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COMPACT TRANSFORMATION GROUPS ON COMPLEX PROJECTIVE SPACES WITH CODIMENSION TWO PRINCIPAL ORBITS

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0. Introduction

In this paper we study compact connected differentiable transformation groups with codimension two principal orbits. Such actions on Euclidean spaces have been studied by Montgomery, Samelson and Yang [5], and such actions on spheres have been studied by Bredon [1].

The first half of this paper is devoted to studying such an action on a closed manifold with non-zero Euler characteristic, and the following result is obtained.

Theorem 0.1 *Let* $\psi: G\times M\to M$ *be a differentiable action of a compact connected Lie group* G *on a closed simply connected manifold* M*, and let* (H) *be the type of principal isotropy subgroups of* ψ*. Suppose that the Euler characteristic of* M *is non-zero, and suppose* $\dim M-\dim G/H=2$.

(i) *If each orbit is of the same dimension, then* M *is equivariantly diffeomorphic to* $G/H\times S^2$*. Here* G *acts trivially on the 2-sphere* S^2.

(ii) *If* ψ *has only one type* (K) *of singular isotropy subgroups, then* H *and* K *are connected, and* M *is equivariantly diffeomorphic to* $G\times_K S^{2t}$*. Here* K *acts on the* $2t$*-sphere* S^{2t} *by an orthogonal representation* $\sigma: K\to \boldsymbol{O}(2t+1)$ *such that* $\sigma(K)\subset\boldsymbol{O}(2t-1)$ *and* $K/H=S^{2t-2}$.

A closed connected $2n$-manifold M is called to be *symplectic* if there is an element $x\in H^2(M;\boldsymbol{Q})$ such that $x^n\neq 0$. A closed manifold is called to be *indecomposable* if it cannot be the direct product of two manifolds of positive dimension. As a result of Theorem 0.1, we have

Corollary 0.2. *Let* M *be a simply connected indecomposable symplectic manifold with non-zero Euler characteristic, and let* ψ *be a differentiable action of a compact connected Lie group* G *on* M *with codimension two principal orbit. Then* ψ *has at least two types of singular isotropy subgroups.*

The second half of this paper is devoted to studying a compact connected differentiable transformation group on a rational cohomology complex projective space with codimension two principal orbit and with isolated singular orbits. Here a closed orientable manifold M is called to be a *rational cohomology complex projective n-space* if the cohomology ring $H^*(M; \boldsymbol{Q})$ is isomorphic to $H^*(\boldsymbol{P}_n(\boldsymbol{C}); \boldsymbol{Q})$, and an orbit is called to be *isolated* if its slice representation has no direct summand of the trivial representation. The following result is obtained.

Theorem 0.3. *Let $\psi: G\times M\to M$ be a differentiable action of a compact connected Lie group G on a simply connected rational cohomology complex projective n-space with codimension two principal orbits. Then ψ has at least two isolated singular orbits, unless $n=5,11$.*

Finally we give examples of compact transformation groups on rational cohomology complex projective spaces with codimension two principal orbits. We conjecture that the number of isolated singular orbits is at most three for a compact connected differentiable transformation group on a simply connected rational cohomology complex projective space with codimension two principal orbit. A similar conjecture has been given for a transformation group on a sphere by Montgomery and Yang (cf. [2], p. 214).

1. Preliminary results

Let G be a compact connected Lie group. For a closed subgroup K of G, we denote by K^0, $N(K)=N(K; G)$ and (K) the identity component of K, the normalizer of K, and the conjugate class of K, respectively. If G acts on a space M, $F(G, M)$ denotes the set of stationary points, and $M_{(K)}$ denotes the set $\{x\in M \mid G_x\in(K)\}$, where G_x denotes the isotropy subgroup at x.

Let $\psi: G\times M\to M$ be a differentiable G-action on a closed connected manifold M. It is known that if ψ has codimension two principal orbit G/H, then the orbit space $M^*=M/\psi$ is a compact connected surface with or without boundary. Moreover if M is simply connected and ψ has a singular orbit, then M^* is a 2-disk, $M^*_{(H)}=\text{int } M^*$ and ψ has no exceptional orbit (cf. [2], p. 211, Theorem 8.6).

If the action ψ has no singular orbit, then M is equivariantly diffeomorphic to $G\underset{N(L)}{\times} F(L,M)$ as G-manifold, and the induced action of $N(L)/L$ on $F(L, M)$ is almost free (cf. [7], Lemma 4.2). Here L is the identity component of an isotropy subgroup, and an action is called to be *almost free* if its all isotropy subgroups are discrete.

Here we prove (i) of Theorem 0.1. By our assumption with G, H and M as in Theorem 0.1, M is equivariantly diffeomorphic to $G/L\underset{W}{\times}F(L, M)$, where $L=H^0$ and $W=N(L)/L$. Let F be a connected component of $F(L, M)$. Then M is equivariantly diffeomorphic to $G/L\underset{W^0}{\times}F$, because M is simply connected.

Hence we have

$$\chi(F)\cdot\chi((G/L)/W^0) = \chi(M) \neq 0 .$$

Thus we have $\dim W=0$, and M is equivariantly diffeomorphic to $G/L\times F$, because $\chi(F)\neq 0$ and W^0 acts almost freely on F(cf. [7], Lemma 4.3). Comparing isotropy groups, we obtain $H=L$. Finally F is a 2-sphere, because F is a simply connected closed surface. This completes the proof of (i) of Theorem 0.1.

2. Proof of (ii) of Theorem 0.1

With the same notation as in Theorem 0.1, the G-action has just two types (H) and (K) of isotropy subgroups, the orbit space M^* is a 2-disk, $M^*_{(H)}=\operatorname{int} M^*$ and $M^*_{(K)}=\partial M^*$, because ψ has no exceptional orbit by our assumption (cf. [2], p. 211, Theorem 8.6). Put

$$k = \dim M - \dim G/K .$$

Then $k>2$, and K acts orthogonally on a (k-1)-sphere S^{k-1} by the slice representation. The K-action on S^{k-1} has a codimension one orbit K/H and two fixed points. Therefore K/H is a (k-2)-sphere.

It is easy to see that $M_{(K)}$ is a fibre bundle over a circle with the fibre G/K. Let U be a closed invariant tubular neighborhood of $M_{(K)}$ in M. Then ∂U is a G-manifold with only one type (H) of isotropy subgroups. Let $q\colon \partial U\to M_{(K)}$ be the projection of normal sphere bundle. Then q induces a diffeomorphism $\bar{q}\colon \partial U/\psi\to M_{(K)}/\psi$, because K/H is a (k-2)-sphere. Put $E=M-\operatorname{int} U$. Then E is a G-manifold with only one type (H) of isotropy subgroups, and the orbit space E^* is a 2-disk. Hence E is equivariantly diffeomorphic to $G/H\times D^2$. Since

$$\begin{aligned}\chi(M) &= \chi(E)+\chi(U)-\chi(\partial E)\\ &= \chi(G/H)+\chi(M_{(K)})-\chi(G/H\times S^1)\,,\end{aligned}$$

the assumption $\chi(M)\neq 0$ implies $\chi(G/H)\neq 0$, because

$$\chi(M_{(K)}) = \chi(G/H\times S^1) = 0 .$$

Therefore we obtain

(*) $\qquad \operatorname{rank} H = \operatorname{rank} K = \operatorname{rank} G .$

In particular, K/H is an even dimensional sphere, and hence $k=2t$ ($t>1$). Then

$$\dim M - \dim M_{(K)} = k-1>2 .$$

This implies that G/H is simply connected and hence H is connected. Then K is also connected, because K/H is a (k-2)-sphere which is connected.

Choose a point $x \in F(H, \partial U)$, and let $F_x(H, \partial U)$ denote the connected component of $F(H, \partial U)$ containing x. Because

$$\partial U = \partial E = G/H \times S^1$$

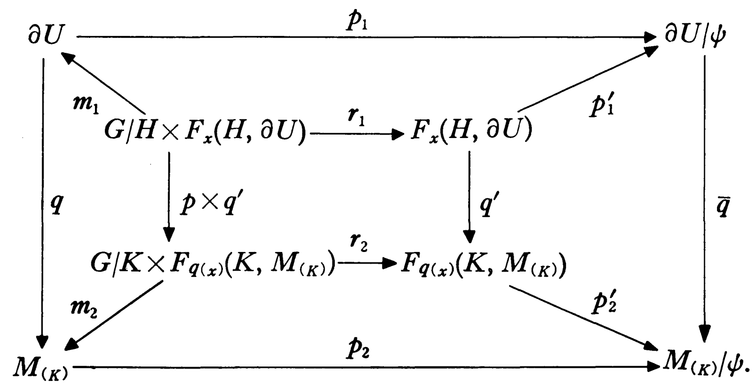
and rank H=rank G, the natural mapping

$$m_1 \colon G/H \times F_x(H, \partial U) \to \partial U$$

given by $m_1(gH, x') = \psi(g, x')$ is a diffeomorphism. Consider the projection $q \colon \partial U \to M_{(K)}$ of normal sphere bundle. By (*), there is a finite subset $\{g_1, \cdots, g_p\}$ of G such that $F(H, M_{(K)})$ is a disjoint union of

$$F(g_1 K g_1^{-1}, M_{(K)}), \cdots, F(g_p K g_p^{-1}, M_{(K)}) .$$

Then $q(F_x(H, \partial U))$ is contained in a connected component of $F(H, M_{(K)})$, say $F_{q(x)}(K, M_{(K)})$. Now we obtain the following commutative diagram:



Here p_s is the natural projection to the orbit space, p_s' is a restriction of p_s, q' is a restriction of q, p is a natural projection, r_s is a natural projection to the second factor, and m_2 is given similarly as m_1. By (*), it is easy to see that q' and p_2' are covering projections from a circle to a circle. But $p_2' q' = \bar{q} p_1'$ is a homeomorphism, and hence q' and m_2 are diffeomorphisms. Therefore the projection q can be identified with $p \times q'$ by the pair (m_1, m_2). Let $\tau \colon K \to \boldsymbol{O}(2t-1)$ be a representation by which K acts transitively on S^{2t-2} with an isotropy group H. It is known that such a representation exists unique up to equivalence (cf. [6], (5.6)). Then it is easy to see that the G-manifold U is equivariantly diffeomorphic to

$$(G \times_K \boldsymbol{D}^{2t-1}) \times S^1 ,$$

where K acts on the disk $\boldsymbol{D}^{2t-1}$ by the representation τ.

Consequently the G-manifold M is equivariantly diffeomorphic to a G-manifold

$$X(f)=(G\underset{K}{\times}D^{2t-1})\times S^1\underset{f}{\cup}G/H\times D^2,$$

where $f\colon (G\underset{K}{\times}K/H)\times S^1\to G/H\times S^1$ is a G-diffeomorphism. Let $c\colon G\underset{K}{\times}K/H\to G/H$ be the natural diffeomorphism given by $c(g, kH)=gkH$. Since $N(H)/H$ is a finite group by (*), there is a diffeomorphism $f'\colon S^1\to S^1$ and there is an element $n\in N(H)$ such that

$$f=(R_n\times f')\cdot(c\times 1).$$

Here $R_n\colon G/H\to G/H$ is given by $R_n(gH)=gnH$. Then we obtain the following commutative diagram of equivariant mappings:

$$\begin{array}{ccccccc}
(G\underset{K}{\times}D^{2t-1})\times S^1 & \xleftarrow{i_1} & (G\underset{K}{\times}K/H)\times S^1 & \xrightarrow{f} & G/H\times S^1 & \xrightarrow{i_2} & G/H\times D^2\\
\downarrow{\scriptstyle 1\times f'} & & \downarrow{\scriptstyle 1\times f'} & & \downarrow{\scriptstyle R_n^{-1}\times 1} & & \downarrow{\scriptstyle R_n^{-1}\times 1}\\
(G\underset{K}{\times}D^{2t-1})\times S^1 & \xleftarrow{i_1} & (G\underset{K}{\times}K/H)\times S^1 & \xrightarrow{c\times 1} & G/H\times S^1 & \xrightarrow{i_2} & G/H\times D^2.
\end{array}$$

Here i_s is the natural inclusion. This shows that $X(f)$ is equivariantly diffeomorphic to $X(c\times 1)$. Therefore the G-manifold M is equivariantly diffeomorphic to $G\underset{K}{\times}S^{2t}$. Here K acts orthogonally on S^{2t} by $\sigma=i\tau$, where $i\colon O(2t-1)\to O(2t+1)$ is the natural inclusion. This completes the proof of (ii) of Theorem 0.1.

REMARK. Actions with two types of orbits have been investigated in general by Hsiang brothers (cf. [3], Theorem 4.1). Actions on spheres with two types of orbits and with codimension two principal orbits have been classified by Bredon [1].

Let $M=G\underset{K}{\times}S^{2t}$ be the G-manifold as in (ii) of Theorem 0.1. Then the Euler class of the sphere bundle

$$S^{2t}\longrightarrow M\xrightarrow{p}G/K$$

is zero. Therefore the Gysin sequence induces the following short exact sequence of cohomology groups with rational coefficient:

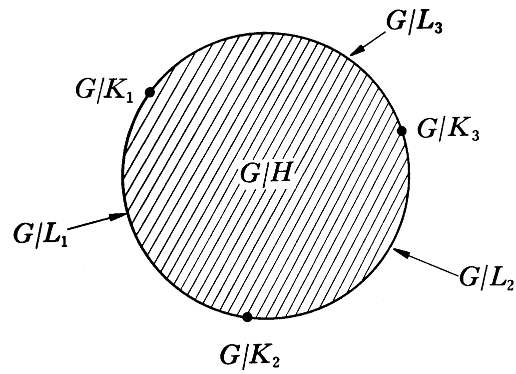
$$0\longrightarrow H^n(G/K)\xrightarrow{p^*}H^n(M)\longrightarrow H^{n-2t}(G/K)\longrightarrow 0.$$

Since $t>1$, the homomorphism $p^*\colon H^2(G/K)\to H^2(M)$ is an isomorphism. This fact shows Corollary 0.2.

3. Orbit structure

Let $\psi\colon G\times M\to M$ be a differentiable action of a compact connected Lie group G on a closed simply connected manifold M. Suppose that ψ has a codimension two principal orbit and at least two types of singular orbits. Then the orbit space M^* is a 2-disk, and there is at least one isolated singular orbit.

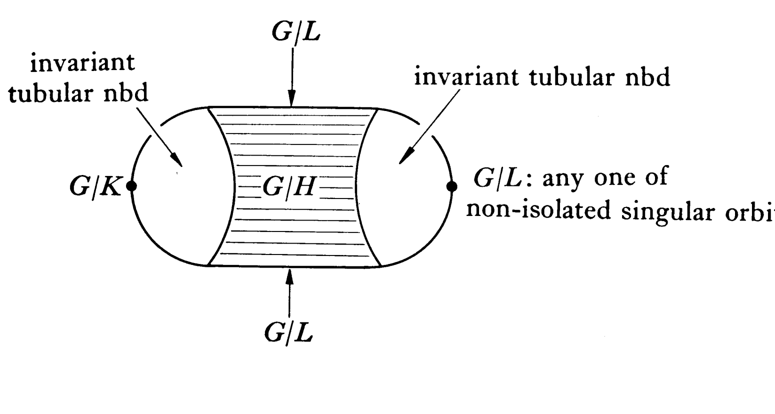
Let c be the number of isolated singular orbits. The orbit structure (for $c=3$) is as shown in the following figure in which the orbit types are indicated (cf. [2], p. 211, Theorem 8.6).



Let $G/K_1, \cdots, G/K_c$ be the isolated singular orbits, let $G/L_1, \cdots, G/L_c$ be the non-isolated singular orbits, and let G/H be the principal orbit as in the figure. Here $(K_i)=(K_j)$ or $(L_i)=(L_j)$ may be allowable. It is easy to see

$$\chi(M) = \chi(G/H)+\sum_{i=1}^{c}\chi(G/K_i)-\sum_{i=1}^{c}\chi(G/L_i)\,.$$

In particular, if $c=1$, then ψ has just three orbit types G/H, G/K and G/L. Here G/H is the principal orbit, G/K is the only one isolated singular orbit, and G/L is the non-isolated singular orbit. It is easy to see that the G-manifold M is equivariantly diffeomorphic to a G-manifold $X_1 \cup_f X_2$, where X_1 is a G-equivariant k-disk bundle over G/K, X_2 is a G-equivariant l-disk bundle over G/L, and $f\colon \partial X_1 \to \partial X_2$ is an equivariant diffeomorphism. Assumption implies $2<l\leqslant k$. The decomposition can be shown by the following figure.



Since $2<l\leqslant k$, the simply connectedness of M implies that G/K and G/L are simply connected, by general position theorem. In particular, K and L are connected.

It is easy to see that, if $c=2$, M has a similar decomposition.

4. Cohomological aspects

To prove Theorem 0.3, we recall the following result which was obtained by Uchida ([8], Theorem 2.2.2).

Let X_i be an orientable k_i-disk bundle over a closed orientable manifold Y_i $(i=1, 2)$, and let $f\colon \partial X_1\to\partial X_2$ be a diffeomorphism. Suppose that a compact connected Lie group G_i acts transitively on Y_i $(i=1, 2)$, and suppose that $X=X_1\underset{f}{\cup} X_2$ is a rational cohomology complex projective n-space. Let $j_i\colon X_i\to X$ be the inclusion, and let $x\in H^2(X; \boldsymbol{Q})$ be a non-zero element. Let n_i denote the non-negative integer defined by $j_i^*(x^{n_i})\neq 0$ but $j_i^*(x^{n_i+1})=0$. Then we have

$$(4.1)\qquad n = n_1+n_2+1 \quad (\text{cf. [8], Lemma 2.1.1})\,.$$

Moreover, (i) if k_1-k_2 is even, then Y_i is a rational cohomology complex projective n_i-space and $k_i=2(n-n_i)$ for $i=1, 2$. (ii) If k_1 is even and k_2 is odd, then $k_1+k_2=n+2$ and there are two cases:

(a) $n_1 = n_2$ and

$$P(Y_1; t) = (1+t^{k_2-1})(1+t^2+\cdots+t^{2n_1})\,,$$
$$P(Y_2; t) = (1+t^{k_1-1})(1+t^2+\cdots+t^{2n_2})\,.$$

(b) $k_1 = 2n_2+2, \quad k_2 = n_1-n_2+1$ and

$$P(Y_1; t) = (1+t^{n_1-n_2})(1+t^2+\cdots+t^{n_1+n_2})\,,$$
$$P(Y_2; t) = (1+t^{n})(1+t^2+\cdots+t^{2n_2})\,.$$

Here $P(\quad; t)$ denotes the Poincaré polynomial of a given space.

5. Proof of Theorem 0.3

Assume that there is a compact connected Lie group G which acts differentiably on a simply connected rational cohomology complex projective n-space M with a codimension two principal orbit G/H and with only one isolated singular orbit G/K. Let G/L denote the non-isolated singular orbit. As shown in section 3, the G-manifold M is equivariantly diffeomorphic to a G-manifold $X_1\underset{f}{\cup} X_2$, where X_1 is a G-equivariant k-disk bundle over G/K, X_2 is a G-equivariant l-disk bundle over G/L, and $f\colon \partial X_1\to\partial X_2$ is an equivariant diffeomorphism. Since $2<l\leqslant k$ and M is simply connected, G/K and G/L are simply connected and hence K and L are connected.

Proposition 5.1. *$n=6r-1$, $k=4r$ and $l=2r+1$ for a positive integer r.*

Proof. It is easy to see

$$n+1=\chi(M)=\chi(X_1)+\chi(X_2)=\chi(G/K)+\chi(G/L).$$

Thus we have rank K=rank G, and hence k is even. Assume that l is even. Then G/K and G/L are rational cohomology complex projective spaces (see the section 4), and hence

$$\operatorname{rank} L=\operatorname{rank} K=\operatorname{rank} G.$$

Therefore the projection $p: G/L \rightarrow G/K$ induces an injection $p^*: H^*(G/K; \boldsymbol{Q}) \rightarrow H^*(G/L; \boldsymbol{Q})$, and hence $K=G$ or $(K)=(L)$. If $K=G$, then the equation (4.1) implies $l=2$. If $(K)=(L)$, then the G-manifold M can not have an isolated singular orbit. Thus the assumption that l is even contradicts our assumption on the G-manifold M. Hence l is odd.

By the slice representation $\sigma: K \rightarrow \boldsymbol{O}(k)$, the compact connected Lie group K acts on the unit sphere S^{k-1} with the codimension one principal orbit K/H. This action has just two singular orbits K/L_1 and K/L_2 where L_1 and L_2 are conjugate to L in G. Because $K \neq L_i$ and K/L_i is orientable ($i=1, 2$), we can prove that

(5.2) K/L_1 and K/L_2 are $(k-l)$-spheres,

(5.3) K/H is homeomorphic with $K/L_1 \times K/L_2$,

(5.4) $H=L_1 \cap L_2$,

by the result of Wang ([9]; (5.2), (11.9) and (4.7)). In particular, (5.3) implies $k-2=2(k-l)$ and hence $k=2l-2$. Since $k+l=n+2$ (see the section 4), we obtain $n=3l-4$. q.e.d.

By the cohomological aspects stated in the section 4, we obtain

(5.5) $$P(G/K; t)=(1+t^{2r})(1+t^2+\cdots+t^{6r-2}),$$

(5.6) $$P(G/L; t)=\begin{cases}(1+t^{4r-1})(1+t^2+\cdots+t^{6r-2}), & \text{if } n_1=n_2=3r-1,\\ (1+t^{6r-1})(1+t^2+\cdots+t^{4r-2}), & \text{if } n_1=4r-1,\ n_2=2r-1.\end{cases}$$

Here n_1 and n_2 are defined similarly as in the section 4.

Proposition 5.7. *Let U be the identity component of*

$$\bigcap_{g \in G} gKg^{-1}.$$

If $r>2$, then U acts non-transitively on K/L_1 by the left translation.

Proof. There is a closed connected normal subgroup G' of G such that

$G=G'\circ U$ (the essentially direct product). It is easy to see that the U-action on G/L_1 has only one orbit type U/U_1 where $U_1=U\cap L_1$. Suppose that U acts transitively on K/L_1. Then $U/U_1=K/L_1=S^{2r-1}$ by (5.2). Since the U-action on G/L_1 has only one orbit type, there is a diffeomorphism

$$G/L_1=U/U_1\underset{W}{\times}F'$$

where $W=N(U_1, U)/U_1$ acts freely on $F'=F(U_1, G/L_1)$. Let F denote a connected component of F' and put $V=W^0$. Since G/L_1 is simply connected, there is a diffeomorphism

$$G/L_1=U/U_1\underset{V}{\times}F\,.$$

Since V acts freely on $U/U_1=S^{2r-1}$ by the right translation, we obtain $\dim V=0$, 1 or 3. If $\dim V=0$, then $G/L_1=S^{2r-1}\times F$ and hence $H^{2r-1}(G/L_1;\boldsymbol{Q})\neq 0$. On the other hand, (5.6) shows $H^{2r-1}(G/L_1;\boldsymbol{Q})=0$. This is a contradiction. Hence $\dim V=1$ or 3.

Put $m=1+\dim V$. Let $y\in H^m(S^{2r-1}/V;\boldsymbol{Q})$ be a non-zero element. It is easy to see that $H^*(S^{2r-1}/V;\boldsymbol{Q})$ is a truncated polynomial ring generated by y. By (5.6) there is a non-zero element $x\in H^2(G/L_1;\boldsymbol{Q})$ such that $x^{2r-1}\neq 0$. Moreover

$$H^2(G/L_1;\boldsymbol{Q})=H^4(G/L_1;\boldsymbol{Q})=\boldsymbol{Q}\,.$$

Now we consider the fibre bundle

$$F\xrightarrow{\ i\ }G/L_1\xrightarrow{\ q\ }S^{2r-1}/V\,.$$

We shall show that the induced homomorphism

$$q^*\colon H^*(S^{2r-1}/V;\boldsymbol{Q})\to H^*(G/L_1;\boldsymbol{Q})$$

is trivial. If q^* is non-trivial, then

$$q^*(y)=ax^{m/2}$$

where a is a non-zero rational number. This implies

$$2r-\mathrm{m}=\dim S^{2r-1}/V\geqslant 4r-m$$

which is a contardiction. Hence q^* is a trivial homorphism. Hence we obtain $H^{m-1}(F;\boldsymbol{Q})\neq 0$ by the spectral sequence of the fibre bundle. It is easy to see that the double coset space $U\backslash G/L_1$ is naturally homeomorphic with G/K. Hence there is a principal bundle

$$V\longrightarrow F\xrightarrow{\ \pi\ }G/K\,.$$

Here π is a composition of $F \xrightarrow{i} G/L_1 \xrightarrow{p} G/K$, where p is a natural projection. Since $V=S^{m-1}$, there is a Gysin exact sequence

$$H^{m-1}(G/K) \xrightarrow{\pi^*} H^{m-1}(F) \longrightarrow H^0(G/K) \xrightarrow{\cdot e(\pi)} H^m(G/K) \xrightarrow{\pi^*} \cdots$$

where the coefficient field is $\boldsymbol{Q}$. Then we obtain $e(\pi)=0$ which is the Euler class of π, and hence the homomorphism

$$\pi^*: H^*(G/K; \boldsymbol{Q}) \to H^*(F; \boldsymbol{Q})$$

is injective. Thus the homomorphism

$$p^*: H^*(G/K; \boldsymbol{Q}) \to H^*(G/L_1; \boldsymbol{Q})$$

is injective. But (5.5) and (5.6) show

$$H^{2r}(G/K; \boldsymbol{Q}) = \boldsymbol{Q}+\boldsymbol{Q},\ H^{2r}(G/L_1; \boldsymbol{Q}) = \boldsymbol{Q}\,.$$

This is a contradiction. This completes the proof of Proposition 5.7.

REMARK 5.8. Put $K'=K\cap G'$. Then K' is a connected subgroup of G', and $K=K'\circ U$ because rank K=rank G. If $r>2$, then K' acts transitively on $K/L_1=S^{2r-1}$ by the left translation (cf. [4], Theorem I′), because the U-action on K/L_1 is non-transitive by Proposition 5.7. It is similarly proved that, if $r>2$, K' acts transitively on $K/L_2=S^{2r-1}$ by the left translation.

Proposition 5.9. *Let G' be the same as in the proof of* Proposition 5.7. *If $r>2$, then G' acts transitively on the principal orbit G/H by the left translation.*

Proof. Let U be the same as in Proposition 5.7. Let U' and U'' be closed connected normal subgroups of $U=U'\circ U''$ defined as follows: each simple normal subgroup of U' is of rank$\leqslant 1$; if $U''\neq\{1\}$, then each simple normal subgroup of U'' is of rank$\geqslant 2$. Since $K=K'\circ U'\circ U''$ and K' acts transitively on $K/L_i=S^{2r-1}$ by the left translation ($i=1, 2$), U'' acts trivially on K/L_i by the left translation (cf. [4], Theorem I). Hence H contains U'' by (5.4). Put $L_1'=L_1\cap G'$. Then it is easy to see $L_1=L_1'\circ\tilde{U}'\circ U''$ where $\tilde{U}'$ is locally isomorphic to U'. If $r>2$, then $\tilde{U}'$ acts non-transitively on $L_1/H=S^{2r-1}$ by the left translation. Moreover, U'' acts trivially on L_1/H, because H contains U''. Hence L_1' acts transitively on L_1/H by the left tarnslation (cf. [4], Theorem I′). Therefore we obtain $L_1=L_1'H$. On the other hand, $K=K'L_1$ by Remark 5.8 and $G=G'K$ by definition of G'. Hence we obtain $G=G'H$. This shows that G' acts transitively on G/H by the left translation. q.e.d.

By Proposition 5.9, the restricted G'-action on M has the codimension two principal orbit $G'/H\cap G'$ and only one isolated singular orbit G'/K'. It is sufficient for our purpose to consider the restricted G'-action on M.

In the following we assume that G acts almost effectively on the isolated singular orbit G/K. Then G is a simple Lie group or an essentially direct product of two simple Lie groups by the structure of $H^*(G/K; \boldsymbol{Q})$.

First we assume $G=N_1\circ N_2$, where N_1 and N_2 are simple normal subgroups of G. Put $N_{(i)}=K\cap N_i$. Then $K=N_{(1)}\circ N_{(2)}$, from rank $K=$rank G. By (5.5) we can assume that $N_1/N_{(1)}$ is a rational cohomology $2r$-sphere and $N_2/N_{(2)}$ is a rational cohomology complex projective $(3r$-$1)$-space. Then $(N_2, N_{(2)})$ is pairwise locally isomorphic to one of the following (cf. [8], Proposition 4.2.2):

$$(\boldsymbol{SU}(3r),\ \boldsymbol{S}(\boldsymbol{U}(3r-1)\times \boldsymbol{U}(1)))\,,$$
$$(\boldsymbol{SO}(3r+1),\ \boldsymbol{SO}(3r-1)\times \boldsymbol{SO}(2))\,,\quad r\colon \text{even}$$
$$(\boldsymbol{Sp}(3r/2),\ \boldsymbol{Sp}(3r/2-1)\times \boldsymbol{U}(1))\,,\quad r\colon \text{even}$$
$$(\boldsymbol{G}_2,\ \boldsymbol{U}(2)),\ r=2\,.$$

Hence, if $r>2$, then $N_{(2)}$ can not act transitively on $(2r-1)$-sphere (cf. [4], [6]). Therefore, if $r>2$, then $N_{(1)}$ acts transitively on $K/L_i=S^{2r-1}$ by the left translation (cf. [4], Theorem I′). On the other hand $(N_1, N_{(1)})$ is pairwise locally isomorphic to one of the following (cf. [8], Proposition 4.2.1):

$$(\boldsymbol{SO}(2r+1),\ \boldsymbol{SO}(2r))\,,$$
$$(\boldsymbol{G}_2,\ \boldsymbol{SU}(3)),\ r=3\,.$$

Put $V^{(i)}=L_i\cap N_{(1)}$. If $(N_1, N_{(1)})$ is pairwise locally isomorphic to $(\boldsymbol{SO}(2r+1), \boldsymbol{SO}(2r))$ for some $r>2$, then $V^{(i)}$ is locally isomorphic to $\boldsymbol{SO}(2r-1)$, and hence the centralizer of $V^{(i)}$ in $N_{(1)}$ is a finite group. Hence we obtain $L_i=V^{(i)}\circ N_{(2)}$ for $i=1, 2$. By (5.4) we obtain $H=(V^{(1)}\cap V^{(2)})\circ N_{(2)}$. Then

$$K/H=N_{(1)}/V^{(1)}\cap V^{(1)}\,,$$

which is not homeomorphic with $S^{2r-1}\times S^{2r-1}$, because $V^{(1)}\cap V^{(2)}$ is locally isomorphic to $\boldsymbol{SO}(2r-1)$ or $\boldsymbol{SO}(2r-2)$. This is a contradiction to (5.3). If $(N_1, N_{(1)})$ is pairwise locally isomorphic to $(\boldsymbol{G}_2, \boldsymbol{SU}(3))$, then $N_{(1)}/V^{(i)}=S^5$, and hence dim $V^{(i)}=3$. Moreover we obtain $L_i=V^{(i)}\circ W^{(i)}$, where $W^{(i)}$ is locally isomorphic to $N_{(2)}$. Then $V^{(i)}$ and $W^{(i)}$ can not act transitively on S^5. Hence L_i can not act transitively on S^5 (cf. [4], Theorem I′). This a contradiction to $L_i/H=S^5$.

Next we assume that G is a simple Lie group. By (5.5) the pair (G,K) is pairwise locally isomorphic to one of the following (cf. [8], Proposition 4.2.3):

$$(\boldsymbol{SU}(3),\ T^2),\ r=1$$
$$(\boldsymbol{SO}(7),\ \boldsymbol{SO}(4)\times \boldsymbol{SO}(2)),\ r=2$$
$$(\boldsymbol{Sp}(3),\ \boldsymbol{Sp}(1)\times \boldsymbol{Sp}(1)\times \boldsymbol{U}(1)),\ r=2$$

$$(F_4, Spin(7)\circ T^1),\ r=4$$
$$(F_4, Sp(3)\circ T^1),\ r=4\,.$$

If $r>2$, then $r=4$ and K is locally isomorphic to $Sp(3)\circ T^1$ or $Spin(7)\circ T^1$. If K is locally isomorphic to $Sp(3)\circ T^1$, then K can not act transitively on S^7. This is a contradiction to $K/L_i=S^7$. If K is locally isomorphic to $Spin(7)\circ T^1$, then L_i is locally isomorphic to $G_2\circ T^1$, which can not act transitively on S^7. This is a contradiction to $L_i/H=S^7$.

Consequently $n=5$ or $n=11$, if a compact connected Lie group G acts on a simply connected rational cohomology complex projective n-space M with a codimension two principal orbit and only one isolated singular orbit. On the other hand, by Corollary 0.2, we see that if a compact connected Lie group G acts on a simply connected rational cohomology complex projective space M with a codimension two principal orbit, then there are at least two types of singular orbits, and hence there is at least one isolated singular orbit.

This completes the proof of Theorem 0.3.

6. Examples

6.1. Consider the natural action of $G=U(p)\times U(q)\times U(r)$ on $P_{p+q+r-1}(C)=P(C^p\oplus C^q\oplus C^r)$. Let $(u\colon v\colon w)$ denote the point of $P_{p+q+r-1}(C)$ represented by $u\in C^p$, $v\in C^q$ and $w\in C^r$. If each of u, v, w is non-zero vector, then the G-orbit through $(u\colon v\colon w)$ is of codimension 2. If only one of u, v, w is non-zero vector, then the G-orbit through $(u\colon v\colon w)$ is an isolated singular orbit. There are just three isolated singular orbits.

6.2. Let μ_n denote the canonical representation of $U(n)$ on C^n and let μ_n^* denote the dual representation of μ_n. Consider the $U(n)$-action on $P_{2n-1}(C)=P(C^n\oplus C^n)$ by the representation $\mu_n\oplus\mu_n^*$. Let $(u\colon v)$ denote the point of $P_{2n-1}(C)$ represented by $u, v\in C^n$. Denote $\bar{v}=(\bar{v}_1, \bar{v}_2, \cdots, \bar{v}_n)$ for $v=(v_1, v_2, \cdots, v_n)$, and denote $u\cdot v=u_1v_1+u_2v_2+\cdots+u_nv_n$ for $u=(u_1, u_2, \cdots, u_n)$, $v=(v_1, v_2, \cdots, v_n)$. Assume $n\geqslant 2$. If $u, \bar{v}$ are linearly independent and $u\cdot v\neq 0$, then the $U(n)$-orbit through $(u\colon v)$ is of codimension 2. If $u, \bar{v}$ are linearly independent and $u\cdot v=0$, then the $U(n)$-orbit through $(u\colon v)$ is of codimension 3. If $u, \bar{v}$ are linearly dependent but u, v are non-zero vectors, then the $U(n)$-orbit through $(u\colon v)$ is of codimension $2n-1$. If $u=0$ or $v=0$, then the orbit through $(u\colon v)$ is an isolated singular orbit of codimension $2n$. There are just two isolated singular orbits.

6.3. Consider the natural action of $G=U(3)\times U(n)$ on $P_{3n-1}(C)=P(C^3\underset{C}{\otimes}C^n)$. If $n\geqslant 3$, then the G-action has a principal orbit of codimension 2 and just three isolated singular orbits. Let $\{e_1, e_2, e_3\}$ and $\{e_1', e_2', \cdots, e_n'\}$ denote the canonical bases of C^3 and C^n respectively. Denote

$$\boldsymbol{D} = \{(u_1, u_2, u_3) \in \boldsymbol{R}^3 \colon u_1^2 + u_2^2 + u_3^2 = 1,\ u_1 \geqslant u_2 \geqslant u_3 \geqslant 0\}\ .$$

Let $s(u_1, u_2, u_3)$ be the complex line through $u_1e_1 \otimes e_1' + u_2e_2 \otimes e_2' + u_3e_3 \otimes e_3'$ which defines a continuous mapping $s\colon \boldsymbol{D} \to \boldsymbol{P}_{3n-1}(\boldsymbol{C})$. It is easy to see that each G-orbit contains one and only one point of $s(\boldsymbol{D})$.

6.4. Consider the natural action of $G = \boldsymbol{SO}(p) \times \boldsymbol{U}(q)$ on $\boldsymbol{P}_{p+q-1}(\boldsymbol{C}) = \boldsymbol{P}(\boldsymbol{C}^p \oplus \boldsymbol{C}^q)$. If $p \geqslant 2$ and $q \geqslant 2$, then the G-action has a principal orbit of codimension 2 and just three isolated singular orbits.

6.5. Let $\boldsymbol{Q}_n = \boldsymbol{SO}(n+2)/\boldsymbol{SO}(n) \times \boldsymbol{SO}(2)$ be the complex quadric. If n is odd, then $\boldsymbol{Q}_n$ is a simply connected rational cohomology complex projective n-space. Consider the action of $G = \boldsymbol{SO}(a) \times \boldsymbol{SO}(b)$ on $\boldsymbol{Q}_{a+b-2}$ by the left translation. If $a \geqslant b > 2$ or $a > b = 2$, then the G-action has a principal orbit of codimension 2 and just three isolated singular orbits.

6.6. Let $\boldsymbol{G}_2$ denote the exceptional Lie group defined as the automorphism group of the Cayley numbers. Let $\omega\colon \boldsymbol{G}_2 \to \boldsymbol{SO}(7)$ be the canonical representation. Denote $G = \omega^{-1}(\boldsymbol{SO}(6))$ and $G' = \omega^{-1}(\boldsymbol{SO}(5))$. Then G is isomorphic to $\boldsymbol{SU}(3)$ and G' is isomorphic to $\boldsymbol{SU}(2)$. Let N be the identity component of the centralizer of G' in $\boldsymbol{G}_2$. Then N is isomorphic to $\boldsymbol{SU}(2)$, but N is not conjugate to G' in $\boldsymbol{G}_2$. Let H denote a closed connected 4-dimensional subgroup of $\boldsymbol{G}_2$ whose semi-simple part is N. Then the G-action on $\boldsymbol{G}_2/H$ by the left translation has a principal orbit of codimension 2 and only one isolated singular orbit. The coset space $\boldsymbol{G}_2/H$ is a simply connected rational cohomology complex projective 5-space with $\pi_5(\boldsymbol{G}_2/H) = \boldsymbol{Z}_2$. This example is a supplement to Theorem 0.3.

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