

ON MEASURE-COMPACTNESS AND BOREL MEASURE-COMPACTNESS

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1. Introduction

Moran has introduced "measure-compactness" and "strongly measure-compactness" in [7], [8]. For measure-compactness, Kirk has independently presented in [4]. For Borel measures, "Borel measure-compactness" and "property B" are introduced by Gardner [1].

We study, defining "a weak Radon space", the relation among these properties at the last of §2. Furthermore we investigate the stability of countable unions in §3 and the hereditariness in §4.

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2. Preliminaries and fundamental results

All spaces considered in this paper are completely regular and Hausdorff. Let $C^b(X)$ be the space of all bounded real-valued continuous functions on a topological space X . By $Z(X)$ we mean the family of *zero sets* $\{f^{-1}(0); f \in C^b(X)\}$ and $U(X)$ the family of *cozero sets* $\{Z^c; Z \in Z(X)\}$. We denote by $\mathcal{B}_a(X)$ the *Baire field*, that is, the smallest σ -algebra containing $Z(X)$ and by $\mathcal{B}(X)$ the σ -algebra generated by all open subsets of X , which is called the *Borel field*.

A *Baire measure* [resp. *Borel measure*] is a totally finite, non-negative countably additive set function defined on $\mathcal{B}_a(X)$ [resp. $\mathcal{B}(X)$]. For a Baire measure μ , the support of μ is

$$\text{supp } \mu = \{x \in X; \mu(U) > 0 \text{ for every } U \in U(X) \text{ containing } x\}.$$

Similarly the support of a Borel measure μ is

$$\text{supp } \mu = \{x \in X; \mu(O) > 0 \text{ for every open } O \text{ containing } x\}.$$

The following is clear.

Lemma 2.1. *Let μ be a Borel measure on a topological space X . If we denote by ν the restriction of μ to the Baire field, then we have*

$$\text{supp } \mu = \text{supp } \nu .$$

A Baire measure μ is τ -smooth if, whenever a net $\{Z_\alpha\}$ of zero sets decreases to the empty set \emptyset , $\lim_{\alpha} \mu(Z_\alpha) = 0$. A Baire measure μ is *tight* if for every $\varepsilon > 0$, there is a compact set K in X such that $\mu^*(K) > \mu(X) - \varepsilon$, where $\mu^*(K) = \inf \{\mu(U); K \subset U \in \mathcal{U}(X)\}$. A topological space is said to be *measure-compact* [resp. *strongly measure-compact*] if every Baire measure is τ -smooth [resp. tight]. For measure-compactness and strongly measure-compactness, the followings are given by Moran:

(I) ([7; Th. 2.1]) A topological space X is measure-compact if and only if every non-zero Baire measure on X has a non-empty support;

(II) ([8; Prop. 4.4]) A topological space X is strongly measure-compact if and only if for every non-zero Baire measure μ there is a compact set K in X such that $\mu^*(K) > 0$.

A Borel measure μ is *weakly* τ -smooth if, whenever a net $\{F_\alpha\}$ of closed sets decreases to the empty set \emptyset , $\lim_{\alpha} \mu(F_\alpha) = 0$. We call a Borel measure μ is τ -smooth if, whenever a net $\{F_\alpha\}$ of closed sets decreases to a closed set F , $\lim_{\alpha} \mu(F_\alpha) = \mu(F)$. A Borel measure μ is *regular* if for every B in $\mathcal{B}(X)$,

$$\mu(B) = \sup \{\mu(F); F \subset B \text{ and } F \text{ is closed}\}.$$

A Borel measure μ is a *Radon measure* if for every Borel set A in $\mathcal{B}(X)$,

$$\mu(A) = \sup \{\mu(K); K \subset A \text{ and } K \text{ is compact}\}.$$

We say a topological space X has the *property B* if every Borel measure μ is weakly τ -smooth. A topological space X is called an *HB-space* if every subspace of X has the property B. A topological space X is *Borel measure-compact* if every regular Borel measure on X is τ -smooth. Gardner has given the following three results:

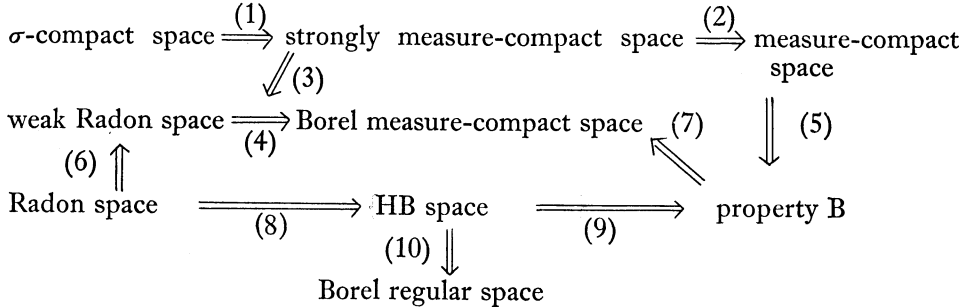
(III) ([1; Th. 4.1]) A topological space X has the property B if and only if every non-zero Borel measure on X has non-empty support;

(IV) ([1; Th. 5.2]) A topological space X is an HB-space if and only if every Borel measure on X is τ -smooth;

(V) ([1; Th. 3.1]) A topological space X is Borel measure-compact if and only if every non-zero regular Borel measure on X has a non-empty support.

A topological space is said to be a *Borel-regular space* if every Borel measure is regular. We call a topological space X is a *weak Radon space* [resp. *Radon space*] if every regular Borel measure [resp. Borel measure] is a Radon measure.

For the properties defined above, we offer the relation among them.



(1), (2), (4), (6), (8) and (9) are obvious from the definitions. (7) is evident from (III) and (V). Since X is a regular space, each point in X has a neighborhood base consisting of closed sets, which means (10) by (IV) (Gardner [1; Th. 6.1]).

Theorem 2.2. (i) *A strongly measure-compact space X is a weak Radon space.*

(ii) *A measure-compact space X has the property B.*

Proof. (i) Let μ be a regular Borel measure on X and ν be the restriction of μ to the Baire field. Since X is strongly measure-compact, ν is a tight measure. By Knowles [6; Th. 2.5] ν is uniquely extended to a Radon measure ϖ . Then μ equals ϖ by Kirk [5; Cor. 1.15].

(ii) Let μ be a Borel measure on X and ν be the restriction of μ to the Baire field. By (I) $\text{supp } \nu$ is non-empty. Since $\text{supp } \mu$ is equal to $\text{supp } \nu$, $\text{supp } \mu$ is non-empty. By (III) X has the property B. This completes the proof.

Let Ω be the first uncountable ordinal. Then $[0, \Omega]$ is an example that the converses of (6) and (9) do not hold and it implies that the property B does not necessarily imply Borel-regularity. Let A be a non-Lebesgue-measurable subset of $[0, 1]$ such that $\mu^*(A) > 0$, where μ is the Lebesgue measure on $[0, 1]$ and μ^* denotes the outer measure derived from μ . Then this space A is an example that the converses of (4), (7) and (8) do not hold. Furthermore the countable product \mathbf{R}^∞ of real lines gives an example that the converse of (1) does not hold.

3. Countable union

Let X be a topological space. Gardner [1; Th. 3.7, Th. 7.4] has shown that if X is the union of a sequence $\{X_n\}$ of Borel measure-compact [resp. C^* -embedded, measure-compact] subsets, then X is Borel measure-compact [resp. measure-compact]. We examine whether other spaces defined in the preceding section have the same property. We recall that a subset A of X is *Baire-embedded*

in X if $\mathcal{B}_a(A) = A \cap \mathcal{B}_a(X)$. It is clear that if A is C^* -embedded, then A is Baire-embedded.

The following lemma is derived from the same argument as in Lemma 3.8 of Gardner [1].

Lemma 3.1. *Let μ be a totally finite measure on a measurable space (X, \mathcal{B}) . If a subset A of X satisfies $\mu^*(A) > 0$, then μ^* is a totally finite measure on $(A, A \cap \mathcal{B})$, where $\mu^*(B) = \inf \{\mu(C); B \subset C \in \mathcal{B}\}$ for a subset B of X .*

We denote by μ_A the above $\mu^*|_{A \cap \mathcal{B}}$, which we call the *restriction* of μ to A .

Theorem 3.2. *Let X be the countable union of a sequence $\{X_n\}$.*

(i) *If every X_n is Baire-embedded measure-compact, then X is measure-compact.*

(ii) *If every X_n is Baire-embedded strongly measure-compact, then X is strongly measure-compact.*

Proof. (i) It follows by the way similar to Gardner [1; Th. 3.7].

(ii) Let μ be a Baire measure on X , then there exists an n such that $\mu^*(X_n) > 0$. Since X_n is Baire-embedded in X , the restriction μ_{X_n} of μ to X_n is non-zero Baire measure on X_n by Lemma 3.1. By (II) in §2, there exists a compact subset K of X_n such that $\mu_{X_n}^*(K) > 0$. For every Baire set B in $\mathcal{B}_a(X)$ containing K , we have

$$\mu(B) \geq \mu_{X_n}(B \cap X_n) \geq \mu_{X_n}^*(K) > 0,$$

which implies $\mu^*(K) > 0$. The theorem is proved.

Theorem 3.3. *Suppose that a space X is the countable union of $\{X_n\}$.*

(i) *If every X_n has the property B, then X also has the property B.*

(ii) *If every X_n is an HB-space, then so is X .*

(iii) *If every X_n is a weak Radon space, then so is X .*

(iv) *If every X_n is a Radon space, then so is X .*

Proof. Let μ be a Borel measure on X . We put $\mu^*(A) = \inf \{\mu(B); A \subset B \in \mathcal{B}(X)\}$ for every subset A .

(i) There is an n such that $\mu^*(X_n) > 0$. Then the restriction μ_{X_n} is a non-zero Borel measure on X_n . Since X_n has the property B, $\text{supp } \mu_{X_n}$ is non-empty, that is, $\text{supp } \mu_{X_n}$ contains an element x . For every open neighborhood U of x , it follows that

$$\mu(U) \geq \mu_{X_n}(U \cap X_n) > 0.$$

Therefore x is contained in $\text{supp } \mu$. By (III) in §2 X has the property B.

(ii) Let A be any subset of X . We shall show A has the property B.

Since every X_n is an HB-space, $A \cap X_n$ has the property B. Then by (i) $A = \bigcup_{n=1}^{\infty} (A \cap X_n)$ has the property B.

(iii) Suppose μ is regular, then for every n the measure μ_{X_n} is regular on X_n . Since μ_{X_n} is a Radon measure, there exists a compact subset K_n of X_n such that $\mu_{X_n}(X_n - K_n) < \varepsilon/2^n$. Then we have

$$\mu(X - \bigcup_{n=1}^{\infty} K_n) \leq \sum_{n=1}^{\infty} \mu_{X_n}(X_n - K_n) < \varepsilon.$$

It follows $\mu(X) = \sup \{ \mu(K); K \text{ is compact} \}$. By the regularity of μ , μ is a Radon measure on X .

(iv) By the argument of (iii), we have $\mu(B) = \sup \{ \mu(K); K \subset B \text{ and } K \text{ is compact} \}$ for every B in $\mathcal{B}(X)$.

4. Hereditariness

Every Baire set of a measure-compact space X is measure-compact by Moran [8; Prop. 5.1] and for closed subsets, by Kirk [4; Th. 3.5]. We show the following:

Theorem 4.1. (i) *Let X be a measure-compact space and A be a subset of X . Suppose that for every closed subset F which is disjoint from A , there exists a Baire set B such that $B \supset A$ and $B \cap F = \emptyset$. Then A is measure-compact.*

(ii) *Let X be a normal and measure-compact space. Then every F_σ -subset is measure-compact.*

Proof. (i) It follows by the same way as in the proof of Moran [8; Prop. 5.1].

(ii) Since a closed subset of a normal space is Baire-embedded in X , every F_σ -subset is measure-compact by Theorem 3.2.

For a strongly measure-compact space X , Baire sets are strongly measure-compact by Moran [8; Prop. 4.5] and closed sets are strongly measure-compact by Mosiman and Wheeler [9; Prop. 3.2]. If X is a normal, strongly measure-compact space, then each F_σ -set is strongly measure-compact by Theorem 3.2.

Next we consider subspaces of a space having the property B.

Theorem 4.2. *Suppose a topological space X has the property B.*

(i) *Every closed subset F has the property B.*

(ii) *If a subset A of X satisfies the condition that for every closed subset F which is disjoint from A , there is a Baire set B such that $B \supset A$ and $B \cap F = \emptyset$, then A has the property B. In particular, every Baire set has the property B.*

Proof. (i) Let μ be a Borel measure on F . We define a Borel measure on X by $\bar{\mu}(B) = \mu(B \cap F)$ for B in $\mathcal{B}(X)$. Then $\bar{\mu}$ is weakly τ -smooth. For a

net $\{F_\alpha\}$ of closed sets in F decreasing to \emptyset , we have

$$\lim_{\alpha} \mu(F_\alpha) = \lim_{\alpha} \bar{\mu}(F_\alpha) = 0,$$

which implies F has the property B.

(ii) For a Borel measure μ on A , $\bar{\mu}$ is the same one on X as in (i). Since $\bar{\mu}$ is weakly τ -smooth, $\text{supp } \bar{\mu}$ is non-empty and $\inf \{\bar{\mu}(U); U \supset \text{supp } \bar{\mu} \text{ and } U \text{ is open}\} = \bar{\mu}(X) = \mu(A)$. If $\text{supp } \bar{\mu} \cap A = \emptyset$, then there is a Baire set B such that $B \supset A$ and $B \cap \text{supp } \bar{\mu} = \emptyset$. It follows

$$\begin{aligned} 0 &= \mu(B^c) = \inf \{\bar{\mu}(V); B^c \subset V \in U(X)\} \\ &\geq \inf \{\bar{\mu}(U); U \supset \text{supp } \bar{\mu} \text{ and } U \text{ is open}\} \\ &= \mu(A), \end{aligned}$$

which is a contradiction. Therefore there is an x in $\text{supp } \bar{\mu} \cap A$. Let U be an open subset of X containing x , then we have

$$\mu(U \cap A) = \bar{\mu}(U) > 0,$$

which shows x belongs to $\text{supp } \mu$. By (III) in §2 A has the property B. The proof of Theorem 4.2 is complete.

We give a necessary and sufficient condition under which a subset of a Borel measure-compact space is Borel measure-compact.

Theorem 4.3. *Let X be a Borel measure-compact space. A subset A of X is Borel measure-compact if and only if for every regular Borel measure μ on A , $\iota(\mu)$ is a regular measure on X , where ι is the natural injection of A into X .*

Proof. The necessity of the condition is derived from Gardner [1; Th. 5.4]. Conversely suppose $\iota(\mu)$ is regular. Then $\iota(\mu)$ is τ -smooth. For a net $\{F_\alpha\}$ of closed subsets of A decreasing to \emptyset , we have

$$\begin{aligned} \inf_{\alpha} \mu(F_\alpha) &= \inf_{\alpha} \mu(\bar{F}_\alpha \cap A) \\ &= \inf_{\alpha} \iota(\mu)(\bar{F}_\alpha) \\ &= \iota(\mu)(\bigcap_{\alpha} \bar{F}_\alpha) = \mu((\bigcap_{\alpha} \bar{F}_\alpha) \cap A) \\ &= \mu(\bigcap_{\alpha} F_\alpha) = \mu(\emptyset) = 0, \end{aligned}$$

which implies μ is a τ -smooth measure on A . We have proved the theorem.

The following lemma is easy to prove, so the proof is omitted.

Lemma 4.4. *Let X and Y are two topological spaces and f be a continuous closed map of X to Y . Provided μ is a regular Borel measure, then the image*

measure $f(\mu)$ defined by $f(\mu)(B) = \mu(f^{-1}(B))$ for B in $\mathcal{B}(Y)$ is a regular Borel measure on Y .

Proposition 4.5. *Every F_σ -subset of a Borel measure-compact space is Borel measure-compact.*

Proof. Every closed subset is Borel measure-compact by Theorem 4.3 and Lemma 4.4. By Gardner [1; Th. 3.7] every F_σ -set is Borel measure-compact.

For a weak Radon space, we have the following proposition by Theorem 3.3 and Lemma 4.4.

Proposition 4.6. *Every F_σ -subset of a weak Radon space is a weak Radon space.*

REMARK 4.7. Let μ be the Dieudonné measure on $[0, \Omega]$ (see Halmos [2; p. 231, (10)]) and ν be the restriction of μ to $[0, \Omega) = [0, \Omega] - \{\Omega\}$. Then ν is a regular Borel measure on $[0, \Omega)$ but $\text{supp } \nu$ is empty. Hence $[0, \Omega)$ is not Borel measure-compact. This implies that an open subset of a Borel measure-compact [resp. strongly measure-compact, measure-compact, weak Radon] space is not necessarily a Borel measure-compact [resp. strongly measure-compact, measure-compact, weak Radon] space. And even if X has the property B, an open subset has not necessarily the property B.

As for a Borel-regular space, the following proposition is easily derived.

Proposition 4.8. *Every subset of a Borel-regular space is also a Borel-regular space.*

Appendix

We study a condition for a locally compact space to be measure-compact. Let X be a locally compact space and μ be a Baire measure on X . As well-known, there exists a Radon measure $\nu(\mu)$ on X such that

$$\int_X f d\mu = \int_X f d\nu(\mu)$$

for every f in $C_0(X)$, where $C_0(X)$ is the space of continuous functions vanishing at infinity.

Lemma. *μ is τ -smooth if and only if $\nu(\mu)(X)$ is identical to $\mu(X)$.*

Proof. For a cozero set U in $U(X)$, we have

$$\begin{aligned} \nu(\mu)(U) &= \sup \left\{ \int_X f d\mu; \chi_U \geq f \in C_0(X) \right\} \\ &\leq \sup \left\{ \int_X f d\mu; \chi_U \geq f \in C^b(X) \right\} \\ &= \mu(U), \end{aligned}$$

where χ_U is the characteristic function of U . Then it holds

$$\nu(\mu) \leq \mu \quad \text{on} \quad \mathcal{B}_a(X).$$

Assume $\nu(\mu)(X) = \mu(X)$. Then we have $\nu(\mu) = \mu$ on $\mathcal{B}_a(X)$, which shows μ is τ -smooth.

Suppose μ is τ -smooth. Let $\{f_\alpha\}$ be a net in $C_0(X)$ increasing to 1 on X . Since μ is τ -smooth, it follows

$$\begin{aligned} \nu(\mu)(X) &= \lim_\alpha \int_X f_\alpha d\nu(\mu) \\ &= \lim_\alpha \int_X f_\alpha d\mu = \mu(X), \end{aligned}$$

which proves the lemma.

Theorem. *Let X be a locally compact space. For X to be measure-compact, it is necessary and sufficient that for every Baire measure μ , $\nu(\mu)(X)$ equals $\mu(X)$.*

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