)\}$. This proves that $\bar{\mu}$ is a valuation. For any $c\in\underset{\sim}{R}$ it follows from $x\cup c\in\underset{\sim}{R}\cap\tilde{R}$ that $\bar{\mu}(c)=\underset{\sim}{\mu}(x\cap c)+\underset{\sim}{\mu}(x\cup c)-\underset{\sim}{\mu}(x)=\underset{\sim}{\mu}(c)$. Hence $\bar{\mu}$ is an extension of $\underset{\sim}{\mu}$ and, dually, of $\tilde{\mu}$. Thus the existence of $\bar{\mu}$ is proved and the uniqueness is obvious.

We denote by $\bar{\mu}$ the valuation in the above proposition.

Proposition 4.5. *If $\underset{\sim}{\mu}$ and $\tilde{\mu}$ are $\delta^+[\delta^-, \delta]$-convergent, then $\bar{\mu}$ is $\delta^+[\delta^-, \delta]$-convergent.*

Proof. Let $\alpha\in\sum(\bar{R})$ be such that $\bar{\alpha}\in\bar{R}$. Then for an $x\in R$ we have $x\cap\alpha\in\sum(\underset{\sim}{R})$ with $\overline{x\cap\alpha}=x\cap\bar{\alpha}\in\underset{\sim}{R}$ and $x\cup\alpha\in\sum(\tilde{R})$ with $\overline{x\cup\alpha}=x\cup\bar{\alpha}\in\tilde{R}$. This implies $\bar{\mu}(\alpha(i))=\underset{\sim}{\mu}((x\cap\alpha)(i))+\tilde{\mu}((x\cup\alpha)(i))-\underset{\sim}{\mu}(x)\to\underset{\sim}{\mu}(x\cap\bar{\alpha})+\tilde{\mu}(x\cup\bar{\alpha})-\underset{\sim}{\mu}(x)=\bar{\mu}(\bar{\alpha})$ as $i\to\infty$ and therefore $\bar{\mu}$ is δ^+-convergent.

Proposition 4.6. *If $\underset{\sim}{\mu}$ and $\tilde{\mu}$ are complete, then $\bar{\mu}$ is complete.*

Proof. For any $a\in\hat{\bar{R}}$, we are to prove that $a\in\bar{R}$ or that $x\cap a\in\underset{\sim}{R}$ and $x\cup a\in\tilde{R}$ for any $x\in R$. For any $U\in\mathcal{U}$, there exist $u, v\in\bar{R}$ with $u\leqq a\leqq v$ such that $\bar{\mu}(s)-\bar{\mu}(t)\in U$ for any $s, t\in\bar{R}$ with $u\leqq s\leqq v$ and $u\leqq t\leqq v$. Thus we have $x\cap u, x\cap v\in\underset{\sim}{R}$ with $x\cap u\leqq x\cap a\leqq x\cap v$. Suppose that $s_i\in\underset{\sim}{R}$ are such that $x\cap u\leqq s_i\leqq x\cap v$ for $i=1, 2$. Since $s_i\cap u=x\cap u$ we have $\underset{\sim}{\mu}(s_i)=\bar{\mu}(s_i)=\bar{\mu}(x\cap u)+\bar{\mu}(s_i\cup u)-\bar{\mu}(u)$ and hence it follows from $u\leqq s_i\cup u\leqq v$ that $\underset{\sim}{\mu}(s_1)-\underset{\sim}{\mu}(s_2)=\bar{\mu}(s_1\cup u)-\bar{\mu}(s_2\cup u)\in U$. Hence the completeness of $\underset{\sim}{\mu}$ implies $x\cap a\in\underset{\sim}{R}$ and dually $x\cup a\in\tilde{R}$.

Assumption 4.3. *M is an l-group and R contains 0. $\tilde{R}$ is a subsemigroup such that $\underset{\sim}{R}=-\tilde{R}$ and such that $q-p\in\tilde{R}$ for any $p, q\in\tilde{R}$ with $0\leqq p\leqq q$.*

Proposition 4.7. *It holds that $\bar{R}=\underset{\sim}{R}+\tilde{R}$.*

Proof. For any $a\in\bar{R}$ it follows from $a_-=0\cap a\in\underset{\sim}{R}$ and $a_+=0\cup a\in\tilde{R}$ that $a=a_-+a_+\in\underset{\sim}{R}+\tilde{R}$, which proves that $\bar{R}\subset\underset{\sim}{R}+\tilde{R}$. Conversely let us show that $r\in\bar{R}$ for any $s\in\underset{\sim}{R}$ and $t\in\tilde{R}$ with $r=s+t$. Let $x\in R$ be any element. Since $-s_-$, s_+, t, and $-x_-$ are elements of $\tilde{R}$, for $p=-s_-$ and $q=p\cup(s_++t-x_-)$ we have $p, q\in\tilde{R}$ with $0\leqq p\leqq q$ so that $\tilde{R}\ni q-p=0\cup(s+t-x_-)=x_-\cup r-x_-$. Since $x_-\in\tilde{R}$ we have $x_-\cup r=(q-p)+x_-\in\tilde{R}$ and hence $x\cup r=x\cup(x_-\cup r)\in\tilde{R}$. Dually $x\cap r\in\underset{\sim}{R}$ and hence $r\in\bar{R}$.

Corollary. *$\bar{R}$ is a subgroup of M.*

Proposition 4.8. *If $\tilde{\mu}$ is a homomorphism such that $\underset{\sim}{\mu}(-x)=-\tilde{\mu}(x)$ for any $x\in\tilde{R}$, then $\bar{\mu}$ is a homomorphism.*

Proof. We note that $\underset{\sim}{\mu}$ also is a homomorphsim. If $a, a' \in \underset{\sim}{R}$ and $b, b' \in \tilde{R}$ are such that $a+b=a'+b'$, then $\{\underset{\sim}{\mu}(a)+\tilde{\mu}(b)\}-\{\underset{\sim}{\mu}(a')+\tilde{\mu}(b')\}=\{\underset{\sim}{\mu}(a)+\underset{\sim}{\mu}(-b')\}-\{\underset{\sim}{\mu}(a')+\underset{\sim}{\mu}(-b)\}=\underset{\sim}{\mu}(a-b')-\underset{\sim}{\mu}(a'-b)=0$. Hence a map ν of $\bar{R}=\underset{\sim}{R}+\tilde{R}$ into G is defined by $\nu(a+b)=\underset{\sim}{\mu}(a)+\tilde{\mu}(b)$ for $a \in \underset{\sim}{R}$ and $b \in \tilde{R}$. It is obvious that ν is a homomorphism and hence a valuation. Further $0 \in \underset{\sim}{R} \cap \tilde{R}$ implies that ν is an extension of $\underset{\sim}{\mu}$ and $\tilde{\mu}$. Hence $\nu=\bar{\mu}$ and this proves the proposition.

5. The set of "measurable" elements

Assumption 5.1. *M is a modular lattice and L is a sublattice of M. Further μ is a map of L into G.*

Let us put $S=\{a \mid a \in L$ and $\mu(x)+\mu(a)=\mu(x \cup a)+\mu(x \cap a)$ for any $x \in L\}$ and denote by μ^S the restriction of μ on S. Then

Lemma 5.1. *Let a be an element of L and put $\Delta(x)=\mu(x \cup a)+\mu(x \cap a)-\mu(x)$ for each $x \in L$. Then $\Delta(x \cap v)=\Delta(x)$ for any $x \in L$ and $v \in S$ such that $a \leqq v$.*

Proof. Since $v \in S$, for $y=x \cap v$ we have $\mu(x)+\mu(v)=\mu(x \cup v)+\mu(y)$. Further the modularity implies $\mu(x \cup a)+\mu(v)=\mu((x \cup a) \cup v)+\mu((x \cup a) \cap v)=\mu(x \cup v)+\mu(y \cup a)$. Hence $\mu(x \cup a)-\mu(x)=\mu(y \cup a)-\mu(y)$ and thus it follows from $x \cap a=y \cap a$ that $\Delta(x)=\Delta(y)=\Delta(x \cap v)$.

Corollary 1. *Suppose that $a \in L$ and $v \in S$ are such that $a \leqq v$. Then, $a \in S$ if and only if $\mu(x)+\mu(a)=\mu(x \cup a)+\mu(x \cap a)$ for any $x \in L$ with $x \leqq v$.*

Proof. If the latter condition is satisfied, the lemma implies $\Delta(y)=\Delta(y \cap v)=\mu(a)$ for any $y \in L$ and hence $a \in S$.

Corollary 2. *Suppose that $a \in L$ and $u, v \in S$ are such that $u \leqq a \leqq v$. Then, $a \in S$ if and only if $\mu(x)+\mu(a)=\mu(x \cup a)+\mu(x \cap a)$ for any $x \in L$ with $u \leqq x \leqq v$.*

Proof. Suppose the latter condition is satisfied. Then the dual of the lemma implies $\Delta(x)=\Delta(u \cup x)=\mu(a)$ for any $x \in L$ with $x \leqq v$ and hence Corollary 1 implies $a \in S$.

Proposition 5.1. *S is a sublattice of M and μ^S is a valuation.*

Proof. Let us show that $a \cap b \in S$ for any $a, b \in S$. By Corollary 1 to Lemma 5.1, it suffices to prove that $\mu(x)+\mu(a \cap b)=\mu(x \cup (a \cap b))+\mu(x \cap a)$ for any $x \in L$ such that $x \leqq b$. Since $b \in S$ we have $\mu(a)+\mu(b)=\mu(a \cup b)+\mu(a \cap b)$ and, by the modularity, $\mu(x \cup a)+\mu(b)=\mu(a \cup b)+\mu(x \cup (a \cap b))$. Hence the relation needed follows from $\mu(x)+\mu(a)=\mu(x \cup a)+\mu(x \cap a)$. Thus we have

$a \cap b \in S$ and hence the duality implies that S is a sublattice. It is obvious that μ^S is a valuation.

Proposition 5.2. *If M is an r.c. distributive lattice and if L is an r.c. sublattice, then S is an r.c. sublattice of M.*

Proof. Suppose that $a \in L$ and $b \in S$ are such that $a \cap b \in S$ and $a \cup b \in S$. Then it is sufficient to prove that $a \in S$ or that $\Delta = 0$ for $\Delta = \mu(x) + \mu(a) - \mu(x \cup a) - \mu(x \cap a)$ with an arbitrary $x \in L$. The relations $\mu(x) + \mu(b) = \mu(x \cup b) + \mu(x \cap b)$ and $\mu(a) + \mu(b) = \mu(a \cup b) + \mu(a \cap b)$ imply $\Delta = \Delta_0 + \Delta_1$ for $\Delta_0 = \mu(x \cup b) + \mu(a \cup b) - \mu(x \cap a) - \mu(b) + \mu(x \cap a \cap b) - \mu(x \cup a \cup b)$ and $\Delta_1 = \mu(x \cap b) + \mu(a \cap b) - \mu(x \cup a) - \mu(b) + \mu(x \cup a \cup b) - \mu(x \cap a \cap b)$. Since $a \cup b \in S$ implies $\mu(x \cup b) + \mu(a \cup b) = \mu(x \cup a \cup b) + \mu((x \cap a) \cup b)$ and since $\mu(x \cap a) + \mu(b) = \mu((x \cap a) \cup b) + \mu(x \cap a \cap b)$ we have $\Delta_0 = 0$ and dually $\Delta_1 = 0$. Hence $\Delta = 0$ and this proves the proposition.

Lemma 5.2. *Let us assume that G is separated, that M is a δ^+-lattice, and that μ is δ^+-convergent. Then, for any $a_i \in S$, $i \in \boldsymbol{N}$, with $\bigcup_{i=1}^{\infty} a_i = a \in L$, it holds that $a \in S$.*

Proof. The sequence may be assumed to be increasing. If $x \in L$, then $a_i \in S$ implies $\mu(x) + \mu(a_i) = \mu(x \cup a_i) + \mu(x \cap a_i)$. Since $a_i \uparrow a$, $x \cup a_i \uparrow x \cup a$, and $x \cap a_i \uparrow x \cap a$ as $i \to \infty$, the δ^+-convergence of μ implies $\mu(x) + \mu(a) = \mu(x \cup a) + \mu(x \cap a)$ and hence $a \in S$.

Corollary. *Under the assumptions in the lemma, if L is a δ^+-sublattice, then S is a δ^+-sublattice of M and μ^S is a δ^+-convergent valuation.*

Proposition 5.3. *If G is separated, if M is an r.c. δ-lattice, if L is convex*[1], *and if μ is δ-convergent, then S is an r.c. δ-sublattice of M and μ^S is a δ-convergent valuation.*

Proof. Since L is an r.c. δ-sublattice, our proposition follows from Proposition 5.2 and Corollary to Lemma 5.2.

Assumption 5.2. *M is an l-group and L is a subsemigroup. Further $0 \in L$ and $\mu(0) = 0$.*

We write $T = \{a \mid a \in L$ and $\mu(x+a) = \mu(x) + \mu(a)$ for any $x \in L\}$ and denote by μ^T the restriction of μ on T.

Proposition 5.4. *T is a subsemigroup of M containing 0 and μ^T is a homomorphism. If L is a subgroup, then T is a subgroup of M.*

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1) A subset A of a (partially) ordered set K is *convex* [4] if $x \in A$ for any $a, b \in A$ and $x \in K$ with $a \leqq x \leqq b$.

Proof. If a and b are elements of T, then for any $x\in L$ we have $\mu(x+(a+b))=\mu((x+a)+b)=\mu(x+a)+\mu(b)=\mu(x)+\mu(a)+\mu(b)=\mu(x)+\mu(a+b)$, which implies that $a+b\in T$ or that T is a subsemigroup. Further $\mu(0)=0$ implies $0\in T$ and it is obvious that μ^T is a homomorphism. Let us assume that L is a subgroup. Then it suffices to show that $-a\in T$ for any $a\in T$. For any $x\in L$ it follows from $\mu(x)=\mu((x-a)+a)=\mu(x-a)+\mu(a)$ that $\mu(x-a)=\mu(x)-\mu(a)$. Considering x to be 0 we have $\mu(-a)=-\mu(a)$ and hence, for any $x\in L$, $\mu(x+(-a))=\mu(x-a)=\mu(x)-\mu(a)=\mu(x)+\mu(-a)$ so that $-a\in T$.

Lemma 5.3. *If G is separated and if μ is δ^+-convergent, then $a\in T$ for any increasing sequence $a_i\in T$, $i\in \boldsymbol{N}$, with $\bigcup_{i=1}^{\infty} a_i=a\in L$.*

Proof. For any $x\in L$, the relation $\mu(x+a_i)=\mu(x)+\mu(a_i)$ yields, when i tends to ∞, the equality $\mu(x+a)=\mu(x)+\mu(a)$ which proves that $a\in T$.

REMARK. Under the assumption that L is a subgroup, the following relation between the sets S and T can be verified: $a+b\in S$ for any $a\in S$ and $b\in T$. This implies that the following three conditions are equivalent: 1) $T\subset S$, 2) $0\in S$, 3) $\mu(x)=\mu(x\cup 0)+\mu(x\cap 0)$ for any $x\in L$.

6. Extension of δ^+-convergent valuations

Assumption II. *R is a sublattice of M and μ is a valuation on R.*

We say that an element $a\in M$ is *μ-inner regular* [*μ-outer regular*] if it satisfies the condition: for any $U\in\mathfrak{V}$ there exists an $x\in R$ with $x\leqq a$ [$a\leqq x$] such that $\mu(s)-\mu(x)\in U$ for any $s\in R$ with $x\leqq s\leqq a$ [$a\leqq s\leqq x$]. Obviously, the part "$\mu(s)-\mu(x)\in U\cdots[a\leqq s\leqq x]$" may be replaced by "$\mu(s)-\mu(t)\in U$ for any s, $t\in R$ with $x\leqq s\leqq a$ and $x\leqq t\leqq a$ [$a\leqq s\leqq x$ and $a\leqq t\leqq x$]".

Let us denote by E, in this section, the set of all μ-inner regular elements of M. Then we have

Proposition 6.1. *It holds that $R\subset\hat{R}\subset E$. Further $a\cap b\in E$ for any $a, b\in E$.*

Proof. The relation $R\subset\hat{R}\subset E$ is obvious. Let a_1, $a_2\in E$ and $U\in\mathfrak{V}$ be any elements. Then there exists an $x_i\in R$, for each $i=1, 2$, with $x_i\leqq a_i$ such that $\mu(s)-\mu(x_i)\in U$ for any $s\in R$ with $x_i\leqq s\leqq a_i$. Thus we have $x_1\cap x_2\in R$ with $x_1\cap x_2\leqq a_1\cap a_2$. Let $s\in R$ be such that $x_1\cap x_2\leqq s\leqq a_1\cap a_2$. Since $x_1\leqq x_1\cup s\leqq a_1$, for $s'=x_1\cap s\in R$ we have $\mu(s)-\mu(s')=\mu(x_1\cup s)-\mu(x_1)\in U$. Similarly $x_1\cap x_2\leqq s'\leqq a_1\cap a_2$ implies $\mu(s')-\mu(x_1\cap x_2)=\mu(s')-\mu(x_2\cap s')\in U$. Hence $\mu(s)-\mu(x_1\cap x_2)\in 2U$, which proves that $a_1\cap a_2\in E$.

The set E is not necessarily a sublattice as is seen in the following

EXAMPLE 6.1. Suppose that $m=\boldsymbol{Z}$ (=the set of all integers) in Example 1.1. Let R be the ring of all finite symmetric subsets of m and μ the valuation defined by $\mu(x)=\mathrm{Card}\,(x)\in\boldsymbol{R}=G$ for each $x\in R$. Then the intervals $a=(-\infty, 0]$ and $b=[0, \infty)$ in $\boldsymbol{Z}$ are elements of E although $a\cup b=m$ does not belong to E.

Proposition 6.2. *If μ is δ^+-fundamental, then E is a convex subset of M. The converse holds under the assumption that M is a δ^+-lattice and μ is δ^+-convergent.*

Proof. It suffices to show a contradiction under each of the following two conditions:

1) μ is δ^+-fundamental and E is not convex.
2) M is a δ^+-lattice and E is convex. μ is δ^+-convergent and not δ^+-fundamental.

First let us prove a lemma: for some $U\in\mathcal{U}$ there exists an increasing sequence $x_i\in R$, $i\in\boldsymbol{N}$, with an upper bound $c\in E$ such taht $\mu(x_{i+1})-\mu(x_i)\notin U$ for any $i\in\boldsymbol{N}$. If 1) is the case, there are elements a, $c\in E$ and $b\in M$ such that $b\notin E$ and $a\leqq b\leqq c$. Since $x_1\leqq a\leqq b$ for some $x_1\in R$, our lemma follows from the fact that $b\notin E$. In case 2), μ is not δ^+-fundamental and hence we have an increasing sequence $x_1'\in R$, $i\in\boldsymbol{N}$, with an upper bound $y\in R$ such that the sequence $\mu(x_i')$, $i\in\boldsymbol{N}$, is not fundamental. It is easy to see that some subsequence $x_i\in R$, $i\in\boldsymbol{N}$, of the sequence and the element $c=\bigcup_{i=1}^{\infty} x_i$, which lies between x_1, $y\in R\subset E$, satisfy the condition in the lemma. Thus the lemma is proved with, in case 2), the additional condition $c=\bigcup_{i=1}^{\infty} x_i$. Let $V\in\mathcal{U}$ be such that $2V\subset U$. Then $c\in E$ implies the existence of an $x\in R$ with $x\leqq c$ such that $\mu(s)-\mu(t)\in V$ for any s, $t\in R$ between x and c. Since μ is δ^+-fundamental (in case 1)) or since μ is δ^+-convergent and $\bigcup_{i=1}^{\infty}(x\cap x_i)=x\cap c=x\in R$ (in case 2)), the sequence $\mu(x\cap x_i)$, $i\in\boldsymbol{N}$, is fundamental. Hence there exists an $n\in\boldsymbol{N}$ such that $\mu(x\cap x_{n+1})-\mu(x\cap x_n)\in V$. Since $x\cup x_i\in R$, $i\in\boldsymbol{N}$, are such that $x\leqq x\cup x_i\leqq c$ we have $\mu(x_{n+1})-\mu(x_n)=\{\mu(x\cap x_{n+1})-\mu(x\cap x_n)\}+\{\mu(x\cup x_{n+1})-\mu(x\cup x_n)\}\in 2V\subset U$ and this is a contradiction.

Lemma 6.1. *Let E_0 be the set of all $a\in M$ such that*

1) *$x\leqq a$ for some $x\in R$,*

2) *if a is an upper bound of an increasing sequence $x_i\in R$, $i\in\boldsymbol{N}$, then the sequence $\mu(x_i)$, $i\in\boldsymbol{N}$, is fundamental.*

Then it holds that $E_0\subset E$.

Proof. Let $a\in E_0$ be such that $a\notin E$. Then some $U\in\mathcal{U}$ satisfies the condition: for any $x\in R$ with $x\leqq a$ there exists an $x'\in R$ such that $x\leqq x'\leqq a$ and $\mu(x')-\mu(x)\notin U$. Thus the condition 1) implies the existence of elements $x_i\in R$, $i\in\boldsymbol{N}$, such that $x_i\leqq x_{i+1}\leqq a$ and $\mu(x_{i+1})-\mu(x_i)\notin U$ for any $i\in\boldsymbol{N}$. This contradicts the condition 2).

Corollary. *If M is a δ^+-lattice, if $\bar{R}$ is a δ^+-sublattice of M such that $R\subset\bar{R}\subset R^{\sigma+}$, and if μ is extended to a δ^+-convergent map $\bar{\mu}$ of $\bar{R}$ into G, then it holds that $\bar{R}\subset E$. If, further, G is separated and if a δ^+-convergent map $\bar{\bar{\mu}}$ of $\bar{R}$ into G is an extension of μ, then $\bar{\bar{\mu}}=\bar{\mu}$.*

Proof. It is sufficient, for the first assertion, to prove that each element $a\in\bar{R}$ satisfies the conditions 1) and 2) in the lemma. 1) being obvious, suppose that a is an upper bound of an increasing sequence $x_i\in R$, $i\in \boldsymbol{N}$. Then our assumptions imply $x=\bigcup_{i=1}^{\infty} x_i\in\bar{R}$ and $\mu(x_i)=\bar{\mu}(x_i)\to\bar{\mu}(x)$ as $i\to\infty$ so that the sequence $\mu(x_i)$, $i\in\boldsymbol{N}$, is fundamental. Thus it is proved that $\bar{R}\subset E_0\subset E$. The second assertion immediately follows from $\bar{R}\subset R^{\sigma+}=\overline{\sum(R)}$.

REMARK. The set E_0 in the above lemma is obviously a convex subset of M such that $a\cap b\in E_0$ for any a, $b\in E_0$. Further it can be verified that the following conditions are equivalent: 1) $R\subset E_0$, 2) $E_0=E$, 3) μ is δ^+-fundamental.

Assumption III. *G is separated and complete.*

Then immediately we have

Proposition 6.3. *If $a\in M$ is μ-inner[outer] regular, then there uniquely exists an element $g\in G$ with the property: for any $U\in\mathfrak{U}$ there is an $x\in R$ with $x\leqq a$ $[a\leqq x]$ such that $\mu(s)-g\in U$ for any $s\in R$ with $x\leqq s\leqq a$ $[a\leqq s\leqq x]$.*

For a μ-inner[outer] regular element $a\in M$, the element $g\in G$ in the proposition will be called the *μ-inner value* [*μ-outer value*] of a.

Let us define a map $\breve{\mu}$ of E into G by assigning the μ-inner value $\breve{\mu}(a)$ to each $a\in E$. Then $\breve{\mu}$ is an extension of $\hat{\mu}$ and hence of μ.

Lemma 6.2. *If $\alpha\in\sum^*(E)$ and if the sequence $\alpha(i)$, $i\in\boldsymbol{N}$, has a lower bound $x\in R$, then for any $U\in\mathfrak{U}$ there exists a $\xi\in\sum^*(R)$ such that $x\leqq\xi(i)\leqq\alpha(i)$ and such that $\mu(\xi(i))-\breve{\mu}(\alpha(i))\in U$ for any $i\in\boldsymbol{N}$.*

Proof. Let $U_0=U$ and $U_i\in\mathfrak{U}$, $i\in\boldsymbol{N}$, be such that $2U_i\subset U_{i-1}$. Then the definition of $\breve{\mu}(\alpha(i))$ implies the existence of a $y_i\in R$ with $x\leqq y_i\leqq\alpha(i)$ such that $\mu(s)-\breve{\mu}(\alpha(i))\in U_{i+1}$ for any $s\in R$ with $y_i\leqq s\leqq\alpha(i)$. Note that any $s\in R$ with $y_i\leqq s\leqq\alpha(i)$ satisfies $\mu(s)-\mu(y_i)\in 2U_{i+1}\subset U_i$ Now putting $\xi(i)=x_i=\bigcap_{j=1}^{i} y_j$ we have a $\xi\in\sum^*(R)$ such that $x\leqq\xi(i)\leqq y_i\leqq\alpha(i)$. Hence it suffices to prove that $\mu(x_i)-\breve{\mu}(\alpha(i))\in U$ for any $i\in\boldsymbol{N}$. Let us write $\Delta_i(z)=\mu(x_i\cup z)-\mu(x_i)$ for each $i\in\boldsymbol{N}$ and $z\in R$. Then we can prove by induction the following: $\Delta_i(z)\in\sum_{j=1}^{i} U_j$ for any $z\in R$ with $z\leqq\alpha(i)$. The case $i=1$ is proved by $\Delta_1(z)=\mu(y_1\cup z)-\mu(y_1)\in U_1$ which follows from $y_1\leqq y_1\cup z\leqq\alpha(1)$. In case $i\geqq 2$, we see that

$\Delta_i(z)=\mu(z)-\mu(x_i\cap z)=\{\mu(z)-\mu(x_{i-1}\cap z)\}+\{\mu(x_{i-1}\cap z)-\mu(y_i\cap x_{i-1}\cap z)\}=\Delta_{i-1}(z)+\{\mu(s)-\mu(y_i)\}$ for $s=y_i\cup(x_{i-1}\cap z)$. Since $y_i\leqq s\leqq y_i\cup z\leqq\alpha(i)$, the relation $z\leqq\alpha(i)\leqq\alpha(i-1)$ and the assumption for induction imply $\Delta_i(z)\in\sum_{j=1}^{i-1}U_j+U_i=\sum_{j=1}^{i}U_j$, which completes the induction. In particular, $x_i\leqq y_i\leqq\alpha(i)$ implies $\mu(y_i)-\mu(x_i)=\Delta_i(y_i)\in\sum_{j=1}^{i}U_j$. The definition of y_i implies $\mu(y_i)-\breve{\mu}(\alpha(i))\in U_{i+1}$ and thus we have $\mu(x_i)-\breve{\mu}(\alpha(i))\in\sum_{j=1}^{i}U_j+U_{i+1}\subset U_0=U$ proving the lemma.

Corollary 1. *If μ is δ^--fundamental, then $\breve{\mu}$ is δ^--fundamental.*

Proof. Let $\alpha\in\sum_0^*(E)$ be any element. Then the lemma implies that for any $U\in\mathcal{U}$ there exists a $\xi\in\sum_0^*(R)$ such that $\mu(\xi(i))-\breve{\mu}(\alpha(i))\in U$ for any $i\in\boldsymbol{N}$. The assumption that μ is δ^--fundamental implies that the sequence $\mu(\xi(i))$, $i\in\boldsymbol{N}$, is fundamental, and therefore so is the sequence $\breve{\mu}(\alpha(i))$, $i\in\boldsymbol{N}$.

Corollary 2. *If M is a δ^--lattice, if R is a δ^--sublattice, and if μ is δ^--convergent, then $\breve{\mu}$ is δ^--convergent.*

Proof. Let $\alpha\in\sum^*(E)$ be such that $\bar{\alpha}\in E$ and $U\in\mathcal{U}$ be any. Then there exists an $x\in R$ with $x\leqq\bar{\alpha}$ such that $\mu(s)-\breve{\mu}(\bar{\alpha})\in U$ for any $s\in R$ with $x\leqq s\leqq\bar{\alpha}$. Hence the lemma implies the existence of a $\xi\in\sum_0^*(R)$ such that $x\leqq\xi(i)\leqq\alpha(i)$ and $\mu(\xi(i))-\breve{\mu}(\alpha(i))\in U$ for any $i\in\boldsymbol{N}$. It follows from $\bar{\xi}\in R$ and $x\leqq\bar{\xi}\leqq\bar{\alpha}$ that $\mu(\bar{\xi})-\breve{\mu}(\bar{\alpha})\in U$. Since μ is δ^--convergent there exists an $n\in\boldsymbol{N}$ such that $\mu(\xi(i))-\mu(\bar{\xi})\in U$ for any $i\geqq n$. Thus for any $i\geqq n$ we have $\breve{\mu}(\alpha(i))-\breve{\mu}(\bar{\alpha})\in 3U$ proving the corollary.

Assumption 6.1. *M is a δ^+-lattice and μ is δ^+-convergent.*

In this section, we write $\tilde{R}=E\cap R^{\sigma+}$ and denote by $\tilde{\mu}$ the restriction of $\breve{\mu}$ on $\tilde{R}$. Then

Proposition 6.4. *$\tilde{R}$ is a sublattice of M containing R. Further, $\tilde{\mu}$ is a δ^+-convergent valuation on $\tilde{R}$ and is an extension of μ. If μ is δ^+-fundamental, then $\tilde{R}$ is a δ^+-sublattice.*

Proof. In order to see that $\tilde{R}$ is a sublattice, let $a_1, a_2\in\tilde{R}$ be any elements and put $a_0=a_1\cap a_2$. Since Proposition 6.1 implies $a_0\in\tilde{R}$ we need only show that $a_1\cup a_2\in E$. For any $U\in\mathcal{U}$, let $V\in\mathcal{U}$ be such that $4V\subset U$. Since $a_k\in\tilde{R}\subset E$, for each $k=0, 1, 2$, there exists an $x_k\in R$ with $x_k\leqq a_k$ such that $\mu(s)-\mu(t)\in V$ for any s, $t\in R$ between x_k and a_k. Here we may assume that $x_0\leqq x_1\cap x_2$. Since $R\ni x_1\cup x_2\leqq a_1\cup a_2$ it suffices to show that $\mu(s)-\mu(x_1\cup x_2)\in U$ for any $s\in R$ with $x_1\cup x_2\leqq s\leqq a_1\cup a_2$. For $k=1, 2$, we can write $a_k=\bar{\alpha}_k$ with some $\alpha_k\in\sum(R)$, where it may be assumed that $x_k\leqq\alpha_k(i)$ for any $i\in\boldsymbol{N}$. Thus we have a $\sigma=s\cap(\alpha_1\cup\alpha_2)\in\sum(R)$ with $\bar{\sigma}=s\cap(a_1\cup a_2)=s$. Putting $s_{ki}=s\cap\alpha_k(i)$ for $k=1, 2$ and $i\in\boldsymbol{N}$ we have

$s_{ki}\in R$ with $x_k\leqq s_{ki}\leqq a_k$. Then $x_0\leqq x_1\cap x_2\leqq s_{1i}\cap s_{2i}\leqq a_0$ implies $\mu(s_{1i}\cup s_{2i})-\mu(x_1\cup x_2)=\sum_{k=1}^{2}\{\mu(s_{ki})-\mu(x_k)\}-\{\mu(s_{1i}\cap s_{2i})-\mu(x_1\cap x_2)\}\in 3V$ for any $i\in \boldsymbol{N}$. Since $\mu(s_{1i}\cup s_{2i})=\mu(s\cap(\alpha_1(i)\cup\alpha_2(i)))=\mu(\sigma(i))\rightarrow\mu(s)$ as $i\rightarrow\infty$, there exists an $n\in\boldsymbol{N}$ such that $\mu(s_{1n}\cup s_{2n})-\mu(s)\in V$. Hence $\mu(s)-\mu(x_1\cup x_2)\in 4V\subset U$ and this proves that $\tilde{R}$ is a sublattice.

Let us prove that $\tilde{\mu}$ is a valuation or that $\tilde{\mu}(a_1)+\tilde{\mu}(a_2)=\tilde{\mu}(a_0)+\tilde{\mu}(a_3)$ for any $a_1, a_2\in\tilde{R}$ and for $a_0=a_1\cap a_2$ and $a_3=a_1\cup a_2$. We can write $a_1=\bar{\alpha}_1$ and $a_2=\bar{\alpha}_2$ for some $\alpha_1, \alpha_2\in\sum(R)$. Putting $\alpha_0=\alpha_1\cap\alpha_2$ and $\alpha_3=\alpha_1\cup\alpha_2$ we have $\alpha_k\in\sum(R)$ with $\bar{\alpha}_k=a_k\in E$ for $k=0, 1, 2$, and 3. Since $\mu(\alpha_1(i))+\mu(\alpha_2(i))=\mu(\alpha_0(i))+\mu(\alpha_3(i))$ for each $i\in\boldsymbol{N}$, the equality needed immediately follows from the lemma: $\mu(\xi(i))\rightarrow\tilde{\mu}(\bar{\xi})$ $(i\rightarrow\infty)$ for any $\xi\in\sum(R)$ such that $\bar{\xi}\in E$. Let us prove this. For any $U\in\mathcal{U}$, there exists an $x\in R$ with $x\leqq\bar{\xi}$ such that $\mu(s)-\tilde{\mu}(\bar{\xi})\in U$ for any $s\in R$ with $x\leqq s\leqq\bar{\xi}$. Since the relations $x\cap\xi\in\sum(R)$ and $\overline{x\cap\xi}=x\cap\bar{\xi}=x$ imply the convergence $\mu((x\cap\xi)(i))\rightarrow\mu(x)$ $(i\rightarrow\infty)$ there is an $n\in\boldsymbol{N}$ such that $\mu(\xi(i))-\mu(x\cup\xi(i))=\mu(x\cap\xi(i))-\mu(x)\in U$ for any $i\geqq n$. Further $x\leqq x\cup\xi(i)\leqq\bar{\xi}$ implies $\mu(x\cup\xi(i))-\tilde{\mu}(\bar{\xi})\in U$ so that $\mu(\xi(i))-\tilde{\mu}(\bar{\xi})\in 2U$ for any $i\geqq n$, which proves the lemma.

To prove the δ^+-convergence of $\tilde{\mu}$, let $\alpha\in\sum(\tilde{R})$ be such that $\bar{\alpha}\in\tilde{R}$ and let $U\in\mathcal{U}$ be any. Then for each $i\in\boldsymbol{N}$ there exists an $x_i\in R$ with $x_i\leqq\alpha(i)$ such that $\mu(s)-\tilde{\mu}(\alpha(i))\in U$ for any $s\in R$ with $x_i\leqq s\leqq\alpha(i)$. Hence, by Lemma 1.3, some $\xi\in\sum(R)$ with $\bar{\xi}=\bar{\alpha}$ satisfies $x_i\leqq\xi(i)\leqq\alpha(i)$ for any $i\in\boldsymbol{N}$. Further the lemma proved above implies the existence of an $n\in\boldsymbol{N}$ such that $\mu(\xi(i))-\tilde{\mu}(\bar{\alpha})\in U$ for any $i\geqq n$. Thus $\mu(\xi(i))-\tilde{\mu}(\alpha(i))\in U$ implies $\tilde{\mu}(\alpha(i))-\tilde{\mu}(\bar{\alpha})\in 2U$ for any $i\geqq n$, proving that $\tilde{\mu}(\alpha(i))\rightarrow\tilde{\mu}(\bar{\alpha})(i\rightarrow\infty)$ or that $\tilde{\mu}$ is δ^+-convergent.

If μ is δ^+-fundamental then Proposition 6.2 implies that $\tilde{R}$ is a δ^+-sublattice and hence the proposition holds.

Corollary 1. *μ is extended to a δ^+convergent valuation $\bar{\mu}$ on a δ^+-sublattice $\bar{R}$ of M such that $R\subset\bar{R}\subset R^{\sigma+}$ if and only if μ is δ^+-fundamental. If such an extension $\bar{\mu}$ on $\bar{R}$ exists then $\bar{R}\subset\tilde{R}$ and $\bar{\mu}$ is the restriction of $\tilde{\mu}$.*

Proof. This follows from Proposition 1.1 and Corollary to Lemma 6.1.

Corollary 2. *If μ is δ^+-fundamental, then μ is uniquely extended to a δ^+-convergent valuation $\mu^{\delta+}$ on $R^{\delta+}$.*

Corollaries 1 and 2 to Lemma 6.2 imply that the extension $\tilde{\mu}$ partly inherits the properties of μ:

Proposition 6.5.

1) *If μ is δ^--fundamental, then $\tilde{\mu}$ is δ^--fundamental.*

2) *If M is a δ-lattice, if R is a δ^--sublattice, and if μ is δ-convergent, then $\tilde{\mu}$ is δ-convergent.*

If R is not a δ^--sublattice, in the above assertion 2), then $\tilde{\mu}$ is not necessarily δ^--convergent. In fact we have

EXAMPLE 6.2. The l-group $M=\boldsymbol{R}$ is a δ-lattice and $R=\{x \mid x\in M \text{ and } x\neq 0\}$ is a sublattice. Putting $\mu(x)=\operatorname{sgn} x$ we have a real-valued valuation on R which is δ-convergent and δ-fundamental. Then $\tilde{R}=R^{\delta+}=M$ and $\tilde{\mu}(0)=-1$. Since the sequence $\tilde{\mu}\left(\frac{1}{n}\right)=1$, $n\in \boldsymbol{N}$, does not converge to $\tilde{\mu}(0)$, the valuation $\tilde{\mu}$ is not δ^--convergent.

7. Extension of r.i. valuations

Assumption IV. *M is a δ^+-lattice. μ is δ^+-convergent and δ-fundamental.*

Assumption V. *$\tilde{R}$ is a δ^+-sublattice of M such that $R\subset\tilde{R}\subset R^{\sigma+}$ and each element of $\tilde{R}$ is μ-inner regular.*

EXAMPLE 7.1. If we denote by $\tilde{R}$ the set of all μ-inner regular elements in $R^{\sigma+}$, then $\tilde{R}$ satisfies Assumption V. (Proposition 6.4.)

EXAMPLE 7.2. If $\tilde{R}=R^{\delta+}$, then $\tilde{R}$ satisfies Assumption V.

Let us denote by $\tilde{\mu}(x)$ the μ-inner value of $x\in\tilde{R}$. Then we have

Proposition 7.1. *$\tilde{\mu}$ is a δ^+-convergent and δ-fundamental valuation on $\tilde{R}$ and is an extension of μ.*

Proof. Since $\tilde{\mu}$ is the restriction of the map $\tilde{\mu}$ in the previous section, our proposition follows from Propositions 6.4, 6.5, and 1.1.

Corollary 1. *$\tilde{\mu}$ is the unique map of $\tilde{R}$ into G such that $\mu(\xi(i))\to\tilde{\mu}(\bar{\xi})$ $(i\to\infty)$ for any $\xi\in\sum(R)$ with $\bar{\xi}\in\tilde{R}$.*

Corollary 2. *If M is a δ-lattice, if R is a δ^--sublattice, and if μ is δ-convergent, then $\tilde{\mu}$ is δ-convergent.*

Proof. This follows from Proposition 6.5.

Corollary 3. *Let F be the set of all $\tilde{\mu}$-outer regular elements of M. Then F is a convex subset of M containing $\tilde{R}$. Further F contains $a\cup b$ for any a, $b\in F$.*

Proof. This follows from the duals of Propositions 6.1 and 6.2.

Corollary 4. *If M is a δ-lattice and if $\tilde{\mu}$ is δ-convergent, then the set of all $\tilde{\mu}$-outer regular elements in $\tilde{R}^{\sigma-}$ is a δ^--sublattice of M containing $\tilde{R}$.*

Proof. This follows from the dual of Proposition 6.4.

Assumption VI. *L is a convex sublattice of M containing $\tilde{R}$ and each element of L is $\tilde{\mu}$-outer regular.*

EXAMPLE 7.3. If we denote by L the convex subset of M generated by $\tilde{R}$, then $L=\{a \mid a\in M$ and $p\leqq a\leqq q$ for some $p,\ q\in\tilde{R}\}=\{a \mid a\in M$ and $x\leqq a\leqq q$ for some $x\in R$ and $q\in\tilde{R}\}$ satisfies Assumption VI. (The $\tilde{\mu}$-outer regularity of $a\in L$ follows from Corollary 3 to Proposition 7.1.)

EXAMPLE 7.4. If $\tilde{R}=R^{\delta+}$ (Example 7.2), then the set L in Example 7.3 is the convex subset of M generated by R, namely, $L=\{a \mid a\in M$ and $x\leqq a\leqq y$ for some $x,\ y\in R\}$.

EXAMPLE 7.5. If M is a δ-lattice and if $\tilde{\mu}$ is δ-convergent, then the set $L=\{a \mid a\in M$ and $t\leqq a\leqq q$ for some $\tilde{\mu}$-outer regular element t in $\tilde{R}^{\sigma-}$ and for some $q\in\tilde{R}\}$ satisfies Assumption VI. (Corollaries 3 and 4 to Proposition 7.1.)

We denote by $\bar{\mu}(a)$ the $\tilde{\mu}$-outer value of $a\in L$. Then

Proposition 7.2. *$\bar{\mu}$ is a δ^{+}-convergent map of L into G and is an extension of $\tilde{\mu}$.*

Proof. The dual of Corollary 2 to Lemma 6.2 implies that $\bar{\mu}$ is δ^{+}-convergent and hence the proposition holds.

Lemma 7.1. *For any $a\in L$ and $U\in\mathcal{U}$, there exists a $p\in\tilde{R}$ with $a\leqq p$ such that $\bar{\mu}(s)-\bar{\mu}(a)\in U$ for any $s\in L$ with $a\leqq s\leqq p$.*

Proof. For a $V\in\mathcal{U}$ such that $2V\subset U$, the definition of $\bar{\mu}(a)$ implies the existence of a $p\in\tilde{R}$ with $a\leqq p$ such that $\tilde{\mu}(t)-\bar{\mu}(a)\in V$ for any $t\in\tilde{R}$ with $a\leqq t\leqq p$. It is easy to see that this element p satisfies the condition in the lemma.

Assumption VII. *M is an r.i. lattice and R is an r.i. sublattice. Further μ is an r.i. valuation.*

EXAMPLE 7.6. If M is an r.c. distributive lattice and if R is an r.c. sublattice, then Assumption VII is satisfied. (Example 2.1.)

EXAMPLE 7.7. If M is an l-group, if R is an r.i. subsemigroup (e.g. if R is a subgroup), and if μ is a homomorphism, then Assumption VII is satisfied. (Example 2.2.)

The following proposition, which follows from Propositions 2.8 and 2.6, implies that the dual of Assumption IV is satisfied.

Proposition 7.3. *M is a δ-lattice and μ is δ-convergent.*

Assumption V*. *$\underset{\sim}{R}$ is a δ^{-}-sublattice of M such that $R\subset\underset{\sim}{R}\subset R^{\sigma-}$ and each element of $\underset{\sim}{R}$ is μ-outer regular.*

We denote by $\underset{\sim}{\mu}(x)$ the μ-outer value of $x \in \underset{\sim}{R}$.

Assumption VI*. *L^* is a convex sublattice of M containing $\underset{\sim}{R}$ and each element of L^* is $\underset{\sim}{\mu}$-inner regular.*

We denote by $\underline{\mu}(a)$ the $\underset{\sim}{\mu}$-inner value of $a \in L^*$.
It is easy to see the following

Proposition 7.4. *L and L^* are r.i. δ-sublattices of M.*

Lemma 7.2. *If $p, q \in \tilde{R}$ and if $x \in R$, then $(q-x)_p \in \tilde{R}$ and $\tilde{\mu}((q-x)_p) + \mu(x) = \tilde{\mu}(q \cup x) + \tilde{\mu}(x \cap p)$.*

Proof. We can write $p = \bar{\alpha}$ and $q = \bar{\beta}$ for some $\alpha, \beta \in \sum(R)$. Putting $z_{ij} = (\beta(j) - x)_{\alpha(i)}$ and $\gamma(i) = z_{ii}$, for each $i, j \in \boldsymbol{N}$, we have an element $\gamma \in \sum(R)$ such that $\gamma(i) \leqq \beta(i) \cup x \leqq q \cup x \in \tilde{R}$, which implies $\bar{\gamma} \in \tilde{R}$. Since $\mu(\gamma(i)) + \mu(x) = \mu({}^{\alpha(i) \cap x}x^{x \cup \beta(i)}) + \mu(x) = \mu((x \cup \beta)(i)) + \mu((x \cap \alpha)(i))$ for any $i \in \boldsymbol{N}$, it follows from Corollary 1 to Proposition 7.1 that $\tilde{\mu}(\bar{\gamma}) + \mu(x) = \tilde{\mu}(x \cup q) + \tilde{\mu}(x \cap p)$. Now, since the double sequence z_{ij} is increasing with respect to each index, Lemma 2.2 implies $(q-x)_p = \bigcup_{j=1}^{\infty} (\beta(j) - x)_p = \bigcup_{j=1}^{\infty} (\bigcup_{i=1}^{\infty} z_{ij}) = \bigcup_{i=1}^{\infty} z_{ii} = \bar{\gamma} \in \tilde{R}$ and this proves the lemma.

Corollary 1. *If $p, q \in \tilde{R}$ and $t \in \underset{\sim}{R}$ are such that $p \leqq t \leqq x \leqq q$ for some $x \in R$, then ${}^p t^q \in \tilde{R}$ and $\tilde{\mu}({}^p t^q) + \underset{\sim}{\mu}(t) = \tilde{\mu}(p) + \tilde{\mu}(q)$.*

Proof. Since $t = \overline{\tau'}$ for some $\tau' \in \sum^*(R)$, we have a $\tau = x \cap \tau' \in \sum^*(R)$ such that $\bar{\tau} = x \cap t = t$ and such that $p \leqq t \leqq \tau(i) \leqq x \leqq q$ for any $i \in \boldsymbol{N}$. For $\lambda(i) = {}^p\tau(i)^q$ the lemma implies $\lambda(i) \in \tilde{R}$ and $\tilde{\mu}(\lambda(i)) + \mu(\tau(i)) = \tilde{\mu}(q) + \tilde{\mu}(p)$ for each $i \in \boldsymbol{N}$. Thus we have $\lambda \in \sum_0(\tilde{R})$ and ${}^p t^q = {}^p(\bigcap_{i=1}^{\infty} \tau(i))^q = \bigcup_{i=1}^{\infty} {}^p\tau(i)^q = \bar{\lambda} \in \tilde{R}$ so that $\tilde{\mu}(\lambda(i)) \to \tilde{\mu}({}^p t^q)$ $(i \to \infty)$. Hence the convergence $\mu(\tau(i)) \to \underset{\sim}{\mu}(t)$ $(i \to \infty)$ implies the equality needed.

Corollary 2. *If $x, y \in R$ and $t \in \underset{\sim}{R}$ are such that $x \leqq t \leqq y$, then ${}^x t^y \in \tilde{R}$ and $\tilde{\mu}({}^x t^y) + \underset{\sim}{\mu}(t) = \mu(x) + \mu(y)$.*

Lemma 7.3. *If $x, y \in R$ and $a \in M$ are such that $x \leqq a \leqq y$, then, $a \in L$, ${}^x a^y \in L^*$, and $\mu(x) + \mu(y) = \bar{\mu}(a) + \underline{\mu}({}^x a^y)$.*

Proof. The relations $a \in L$ and ${}^x a^y \in L^*$ are obvious. For any $U \in \mathcal{U}$, there exists a $p \in \underset{\sim}{R}$ with $x \leqq p \leqq {}^x a^y$ such that $\underset{\sim}{\mu}(t) - \underline{\mu}({}^x a^y) \in U$ for any $t \in \underset{\sim}{R}$ with $p \leqq t \leqq {}^x a^y$. Similarly there is a $q \in \tilde{R}$ with $a \leqq q \leqq y$ such that $\tilde{\mu}(s) - \bar{\mu}(a) \in U$ for any $s \in \tilde{R}$ with $a \leqq s \leqq q$. Since the dual of Corollary 2 to Lemma 7.2 implies ${}^x q^y \in \underset{\sim}{R}$ we have an element $t = p \cup {}^x q^y \in \underset{\sim}{R}$ such that $p \leqq t \leqq {}^x a^y$ and hence $\underset{\sim}{\mu}(t) - \underline{\mu}({}^x a^y) \in U$. Then, for $s = {}^x t^y$, Corollary 2 to Lemma 7.2 implies $s \in \tilde{R}$ and

$\mu(x)+\mu(y)=\tilde{\mu}(s)+\underset{\sim}{\mu}(t)$. Since $a\leqq s\leqq q$ we have $\tilde{\mu}(s)-\overline{\mu}(a)\in U$ and hence $\mu(x)+\mu(y)=\overline{\mu}(a)+\underline{\mu}({}^{x}a^{y})+2U$ for any $U\in\mathcal{U}$, which proves the lemma.

We put $S=\{a\,|\,a\in L$ and $\overline{\mu}(x)+\overline{\mu}(a)=\overline{\mu}(x\cup a)+\overline{\mu}(x\cap a)$ for any $x\in L\}$ and $S^*=\{a\,|\,a\in L^*$ and $\underline{\mu}(x)+\underline{\mu}(a)=\underline{\mu}(x\cup a)+\underline{\mu}(x\cap a)$ for any $x\in L^*\}$. Further we denote by $\mu^S[\mu^{S^*}]$ the restriction of $\overline{\mu}[\underline{\mu}]$ on $S[S^*]$.

Proposition 7.5. *S is a δ^+-sublattice of M such that $L\cap S^{\sigma+}\subset S\subset L$. Moreover μ^S is a δ^+-convergent valuation.*

Proof. This follows from Lemma 5.2 and its Corollary.

Proposition 7.6. *It holds that $R\subset\tilde{R}\subset S$.*

Proof. First for any $x\in R$ let us prove that $x\in S$ or that $\Delta\in 3U$ for any $a\in L$, $U\in\mathcal{U}$ and for $\Delta=\mu(x)+\overline{\mu}(a)-\overline{\mu}(x\cup a)-\overline{\mu}(x\cap a)$. Since $x\cup a\in L$ there is a $p\in\tilde{R}$ with $x\cup a\leqq p$ such that $\tilde{\mu}(s)-\overline{\mu}(x\cup a)\in U$ for any $s\in\tilde{R}$ with $x\cup a\leqq s\leqq p$. Since $a\leqq p$ there is a $q\in\tilde{R}$ with $a\leqq q\leqq p$ such that $\tilde{\mu}(t)-\overline{\mu}(a)\in U$ for any $t\in\tilde{R}$ with $a\leqq t\leqq q$. Further we have an $r\in\tilde{R}$ such that $x\cap a\leqq r\leqq x\cap q$ and $\tilde{\mu}(r)-\overline{\mu}(x\cap a)\in U$. Then, for $t={}^{r}x^{x\cup q}$, Lemma 7.2 implies $t=(q-x)_r\in\tilde{R}$ and $\tilde{\mu}(t)+\mu(x)=\tilde{\mu}(x\cup q)+\tilde{\mu}(r)$. Since $x\cup a\leqq x\cup q\leqq p$ we have $\tilde{\mu}(x\cup q)-\overline{\mu}(x\cup a)\in U$. Further $a={}^{x\cap a}x^{x\cup a}\leqq{}^{r}x^{x\cup q}=t\leqq{}^{x\cap q}x^{x\cup q}=q$ implies $\tilde{\mu}(t)-\overline{\mu}(a)\in U$. Thus we have $\Delta\in\mu(x)+\{\tilde{\mu}(t)+U\}-\{\tilde{\mu}(x\cup q)+U\}-\{\tilde{\mu}(r)+U\}=3U$ and hence it is proved that $R\subset S$. Then Proposition 7.5 implies $\tilde{R}\subset L\cap R^{\sigma+}\subset L\cap S^{\sigma+}\subset S$ and this proves the proposition.

In the above proof, Assumption VII is essentially used as is seen in the following:

EXAMPLE 7.8. Suppose that M is the complemented δ-lattice (Example 1.1) consisting of all subsets of the set $m=\{1, 2\}$. Putting $\mu(x)=\text{Card}\,(x)$ we have an $\boldsymbol{R}$-valued function μ defined on a sublattice $R=\{m, \{1\}, \phi\}$ of M. Then all assumptions but Assumption VII are satisfied for $\tilde{R}=\underset{\sim}{R}=R$ and $L=L^*=M$. However, the sublattice $S=\{m, \phi\}$ does not contain R.

Proposition 7.7. *Let $x, y\in R$ and $a\in M$ be such that $x\leqq a\leqq y$. Then $a\in S$ if and only if ${}^{x}a^{y}\in S^*$.*

Proof. Under the assumption ${}^{x}a^{y}\in S^*$ let us prove that $a\in S$. Let us put $a_1=a$, $a_3=a_1\cup a_2$, and $a_4=a_1\cap a_2$ for any $a_2\in L$. Then it suffices to show that $\Delta=0$ for $\Delta=\overline{\mu}(a_1)+\overline{\mu}(a_2)-\overline{\mu}(a_3)-\overline{\mu}(a_4)$. Corollary 2 to Lemma 5.1 implies that we may assume $x\leqq a_2\leqq y$. Then $x\leqq a_k\leqq y$ and hence we can put $a_k'={}^{x}a_k{}^{y}$ for each $k=1, 2, 3$, and 4. Thus it follows from $a_1'\in S^*$ and $a_2'\in L^*$ that $\underline{\mu}(a_1')+\underline{\mu}(a_2')=\underline{\mu}(a_1'\cap a_2')+\underline{\mu}(a_1'\cup a_2')=\underline{\mu}(a_3')+\underline{\mu}(a_4')$. Now Lemma 7.3 implies $\overline{\mu}(a_k)=\mu(x)+\mu(y)-\underline{\mu}(a_k')$ for each k and hence we have $\Delta=0$. Dually ${}^{x}a^{y}\in S^*$ follows from $a\in S$.

Lemma 7.4. *Let I be the convex subset of M generated by R and suppose that an element $a\in I$ satisfies the conditions:*

1) $\underline{\mu}(a)=\overline{\mu}(a)$

2) $\underline{\mu}(r)=\overline{\mu}(r)$ *and* $\underline{\mu}(a\cap r)+\overline{\mu}(a\cup r)=\underline{\mu}(a)+\underline{\mu}(r)$ *for any* $r\in(\underset{\sim}{R}\cup\tilde{R})\cap I$.

Then, for any $U\in\mathcal{U}$ there exist $p\in\underset{\sim}{R}$ and $q\in\tilde{R}$ with $p\leqq a\leqq q$ such that $\underline{\mu}(s)-\overline{\mu}(t)\in U$ for any s, $t\in I$ with $p\leqq s\leqq q$ and $p\leqq t\leqq q$.

Proof. We can write $x\leqq a\leqq y$ for some x, $y\in R$. For a $V\in\mathcal{U}$ such that $10V\subset U$ we have a $p\in\underset{\sim}{R}$ with $x\leqq p\leqq a$ such that $\underset{\sim}{\mu}(u)-\underline{\mu}(a)\in V$ for any $u\in\underset{\sim}{R}$ with $p\leqq u\leqq a$ and dually we have a $q\in\tilde{R}$ with $a\leqq q\leqq y$ such that $\tilde{\mu}(v)-\overline{\mu}(a)\in V$ for any $v\in\tilde{R}$ with $a\leqq v\leqq q$. Thus we have $p\in\underset{\sim}{R}$ and $q\in\tilde{R}$ with $x\leqq p\leqq a\leqq q\leqq y$. Let s, $t\in I$ be such that $p\leqq s\leqq q$ and $p\leqq t\leqq q$. Then $p\leqq s$ implies the existence of an $r\in\underset{\sim}{R}$ such that $p\leqq r\leqq s$ and $\underset{\sim}{\mu}(r)-\underline{\mu}(s)\in V$. Likewise $\underset{\sim}{\mu}(u)-\underline{\mu}(a\cap r)\in V$ for some $u\in\underset{\sim}{R}$ with $p\leqq u\leqq a\cap r$ and further $\tilde{\mu}(v)-\overline{\mu}(a\cup r)\in V$ for some $v\in\tilde{R}$ with $a\cup r\leqq v\leqq q$. Then $p\leqq u\leqq a\leqq v\leqq q$ implies $\underset{\sim}{\mu}(u)-\underline{\mu}(a)\in V$ and $\tilde{\mu}(v)-\overline{\mu}(a)\in V$ and thus we have $\underline{\mu}(a\cap r)+\overline{\mu}(a\cup r)-\underline{\mu}(a)-\overline{\mu}(a)\in 4V$. Now the condition 2) implies $\underline{\mu}(a\cap r)+\overline{\mu}(a\cup r)-\underline{\mu}(a)=\underline{\mu}(r)=\underset{\sim}{\mu}(r)\in\underline{\mu}(s)+V$ so that $\underline{\mu}(s)-\overline{\mu}(a)\in 5V$. Dually we have $\overline{\mu}(t)-\underline{\mu}(a)\in 5V$ and hence the condition 1) implies $\underline{\mu}(s)-\overline{\mu}(t)\in 10V\subset U$, proving the lemma.

Assumption 7.1. *R is a non-empty δ-sublattice of M.*

Proposition 7.8. *$\underset{\sim}{R}$ and $\tilde{R}$ are r.i. δ-sublattices of M. Moreover $a\cap b\in\underset{\sim}{R}$ and $a\cup b\in\tilde{R}$ for any $a\in\underset{\sim}{R}$ and $b\in\tilde{R}$.*

Proof. Lemma 2.3 implies that $\tilde{R}$ is an r.i. δ-sublattice of M and, dually, so is $\underset{\sim}{R}$. Since $b=\overline{\beta}$ for some $\beta\in\sum(R)$, we have $a\cap\beta\in\sum_0(\underset{\sim}{R})$ so that $a\cap b=\overline{a\cap\beta}\in\underset{\sim}{R}$. Dually we have $a\cup b\in\tilde{R}$.

Proposition 7.9. *$\underset{\sim}{\mu}$ and $\tilde{\mu}$ are δ-convergent r.i. valuations.*

Proof. The δ-convergence of $\tilde{\mu}$ follows from Corollary 2 to Proposition 7.1. Suppose that p, x, $q\in\tilde{R}$ are such that $p\leqq x\leqq q$. Then $x=\overline{\xi}$ for some $\xi\in\sum(R)$. Let us put $\eta(i)=(q-\xi(i))_p$ for each $i\in\boldsymbol{N}$. Then Lemma 7.2 implies $\eta(i)\in\tilde{R}$ and $\tilde{\mu}(\eta(i))+\mu(\xi(i))=\tilde{\mu}(q\cup\xi(i))+\tilde{\mu}(\xi(i)\cap p)=\tilde{\mu}(q)+\tilde{\mu}((p\cap\xi)(i))$. Further Lemma 2.1 implies $\eta\in\sum^*(\tilde{R})$ and $\overline{\eta}=\bigcap_{i=1}^{\infty}(q-\xi(i))_p=(q-x)_p={}^px^q\in\tilde{R}$ so that $\tilde{\mu}(\eta(i))\to\tilde{\mu}({}^px^q)$ $(i\to\infty)$. Since $\overline{\xi}=x$ and since $\overline{p\cap\xi}=p\cap x=p$ we have $\tilde{\mu}({}^px^q)+\tilde{\mu}(x)=\tilde{\mu}(q)+\tilde{\mu}(p)$, which proves that $\tilde{\mu}$ is an r.i. valuation. Dually the assertion on $\underset{\sim}{\mu}$ holds.

Corollary. $\underset{\sim}{\mu}(u)=\tilde{\mu}(u)$ for any $u\in\underset{\sim}{R}\cap\tilde{R}$.

Now we can see that Assumptions 4.1 and 4.2 are satisfied. Let us consider the sublattice $\overline{R}$ of M and the valuation $\overline{\mu}$ on $\overline{R}$ defined in section 4. Here we write $\overline{\overline{\mu}}$ for the valuation $\overline{\mu}$. Then

Proposition 7.10. *$\bar{R}$ is a δ-sublattice of M such that $\underset{\sim}{R} \cup \tilde{R} \subset \bar{R} \subset R^{\sigma}$. Further $\bar{\bar{\mu}}$ is a δ-convergent valuation on $\bar{R}$ and is a common extension of $\underset{\sim}{\mu}$ and $\tilde{\mu}$.*

Proof. This follows from Propositions 4.2 and 4.4,Corollary to Lemma 4.1, and Proposition 4.5.

Proposition 7.11. *If μ is complete, then $\bar{\bar{\mu}}$ is complete.*

Proof. This follows from Lemma 3.1 and Proposition 4.6.

Proposition 7.12. *If M is an r.c. lattice, then $\bar{R}$ is an r.c. sublattice.*

Proof. This follows from Proposition 4.3.

Assumption 7.2. *$\tilde{R}$ is the set of all μ-inner regular elements in $R^{\sigma+}$ (Example 7.1) and $\underset{\sim}{R}$ is the set of all μ-outer regular elements in R^{σ^-}.*

Proposition 7.13. *Let P be a δ-sublattice of M such that $R \subset P \subset R^{\sigma}$ and suppose that μ is extended to a δ-convergent valuation ν on P. Then P is contained in $\bar{R}$ and ν is the restriction of $\bar{\bar{\mu}}$.*

Proof. Let us consider the set $A = \{x \cup a \mid x \in R \text{ and } a \in P\}$. Since $P \subset R^{\sigma}$, Corollary to Lemma 4.1 implies that $A \subset R^{\sigma+}$. We can write $A = \{a \mid a \in P$ and $x \leqq a$ for some $x \in R\}$ so that A is a δ^+-sublattice of M containing R. Hence Corollary 1 to Proposition 6.4 implies that $A \subset \tilde{R}$ and that the restriction of ν on A is the restriction of $\tilde{\mu}$. Dually the subset $B = \{x \cap a \mid x \in R \text{ and } a \in P\}$ of P is contained in $\underset{\sim}{R}$ and the map ν coincides with $\underset{\sim}{\mu}$ on B. Hence each element $a \in P$ is contained in $\bar{R}$ and, for an $x \in R$, $\nu(a) = \nu(x \cup a) + \nu(x \cap a) - \nu(x) = \tilde{\mu}(x \cup a) + \underset{\sim}{\mu}(x \cap a) - \mu(x) = \bar{\bar{\mu}}(a)$.

8. Valuations on r.c. sublattices

Assumption 8.1. *M is an r.c. lattice.*

For S and μ^S defined in the preceding section we have

Proposition 8.1. *S is an r.c. δ-sublattice of M and μ^S is a δ-convergent valuation on S.*

Proof. Proposition 5.2 implies that S is an r.c. sublattice and hence our proposition follows from Propositions 7.5, 2.9, and 2.6.

Corollary. *It holds that $R^{\delta} \subset S$. Moreover μ is uniquely extended to a δ-convergent valuation μ^{δ} on R^{δ} and μ^{δ} is the restriction of μ^S.*

Proof. Proposition 7.6 implies $R^{\delta} \subset S$. The uniqueness of μ^{δ} follows from Corollary to Lemma 1.6.

Proposition 8.2. *If $a\in S\cup S^*$ is such that $x\leqq a\leqq y$ for some $x, y\in R$, then $a\in S\cap S^*$ and $\mu^S(a)=\mu^{S^*}(a)$.*

Proof. Suppose that $a\in S$ satisfies $x\leqq a\leqq y$ for some $x, y\in R$. Then Proposition 7.7 implies ${}^x a^y\in S^*$ and therefore the dual of Proposition 8.1 implies $a={}^x({}^x a^y)^y\in S^*$. Further Lemma 7.3 implies $\mu^S(a)+\mu^{S^*}({}^x a^y)=\mu(x)+\mu(y)=\mu^{S^*}(a)+\mu^{S^*}({}^x a^y)$ so that $\mu^S(a)=\mu^{S^*}(a)$. Thus the proposition follows from the duality.

Let us put $S_0=\{a\,|\,a\in S$ and $x\leqq a\leqq y$ for some $x, y\in R\}$ and denote by μ_0 the restriction of μ^S on S_0.

If $\tilde{R}$ and L are the sets stated in Example 7.4, then we see that $S_0=S$ and $\mu_0=\mu^S$.

Lemma 8.1. *For any $a\in S_0$ and for any $U\in\mathcal{U}$ there exist $p\in\underset{\sim}{R}\cap R^\delta$ and $q\in\tilde{R}\cap R^\delta$ with $p\leqq a\leqq q$ such that $\underline{\mu}(s)-\overline{\mu}(t)\in U$ for any $s\in L^*$ with $p\leqq s\leqq q$ and for any $t\in L$ with $p\leqq t\leqq q$.*

Proof. Proposition 8.2 implies that the element a satisfies the conditions 1) and 2) in Lemma 7.4 (Note that $(\underset{\sim}{R}\cup\tilde{R})\cap I\subset R^\delta\subset S_0$). Since $x\leqq a\leqq y$ for some $x, y\in R$, we may assume that the elements $p\in\underset{\sim}{R}$ and $q\in\tilde{R}$ in the lemma satisfy $x\leqq p\leqq a\leqq q\leqq y$. Then $p, q\in R^\delta$ and hence our lemma holds.

For the valuation μ^δ in Corollary to Proposition 8.1, we have

Proposition 8.3. *$S_0=\widehat{R^\delta}$ and μ_0 coincides with the completion $\widehat{\mu^\delta}$ of μ^δ.*

Proof. Let us show that $a\in S_0$ for any $a\in\widehat{R^\delta}$. Since $x\leqq a\leqq y$ for some $x, y\in R^\delta$ we have $a\in L$. Hence it suffices to show that $\Delta(u)=\overline{\mu}(a)$ for any $u\in L$ and for $\Delta(u)=\overline{\mu}(u\cup a)+\overline{\mu}(u\cap a)-\overline{\mu}(u)$. For any $U\in\mathcal{U}$, the assumption $a\in\widehat{R^\delta}$ implies the existence of $p, q\in R^\delta$ with $p\leqq a\leqq q$ such that $\mu^\delta(s)-\mu^\delta(t)\in U$ for any $s, t\in R^\delta$ with $p\leqq s\leqq q$ and $p\leqq t\leqq q$. Putting $v_0=a$, $v_1=p\cup(u\cap q)$, $v_2=v_1\cup a$, and $v_3=v_1\cap a$, we have $v_i\in L$ with $p\leqq v_i\leqq q$. Since $v_i\leqq q\leqq z$ for some $z\in R$, Lemma 7.1 implies the existence of an $r_i\in\tilde{R}$ with $v_i\leqq r_i\leqq z$ such that $\overline{\mu}(s)-\overline{\mu}(v_i)\in U$ for any $s\in L$ with $v_i\leqq s\leqq r_i$. Then, for $s_i=q\cap r_i\in R^\delta$ we have $v_i\leqq s_i\leqq q$ and $\mu^\delta(s_i)-\overline{\mu}(v_i)\in U$. Hence $p\leqq v_i\leqq s_i\leqq q$ implies $\overline{\mu}(v_i)-\overline{\mu}(v_0)\in\mu^\delta(s_i)-\mu^\delta(s_0)+2U\subset 3U$. Since $p, q\in R^\delta\subset S$ are such that $p\leqq a\leqq q$, Lemma 5.1 and its dual imply $\Delta(u)=\Delta(u\cap q)=\Delta(v_1)=\overline{\mu}(v_2)+\overline{\mu}(v_3)-\overline{\mu}(v_1)\in\overline{\mu}(v_0)+9U=\overline{\mu}(a)+9U$. Since $U\in\mathcal{U}$ is arbitrary, we have $\Delta(u)=\overline{\mu}(a)$ proving that $a\in S_0$ for any $a\in\widehat{R^\delta}$. It is easy to see that the relations $S_0\subset\widehat{R^\delta}$ and $\mu_0=\widehat{\mu^\delta}$ follow from Lemma 8.1.

9. Valuations on l-groups

Assumption 9.1. *M is an l-group, R is a subgroup of M, and μ is a homomorphism.*

Proposition 9.1. *$R^{\sigma+}$ and $R^{\delta+}$ are subsemigroups of M.*

Proof. For any ξ, $\eta\in\sum(R)$ we can define an element $\xi+\eta$ of $\sum(R)$ by $(\xi+\eta)(i)=\xi(i)+\eta(i)$. Then the identity $\overline{\xi+\eta}=\bar{\xi}+\bar{\eta}$ and Lemma 1.4 imply the proposition.

Lemma 9.1. *Assume that $\tilde{R}[\underset{\sim}{R}]$ is the set of all μ-inner[outer] regular elements in $R^{\sigma+}[R^{\sigma-}]$ (Example 7.1). Then*

1) $\underset{\sim}{R}=-\tilde{R}$.
2) $\underset{\sim}{\mu}(-x)=-\tilde{\mu}(x)$ *for any* $x\in\tilde{R}$.

Proof. Let us consider the dual automorphism φ of the l-group M defined by $\varphi(x)=-x$. Then the set $\underset{\sim}{R}$ is the inverse image $\varphi^{-1}(\tilde{\tilde{R}})$ of the set $\tilde{\tilde{R}}$ of all $\mu\circ\varphi^{-1}$-inner regular elements in $\varphi(R)^{\sigma+}=R^{\sigma+}$. Since we can write $\mu\circ\varphi^{-1}=\psi\circ\mu$ for the automorphism ψ of the topological group G defined by $\psi(g)=-g$, the $\mu\circ\varphi^{-1}$-inner regularity coincides with the μ-inner regularity. Hence we have $\tilde{\tilde{R}}=\tilde{R}$ and this proves that $\underset{\sim}{R}=\varphi^{-1}(\tilde{R})=-\tilde{R}$. Further $x\in\tilde{R}$ implies $\underset{\sim}{\mu}(-x)=(\underset{\sim}{\mu}\circ\varphi^{-1})(x)=\widetilde{\mu\circ\varphi^{-1}}(x)=\widetilde{\psi\circ\mu}(x)=(\psi\circ\tilde{\mu})(x)=-\tilde{\mu}(x)$.

Lemma 9.2. *Assume that R is a δ-sublattice and that $\tilde{R}$ is the set of all μ-inner regular elements in $R^{\sigma+}$. Then $\tilde{R}$ is an r.i. sublattice and subsemigroup of M and $\tilde{\mu}$ is a homomorphism.*

Proof. To prove that $\tilde{R}$ is a subsemigroup, let us put $a=a_1+a_2$ for any a_1, $a_2\in\tilde{R}$. Since the relation $a\in R^{\sigma+}$ follows from Proposition 9.1 we need only show the μ-inner regularity of a. For any $U\in\mathcal{U}$ and for a $V\in\mathcal{U}$ such that $2V\subset U$, the μ-inner regularity of a_i, for each $i=1$, 2, implies the existence of an $x_i\in R$ with $x_i\leqq a_i$ such that $\mu(s_i)-\mu(x_i)\in V$ for any $s_i\in R$ with $x_i\leqq s_i\leqq a_i$. Thus we have an element $x=x_1+x_2\in R$ such that $x\leqq a$. Let $s\in R$ be such that $x\leqq s\leqq a$. Since $s-x_2\in R$ and since $a_1\in\tilde{R}\subset R^{\sigma+}$ we have elements $s_1=(s-x_2)\cap a_1\in R^{\delta+}\subset R$ and $s_2=s-s_1\in R$. The inequality $x_i\leqq s_i\leqq a_i$ being easily verified we have $\mu(s_i)-\mu(x_i)\in V$ for $i=1$, 2. Hence $\mu(s)-\mu(x)=\sum_{i=1}^{2}\{\mu(s_i)-\mu(x_i)\}\in 2V\subset U$, which implies the μ-inner regularity of a proving that $\tilde{R}$ is a subsemigroup. Propositions 7.8 and 7.9 imply that $\tilde{R}$ is an r.i. sublattice and $\tilde{\mu}$ is an r.i. valuation, and hence Proposition 2.7 implies that $\tilde{\mu}$ is a homomorphism.

Now we can prove Theorem 2.

Proof of Theorem 2. It is obvious that Assumptions I, II, III, and VII are satisfied and Assumption IV follows from Proposition 1.1. For the sets $\tilde{R}$ and $\underset{\sim}{R}$ stated in Lemma 9.1, Assumptions V and V* are satisfied. Further Assumptions VI and VI* hold for some L and L^* (Example 7.3). We may assume that R is non-empty and then follow Assumptions 7.1 and 7.2. Hence Propositions 7.10 and 7.13 imply that the sublattice $\bar{R}$ and the valuation $\bar{\bar{\mu}}$ in section 7 satisfy the conditions in the theorem. If M is an r.c. lattice, then Proposition 7.12 implies that $\bar{R}$ is an r.c. sublattice. Let us assume that M is an l-group, that R is a subgroup of M, and that $\mu(0)=0$. It has been verified in section 7 that Assumptions 4.1 and 4.2 are satisfied. Since Assumption 9.1 follows from Proposition 2.7, we see that Assumption 4.3 follows from Lemmas 9.1 and 9.2. In fact, for any $p, q \in \tilde{R}$ with $0 \leqq p \leqq q$ we have $q-p={}^0p^q \in \tilde{R}$. Hence Corollary to Proposition 4.7 proves that $\bar{R}$ is a subgroup. Finally the above two lemmas and Proposition 4.8 imply that $\bar{\bar{\mu}}$ is a homomorphism.

REMARK. In the above arguments we see that Proposition 7.11 implies the following: *If μ is complete, in Theorem 2, then $\bar{\mu}$ is complete.*

Assumption 9.2. *$\tilde{R}$ and $\underset{\sim}{R}$ are subsemigroups of M.*

EXAMPLE 9.1. If $\tilde{R}=R^{\delta^+}$ (Example 7.2) and if $\underset{\sim}{R}=R^{\delta^-}$, then Assumption 9.2 is satisfied (Proposition 9.1).

Proposition 9.2. *$\tilde{\mu}$ is a homomorphism.*

Proof. For each $a_k \in \tilde{R}$, $k=1, 2$, we can write $a_k=\bar{\alpha}_k$ for some $\alpha_k \in \sum(R)$. For the element $\alpha_0=\alpha_1+\alpha_2 \in \sum(R)$ defined in the proof of Proposition 9.1, we have an element $a_0=\bar{\alpha}_0=\bar{\alpha}_1+\bar{\alpha}_2=a_1+a_2 \in \tilde{R}$. Hence our proposition follows from Assumption 9.1 and Corollary 1 to Proposition 7.1.

Assumption 9.3. *L and L^* are subsemigroups of M.*

EXAMPLE 9.2. If $L[L^*]$ is the convex subset of M generated by $\tilde{R}[\underset{\sim}{R}]$ (Example 7.3), then Assumption 9.3 is satisfied.

We put $T=\{a \mid a \in L \text{ and } \bar{\mu}(x+a)=\bar{\mu}(x)+\bar{\mu}(a) \text{ for any } x \in L\}$ and $T^*=\{a \mid a \in L^* \text{ and } \underline{\mu}(x+a)=\underline{\mu}(x)+\underline{\mu}(a) \text{ for any } x \in L^*\}$. Further we denote by $\mu^T[\mu^{T^*}]$ the restriction of $\bar{\mu}[\underline{\mu}]$ on $T[T^*]$.

Then Proposition 5.4 implies

Proposition 9.3. *T is a subsemigroup of M and μ^T is a homomorphism.*

Further Lemma 5.3 implies

Lemma 9.3. $L \cap \overline{\sum(T)} \subset T$.

Corollary. *If T is a sublattice of M, then T is a δ^+-sublattice.*

Proposition 9.4. *It holds that $R \subset \tilde{R} \subset T$.*

Proof. First let us show that $T \subset R$ or that $\bar{\mu}(a+x) \in \bar{\mu}(a)+\mu(x)+2U$ for any $x \in R$, $a \in L$, and $U \in \mathcal{U}$. There exists a $p \in \tilde{R}$ with $a \leqq p$ such that $\tilde{\mu}(s)-\bar{\mu}(a) \in U$ for any $s \in \tilde{R}$ with $a \leqq s \leqq p$. Likewise for some $q \in \tilde{R}$ with $a+x \leqq q$ it holds that $\tilde{\mu}(t)-\bar{\mu}(a+x) \in U$ for any $t \in \tilde{R}$ with $a+x \leqq t \leqq q$. Since $q-x \in$

$\tilde{R}+R\subset\tilde{R}$ we have an element $s=p\cap(q-x)\in\tilde{R}$ such that $a\leqq s\leqq p$, which implies that $\tilde{\mu}(s)-\overline{\mu}(a)\in U$. Thus it follows from $s+x\in\tilde{R}$ and $a+x\leqq s+x\leqq q$ that $\overline{\mu}(a+x)\in\tilde{\mu}(s+x)+U=\tilde{\mu}(s)+\tilde{\mu}(x)+U\subset\overline{\mu}(a)+\mu(x)+2U$, which proves that $R\subset T$. Hence Lemma 9.3 implies the proposition: $R\subset\tilde{R}\subset L\cap\overline{\sum(R)}\subset L\cap\overline{\sum(T)}\subset T$.

Lemma 9.4. *An element $a\in L$ is contained in T if: for any $U\in\mathcal{U}$ there exist $p\in-\tilde{R}$ and $q\in\tilde{R}$ with $p\leqq a\leqq q$ such that $\mu(x)\in U$ for any $x\in R$ with $0\leqq x\leqq q-p$.*

Proof. It is sufficient to show that $\overline{\mu}(a)+\overline{\mu}(b)-\overline{\mu}(a+b)\in 6U$ for any $b\in L$ and $U\in\mathcal{U}$. Suppose that $p\in-\tilde{R}$ and $q\in\tilde{R}$ satisfy the conditions in the lemma. Then there exists a $u\in\tilde{R}$ with $a\leqq u\leqq q$ and $\tilde{\mu}(u)-\overline{\mu}(a)\in U$. There exists a $w\in\tilde{R}$ with $a+b\leqq w$ such that $\tilde{\mu}(w')-\overline{\mu}(a+b)\in U$ for any $w'\in\tilde{R}$ with $a+b\leqq w'\leqq w$. Further $b=(a+b)-a\leqq w-p\in\tilde{R}$ implies the existence of a $v\in\tilde{R}$ with $b\leqq v\leqq w-p$ and $\tilde{\mu}(v)-\overline{\mu}(b)\in U$. Since $s=u+v\in\tilde{R}$ satisfies $a+b\leqq w\cap s\leqq w$ we have $\tilde{\mu}(w\cap s)-\overline{\mu}(a+b)\in U$ so that $\overline{\mu}(a)+\overline{\mu}(b)-\overline{\mu}(a+b)\in\tilde{\mu}(u)+\tilde{\mu}(v)-\tilde{\mu}(w\cap s)+3U=\tilde{\mu}(s)-\tilde{\mu}(w\cap s)+3U$.

Now it suffices to prove that $\tilde{\mu}(s)-\tilde{\mu}(w\cap s)\in 3U$. Let us put $c=w\cup s-w$. Since $z\leqq w$ for some $z\in R$ we have $0\leqq c\leqq w\cup s-z\in\tilde{R}+R\subset L$ so that $c\in L$. Since $c=0\cup(u+v-w)\leqq 0\cup(q-p)=q-p\in\tilde{R}$ there is a $t\in\tilde{R}$ such that $c\leqq t\leqq q-p$ and $\tilde{\mu}(t)-\overline{\mu}(c)\in U$. Further $0\leqq c\leqq t$ implies the existence of an $x\in R$ with $0\leqq x\leqq t$ and $\mu(x)-\tilde{\mu}(t)\in U$. Thus $0\leqq x\leqq t\leqq q-p$ implies $\mu(x)\in U$ and hence it follows from $w\in\tilde{R}\subset T$ that $\tilde{\mu}(s)-\tilde{\mu}(w\cap s)=\tilde{\mu}(w\cup s)-\tilde{\mu}(w)=\overline{\mu}(c+w)-\overline{\mu}(w)=\overline{\mu}(c)\in\tilde{\mu}(t)+U\subset\mu(x)+2U\subset 3U$, which proves the lemma.

Assumption 9.4. $L=L^*=\{a\,|\,a\in M$ *and* $x\leqq a\leqq y$ *for some* $x, y\in R\}$.

EXAMPLE 9.3. If $\tilde{R}$ and $\underset{\sim}{R}$ are the sets in Example 9.1 and if L and L^* are the sets in Example 9.2, then Assumption 9.4 is satisfied. Conversely, the following lemma shows that these conditions are necessary under the assumption.

Lemma 9.5. $\tilde{R}=R^{\delta+}$ *and* $\underset{\sim}{R}=R^{\delta^-}=-\tilde{R}$.

Proof. Assumptions V and VI imply $R^{\delta+}\subset\tilde{R}\subset R^{\sigma+}\cap L\subset R^{\delta+}$ and dually we have $\underset{\sim}{R}=R^{\delta^-}=\overline{\sum_0^*(R)}=-\overline{\sum_0(-R)}=-\overline{\sum_0(R)}=-\tilde{R}$.

Proposition 9.5. *L and T are subgroups of M.*

Proof. This follows from Proposition 5.4.

REMARK. We can verify that $T\subset S$ for the set S in section 7, or equivalently (Remark in section 5) that $\overline{\mu}(x_0)-\overline{\mu}(x_1)-\overline{\mu}(x_2)=0$ for any $x_0\in L$ and for $x_1=x_0\cup 0$, $x_2=x_0\cap 0$. In fact, for any $U\in\mathcal{U}$ there exists a $p_i\in\tilde{R}$, for $i=0, 1, 2$, with $x_i\leqq p_i$ such that $\tilde{\mu}(s_i)-\overline{\mu}(x_i)\in U$ for any $s_i\in\tilde{R}$ with $x_i\leqq s_i\leqq p_i$. Putting

$s_1=p_1\cap(p_0\cup 0)$, $s_2=p_2\cap(p_0\cap 0)$, and $s_0=s_1+s_2$, we have $x_i\leqq s_i\leqq p_i$ so that $\overline{\mu}(x_0)-\overline{\mu}(x_1)-\overline{\mu}(x_2)\in\tilde{\mu}(s_0)-\tilde{\mu}(s_1)-\tilde{\mu}(s_2)+3U=3U$.

Lemma 9.6. *$\underline{\mu}(a)=-\overline{\mu}(-a)$ for any $a\in L$.*

Proof. For the dual automorphism φ of M and the automorphism ψ of G considered in the proof of Lemma 9.1, the identity $\underset{\sim}{\mu}\circ\varphi^{-1}=\psi\circ\tilde{\mu}$ is easily verified (see the proof of Lemma 9.1, 2)). Hence we have $\underline{\mu}(a)=(\underset{\sim}{\mu}$-inner value of $a)=(\underset{\sim}{\mu}\circ\varphi^{-1}$-outer value of $\varphi(a))=(\psi\circ\tilde{\mu}$-outer value of $-a)=\psi(\tilde{\mu}$-outer value of $-a)=-\overline{\mu}(-a)$.

Corollary 1. *$T=T^*$ and $\mu^T=\mu^{T^*}$.*

Proof. Let a be any element of T. Then $a\in L^*$ is obvious and for any $x\in L^*$ it follows from $-a\in T$ that $\underline{\mu}(x+a)=-\overline{\mu}(-x-a)=-\overline{\mu}(-x)-\overline{\mu}(-a)=\underline{\mu}(x)+\underline{\mu}(a)$. Thus we have $T\subset T^*$ and dually $T^*\subset T$. Further $a\in T$ implies $\mu^{T^*}(a)=\underline{\mu}(a)=-\overline{\mu}(-a)=-\mu^T(-a)=\mu^T(a)$ so that $\mu^T=\mu^{T^*}$.

Corollary 2. *$\underline{\mu}(a)+\overline{\mu}(b)=\mu^T(a+b)$ for any a, $b\in L$ such that $a+b\in T$.*

Proof. This follows from $\overline{\mu}(b)=\overline{\mu}(-a+(a+b))=\overline{\mu}(-a)+\overline{\mu}(a+b)=-\underline{\mu}(a)+\mu^T(a+b)$.

Lemma 9.7. *For any $a\in T$ and for any $U\in\mathcal{U}$ there exist $p\in\underset{\sim}{R}$ and $q\in\tilde{R}$ with $p\leqq a\leqq q$ such that $\underline{\mu}(s)-\overline{\mu}(t)\in U$ for any s, $t\in L$ with $p\leqq s\leqq q$ and $p\leqq t\leqq q$.*

Proof. It suffices to see that the element $a\in T\subset L=I$ satisfies the conditions 1) and 2) in Lemma 7.4 and this follows from the above two corollaries and Proposition 9.4. In fact, for any $r\in(\underset{\sim}{R}\cup\tilde{R})\cap I$ we have $r\in T=T^*$ and $a\cap r+a\cup r=a+r\in T$ so that $\underline{\mu}(a\cap r)+\overline{\mu}(a\cup r)=\mu^T(a+r)=\mu^T(a)+\mu^T(r)=\underline{\mu}(a)+\underline{\mu}(r)$.

Corollary. *An element $a\in M$ is contained in T if and only if: for any $U\in\mathcal{U}$ there exist $p\in\underset{\sim}{R}$ and $q\in\tilde{R}$ with $p\leqq a\leqq q$ such that $\mu(x)\in U$ for any $x\in R$ with $0\leqq x\leqq q-p$.*

Proof. If $a\in T$, then for any $U\in\mathcal{U}$ the elements p and q in the lemma satisfy the conditions in the corollary as is easily verified by putting $s=x+p$ and $t=p$. The converse follows from Lemma 9.4.

Proposition 9.6. *T is a δ-sublattice of M and μ^T is a δ-convergent homomorphism.*

Proof. First we show that $a_+=a\cup 0\in T$ for any $a\in T$. For any $U\in\mathcal{U}$, we have the elements $p\in\underset{\sim}{R}$ and $q\in\tilde{R}$ stated in the above corollary. Then the elements $p_+\in\underset{\sim}{R}$ and $q_+\subset\tilde{R}$ satisfy $p_+\leqq a_+\leqq q_+$ and for any $x\in R$ with $0\leqq x\leqq q_+-p_+$ it follows from $0\leqq x\leqq q_+-p_+=(q-p)-(q\cap 0-p\cap 0)\leqq q-p$ that

$\mu(x)\in U$. Thus the corollary implies $a_+\in T$. This proves that the subgroup T is an l-subgroup and hence Corollary to Lemma 9.3 and Proposition 2.9 imply that T is a δ-sublattice. The δ-convergence of the homomorphism μ^T follows from Propositions 7.2 and 2.6.

Corollary. *It holds that $\underset{\sim}{R}\cup\tilde{R}\subset R^\delta\subset T$. The homomorphism μ is uniquely extended to a δ-convergent map μ^δ on R^δ and μ^δ is the restriction of μ^T. Further R^δ is a subgroup of M and μ^δ is a homomorphism.*

Proof. The inclusion $R^\delta\subset T$ is a consequence of the proposition. The uniqueness of μ^δ follows from Corollary to Lemma 1.6. Finally Proposition 2.11 implies that R^δ is a subgroup of M and thus the corollary holds.

For the homomorphism μ^δ in the corollary we have

Proposition 9.7. *$T=\widehat{R^\delta}$ and μ^T coincides with the completion $\widehat{\mu^\delta}$ of μ^δ.*

Proof. Let us prove that $a\in T$ for any $a\in\widehat{R^\delta}$. For a given $U\in\mathcal{U}$, let $V\in\mathcal{U}$ be such that $6V\subset U$. Since $a\in\widehat{R^\delta}$ there are $x, y\in R^\delta$ with $x\leqq a\leqq y$ such that $\mu^\delta(w)-\mu^\delta(w')\in V$ for any $w, w'\in R^\delta$ with $x\leqq w\leqq y$ and $x\leqq w'\leqq y$. Since $x\in R^\delta\subset L^*$ there exists a $p\in\underset{\sim}{R}$ with $p\leqq x$ such that $\underset{\sim}{\mu}(u)-\underset{\sim}{\mu}(x)\in V$ for any $u\in\underset{\sim}{R}$ with $p\leqq u\leqq x$. Dually there exists a $q\in\tilde{R}$ with $y\leqq q$ such that $\tilde{\mu}(v)-\overline{\mu}(y)\in V$ for any $v\in\tilde{R}$ with $y\leqq v\leqq q$. Thus we have $p\in\underset{\sim}{R}$ and $q\in\tilde{R}$ with $p\leqq x\leqq a\leqq y\leqq q$. Hence, by Corollary to Lemma 9.7, it suffices to verify that $\mu(r)\in U$ for any $r\in R$ with $0\leqq r\leqq q-p$. Suppose that an element $s\in R^\delta$ satisfies the condition $p\leqq s\leqq q$. Since $p\leqq s\cap x$ there exists a $u\in\underset{\sim}{R}$ with $p\leqq u\leqq s\cap x$ and $\underset{\sim}{\mu}(u)-\underset{\sim}{\mu}(s\cap x)\in V$. Then $p\leqq u\leqq s\cap x\leqq x$ implies $\underset{\sim}{\mu}(u)-\underset{\sim}{\mu}(x)\in V$ so that $\mu^\delta(s)-\mu^\delta(s\cup x)=\underset{\sim}{\mu}(s\cap x)-\underset{\sim}{\mu}(x)\in 2V$. Dually, for any $t\in R^\delta$ with $p\leqq t\leqq q$ we have $\mu^\delta(t)-\mu^\delta(t\cap y)\in 2V$. Putting $s=r+p$ and $t=s\cup x$ we have s, $t\in R^\delta$ with $p\leqq s\leqq t\leqq q$ and hence $\mu^\delta(s)-\mu^\delta(t\cap y)=\{\mu^\delta(s)-\mu^\delta(s\cup x)\}+\{\mu^\delta(t)-\mu^\delta(t\cap y)\}\in 4V$. Since $x\leqq t\cap y\leqq y$ implies $\mu^\delta(t\cap y)-\mu^\delta(x)\in V$ and since $\mu^\delta(p)-\mu^\delta(x)=\underset{\sim}{\mu}(p)-\underset{\sim}{\mu}(x)\in V$ we have $\mu(r)=\mu^\delta(s)-\mu^\delta(p)\in 6V\subset U$. Thus it is proved that $a\in T$ for any $a\in\widehat{R^\delta}$ or that $\widehat{R^\delta}\subset T$. The converse $T\subset\widehat{R^\delta}$ and the relation $\mu^T=\widehat{\mu^\delta}$ immediately follow from Lemma 9.7 and thus the proposition is proved.

Proof of Theorem 1. We have seen that the uniqueness of μ^δ follows from Corollary to Lemma 1.6. In either case 1) or 2) we see that Assumptions I, II, III, IV, and VII are satisfied. Let us put $\tilde{R}=R^{\delta+}$ and $\underset{\sim}{R}=R^{\delta-}$, and denote by $L=L^*$ the convex subset of M generated by R. Then Assumptions V, V*, VI, and VI* are satisfied (Examples 7.2, 7.3, and 7.4). In case 1), Assumption 8.1 is satisfied and hence Corollary to Proposition 8.1 and Proposition 2.10 imply the theorem. In case 2), all assumptions in section 9 are satisfied so that Corollary to Proposition 9.6 implies the theorem. Thus the proof is completed.

REMARK. The following problems are unsolved:

1) Can the conditions 1) and 2) in Theorem 1 be unified in a general form: 0) M is an r.i. lattice, R is an r.i. sublattice, and μ is an r.i. valuation?

2) In Theorem 2 can we say that $\bar{R}$ is an r.i. sublattice and $\bar{\mu}$ is an r.i. valuation?

3) Does it hold that $T=S$ under the assumptions in section 9?

4) Does the valuation μ^S in Proposition 8.1 become an extension of the maximal extension, in the sense of Theorem 2, of μ^s when the sublattices $\tilde{R}$ and L are chosen sufficiently large?

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