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SYMMETRIC GROUPOIDS

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Introduction

Loos has shown in [3] that a symmetric space can be defined as a manifold carrying a diffeomorphic binary operation that satisfies three algebraic and one topological condition. This algebraic approach to symmetric spaces has been explored by Loos in [4], and by various other workers, for example Kikkawa in the series of papers [2]. Abstracting the algebraic properties of a symmetric space, Nobusawa introduced in [6] the concept of symmetric structure on a set. In that paper, and a sequel to it [1], the structure of finite symmetric sets satisfying a certain transitivity condition has been invesitgated. In particular, it was shown in [1] that there is a close relationship between symmetric sets and groups that are generated by involutions.

The purpose of this paper is to lay the foundations of a general theory of symmetric sets. The principal emphasis of this program is the connection between symmetric sets and groups that are generated by involutions. For the most part, we use the resources of group theory to gain insight into the structure of symmetric sets. It is to be hoped that in the future the flow of ideas will move the other way.

Our viewpoint in this paper is influenced by the ideas of universal algebra and category theory. Symmetric sets are looked upon as members of a particular variety of groupoids. For this reason, it seems appropriate to break a tradition by using the term "symmetric groupoid" rather than "symmetric set." Henceforth, this convention will be followed. Also, we will use the abbreviation "*GI* Group" for a group that is generated by the set of its involutions. Other than these idiosyncrasies our terminology in the paper is generally standard.

A brief outline of this work follows. The first section introduces the principal concepts that form the subject of the paper. Standard notation is established, and a few elementary facts are noted. Section two is devoted to categorical matters. Special kinds of morphisms of symmetric groupoids and *GI* groups are introduced in such a way that the natural correspondence between

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symmetric groupoids and GI groups is functorial. The third section further explores the correspondence between symmetric groupoids and GI groups. A method of constructing all symmetric groupoids from their associated GI groups is developed in this section. The last section of the paper is concerned with the semantics of symmetric groupoids and GI groups. Explicit constructions of the free objects in these categories are given, and the free algebras are used to investigate certain closure properties of the classes of GI groups and symmetric groupiods.

1. Basic concepts

DEFNITION 1.1. A *symmetric groupoid* is a groupoid $\langle A, \circ\rangle$ that satisfies the identities:

1.1.1. $a\circ a = a$;

1.1.2. $a\circ(a\circ b) = b$;

1.1.3. $a\circ(b\circ c) = (a\circ b)\circ(a\circ c)$.

The algebraic analogues of the symmetric groupoids that arise in the study of symmetric spaces can be described in the following way.

EXAMPLE 1.2. Let G be a group, f an involution in aut G, and H a subgroup of G such that $f(x)=x$ for all $x\in H$. Let A be the left coset space G/H. Define $xH\circ yH=xf(x)^{-1}f(y)H$. A straightforward calculation shows that $\circ$ is a well defined binary operation under which G/H is a symmetric groupoid.

For the purpose of this paper, the following example of a symmetric groupoid is of fundamental inportance.

Proposition 1.3. *Let G be a group. Denote $I(G)=\{a\in G: a^2=1\}$, the set of involutions of G, including 1. For a and b in $I(G)$, define $a\circ b=aba$. Then $\langle I(G), \circ\rangle$ is a symmetric groupoid. If $f: G\rightarrow H$ is a homomorphism of groups, then $f(I(G))\subseteq I(H)$, and $f\mid I(G)$ is a groupoid homomorphism. The maps $G\rightarrow I(G)$, $f\rightarrow f\mid I(G)=I(f)$ define a functor from the categroy of groups to the category of symmetric groupoids.*

The straightforward proof of Proposition 1.3 is omitted.

DEFINITION 1.4. A symmetric groupoid A is called *special* if A is isomorphic to a subgroupoid of $I(G)$ for some group G. A homomorphism $f: A\rightarrow B$ of special symmetric groupoids is called *special* if it preserves the partial product operation that A inherits from G. That is, if $A< I(G)$, $B< I(H)$, and if $a_1, \cdots, a_n$, and b in A satisfy $b=a_1\cdots a_n$ in G, then

$$f(b) = f(a_1)\cdots f(a_n) \text{ in } H.$$

It is obvious that if $f\colon G\to H$ is a group homomorphism, then

$$I(f)\colon I(G)\to I(H)$$

is a special homomorphism of symmetric groupoids.

In general, the groupoid operation in a special symmetric groupoid does not determine the multiplication in the ambient group, so that the definition of a special homomorphism presupposes fixed embeddings into $I(G)$ and $I(H)$.

Special groupoids will be studied in Section 4. They will also play a minor part in the considerations of Section 2.

The following property is an easy consequence of Definition 1.1.

Lemma 1.5. *Every symmetric groupoid satisfies the identity*

$$(a\circ b)\circ c = a\circ(b\circ(a\circ c)).$$

The observation Lemma 1.5, together with 1.1.2 and 1.1.3 yields the next result.

Proposition 1.6. *Let A be a symmetric groupoid. For elements a and b of A, define $\lambda_a(b)=a\circ b$. Then $\lambda_a\in aut\ A$, and the mapping $p_A\colon a\to\lambda_a$ is a groupoid homomorphism from A to $I(aut\ A)$.*

Corollary 1.7. $Z(A)=\{(a, b)\in A\times A\colon \lambda_a=\lambda_b\}$ *is a congruence relation on the symmetric groupoid A.*

We will call $Z(A)$ the *central congruence* of A. This concept is different from the notion of the center of a symmetric space that was introduced in [4]. Note that A is effective (in the terminology of [6]) if and only if $Z(A)$ is the identity congruence on A.

NOTATION 1.8. Let A be a symmetric groupoid. Denote $M(A)=\{\lambda_a\colon a\in A\}$, and define $\Lambda(A)$ to be $\langle M(A)\rangle$, the subgroup of aut A that is generated by $M(A)$.

Since $\lambda_a^{-1}=\lambda_a$, every element of $\Lambda(A)$ can be written in the form $\xi=\lambda_{a_1}\lambda_{a_2}\cdots\lambda_{a_m}$, $a_i\in A$. Moreover, $\xi^{-1}=\lambda_{a_m}\cdots\lambda_{a_2}\lambda_{a_1}$. Obviously $\Lambda(A)$ is GI group and $p_A(A)=M(A)\subseteq I(\Lambda(A))$.

Corollary 1.9. *If A is a symmetric groupoid, then $p_A\colon A\to I(\Lambda(A))$ induces an injective homomorphism $\bar{p}_A\colon A/Z(A)\to I(\Lambda(A))$, with $Im\,\bar{p}_A=M(A)$. In particular, $A/Z(A)$ is a special symmetric groupoid.*

Lemma 1.10. *Let G be a GI group. Let $u_1, u_2, \cdots, u_n, v\in I(G)$. Denote $x=u_1u_2\cdots u_n$. Then $\lambda_{u_1}\lambda_{u_2}\cdots\lambda_{u_n}(v)=v$ if and only if $x\in C_G(v)$, the centralizer of v in G. In particular, $\lambda_{u_1}\lambda_{u_2}\cdots\lambda_{u_n}$ is the identity automorphism of $I(G)$ if and only if $x\in C(G)$, the center of G.*

Proof. By definition, $\lambda_{u_1}\lambda_{u_2}\cdots\lambda_{u_n}(v)=xvx^{-1}$, from which the first statement follows. Since G is a GI group, $C_G(I(G))=C_G(G)=C(G)$, which proves the second assertion.

Proposition 1.11. *Let G be a GI group. Then there is a unique epimorphism q_G: $G\to\Lambda(I(G))$ satisfying $q_G(u)=\lambda_u$ for all $u\in I(G)$. The kernel of q_G is $C(G)$, so that q_G induces an isomorphism $\bar{q}_G$: $G/C(G)\to\Lambda(I(G))$.*

Proof. If $x\in G$, then $x=u_1u_2\cdots u_n$ for some $u_i\in I(G)$. Define $q_G(x)=\lambda_{u_1}\lambda_{u_2}\cdots\lambda_{u_n}$. This definition is well posed since $x=u_1u_2\cdots u_n=v_1v_2\cdots v_m$ implies $v_1v_2\cdots v_mu_n\cdots u_2u_1=1$, so that $\lambda_{u_1}\lambda_{u_2}\cdots\lambda_{u_n}=\lambda_{v_1}\lambda_{v_2}\cdots\lambda_{v_m}$ by 1.10. It follows from our definition, that, q_G is a group epimorphism from G to $\Lambda(I(G))$. By 1.10, Ker $q_G=C(G)$.

In the next section, we will extend the object maps $A\to\Lambda(A)$, $G\to I(G)$ to functors. There is no natural way to do this on the full categories of symmetric groupoids and GI groups; it is necessary to restrict the allowable morphisms. The foundation for this work will be laid in the rest of this section. It is economical to introduce a convention for dropping parentheses.

NOTATION 1.12. If $a_1, \cdots, a_{n-1}, a_n$ are elements of a symmetric groupoid, denote

$$a_1\circ\cdots\circ a_{n-1}\circ a_n = a_1\circ(\cdots\circ(a_{n-1}\circ a_n)\cdots)\,.$$

Lemma 1.13. *Let $a_1, \cdots, a_n$, and a be elements of a symmetric groupoid. Then*:

1.13.1. $(\lambda_{a_1}\cdots\lambda_{a_n})(a) = a_1\circ\cdots\circ a_n\circ a$;

1.13.2. $\lambda_{a_1\circ\cdots\circ a_n\circ a} = \lambda_{a_1}\circ\cdots\circ\lambda_{a_n}\circ\lambda_a$;

1.13.3. if $\xi\in\Lambda(A)$, *then* $\xi\lambda_a\xi^{-1} = \lambda_{\xi(a)}$.

These equations are direct consequences of the definition of λ_a. Note that 1.13.3 is a reformulation of 1.13.2.

DEFINITION 1.14. Let A be a symmetric groupoid. The *extended center* of A is the set $\mathcal{Z}(A)$ of all n-tuples $(a_1, \cdots, a_n)\in A^n$, $n=1, 2, \cdots$, such that

$$a_1\circ\cdots\circ a_n\circ a = a$$

for all $a\in A$.

By 1.13, $(a_1,\cdots,a_n)\in\mathcal{Z}(A)$ if and only if $\lambda_{a_1}\cdots\lambda_{a_n}=1$. In particular $Z(A)=\mathcal{Z}(A)\cap A^2$.

Proposition 1.15. *Let f: $A\to B$ be a homomorphism of symmetric groupoids. Then there is a group homomorphism $\Lambda(f)$: $\Lambda(A)\to\Lambda(B)$ satisfying $\Lambda(f)p_A=p_Af$ if and only if $f(\mathcal{Z}(A))=\{(f(a_1), \cdots, f(a_n))\colon (a_1, \cdots, a_n)\in\mathcal{Z}(A)\}\subseteq\mathcal{Z}(B)$.*

Proof. The condition $\Lambda(f)p_A=p_Bf$ is equivalent to $\Lambda(f)(\lambda_a)=\lambda_{f(a)}$ for all $a\in A$. Thus, $\Lambda(f)$ can be defined by $\Lambda(f)(\lambda_{a_1}\cdots\lambda_{a_n})=\lambda_{f(a_1)}\cdots\lambda_{f(a_n)}$ if and only if $f(\mathcal{Z}(A))\subseteq\mathcal{Z}(B)$.

Lemma 1.6. *Let $f: A\to B$ be a homomorphism of symmetric groupoids such that $f(\mathcal{Z}(A))\subseteq\mathcal{Z}(B)$. If f is injective (surjective), then $\Lambda(f)$ is injective (surjective).*

Proof. If $\xi=\lambda_{a_1}\cdots\lambda_{a_n}\in\Lambda(A)$ satisfies $\Lambda(f)(\xi)=1$, then $f(a_1)\circ\cdots\circ f(a_n)\circ b=b$ for all $b\in B$. In particular, $f(\xi(a))=f(a_1\circ\cdots\circ a_n\circ a)=f(a_1)\circ\cdots\circ f(a_n)\circ f(a)=f(a)$ for all $a\in A$. Thus, if f is injective, then $\xi=1$. It is obvious that if f is surjective then so is $\Lambda(f)$.

REMARK. If $f: A\to B$ is surjective homomorphism of symmetric groupoids, then $f(\mathcal{Z}(A))\subseteq\mathcal{Z}(B)$ is certainly satisfied, because $\lambda_{f(a_1)}\cdots\lambda_{f(a_n)}(f(a))=f(\lambda_{a_1}\cdots\lambda_{a_n}(a))$.

Proposition 1.17. *Let G be a GI group. Then $\mathcal{Z}(I(G))=\{(u_1,\cdots,u_n)\in I(G)^n: u_1\cdots u_n\in C(G)\}$. If $f: G\to H$ is a homomorphism of GI groups, then $f(\mathcal{Z}(I(G)))\subseteq\mathcal{Z}(I(H))$ if and only if $f(C(G))\subseteq C(H)$.*

This proposition is a corollary of 1.10.

The extended center of a symmetric groupoid has properties that are analogous to the conditions that define a congruence relation. In particular, the following fact will be used in Section 4.

Lemma 1.18. *Let $a_1,\cdots,a_i,\cdots,a_n$, b, and c be elements of the symmetric groupoid A. Assume that $a_i=b\circ c$. Then $(a_1,\cdots,a_{i-1},a_i,a_{i+1},\cdots,a_n)\in\mathcal{Z}(A)$ if and only if $(a_1,\cdots,a_{i-1},b,c,b,a_{i+1},\cdots,a_n)\in\mathcal{Z}(A)$.*

Proof. By 1.13.2, $\lambda_{a_i}=\lambda_b\circ\lambda_c=\lambda_b\lambda_c\lambda_b$, which clearly implies the lemma.

2. Categorical imperatives

The goal for this section is to extend the object maps Λ and I to functors. The fact that Λ is not functorial in a naive way is shown by Proposition 1.15. At the same time, 1.15 suggests that the right solution to this extension problem lies in the direction of restricting the classes of morphisms of symmetric groupoids and *GI* groups.

We begin with purely categorical considerations. If $\mathcal{A}$ is a category, let ob $\mathcal{A}$ denote the class of all objects of $\mathcal{A}$. It will sometimes be convenient to identify ob $\mathcal{A}$ with the identity morphisms of $\mathcal{A}$. The notation $f\in\mathcal{A}$ abbreviates "f is a morphism of $\mathcal{A}$." When the domain A and range B of a morphism $f\in\mathcal{A}$ have to be specified, we will write $f\in\mathcal{A}(A, B)$.

Proposition 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be categories, $\mathcal{A}_0$ and $\mathcal{B}_0$ subcategories of $\mathcal{A}$ and $\mathcal{B}$ respectively such that* ob $\mathcal{A}_0=$ ob $\mathcal{A}$ *and* ob $\mathcal{B}_0=$ ob $\mathcal{B}$. *Assume that*

$\sum$: $\mathcal{A}_0 \to \mathcal{B}$ and T: $\mathcal{B}_0 \to \mathcal{A}$ are functors. Define recursively:

$$\mathcal{A}_{n+1} = \{f \in \mathcal{A}_0 \colon \textstyle\sum f \in \mathcal{B}_n\},\ \mathcal{B}_{n+1} = \{g \in \mathcal{B}_0 \colon Tg \in \mathcal{A}_n\}\ .$$

Let $\mathcal{A}_\omega = \bigcap_{n<\omega} \mathcal{A}_n$, $\mathcal{B}_\omega = \bigcap_{n<\omega} \mathcal{B}$. Then for every $n \leq \omega$, $\mathcal{A}_n$ and $\mathcal{B}_n$ are subcategories of $\mathcal{A}_0$ and $\mathcal{B}_0$ respectively, with ob A_n=ob $\mathcal{A}$ *and* ob $\mathcal{B}_n$=ob $\mathcal{B}$. *Moreover, the restriction of $\sum$ to $\mathcal{A}_\omega$ is a functor to $\mathcal{B}_\omega$ and the restriction of T to $\mathcal{B}_\omega$ is a functor to A_ω.*

Proof. Induction on n shows that $\mathcal{A}_{n+1}$ is a subcategory of $\mathcal{A}_n$, $\mathcal{B}_{n+1}$ a subcategory of $\mathcal{B}_n$, with ob $\mathcal{A}_{n+1}$=ob $\mathcal{A}_n$, ob $\mathcal{B}_{n+1}$=ob $\mathcal{B}_n$. Thus, $\mathcal{A} \supset \mathcal{A}_0 \supseteq \mathcal{A}_1 \supseteq A_2 \supseteq \cdots$, and $\mathcal{B} \supseteq \mathcal{B}_0 \supseteq \mathcal{B}_1 \supseteq \mathcal{B}_2 \supseteq \cdots$. Hence $\mathcal{A}_\omega$ is a subcategory of $\mathcal{A}_0$ such that ob $\mathcal{A}_\omega$=ob $\mathcal{A}$, and $\mathcal{B}_\omega$ is a subcategory of $\mathcal{B}_0$ such that ob $\mathcal{B}_\omega$=ob $\mathcal{B}$. By definition, $f \in \mathcal{A}_\omega$ implies $f \in \mathcal{A}_{n+1}$ for all $n < \omega$. Thus, $\sum f \in \mathcal{B}_n$ for all $n < \omega$, so that $\sum f \in \mathcal{B}_\omega$. Similarly, $T\mathcal{B}_\omega \subseteq \mathcal{A}_\omega$.

REMARKS 2.2. (Corollaries of the proof of 2.1).

2.2.1. $f \in \mathcal{A}_0$ and $\sum f \in \mathcal{B}_\omega$ implies $f \in \mathcal{A}_\omega$; $g \in \mathcal{B}_0$ and $Tg \in \mathcal{A}_\omega$ implies $g \in \mathcal{B}_\omega$.

2.2.2. For $m < \omega$, $\mathcal{A}_{2(m+1)} = \{f \in \mathcal{A}_0 \colon \sum f \in \mathcal{B}_0$ and $T \sum f \in \mathcal{A}_{2m}\}$,
$\mathcal{B}_{2(m+1)} = \{g \in \mathcal{B}_0 \colon Tg \in \mathcal{A}_0$ and $\sum Tg \in \mathcal{B}_{2m}\}$.

Lemma 2.3. *Let $\mathcal{A}$ be a category, and let $\mathcal{A}_0$ be a subcategory of $\mathcal{A}$ such that* ob $\mathcal{A}_0$=ob $\mathcal{A}$. *Let $\mathcal{K}$ be a class of commutative squares*

$$Sq(f, g; h_1, h_2) = \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle h_1}\downarrow & & \downarrow{\scriptstyle h_2} \\ C & \xrightarrow{g} & D \end{array}$$

in $\mathcal{A}$ with the properties: $h_1 \in \mathcal{A}_0$, $h_2 \in \mathcal{A}_0$, and $f \in \mathcal{A}_0$ if and only if $g \in \mathcal{A}_0$. Let Φ: $\mathcal{A}_0 \to \mathcal{A}$ and Ψ: $\mathcal{A}_0 \to \mathcal{A}$ be functors that satisfy:

2.3.1. if $Sq(f, g; h_1, h_2) \in \mathcal{K}$, with $f, g \in \mathcal{A}_0$, then $Sq(\Psi f, \Psi g; \Psi h_1, \Psi h_2) \in \mathcal{K}$;

2.3.2. there is a natural transformation $\{h_A\} \colon \Phi \to \Psi$ such that if $f \in \mathcal{A}_0(A, B)$, then

$$\begin{array}{ccc} \Phi A & \xrightarrow{\Phi f} & \Phi B \\ {\scriptstyle h_A}\downarrow & & \downarrow{\scriptstyle h_B} \\ \Psi A & \xrightarrow{\Psi f} & \Psi B \end{array}$$

belongs to $\mathcal{K}$. For $n \geq 0$, define recursively

$$\mathcal{A}_{n+1} = \{f \in \mathcal{A}_0 \colon \Psi f \in \mathcal{A}_n\}\ .$$

For all $n<\omega$, it follows that:

(a) if $Sq(f, g; h_1, h_2)\in\mathcal{K}$, then $f\in\mathcal{A}_n$ if and only if $g\in\mathcal{A}_n$;

(b) if $f\in\mathcal{A}_0$, then $\Phi f\in\mathcal{A}_n$ if and only if $\Psi f\in\mathcal{A}_n$;

(c) $f\in\mathcal{A}_0$, $\Psi f\in\mathcal{A}_0$, $\Psi^2 f\in\mathcal{A}_0$, $\cdots$, $\Psi^n f\in\mathcal{A}_0$ implies $f\in\mathcal{A}_{n+1}$;

(d) $\mathcal{A}_0\supseteq\mathcal{A}_1\supseteq\mathcal{A}_2\supseteq\cdots$.

Proof. The implication (a) follows by induction from 2.3.1; (c) and (d) are similarly obtained by induction. The case $n=0$ of (b) is a consequence of 2.3.2. Assume that (b) holds for n. If $f\in\mathcal{A}_0(A, B)$, then $\Phi f\in\mathcal{A}_0$ if and only if $\Psi f\in\mathcal{A}_0$, so that it will suffice to prove: $\Psi\Phi f\in\mathcal{A}_n$ if and only if $\Psi^2 f\in\mathcal{A}_n$, under the assumption that $\Phi f\in\mathcal{A}_0$ and $\Psi f\in\mathcal{A}_0$. By 2.3.2

$$\begin{array}{ccc} \Phi A & \xrightarrow{\Phi f} & \Phi B \\ {\scriptstyle h_A}\downarrow & & \downarrow{\scriptstyle h_B} \\ \Psi A & \xrightarrow{\Psi f} & \Psi B \end{array}$$

is in $\mathcal{K}$. Therefore, by 2.3.1, so is

$$\begin{array}{ccc} \Psi\Phi A & \xrightarrow{\Psi\Phi f} & \Psi\Phi B \\ {\scriptstyle \Psi h_A}\downarrow & & \downarrow{\scriptstyle \Psi h_B} \\ \Psi^2 A & \xrightarrow{\Psi^2 f} & \Psi^2 B\,. \end{array}$$

Consequently, by (a), $\Psi\Phi f\in\mathcal{A}_n$ if and only if $\Psi^2 f\in\mathcal{A}_n$.

In the first application of 2.3, let $\mathcal{A}=\mathcal{G}$ be the category of all GI groups and homomorphisms. Let $\mathcal{A}_0=\mathcal{G}_0=\{f\in\mathcal{G}(G, H): f(C(G))\subseteq C(H)\}$. Define $\mathcal{K}$ to be the class of all commutative squares

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ {\scriptstyle h_1}\downarrow & & \downarrow{\scriptstyle h_2} \\ K & \xrightarrow{g} & L \end{array}$$

such that h_1 and h_2 are isomorphisms. Plainly, $h_1\in\mathcal{G}_0$, $h_2\in\mathcal{G}_0$, and $f\in\mathcal{G}_0$ if and only if $g\in\mathcal{G}_0$. Let $\Phi=\Gamma$: $\mathcal{G}_0\rightarrow\mathcal{G}$ be the functor defined by $\Gamma G=G/C(G)$, $\Gamma(f)(xC(G))=f(x)C(G)$ for $f\in\mathcal{G}_0(G, H)$. Thus, the square

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ {\scriptstyle r_G}\downarrow & & \downarrow{\scriptstyle r_H} \\ \Gamma G & \xrightarrow{\Gamma f} & \Gamma H \end{array}$$

commutes, where r_G and r_H are the natural projection homomorphisms. Let $\Psi=\Lambda I$: $\mathcal{G}_0\rightarrow\mathcal{G}$. By 1.15 and 1.17, Ψ is well defined. Note that if $f\in\mathcal{G}_0(G, H)$, then

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ {\scriptstyle q_G}\downarrow & & \downarrow{\scriptstyle q_H} \\ \Lambda IG & \xrightarrow{\Lambda If} & \Lambda IH \end{array}$$

commutes, where q_G and q_H are defined as in 1.11. It follows from 1.11 that $\{\bar{q}_G\}: \Gamma \rightarrow \Lambda I$ is a natural equivalence of functors. The hypothesis 2.3.2 is automatically satisfied because the vertical maps are isomorphisms. It follows from 2.3 that for all $n<\omega$, the inductive definitions $\mathcal{G}_{n+1}=\{f\in\mathcal{G}_0: \Gamma f\in\mathcal{G}_n\}$ and $\mathcal{G}_{n+1}=\{f\in\mathcal{G}_0: \Lambda If\in\mathcal{G}_n\}$ are equivalent. As in 2.1, denote $\mathcal{G}_\omega=\bigcap_{n<\omega}\mathcal{G}_n$.

Lemma 2.4. *Let G and H be GI groups, and let f be a group homomorphism from G to H. Then $f\in\mathcal{G}_\omega$ if and only if $f(C^n(G))\subseteq C^n(H)$ for all natural numbers n, where $C^n(G)$ and $C^n(H)$ are the n'th terms of the upper central series of G and H respectively.*

Proof. It suffices to prove by induction on n that $f\in\mathcal{G}_n$ if and only if $f(C^k(G))\subseteq C^k(H)$ for all $k\leq n+1$. For $n=0$, this equivalence is the definition of $\mathcal{G}_0$, since $C^1(G)=C(G)$. Assume that the equivalence is valid at level n. By the remarks above and 2.3 (d), $f\in\mathcal{G}_{n+1}$ if and only if $f\in\mathcal{G}_0$, $f\in\mathcal{G}_n$, and $\Gamma f\in\mathcal{G}_n$. Thus, by the induction hypothesis, $f\in\mathcal{G}_{n+1}$ is equivalent to $f(C^k(G))\subseteq C^k(H)$ and $\Gamma f(C^k(\Gamma G))\subseteq C^k(\Gamma H)$ for all $k\leq n+1$. It follows from the commutativity of

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ {\scriptstyle r_G}\downarrow & & \downarrow{\scriptstyle r_H} \\ \Gamma G & \xrightarrow{\Gamma f} & \Gamma H, \end{array}$$

the fact that r_H is surjective, and the definitions $C^{n+2}(G)=r_G^{-1}(C^{n+1}(G/C(G)))=r_G^{-1}(C^{n+1}(\Gamma G))$, $C^{n+2}(H)=r_H^{-1}(C^{n+1}(\Gamma H))$ that $\Gamma f(C^{n+1}(\Gamma G))\subseteq C^{n+1}(\Gamma H)$ if and only if $f(C^{n+2}(G))\subseteq C^{n+2}(H)$. This completes the induction.

For the second application of 2.3, let $\mathcal{A}$ be the full category $\mathcal{S}$ of symmetric groupoids and groupoid homomorphisms. Let $\mathcal{S}_0$ be the subcategory of homomorphisms that preserve the extended center, that is, $f\in\mathcal{S}_0(A, B)$ if and only $f(\mathcal{Z}(A))\subseteq\mathcal{Z}(B)$. For the class $\mathcal{K}$, we take all squares in

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle h_1}\downarrow & & \downarrow{\scriptstyle h_2} \\ C & \xrightarrow{g} & D \end{array}$$

satisfying:

2.5.1. h_1 and h_2 are injective;

2.5.2. C and D are special symmetric groupoids and g is a special homomorphism;

2.5.3. every element of C (of D) can be written as a group product of elements of $h_1(A)$ (respectively, of $h_2(B)$).

It is a consequence of 2.5.3 that h_1 and h_1 are members of $\mathcal{S}_0$. In fact, suppose that $(a_1, \cdots, a_n) \in \mathscr{Z}(A)$. By 1.10, the group product $h_1(a_1) \cdots h_1(a_n)$ centralizes every $h_1(a) \in h_1(A)$. Consequently, $h_1(a_1) \cdots h_1(a_n)$ is central by 2.5.3, so that $(h_1(a_1), \cdots, h_1(a_n)) \in \mathscr{Z}(C)$ according to 1.17.

In order to prove that the class $\mathcal{K}$ satisfies the conditions imposed in 2.3, it remains to show that $f \in \mathcal{S}_0$ if and only if $g \in \mathcal{S}_0$. If $g \in \mathcal{S}_0$, then $h_1 \circ f = g \circ h_1 \in \mathcal{S}_0$. Consequently, since h_2 is injective $f \in \mathcal{S}_0$. Conversely, assume that $f \in \mathcal{S}_0$. Let $(c_1, \cdots, c_n) \in \mathscr{Z}(C)$. By 2.5.3, $c_i = h_1(a_{i1}) \cdots h_1(a_{ik(i)})$. It follows from 1.17 that $(h_1(a_{11}), \cdots, h_1(a_{nk(n)})) \in \mathscr{Z}(C)$, so that since h_1 is injective, $(a_{11}, \cdots, a_{nk(n)}) \in \mathscr{Z}(A)$. Consequently, $(h_2 f(a_{11}), \cdots, h_2 f(a_{nk(n)})) \in \mathscr{Z}(D)$, because $f \in \mathcal{S}_0$ and $h_2 \in \mathcal{S}_0$. Using 1.17 again, together with the hypothesis that g is special, it follows that $g(c_1) \cdots g(c_n) = g(h_1(a_{11})) \cdots g(h_1(a_{1k(1)})) \cdots g(h_1(a_{n1})) \cdots g(h_1(a_{nk(n)})) = h_2(f(a_{11})) \cdots h_2(f(a_{nk(n)}))$ is central. Hence, $(g(c_1), \cdots, g(c_n)) \in \mathscr{Z}(D)$.

The role of the functor Φ in 2.3 is taken by Δ, where $\Delta(A) = A/Z(A)$, with $Z(A)$ the central congruence of A. If $f \in \mathcal{S}_0(A, B)$, then $f(Z(A)) = f(\mathscr{Z}(A) \cap A^2) \subseteq \mathscr{Z}(B) \cap B^2 = Z(B)$, so that f induces a unique homomorphism $\Delta f \colon \Delta A \to \Delta B$ such that

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle s_A}\downarrow & & \downarrow{\scriptstyle s_B} \\ \Delta A & \xrightarrow{\Delta f} & \Delta B \end{array}$$

commutes, with s_A and s_B defined to be the natural projection homomorphisms.

For the functor Ψ in 2.3 take $I\Lambda \colon \mathcal{S}_0 \to \mathcal{S}$. This functor is defined by virtue of 1.15. By 1.9, there exist injective homomorphisms $\bar{p}_A \colon \Delta A \to I\Lambda A$ such that $p_A = \bar{p}_A \circ s_A$. Since $\Delta f \circ s_A = s_B \circ f$ and $I\Lambda f \circ p_A = p_B \circ f$ for $f \in \mathcal{S}_0(A, B)$, it follows that $\{\bar{p}_A\}$ is a natural transformation from Δ to $I\Lambda$. Moreover, the squares

$$\begin{array}{ccc} \Delta A & \xrightarrow{\Delta f} & \Delta B \\ {\scriptstyle \bar{p}_A}\downarrow & & \downarrow{\scriptstyle \bar{p}_B} \\ I\Lambda A & \xrightarrow{I\Lambda f} & I\Lambda B \end{array}$$

plainly satisfy 2.5.1, 2.5.2, and 2.5.3. Thus, 2.3.2 is satisfied. To show that 2.3.1 holds, let

$$Sq(f, g; h_1, h_2) = \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle h_1}\big\downarrow & & \big\downarrow{\scriptstyle h_2} \\ C & \xrightarrow{g} & D \end{array}$$

belong to K, with f and g in $\mathcal{S}_0$. Then $Sq(I\Lambda f, I\Lambda g; I\Lambda h_1, I\Lambda h_2)$ satisfies 2.5.1 (by 1.16) and 2.5.2 (by definition). If $c \in C$, then since $Sq \in \mathcal{K}$, there exist $a_1, \cdots, a_n$ in A such that $c = h_1(a_1) \cdots h_1(a_n)$. It follows from 1.13 that $\lambda_c = \lambda_{h_1(a_1)} \cdots \lambda_{h_1(a_n)} = I\Lambda h_1(\lambda_{a_1}) \cdots I\Lambda h_1(\lambda_{a_n})$. Thus, $Sq(I\Lambda f, I\Lambda g; I\Lambda h_1, I\Lambda h_2)$ also satisfies 2.5.3, and is therefore a member of $\mathcal{K}$.

Since the conditions of 2.3 are satisfied, we conclude that for all $n < \omega$, the inductive definitions

$$\mathcal{S}_{n+1} = \{f \in \mathcal{S}_0 \colon \Delta f \in \mathcal{S}_n\} \quad \text{and} \quad \mathcal{S}_{n+1} = \{f \in \mathcal{S}_0 \colon I\Lambda f \in \mathcal{S}_n\}$$

are equivalent. Define $\mathcal{S}_\omega = \bigcap_{n<\omega} \mathcal{S}_n$ as in 2.1. Using the definition of $\mathcal{S}_\omega$ in terms of Δ, it is possible to characterize $\mathcal{S}_\omega$ in a form that is analogous to the description of $\mathcal{G}_\omega$ in 2.4.

DEFINITION 2.5. Let A be a symmetric groupoid. The sequence of *higher extended centers* of A is defined inductively by $\mathcal{Z}^1(A) = \mathcal{Z}(A)$ and $\mathcal{Z}^{n+1}(A) = s_A^{-1}(\mathcal{Z}^n(A/Z(A)))$, where $s_A \colon A \to A/Z(A)$ is the natural projection homomorphism.

Lemma 2.6. *Let A and B be symmetric groupoids, and let f be a groupoid homomorphism from A to B. Then $f \in \mathcal{S}_\omega$ if and only if $f(\mathcal{Z}^n(A)) \subseteq \mathcal{Z}^n(B)$ for all natural numbers n.*

The proof of 2.6 runs parallel to the proof of 2.4, so that it can be omitted.

Proposition 2.7. *If $f \in \mathcal{S}$ is surjective, then $f \in \mathcal{S}_\omega$. If $g \in \mathcal{G}$ is surjective, then $g \in \mathcal{G}_\omega$.*

Proof. Let $f \in \mathcal{S}(A, B)$ be surjective. By the remark following 1.16, $f \in \mathcal{S}_0$. Since $f \circ s_A = s_B \circ f$ and s_B is surjective, it follows that Δf is surjective. By induction, $\Delta^n f \in \mathcal{S}_0$ for all $n < \omega$. Hence, $f \in \mathcal{S}_\omega$ by 2.3(c).

Corollary 2.8. *For all $A \in$ ob $\mathcal{S}$, the homomorphism $p_A \colon A \to I\Lambda A$ belongs to $\mathcal{S}_\omega$. Moreover, $\Lambda p_A = q_{\Lambda A}$, and $Iq_G = p_{IG}$ for all $A \in$ ob $\mathcal{S}$ and $G \in$ ob $\mathcal{G}$.*

Proof. As we noted above, $\bar{p}_A \in \mathcal{S}_0$. Thus, since $p_A = \bar{p}_A \circ s_A$, and s_A is surjective, it follows that $p_A \in \mathcal{S}_0$. Moreover $\Lambda p_A(\lambda_a) = \lambda_{\lambda_a} = q_{\Lambda A}(\lambda_a)$ for all $a \in A$, so that $\Lambda p_A = q_{\Lambda A}$. By 2.7, $q_{\Lambda A} \in \mathcal{G}_\omega$, from which it follows that $p_A \in \mathcal{S}_\omega$ by 2.2.1. Finally, if $u \in IG$, then $q_G(u) = \lambda_u = p_{IG}(u)$. Thus, $Iq_G = p_{IG}$.

Collecting the results of 2.1 through 2.8, we obtain the main theorem of this section.

Theorem 2.9. *Let $\mathcal{S}_\omega$ be the category whose objects are symmetric groupoids, and whose morphisms are groupoid homomorphisms $f\colon A \to B$ such that $f(\mathcal{Z}^n(A)) \subseteq \mathcal{Z}^n(B)$ for all natural numbers n. Let $\mathcal{G}_\omega$ be the category whose objects are GI groups, and whose morphisms are group homomorphisms $g\colon G \to H$ such that $g(C^n(G)) \subseteq C^n(H)$ for all natural numbers n. Then Λ is a functor from $\mathcal{S}_\omega$ to $\mathcal{G}_\omega$ and I is a functor from $\mathcal{G}_\omega$ to $\mathcal{S}_\omega$. Moreover, the class $\{p_A\colon A \in \text{ob } \mathcal{S}_\omega\}$ is a natural transformation in $\mathcal{S}_\omega$ from the identity functor on $\mathcal{S}_\omega$ to $I\Lambda$, and the class $\{q_G\colon G \in \text{ob } \mathcal{G}_\omega\}$ is a natural transformation in $\mathcal{G}_\omega$ from the identity functor on $\mathcal{G}_\omega$ to ΛI.*

Corollary 2.10. *Let $\mathcal{G}_C$ be the full subcategory of $\mathcal{G}$ whose objects are the GI groups with trivial center, and let $\mathcal{S}_Z$ be the full subcategory of $\mathcal{S}_0$ whose objects are the symmetric groupoids A such that $Z(A)$ is the identity congruence on A. Then $\Lambda(\mathcal{S}_Z) \subseteq \mathcal{G}_C$ and $I(\mathcal{G}_C) \subseteq \mathcal{S}_Z$. Moreover, the identity functor on $\mathcal{G}_Z$ is naturally equivalent to ΛI, and the identity functor on $\mathcal{S}_Z$ is naturally equivalent to a subfunctor of $I\Lambda$.*

Proof. If $C(G) = \{1\}$, then $C^n(G) = \{1\}$ for all n, so that $\mathcal{G}_\omega(G, H) = \mathcal{G}(G, H)$ by 2.4. Moreover, by 1.17, $Z(IG) = 1_{IG}$. Similarly, if $Z(A) = 1_A$, then $\mathcal{S}_\omega(A, B) = \mathcal{S}_0(A, B)$ by 2.5 and 2.6. Also, $C(\Lambda A)$ is trivial. In fact, by 1.13.3, $\xi \in C(\Lambda A)$ if and only if $(\xi(a), a) \in Z(A)$ for all $a \in A$. The corollary now follows from 2.9.

Corollary 2.11. *The functor I is faithful and full on $\mathcal{G}_C$. The functor Λ is faithful on $\mathcal{S}_Z$ and full on the subcategory $I(\mathcal{G}_C)$ of $\mathcal{S}_Z$.*

The corollary is a straightforward consequence of 2.10 and 2.8. Notice that $\Lambda\colon \mathcal{S}_Z \to \mathcal{G}_C$ is also representative. It will follow from the results of Section 3 that $\Lambda\colon \mathcal{S} \to \mathcal{G}$ is representative as well.

The implication of 2.10 and 2.11 is that the bond between centerless *GI* groups and their involution groupoids is so tight that the two concepts are virtually interchangeable. For instance, the following observation is a special case of 2.11.

Corollary 2.12. *Let G and H be centerless GI groups.*

2.12.1. *$G \cong H$ if and only if $IG \cong IH$.*

2.12.2. aut $G \cong$ aut IG *by the restriction map.*

EXAMPLE 2.13. The functor Λ is not full on $\mathcal{S}_Z$. To see this, let G be a finite simple group with at least two conjugate classes of involutions, say G is the alternating group on 5 letters. Let $A = IG$, and let B be a single conjugate class of involutions in G. Since G is simple, $C(G) = \{1\}$ and $\langle A \rangle = \langle B \rangle = G$. By 1.17, $\mathcal{Z}(B) \subseteq \mathcal{Z}(A)$, so that the inclusion map $i\colon B \to A$ is a member of

$\mathcal{S}_0(B, A)$. By 2.11, Λi: $\Lambda B \to \Lambda A = \Lambda IG \cong G$ is injective. In fact, since $\langle B \rangle = G$, Λi is an isomorphism. Let $f = (\Lambda i)^{-1}$: $\Lambda A \to \Lambda B$. If $f = \Lambda g$, where $g \in \mathcal{S}_0(A, B)$, then $If = I\Lambda g$ is injective, so that g is also injective. This is impossible because $|B| < |A|$. It is also worth noting that B cannot be isomorphic to IH for any $H \in \mathcal{G}_C$. Otherwise, $G \cong \Lambda B \cong \Lambda IH \cong H$, so that $B \cong IG = A$.

As a final remark, note that 2.12.1 makes essential use of the hypothesis that G and H are centerless. In fact, if G is a finite GI group such that $|C(G)|$ is odd (for instance, if $G = SL_3(GF(25))$), then it is easy to check that the natural projection $G \to G/C(G)$ induces an isomorphism $I(G) \cong I(G/C(G))$.

3. Symmetry systems

The results in Section 2 show that centerless GI groups are faithfully represented by their associated symmetric groupoids; and vice versa, any symmetric groupoid whose central congruence is trivial can be realized as a subgroupoid of $I(G)$ for some centerless GI group G. This circummstance suggests that the central congruence may be one of the most important aspects of the theory of symmetric groupoids. In this section, we will see how much extra data is needed to recover a symmetric groupoid A from $\Lambda(A)$ and $M(A)$. The results provide a new way to look at $Z(A)$. Our construction is somewhat like Nagata's "idealization" of a module (see [5], for example).

DEFINTION 3.1. A symmetry system is an ordered quadruple $\mathfrak{S} = \langle G; M; \{X_u: u \in M\}; \{\theta(x, u): x \in G, u \in M\} \rangle$ such that:

3.1.1. G is a GI group;

3.1.2. M is a subgroupoid of $I(G)$ satisfying $\langle M \rangle = G$;

3.1.3. each X_u is a non-empty set, and $X_u \cap X_v = \emptyset$ for $u \neq v$;

3.1.4. $\theta(x, u)$ is a bijection from X_u to $X_{xux^{-1}}$ satisfying

(a) $\theta(x_1 x_2, u) = \theta(x_1, x_2 u x_2^{-1}) \theta(x_2, u)$, and

(b) $\theta(u, u) = 1_{X_u}$.

Henceforth, we will use the simpler notation $\langle G; M; \{X_u\}; \{\theta(x, u)\} \rangle$ to designate a symmetry system.

Proposition 3.2. *Let A be a symmetric groupoid. For $\mu \in M(A)$, denote $X_\mu = \{a \in A: \lambda_a = \mu\}$, and for $\xi \in \Lambda(A)$, $\mu \in M(A)$, define $\theta_A(\xi, \mu) = \xi | X_\mu$. Then $\mathfrak{S}(A) = \langle \Lambda(A); M(A); \{X_\mu\}; \theta_A(\xi, \mu)\} \rangle$ is a symmetry system.*

This observation is just a short calculation beyond 1.8 and 1.13.

We will presently associate a symmetric groupoid with each symmetry system; first it is convenient to assemble some properties of the mappings $\theta(x, \mu)$.

Lemma 3.3. *Let* $\{\theta(x, u): x\in G, u\in M\}$ *satisfy* 3.1.4. *Then*:

3.3.1. $\theta(1, u) = 1_{X_u}$ *for all* $u\in M$;

3.3.2. $\theta(x, u)^{-1} = \theta(x^{-1}, xux^{-1})$ *for all* $x\in G, u\in M$;

3.3.3. $\theta(u_1, u_2\circ\cdots\circ u_n\circ w)\theta(u_2, u_3\circ\cdots\circ u_n\circ w)\cdots\theta(u_{n-1}, u_n\circ w)\theta(u_n, w)$
$= \theta(u_1u_1\cdots u_n, w)$ *for* $u_i\in M$ *and* $w\in M$.

Proof. $\theta(1, u)=\theta(u, uuu^{-1})\theta(u, u)=1_{X_u}$. Also, $\theta(x^{-1}, xux^{-1})\theta(x, u)=\theta(x^{-1}x, u)=1_{X_u}$. Finally, 3.3.3 follows from 3.1.4 by induction on n.

Proposition 3.4. *Let* $\mathfrak{S}=\langle G; M; \{X_u\}; \{\theta(x, u)\}\rangle$ *be a symmetry system. Define*:

3.4.1. $A(\mathfrak{S}) = \bigcup_{u\in M}X_u$;

3.4.2. *for* $a\in X_u$, $b\in X_v$, *denote* $a\circ b=\theta(u, v)(b)$.

Then $\langle A(\mathfrak{S}), \circ\rangle$ is a symmetric groupoid.

Proof. If $b\in X_v$, then $\theta(u, v)(b)\in X_{uvu^{-1}}=X_{u\circ v}$. Hence, $a\circ b\in X_{u\circ v}$. By 3.1.4(b), $a\circ a=\theta(u, u)(a)=a$. By 3.3, $a\circ(a\circ b)=\theta(u, u\circ v)\theta(u, v)(b)=\theta(u^2, v)(b)=\theta(1, v)(b)=b$. Finally, if $c\in X_w$, then $(a\circ b)\circ(a\circ c)=\theta(u\circ v, u\circ w)\theta(u, w)(c)=\theta((u\circ v)u, w)(c)=\theta(uv, w)(c)=\theta(u, v\circ w)\theta(v, w)(c)=a\circ(b\circ c)$.

Lemma 3.5. *Let* $\mathfrak{S}=\langle G; M; \{X_u\}; \{\theta(x, u)\}\rangle$ *be a symmetry system. Let* $a_i\in X_{u_i}$ *for* $1\leq i\leq n$. *Then* $(a_1, \cdots, a_n)\in \mathcal{Z}(A(\mathfrak{S}))$ *if and only if* $\theta(u_1\cdots u_n, w)=1_{X_w}$ *for all* $w\in M$.

This lemma is a direct consequence of 3.3.3.

DEFINITION 3.6. A symmetry system $\mathfrak{S}=\langle G; M; \{X_u\}; \{\theta(x, u)\}\rangle$ is *reduced* if, for every $x\neq 1$ in G, there exists $u\in M$ such that $\theta(x, u)\neq 1_{X_u}$.

If $x\notin C(G)$, then $xux^{-1}\neq u$ for some $u\in M$. In this case, $\theta(x, u)$ maps X_u to a disjoint set $X_{xux^{-1}}$.

Corollary 3.7. *If* $\mathfrak{S}=\langle G; M; \{X_u\}; \{\theta(x, u)\}\rangle$ *is a reduced symmetry system, then* $Z(A(\mathfrak{S}))=\bigcup_{u\in M} X_u\times X_u$.

Proposition 3.8. *If* A *is a symmetric groupoid, then* $\mathfrak{S}(A)$ *is reduced, and* $A(\mathfrak{S}(A))=A$.

Proof. If $\xi\neq 1_A$, then $\xi(a)\neq a$ for some $a\in A$. Hence, $\theta(\xi, \lambda_a)(a)\neq a$, so that $\mathfrak{S}(A)$ is reduced. By definition, $A(\mathfrak{S}(A))=\bigcup_{\mu\in M}X_\mu=A$ as a set. An easy calculation shows that products in A and $A(\mathfrak{S}(A))$ are identical.

DEFINITION 3.9. Let $\mathfrak{S}_1=\langle G_1; M_1; \{X_{1u}\}; \{\theta_1(x, u)\}\rangle$ and $\mathfrak{S}_2=\langle G_2; M_2;$

$\{X_{2v}\}$; $\{\theta_2(y, v)\}\rangle$ be symmetry systems. A *morphism* from $\mathfrak{S}_1$ to $\mathfrak{S}_2$ is a pair $F=\langle f; \{e_u: u\in M_1\}\rangle$, such that:

3.9.1. $f\in \mathcal{G}_\omega(G_1, G_2)$;

3.9.2. $f(M_1)\subseteq M_2$;

3.9.3. $e_u: X_{1u}\to X_{2f(u)}$ satisfies $\theta_2(f(x), f(u))\circ e_u = e_{xux^{-1}}\circ\theta_1(x, u)$ for all $x\in G_1$, $u\in M_1$.

Our next two observations are direct consequences of this definition.

Lemma 3.10

3.10.1. *Let* $\mathfrak{S}_1$, $\mathfrak{S}_2$, *and* $\mathfrak{S}_3$ *be symmetry systems, and let* $F_i=\langle f_i; \{e_{iu}: u\in M_i\}\rangle: \mathfrak{S}_i\to\mathfrak{S}_{i+1}$ *be morphisms for* $i=1, 2$. *Define* $F_2\circ F_1=\langle f_2\circ f_1; \{e_{2f_1(u)}\circ e_{1u}: u\in M_1\}\rangle$. *Then* $F_2\circ F_1: \mathfrak{S}_1\to\mathfrak{S}_3$ *is a morphism of symmetry systems.*

3.10.2. $I_{\mathfrak{S}}=\langle 1_G; \{1_{X_u}\}\rangle$ *is an endomorphism of* $\mathfrak{S}=\langle\{G; M; \{X_u\}; \theta(x, u)\}\rangle$.

3.10.3. *The class of all symmetry systems and their morphisms forms a category in which composition is defined as in* 3.10.1 *and the identity morphism of* $\mathfrak{S}$ *is* $I_{\mathfrak{S}}$.

Lemma 3.11 *Let* $F=\langle f; \{e_u\}\rangle: \mathfrak{S}_1\to\mathfrak{S}_2$ *be a morphism of symmetry systems. Then F is an isomorphism if and only if f is a group isomorphism such that* $f(M_1)=M_2$, *and each map* e_u *is bijective. In this case,* $F^{-1}=\langle f^{-1}; \{(e_{f^{-1}(v)})^{-1}\}\rangle$.

NOTATION. Denote the full category of all *reduced* symmetry systems by $\mathcal{R}$.

Proposition 3.12. *If* $\mathfrak{S}=\langle G; M; \{X_u\}; \{\theta(x, u)\}\rangle$ *is a reduced symmerty system, then there is an isomorphism* $F(\mathfrak{S}): \mathfrak{S}\to\mathfrak{S}(A(\mathfrak{S}))$.

Proof. For $x\in G$, define $f(x)=\bigcup_{u\in M}\theta(x, u)$. Then $f(x)$ maps $A(\mathfrak{S})=\bigcup_{u\in M}X_u$ to itself, and $f(x_1x_2)=f(x_1)f(x_2)$ by 3.1.4(a). If $v\in M$ and $a\in X_u$, then $f(v)(a)=\theta(v, u)(a)=b\circ a$ for every $b\in X_v$. Hence, $\lambda_b=f(v)$ for all $b\in X_v$. Therefore, $M(A(\mathfrak{S}))=\{\lambda_b: b\in A(\mathfrak{S})\}=f(M)$, and $f(G)=\Lambda(A(\mathfrak{S}))$. Since $\mathfrak{S}$ is reduced, $f(x)=1_{A(\mathfrak{S})}$ implies $x=1$. Thus, f is an isomorphism of G to $\Lambda(A(\mathfrak{S}))$. For $a\in X_u$ and $b\in X_v$, we have $\lambda_a=\lambda_b$ if and only if $u=v$ (by 3.7). Thus, $X_{\lambda_a}=X_u$. Let e_u be the identity map on $X_u=X_{f(u)}$. By Definition 3.2, $\theta(f(x), f(u))=f(x)\,|\,X_{f(u)}=\theta(x, u)$. Hence, $F(\mathfrak{S})=\langle f; \{e_u\}\rangle: \mathfrak{S}\to\mathfrak{S}(A(\mathfrak{S}))$ is an isomorphism in the category $\mathcal{R}$ by 3.11.

Our next objective is to extend the object maps $A\to\mathfrak{S}(A)$ and $\mathfrak{S}\to A(\mathfrak{S})$ to functors.

Lemma 3.13. *Let* A *and* B *be symmetric groupoid, and let* $f\in\mathcal{S}_\omega(A, B)$.

Then $\mathfrak{S}(f)=\langle\Lambda(f);\ \{f\,|\,X_\mu\colon \mu\in M(A)\}\rangle$ *is a morphism of* $\mathfrak{S}(A)$ *to* $\mathfrak{S}(B)$. *Moreover, if* $g\in\mathcal{S}_\omega(B, C)$, *then* $\mathfrak{S}(g\circ f)=\mathfrak{S}(g)\circ\mathfrak{S}(f)$.

Proof. By 2.9, $\Lambda(f)\in\mathcal{G}_\omega(\Lambda(A), \Lambda(B))$, and if $\lambda_a\in M(A)$, then $\Lambda(f)(\lambda_a)=\lambda_{f(a)}\in M(B)$. Thus, 3.9.1 and 3.9.2 are satisfied. A calculation shows that if $\xi\in\Lambda(A)$, $\mu\in M(A)$, and $a\in X_\mu$ (*i.e.*, $\lambda_a=\mu$), then $\theta_B((\Lambda f)(\xi), (\Lambda f)(\mu))(e_\mu(a))=f(\xi(a))=e_{\xi\mu\xi^{-1}}(\theta_A(\xi, \mu)(a))$. Hence, $\mathfrak{S}(f)$ is a morphism. The equality $\mathfrak{S}(g\circ f)=\mathfrak{S}(g)\circ\mathfrak{S}(f)$ is a consequence of the functorial nature of Λ.

Obviously, $\mathfrak{S}(1_A)=I_{\mathfrak{S}(A)}$. Thus, $\mathfrak{S}$ is a functor from $\mathcal{S}_\omega$ to $\mathcal{R}$.

Lemma 3.14. *Let* $\mathfrak{S}_i=\langle G_i;\ M_i;\ \{X_{iu}\};\ \{\theta_i(x, u)\}\rangle$ *be reduced symmetry systems for* $i=1, 2$. *Let* $F=\langle f;\ \{e_u\}\rangle\colon \mathfrak{S}_1\to\mathfrak{S}_2$ *be a morphism. Define* $A(F)=\bigcup_{u\in M_1} e_u$. *Then* $A(F)\in\mathcal{S}_\omega(A(\mathfrak{S}_1), A(\mathfrak{S}_2))$. *Moreover, if* $G\in\mathcal{R}(\mathfrak{S}_2, \mathfrak{S}_3)$ *then* $A(G\circ F)=A(G)\circ A(F)$.

Proof. If $a\in X_u$ and $b\in X_v$, then $a\circ b=\theta(u, v)(b)\in X_{u\circ v}$, and $A(F)(a\circ b)=e_{uvu}\theta_1(u, v)(b)=\theta_2(f(u), f(v))e_v(b)=A(F)(a)\circ A(F)(b)$ by 3.9.3. Thus $A(F)$ is a groupoid homomorphism. If $(a_1, \cdots, a_n)\in\mathcal{Z}(A(\mathfrak{S}_1))$, where $a_j\in X_{u_j}$, then by 3.5 and the assumption that $\mathfrak{S}_1$ is reduced, $u_1\cdots u_u=1$. Consequently, $f(u_1)\cdots f(u_n)=1$, so that since $A(F)(a_j)\in X_{f(u_j)}$, it follows that $(A(F)(a_1), \cdots, A(F)(a_n))\in\mathcal{Z}(A(\mathfrak{S}_2))$. This shows that $A(F)\in\mathcal{S}_0$. Let $f_i\colon G_i\to\Lambda(A(\mathfrak{S}_i))$ be the isomorphism that was defined in 3.12. By the proof of 3.12, $a\in X_u$ implies $f_1(u)=\lambda_a$ and $f_2(f(u))=\lambda_{e_u(a)}$. Thus, $\Lambda(A(F))\circ f_1=f_2\circ f$. Since $f\in\mathcal{G}_\omega$, it follows that $\Lambda(A(F))\in\mathcal{G}_\omega$. Consequently, $A(F)\in\mathcal{S}_\omega$ by 2.2.1. A calculation proves the last assertion of 3.14.

Plainly, $A(I_{\mathfrak{S}})=1_{A(\mathfrak{S})}$, so that A is a functor from $\mathcal{R}$ to $\mathcal{S}_\omega$.

Lemma 3.15. *Let* $f\in\mathcal{S}_\omega(A, B)$ *be a homomorphism of symmetric groupoids. Then* $A(\mathfrak{S}(f))=f$. *Thus,* $A\circ\mathfrak{S}$ *is the identity functor on* $\mathcal{S}_\omega$.

Proof. By definition, $\mathfrak{S}(f)=\langle\Lambda(f);\ \{f\,|\,X_\mu\colon \mu\in M(A)\}\rangle$. Hence, $A(\mathfrak{S}(f))=\bigcup_{\mu\in M(A)} f\,|\,X_\mu=f$.

We can now prove the principal result of this section.

Theorem 3.16. *The category* $\mathcal{S}_\omega$ *of all symmetric groupoids and morphisms that preserve the higher extended centers is naturally equivalent to the category* $\mathcal{R}$ *of all reduced symmetry systems and their morphisms.*

Proof. By 3.15, it suffices to prove that $\mathfrak{S}\circ A$ is naturally equivalent to the the identity functor on $\mathcal{R}$. This is accomplished by showing that if $F=\langle f;\ \{e_n\}\rangle\in\mathcal{R}(\mathfrak{S}_1, \mathfrak{S}_2)$, then the square

$$\begin{array}{ccc}
\mathfrak{S}_1 & \xrightarrow{\quad F \quad} & \mathfrak{S}_2 \\
{\scriptstyle F(\mathfrak{S}_1)}\big\downarrow & & \big\downarrow{\scriptstyle F(\mathfrak{S}_2)} \\
\mathfrak{S}(A(\mathfrak{S}_1)) & \xrightarrow{\mathfrak{S}(A(F))} & \mathfrak{S}(A(\mathfrak{S}_2))
\end{array}$$

commutes, where $F(\mathfrak{S}_1)$ and $F(\mathfrak{S}_2)$ are the isomorphisms that were defined in 3.12. By definition, $F(\mathfrak{S}_2)\circ F=\langle g;\ \{e_n\}\rangle$, where $g=\bigcup_{w\in M_2}\theta_2(*,\ w)\circ f$. Thus, if $x\in G_1$, then $g(x)=\bigcup_{w\in M_2}\theta_2(f(x),\ w)$. On the other hand, $\mathfrak{S}(A(F))\circ F(\mathfrak{S}_1)=\langle h;\ \{e_u\}\rangle$, where $h=\Lambda(\bigcup_{u\in M_1}e_u)\circ(\bigcup_{u\in M_1}\theta_1(*,\ u))$. Let $v\in M_1$, and choose $c\in X_{1v}$. Then $h(v)=\Lambda(\bigcup_{u\in M_1}e_u)(\bigcup_{u\in M_1}\theta_1(v,\ u))=\Lambda(\bigcup_{u\in M_1}e_u)(\lambda_c)=\lambda_{e_v(c)}=\bigcup_{w\in M_2}\theta_2(f(v),\ w)=g(v)$. Therefore, $g\,|\,M_1=h\,|\,M_1$; consequently, $g=h$.

This theorem shows that the theory of symmetric groupoids is substantially equivalent to the theory of symmetry systems. The latter objects have the virtue that they can be constructed from familiar algebraic structures. The rest of this section is concerned with the fabrication of symmetry systems.

For any set X, we denote by $S(X)$ the group of all permutations of X, that is, bijections of X to itself.

DEFINITION 3.17. A *partial symmetry system* is a 5-tuple

$$\mathfrak{P}=\langle\{G;\ M;\ \{v_i\colon i\in J\};\ \{X_i\colon i\in J\};\ \{\theta_i\colon i\in J\}\rangle,$$

such that:

3.17.1. G a GI group;

3.17.2. M is a subgroupoid of $I(G)$ such that $G=\langle M\rangle$; and $M=\bigcup_{i\in J}K_i$, where K_i are distinct conjugate classes of involutions;

3.17.3. $v_i\in K_i$ for all $i\in J$;

3.17.4. X_i is a non-empty set, and $X_i\cap X_j=\emptyset$ for $i\neq j$ in J;

3.17.5. θ_i is a homomorphism from $C_G(v_i)$ to $S(X_i)$ such that $v_i\in\operatorname{Ker}\theta_i$.

As in the case of symmetry systems, we will abbreviate the notation for a partial symmetry system to $\langle G;\ M;\ \{v_i\};\ \{X_i\};\ \{\theta_i\}\rangle$.

REMARK. Since M is a subgroupoid of $I(G)$ and $\langle M\rangle=G$, it follows that M is closed under conjugation by elements of G. Thus, M is indeed a union of conjugate classes of G.

Every symmetry system gives rise to a partial symmetry system. It is the converse of this observation that is most interesting however.

Lemma 3.18. *Let* $\mathfrak{S}=\langle G;M;\ \{X_u\};\ \{\theta(x,\ u)\}\rangle$ *be a symmetry system. Let* $M=\bigcup_{i\in J}K_i$, *where the* K_i *are distinct conjugate classes. For each* $i\in J$, *let* $v_i\in K_i$. *Denote* $X_i=X_{v_i}$, *and* $\theta_i=\theta(*,\ v_i)\,|\,C_G(v_i)$. *Then* $\langle(G;\ M;\ \{v_i\};\ \{X_i\},\ \{\theta_i\}\rangle$ *is a partial symmetry system.*

Proof. By 3.1.4, θ_i is a homomorphism of $C_G(v_i)$ to $S(X_i)$ such that $\theta_i(v_i)=1$.

Construction 3.19. Let $\langle G\colon M;\ \{v_i\};\ \{X_i\};\ \{\theta_i\}\rangle$ be a partial symmetry system. Write $M=\bigcup_{i\in J} K_i$, a disjoint union of conjugate classes. For $i\in J$, choose a set Y_i of representatives of the left cosets of $C_G(v_i)$ in G. Define $\pi_i\colon G\to Y_i$ and $\rho_i\colon G\to C_G(v_i)$ by the condition

3.19.1. $$x=\pi_i(x)\rho_i(x) \qquad \text{for all } x\in G .$$

Define $\gamma_i\colon K_i\to Y_i$ by the conditions

3.19.2 $$u=\gamma_i(u)v_i\gamma_i(u)^{-1},\ \gamma_i(u)\in Y_i \qquad \text{for all } u\in K_i .$$

For $u\in K_i$, define $X_u=\{u\}\times X_i$, and for $x\in G$, $u\in K_i$, define $\theta(x, u)\colon X_u\to X_{xux^{-1}}$ by $\theta(x, u)(u, a)=(xux^{-1}, \theta_i(\rho_i(x\gamma_i(u)))(a))$.

Proposition 3.20. *With the notation of* 3.19 $\mathfrak{S}=\langle G; M; \{X_u\colon u\in M\}; \{\theta(x, u)\colon x\in G, u\in M\}\rangle$ *is a symmetry system. For* $\mathfrak{S}$ *to be reduced, it is necessary and sufficient that* $C(G)\cap\bigcap_{i\in J}\operatorname{Ker}\theta_i=\{1\}$.

Proof. The verification of 3.1.4 uses two simple identities whose proofs we omit:

(1) $\rho_i(xy)=\rho_i(x\pi_i(y))\rho_i(y)$;
(2) $\gamma_i(xux^{-1})=\pi_i(x\gamma_i(u))$.

To prove 3.1.4(a), let $u\in K_i$, $a\in X_i$, $x, y\in G$. Then

$$\begin{aligned}
&\theta(x, yuy^{-1})\theta(y, u)(u, a)=\theta(x, yuy^{-1})(yuy^{-1}, \theta_i(\rho_i(y\gamma_i(u)))(a))\\
&=(xyuy^{-1}x^{-1}, \theta_i(\rho_i(x\gamma_i(yuy^{-1})))(\theta_i(\rho_i(y\gamma_i(u)))(a)))\\
&=(xyuy^{-1}x^{-1}, \theta_i(\rho_i(x\gamma_i(yuy^{-1}))\rho_i(y\gamma_i(u)))(a))\\
&=(xyuy^{-1}x^{-1}, \theta_i(\rho_i(x\pi_i(y\gamma_i(u)))\rho_i(\gamma_i(u)))(a))\\
&=(xyuy^{-1}x^{-1}, \theta_i(\rho_i(xy\gamma_i(u)))(a))=\theta(xy, u)(u, a) .
\end{aligned}$$

Moreover,

$$\begin{aligned}
\theta(u, u)(u, a)&=(uuu^{-1}, \theta_i(\rho_i(u\gamma_i(u)))(a))=(u, \theta_i(\rho_i(\gamma_i(u)v_i))(a)))\\
&=(u, \theta_i(v_i)(a))=(u, a) ,
\end{aligned}$$

by 3.19.2, 3.19.1, and 3.17.5. Thus, 3.1.4(b) also holds. Finally, note that $\theta(x, u)=1_{X_u}$ for all $u\in M$ if and only if $xux^{-1}=u$ for all $u\in M$, and $\theta_i(\rho_i(x\gamma_i(u)))=1_{X_i}$ for all $u\in K_i$. Since $\langle M\rangle=G$, $xux^{-1}=u$ for all $u\in M$ is equivalent to $x\in C(G)$, in which case $\rho_i(x\gamma_i(u))=\rho_i(\gamma_i(u)x)=x$. Hence, $\mathfrak{S}$ is reduced if and only if $\theta_i(x)=1_{X_i}$ for all $i\in J$ and $x\in C(G)$ implies $x=1$. That is, $C(G)\cap\bigcap_{i\in J}\operatorname{Ker}\theta_i=\{1\}$.

It can be shown that different choices of the sets Y_i in 3.19 will lead to isomorphic symmetry systems. We omit this verification.

Corollary 3.21. *Let G be a GI group, and let M be subgroupoid of* $I(G)$

such that $\langle M \rangle = G$. *Moreover, if* $|G|=2$, *assume that* $M=G$. *Then there is a symmetric groupoid* A *and an isomorphism* $f\colon \Lambda(A) \to G$ *such that* $M=f(M(A))$.

Proof. Write $M = \bigcup_{i \in J} K_i$, where the K_i are distinct conjugate classes. For each $i \in J$, choose $v_i \in K_i$. Define $X_i = C_G(v_i)/\langle v_i \rangle$, and let θ_i be the left regular representation of $C_G(v_i)$ on X_i, so that $\operatorname{Ker}\theta_i = \langle v_i \rangle$. Then $\mathfrak{P} = \langle G; M; \{v_i\}; \{X_i\}; \{\theta_i\} \rangle$ is a partial symmetry system. If $|J|>1$, or if $|J|=1$ and $v_i \notin C(G)$, then clearly $C(G) \cap \bigcap_{i \in J} \operatorname{Ker}\theta_i = \{1\}$. The alternative to these cases is $|G|=2$ and $M=\{v_i\}$, which was excluded by hypothesis. Therefore, the symmetry system $\mathfrak{S}$ associated with $\mathfrak{P}$ is reduced. By 3.12., there is an isomorphism $f\colon G \to \Lambda(A(\mathfrak{S}))$ such that $f(M)=M(A(\mathfrak{S}))$.

REMARK. If A is a symmetric groupoid such that $|\Lambda(A)|=2$, then necessarily $M(A)=\Lambda(A)$. In fact, if $|M(A)|=1$, then $\lambda_a=\lambda_b$ for all a, b in A. Hence, $\lambda_a(b)=\lambda_b(b)=b$ for all b, so that $\lambda_a=1_A$ for all a. Consequently, $\Lambda(A)=\{1_A\}$.

EXAMPLE 3.22. Let G be an abelian GI group. Then G is an elementary 2-group, since any product of involutions is an involution. A subset M of G satisfies 3.17.2 provided $\langle M \rangle = G$. The conjugate classes being singletons, the set M itself can serve as the indexing set J in the notation of 3.17. With this convention, $v_u=u$ and $C_G(u)=G$ for $u \in M$, so that $Y_u=\{1\}$ is a set of coset representatives of $C_G(u)$ for the construction 3.19. With this choice of Y_u, we have $\pi_u(x)=1$, $\rho_u(x)=x$ for $x \in G$, and $\gamma_u(u)=1$ for $u \in M$. Let $\{X_u : u \in M\}$ be a set of non-empty sets such that $X_u \cap X_v = \emptyset$ for $u \neq v$ in M. For each $u \in M$, let $\theta_u\colon G \to S(X_u)$ be a homomorphism such that $u \in \operatorname{Ker}\theta_u$. Then $\langle G; M; M; \{X_u : u \in M\}; \{\theta_u : u \in M\} \rangle$ is a partial symmetry system whose associated symmetry system $\mathfrak{S} = \langle G; M; \{\{u\} \times X_u\}; \{\theta(x, u)\} \rangle$ is defined by $\theta(x, u)(v, b) = (v, \theta_v(u)(b))$. Moreover, the corresponding symmetric groupoid $A(\mathfrak{S})$ can be identified with $\bigcup_{u \in M} X_u$, where $a \circ b = \theta_v(u)(b)$ if $a \in X_u$ and $b \in X_v$. Note that $\mathfrak{S}$ is reduced if and only if $\bigcap_{u \in M} \operatorname{Ker}\theta_u = \{1\}$.

4. Semantical matters

Our attention in this section is on the classes of GI groups, symmetric groupoids, and special symmetric groupoids. Closure properties of these classes are studied. Free GI groups and free symmetric groupoids are constructed, and the relation between them is exhibited. The section closes with a characterization of the class of special symmetric groupoids by means of a set of Horn formulas.

Lemma 4.1. *Let* $\{G_j : j \in J\}$ *be a set of subgroups of the group* G, *such that each* G_j *is a GI group, and* $G = \langle \bigcup_{j \in J} G_j \rangle$. *Then* G *is a GI group.*

Proof. $G=\langle \cup_{j\in J} G_j\rangle=\langle \cup_{j\in J}\langle I(G_j)\rangle\rangle=\langle \cup_{j\in J} I(G_j)\rangle\subseteq\langle I(G)\rangle$.

Corollary 4.2. *The class $\mathcal{G}$ is closed under free products, direct limits, finite products, and split extensions. Any homomorphic image of a GI group is a GI group.*

Of course, $\mathcal{G}$ is not closed under the formation of subgroups. In fact, every group can be embedded in a group of the form $S(X)$, the permutations of X, and $S(X)$ is a *GI* group (see [8], p. 306). We will prove shortly that $\mathcal{G}$ is not closed under the formation of ultrapowers.

Proposition 4.3. *Let α be a cardinal number. Then there is a GI group G_α containing a set L of α non-identity involutions such that:*

4.3.1. *every $x\in G_\alpha$ has a unique representation*

$$x = u_0u_1\cdots u_{k-1}, \quad u_j\in L,\ u_j\neq u_{j+1} \quad \textit{for all } j<k-1;$$

4.3.2. *If G is any group, and f is a mapping from L to $I(G)$, then f has a unique extension to a group homomorphism of G_α to G.*

The group G is uniquely determined by either of the properties 4.3.1 or 4.3.2.

Proof. For each ordinal $\xi<\alpha$, let $D_\xi=\{1, v_\xi\}$ be a cyclic group of order two. Define G_α to be the free product of $\{D_\xi\colon \xi<\alpha\}$, and let L consist of the images in G_α of the generators v_ξ of D_ξ. The proposition is just a restatement of standard properties of free products ([8], pp. 175–6), together with 4.2.

We will call G_α the *free GI group* on L, or the free *GI* group on α generators. A representation

$$x = u_0u_1\cdots u_{k-1}, \qquad u_j\in I,$$

of $x\in G_\alpha$ will be called *reduced* if $u_j\neq u_{j+1}$ for all $j<k-1$.

Lemma 4.4. *Every element of $I(G_\alpha)-\{1\}$ is conjugate in G_α to some $u\in L$.*

Proof. Let $a=u_0u_1\cdots u_{k-1}\in I(G_\alpha)-\{1\}$ be a reduced representation of a. Then $k\geq 1$, because $a\neq 1$. We argue by induction on k that a is conjugate to some $u\in L$. This is obvious if $k=1$. Assume $k>1$. Then $1=a^2=u_0u_1\cdots u_{k-1}u_0u_1\cdots u_{k-1}$. By 4.3.1, $u_{k-1}=u_0$. Thus, $b=u_0au_0=u_1\cdots u_{k-2}\in I(G_\alpha)$, and $b\neq 1$ (otherwise, $a=u_0u_{k-1}=u_0^2=1$). By the induction hypothesis, $b=xux^{-1}$ for some $x\in G_\alpha$, $u\in L$ and $a=u_0xu(u_0x)^{-1}$.

Theorem 4.5. *The class of all GI groups is not closed under the formation of ultrapowers.*

Proof. Let $G=G_{\aleph_0}$ be the free *GI* group on the countably infinite set $L=\{u_n\colon n<\omega\}$ of distinct involutions. We will prove that if $\mathcal{F}$ is any nonprincipal filter on ω, then the reduced power $G^{\omega}/\mathcal{F}$ is not a GI group. The proof is based on the following observation:

(1) if $a_0, a_1, \cdots, a_{m-1}$ are elements of $I(G)$ satisfying $a_0a_1\cdots a_{m-1}=u_0u_1\cdots u_{k-1}$, then $m\geq k$.

To prove (1), note that by 4.4, each a_i can be written in the form $v_{i0}v_{i1}\cdots v_{ir_i-1}w_iv_{ir_i-1}\cdots v_{i1}v_{i0}$, where the v_{ij} and w_i belong to L. By 4.3.1, each u_l with $l<k$ occurs an odd number of times in the product

$$v_{00}\cdots v_{0r_0-1}w_0v_{0r_0-1}\cdots v_{00}\cdots v_{m-10}\cdots v_{m-1r_{m-1}-1}w_{m-1}v_{m-1r_{m-1}-1}\cdots v_{m-10}\,.$$

Consequently, each u_l occurs an odd number of times in the list $w_0, \cdots, w_{m-1}$. In particular, $m\geq k$. Returning to the main part of the proof, define $f\in G^{\omega}$ by $f(k)=u_0u_1\cdots u_{k-1}$. It will suffice to show that the equivalence class of f in $G^{\omega}/\mathcal{F}$ is not product of involutions. Suppose otherwise: there exist $g_0, g_1, \cdots, g_{m-1}$ in G^{ω} such that the sets $Q_i=\{j<\omega: g_i(j)^2=1\}$, $i<m$, and $R=\{j<\omega: f(j)=g_0(j)g_1(j)\cdots g_{m-1}(j)\}$ are members of $\mathcal{F}$. Then $R\cap Q_0\cap Q_1\cap\cdots\cap Q_{m-1}\in\mathcal{F}$, and since $\mathcal{F}$ is not principal, there exists $k>m$ such that $k\in R\cap Q_0\cap Q_1\cap\cdots\cap Q_{m-1}$. Hence, $u_0u_1\cdots u_{k-1}=f(k)=g_0(k)g_1(k)\cdots g_{m-1}(k)$, and $g_i(k)\in I(G)$ for all $i<m$. Since $k>m$, this contradicts (1).

Corollary 4.6. *The class $\mathcal{G}$ is not axiomatic: there is no set $\mathcal{E}$ of first order sentences in the language of group theory such that $\mathcal{G}$ is the class of all models of $\mathcal{E}$.*

Indeed, by the theorm of Loš, every axiomatic class is closed under ultraproducts.

We wish now to characterize the extended center of the symmetric groupodis $I(G_{\alpha})-\{1\}$. A definition is needed.

DEFINITION 4.6. Let $Q=\{k_0, k_1, \cdots, k_{2m+1}\}$ be a subset of ω listed in strictly increasing order. A *nested pairing* of Q is a partition Π of Q into two element subsets that satisfies the inductive condition:

4.6.1. there exists $i<2m+1$ such that $\{k_i, k_{i+1}\}\in\Pi$ and $\Pi-\{\{k_i, k_{i+1}\}\}$ is a nested pairing of $Q-\{k_i, k_{i+1}\}$.

Let $\mathcal{P}_m$ denote the set of all nested parings of $\{0, 1, \cdots, 2m+1\}$.

DEFINITION 4.7. Let A be a symmetric groupoid. A sequence $(a_0, a_1, \cdots, a_{2m+1})\in A^{2(m+1)}$ is *collapsible* if there exists $\Pi\in\mathcal{P}_m$ such that $\{i, j\}\in\Pi$ implies $a_i=a_j$.

Lemma 4.8. *If $(a_0, a_1, \cdots, a_{2m+1})$ is a collapsible sequence of elements in the symmetric groupoid A, then $(a_0, a_1, \cdots, a_{2m+1})\in\mathcal{Z}(A)$.*

Proof. If $m=0$, the assertion is obvious, since $\mathcal{P}_0=\{\{\{0, 1\}\}\}$. Assume that $m>0$. By 4.6.1, there exists $\{j, j+1\}\in\Pi$ such that $\Pi-\{\{j, j+1\}\}$ is a nested pairing of $2(m+1)-\{j, j+1\}$. Then $\lambda_{a_0}\lambda_{a_1}\cdots\lambda_{a_{2m+1}}=\lambda_{a_0}\lambda_{a_1}\cdots\lambda_{a_{j-1}}\lambda_{a_{j+2}}\cdots\lambda_{a_{2m+1}}$. The lemma follows by induction on m.

DEFINITION 4.9. Let A be a symmetric groupoid. Denote by $\mathcal{Z}_0(A)$ the set of all sequences $(a_0, a_1, \cdots, a_k)$ of elements of A for which there is a representation $a_i=b_{i0}\circ b_{i1}\circ\cdots\circ b_{ir(i)-1}\circ b_i$ such that the composite sequence $(\beta_0, \beta_1, \cdots, \beta_k)$ is collapsible, where $\beta_i=(b_{i0}, b_{i1}, \cdots, b_{ir(i)-1}, b_i, b_{ir(i)-1}, \cdots, b_{i1}, b_{i0})$. The symmetric groupoid A is called centerless if $Z(A)=1_A$ and $\mathcal{Z}_0(A)=\mathcal{Z}(A)$.

REMARKS. (1) It follows by an inductive argument from 1.18 that $\mathcal{Z}_0(A)\subseteq\mathcal{Z}(A)$ for all symmetric groupoids A.

(2) If f: $A\to B$ is a groupoid homomorphism of symmetric groupoids, then $f(\mathcal{Z}_0(A))\subseteq\mathcal{Z}_0(B)$. Consequently, if A is centerless (so that $\mathcal{Z}^n(A)=\mathcal{Z}(A)=\mathcal{Z}_0(A)$ for all $n<\omega$), then $\mathcal{S}_\omega(A, B)=\mathcal{S}(A, B)$.

We will show that for all α, $I(G_\alpha)-\{1\}$ is centerless. The proof is based on a property of G_α.

Lemma 4.10. *Let G_α be the free GI group on a set L of α involutions. If $\alpha>1$, then $C(G_\alpha)=\{1\}$. Moreover, if $(u_0, u_1, \cdots, u_n)\in L^{n+1}$ satisfies $u_0u_1\cdots u_n=1$, then $(u_0, u_1, \cdots, u_n)$ is collapsible.*

Proof. Assume that $\alpha>1$. Let $x\in G_\alpha-\{1\}$ have the reduced representation $u_0u_1\cdots u_r$. Since $\alpha>1$, there exists $u\in L$ such that either $u\neq u_0$ or $u\neq u_r$. In both cases, it follows from 4.3.1 that $ux\neq xu$. Hence, $C(G_\alpha)=\{1\}$. The second assertion is obtained by induction on n. By 4.3.1, $u_0u_1\cdots u_n=1$ implies that $u_j=u_{j+1}$ for some $j<n$. Consequently, $u_0u_1\cdots u_{j-1}u_{j+2}\cdots u_n=1$.

Proposition 4.11. *For $\alpha\geq1$, the symmetric groupoid $A_\alpha=I(G_\alpha)-\{1\}$ is centerless, where G_α is the free GI group on α involutions.*

Proof. If $\alpha=1$, then G_α is cyclic of order 2, and $|A_\alpha|=1$. In this case, the assertion is trivially true. Assume that $\alpha>1$, so that $C(G_\alpha)=1$ by by 4.10. By 1.17, $(a_0, a_1, \cdots, a_k)\in\mathcal{Z}(A_\alpha)$ implies $a_0a_1\cdots a_k=1$. Thus, if $k=1$, then $a_0=a_1$. Hence $Z(A_\alpha)=1_{A_\alpha}$. Moreover, it follows from 4.4 and 4.10 that $\mathcal{Z}(A_\alpha)\subseteq\mathcal{Z}_0(A_\alpha)$. By the first remark following 4.9, A_α is centerless.

Not all sequences in $\mathcal{Z}(A_\alpha)$ are collapsible. For instance, if $a_0=u_0u_1u_0$, $a_1=u_0u_1u_0u_1u_0u_1u_0$, and $a_2=u_1$, then $(a_0, a_1, a_0, a_2)\in\mathcal{Z}(A_2)$.

Theorem 4.12. *Let G_α be the free GI group on the set L of α involutions. Denote the symmetric groupoid $I(G_\alpha)-\{1\}$ by A_α. Then A_α is the free symmetric groupoid on L.*

Proof. By 4.3 and 4.4, every $a\in I(G_\alpha)-\{1\}$ has a unique reduced representation $a=u_0\circ u_1\circ\cdots\circ u_{k-1}$, with $k\geq 1$, $u_j\in L$ and $u_j\neq u_{j+1}$ for $j<k-1$. Denote by $l(a)$ the number k of terms in the reduced representation of a. Let f be a mapping of L to a symmmetric groupoid A. Extend f to A_α by defining $f(a)=f(u_0)\circ f(u_1)\circ\cdots\circ f(u_{k-1})$, where $a=u_0\circ u_1\circ\cdots\circ u_{k-1}$ is reduced. This definition is well posed by the uniquess of reduced representations. We argue by induction on $l(a)$ that $f(a\circ b)=f(a)\circ f(b)$ for all $a, b\in A$. Let $a=u_0\circ u_1\circ\cdots\circ u_{k-1}$ and $b=v_0\circ v_1\circ\cdots\circ v_{m-1}$ be the reduced representations of a and b. Assume that $k=l(a)=1$. If $u_0\neq v_0$, then $u_0\circ v_0\circ v_1\circ\cdots\circ v_{m-1}$ is the reduced representation of $a\circ b$, so that $f(a\circ b)=f(u_0)\circ f(v_0)\circ f(v_1)\circ\cdots\circ f(v_{m-1})=f(a)\circ f(b)$. If $u_0=v_0$, then $a\circ b=v_1\circ\cdots\circ v_{m-1}$ by 1.1.2. Thus, $f(a\circ b)=f(v_1)\circ\cdots\circ f(v_{m-1})=f(u_0)\circ f(v_0)\circ f(v_1)\circ\cdots\circ f(v_{m-1})=f(a)\circ f(b)$. Assume that $l(a)>1$. Then $a=u_0\circ c$, where $c=u_1\circ\cdots\circ u_{k-1}$ satisfies $l(c)=l(a)-1$. By the induction hypothesis and 1.5, $f(a\circ b)=f((u_0\circ c)\circ b)=f(u_0\circ c\circ u_0\circ b)=f(u_0)\circ f(c)\circ f(u_0)\circ f(b)=(f(u_0)\circ f(c))\circ f(b)=f(a)\circ f(b)$.

REMARK. As we noted in the comment after 4.9, every homomorphism of a centerless symmetric groupoid is a member of $\mathcal{S}_\omega$. Thus, A_α is free in either of the categories $\mathcal{S}$ or $\mathcal{S}_\omega$.

The rest of this section is concerned with the class of special symmetric groupoids: those groupoids that are isomorphic to a subgroupoid of $I(G)$ for some GI group G. An example shows that the special symmetric groupoids constitute a proper subclass of $\mathcal{S}$.

EXAMPLE 4.13. Let $A=\{a, b, c\}$, where a, b, and c are distinct. Define $a\circ x=c\circ x=x$ for all $x\in A$, and $b\circ a=c$, $b\circ b=b$, $b\circ c=a$. Then A is a symmetric groupoid, but A is not special. In fact, if G is a group, then any subgroupoid of $I(G)$ satisfies: $x\circ y=y$ implies $y\circ x=x$. This implication obviously does not hold in A.

It follows from a theorem of A. I. Omarov [7] that the class of special symmetric groupoids is a quasivariety. In particular, this class is hereditary, and closed under the formation of products and ultraproducts. By 4.12 and 4.13 homomorphic image of a special symmetric groupoid needn't be speical.

We proceed to give an explicit construction of the universal special symmetric groupoid asscoiated with an arbitrary symmetric groupoid A. This will make it possible to exhibit a recursive set of Horn formulas that axiomatize the class of all special symmetric groupoids.

Proposition 4.14. *Let A be a symmetric groupoid. Let $u\colon A\to L$ be a bijective map. Let G_α be the free GI group on L where $\alpha=|A|$. Let N_A be the normal subgroup of G_α that is generated by $\{u(a\circ b)u(a)u(b)u(a)\colon a, b\in A\}$. Denote $E_A=G_\alpha/N_A$, with $t\colon G_\alpha\to E_A$ the natural projection. Define $f_A=t\circ u\colon A\to I(E_A)$. Then*:

4.14.1. E_A is a GI group;

4.14.2. f_A is a groupoid homomorphism;

4.14.3. $f_A(A)$ generates E_A as a group;

4.14.4. if H is a group, and $g: A \to I(H)$ is a homomorphism, then there is a group homomorphism $h: E_A \to H$ such that $g=(h \mid I(E_A)) \circ f_A$.

The pair (E_A, f_A) is uniquely determined by 4.12.1-4.12.4.

Proof. The properties 4.14.1, 4.14.2, and 4.14.3 are direct consequences of the definitions. To prove 4.14.4, define $f: L \to I(H)$ by $f(v)=g(u^{-1}(v))$. By 4.3.2, f extends to a group homomorphism of G_α to H. If a, $b \in A$, then $f(u(a \circ b)u(a)u(b)u(a))=g(a \circ b)g(a)g(b)g(a)=1$, so that $N_A \subseteq \operatorname{Ker} f$. Thus, there is a group homomorphism $h: E_A \to H$ such that $f=h \circ t$. Then $h(f_A(a))=h(t(u(a)))$ $=f(u(a))=g(a)$. The uniqueness is a categorical fact.

Corollary 4.15. *A symmetric groupoid A is special if and only if f_A is injective.*

A more explicit description of the normal subgroup N_A that was defined in 4.14 is needed.

Lemma 4.16. *Let the notation and hypotheses be as in* 4.14. *For $a_0, a_1, \cdots, a_r, b$ in A, denote $w(a_0, a_1, \cdots, a_r; b)=u(a_0 \circ a_1 \circ \cdots \circ a_r \circ b)u(a_0)u(a_1)\cdots u(a_r)u(b)u(a_r)\cdots u(a_1)u(a_0)$. Then N_A consists of the set of all products of elements of the form $w(a_0,a_1,\cdots,a_r;b)$, where $a_0 \neq a_1 \neq \cdots \neq a_r \neq b$ in A.*

Proof. Using the identities of 1.1 and the fact $u(a)^2=1$ in G_α, it is easily seen that:

4.16.1. if $a_{i-1}=a_i$ for $i \leq r$, then $w(a_0, \cdots, a_r; b)=w(a_0, \cdots, a_{i-2}, a_{i+1}, \cdots, a_r; b)$, and if $a_r=b$, then $w(a_0, \cdots, a_r; b)=w(a_0, \cdots, a_{r-1}; a_r)$;

4.16.2. $w(a_0, \cdots, a_r; b)^{-1}=w(a_0 \circ \cdots \circ a_r \circ b, a_0, \cdots, a_r; b)$;

4.16.3. $w(a_0, a_1, \cdots, a_r; b)=w(a_0, a_1 \circ \cdots \circ a_r \circ b)u(a_0)w(a_1, \cdots, a_r; b)u(a_0)$.

Consequently, the set N of all products of elements of the form $w(a_0, a_1, \cdots, a_r; b)$ with $a_0 \neq a_1 \neq \cdots \neq a_r \neq b$ is a normal subgroup of G_α that includes all products of the form $u(a \circ b)u(a)u(b)u(a)$. Thus $N \supseteq N_A$. On the other hand, it follows from 4.16.3 by induction on r that every $w(a_0, \cdots, a_r; b)$ is a member of N_A.

Corollary 4.17. *The symmetric groupoid A is special if and only if every relation of the form*

$$w(a_{00}, \cdots, a_{0r(0)}; b_0)w(a_{10}, \cdots, a_{1r(1)}; b_1)\cdots w(a_{k0}, \cdots, a_{kr(k)}; b_k) = u(c)u(d)$$

in G entails $c=d$ in A.

Proof. $f_A(c)=f_A(d)$ if and only if $u(c)u(d) \in N_A$.

It is reasonably clear from 4.10 that the criterion of 4.17 can be formalized. The details follow.

Lemma 4.18. *Let G_α be the free GI group on the set L of involutions. Let $(u_0, u_1, \cdots, u_n) \in L^{n+1}$. Then $u_0 u_1 \cdots u_n = vw$, where $v, w \in L$ if and only if either $v = w$ and $(u_0, u_1, \cdots, u_n)$ is collapsible, or there exist $i < j \leq n$ such that $v = u_i$, $w = u_j$, and the sequences $(u_0, \cdots, u_{i-1})$, $(u_{i+1}, \cdots, u_{j-1})$, $(u_{j+1}, \cdots, u_n)$ are collapsible or empty.*

Proof. These conditions obviously imply $u_0 u_1 \cdots u_n = vw$. For the proof of the converse, it can be assumed by 4.10 that $v \neq w$ and $n > 2$. By 4.3.1 there exists $j < n$ such that $u_j = u_{j+1}$. Moreover, if $u_j = v$, then $v = u_i$ for some $i \neq j$, $j+1$. The same is true if $u_j = w$. The result then follows by induction on n.

NOTATION 4.19. Let L be the first order language of symmetric groupoids with a countable sequence $\{z_n : n < \omega\}$ of distinct variables. Thus, in addition to the usual logical symbols $\wedge$, $\vee$, $\sim$, $\rightarrow$, $\exists$, $\forall$, $=$ of the first order predicate calculus with equality, L includes a binary operation symbol $\circ$. It is convenient to add to the operation symbols of L the n-fold composition of $\circ$, grouped according to the convention of 1.12. Of course, these operations are definable in L:

4.19.1. for $r+1 < s$, denote by $W(r, s)$ the formula

$$(z_{2r+1} = z_{2s-1}) \wedge (z_{2r+2} = z_{2s-2}) \wedge \cdots \wedge (z_{r+s-1} = z_{r+s+1})$$
$$\wedge (z_{2r} = z_{2r+1} \circ z_{2r+2} \circ \cdots \circ z_{r+s-1} \circ z_{r+s});$$

4.19.2. for a nested pairing $\Pi = \{\{i_1, j_1\}, \{i_2, j_2\}, \cdots, \{i_s, j_s\}\}$ of a finite subset of ω, denote by $V(\Pi)$ the formula

$$(z_{i_1} = z_{j_1}) \wedge (z_{i_2} = z_{j_2}) \wedge \cdots \wedge (z_{i_s} = z_{j_s});$$

let $V(\Pi)$ be the empty formula when $\Pi = \phi$.

Theorem 4.20. *Let $\mathcal{H}$ be the set of all formulas in L that are of the form*

$$W(0, r_1) \wedge W(r_1, r_2) \wedge \cdots \wedge W(r_{k-1}, m) \wedge V(\Pi_1) \wedge V(\Pi_2) \wedge V(\Pi_3) \rightarrow (z_i = z_j),$$

where $1 < r_1$, $r_1 + 1 < r_2$, $\cdots$, $r_{k-1} + 1 < m$, $0 \leq i < j \leq 2m-1$, Π_1 is a nested pairing of $\{0, 1, \cdots, i-1\}$, Π_2 is a nested pairing of $\{i+1, i+2, \cdots, j-1\}$, and Π_3 is a nested pairing of $\{j+1, j+2, \cdots, 2m-1\}$. Then the class of symmetric groupoids that satisfy all of the formulas of $\mathcal{H}$ coincides with the class of special symmetric groupoids.

Proof. Let A be a symmetric groupoid, and suppose that $(a_0, a_1, \cdots, a_{2m-1}) \in A^{2m}$, where $m \geq 1$. By 4.16, $u(a_0)u(a_1) \cdots u(a_{2m-1}) \in N_A$ (in the notation of 4.14) if and only if $(a_0, a_1, \cdots, a_{2m-1})$ satisfies $W(0, r_1) \wedge W(r_1, r_2) \wedge \cdots \wedge W(r_{k-1}, m)$ for a suitable choice $0<r_1, r_1+1<r_2, \cdots, r_{k-1}+1<m$. For c and d in A, it follows from 4.18 that $u(c)u(d)\in N_A$ if and only if c occurs as a_i and d occurs as a_j in a sequence $(a_0, a_1, \cdots, a_{2m-1})\in A^{2m}$ that satisfies $V(\Pi_1)\wedge V(\Pi_2)\wedge V(\Pi_3)$ for suitable nested pairing Π_1 of $\{0, 1, \cdots, i-1\}$, Π_2 of $\{i+1, i+2, \cdots, j-1\}$, and Π_3 of $\{j+1, \cdots, 2m-1\}$, and $u(a_0)u(a_1)\cdots u(a_{2m-1})\in N_A$. On the basis of these observations and 4.17, it is clear that A is special if and only if A satisfies all formulas in $\mathcal{H}$.

It is evident that the set $\mathcal{H}$ is recursive with respect to a Gödel numbering of L. However, the construction process will frequently produce sentences that are deducible from the identities of the class of symmetric groupoids. Example 4.13 shows that there is at least one formula in $\mathcal{H}$ that is not a consequence of the theory of symmetric groupoids. The following example shows that $\mathcal{H}$ is effectively infinite.

EXAMPLE 4.21. Let n be a positive integer. For $k<\omega$, denote by (k) the least non-negative residue of k modulo $n+1$. Let $U(k)$ denote the formula $z_{(k)}\circ z_{(k+1)}\circ\cdots\circ z_{(k+n)}=z_{(k+n)}$. It is easy to see $U(1)\wedge U(2)\wedge\cdots\wedge U(n)\rightarrow U(0)$ is equivalent to a formula of $\mathcal{H}$. For example, if $n=3$, then $U(1)\wedge U(2)\wedge U(3)\rightarrow U(0)$ can be obtained by the rule of substitution from $W(0, 4)\wedge W(4, 8)\wedge W(8, 12)\wedge W(12, 16)\wedge V(\Pi)\rightarrow(z_0=z_{31})$, where Π is the nested pairing $\{7, 8\}$, $\{6, 9\}$, $\{5, 10\}$, $\{4, 11\}$, $\{15, 16\}$, $\{14, 17\}$, $\{13, 18\}$, $\{12, 19\}$, $\{23, 24\}$, $\{22, 25\}$, $\{21, 26\}$, $\{20, 27\}$, $\{3, 28\}$, $\{2, 29\}$, $\{1, 30\}$. Assume now that $n\geq 3$. We will construct a symmetric groupoid that satisfies $U(1)\wedge U(2)\wedge\cdots\wedge U(m)\rightarrow U(0)$ for all $m<n$, but does not satisfy this formula for $m=n$. Let $G=\langle u_0\rangle\times\langle u_1\rangle\times\cdots\times\langle u_n\rangle$ be a direct product of $n+1$ copies $\langle u_i\rangle$ of the cyclic group of order 2. Denote $M=\{u_0, u_1, \cdots, u_n\}$. Then $\langle M\rangle=G$. Define subgroups H_i of G by $H_0=\langle u_0\rangle$, $H_i=\langle u_i, w\rangle$ for $1\leq i\leq n$, where $w=u_0u_1\cdots u_n$. Let $X_i=X_{u_i}$ be the coset space G/H_i. Finally, define $\theta_i=\theta_{u_i}\colon G\rightarrow S(X_i)$ by $\theta_i(x)(yH_i)=xyH_i$. Plainly, $\theta_i(x)(yH_i)=yH_i$ for some $y\in G$ if and only if $x\in H_i$. By 3.22, the partial symmetry system $\langle G; M; M; \{X_i\}; \{\theta_i\}\rangle$ determines a symmetric groupoid A in which

$$a_0\circ a_1\circ\cdots\circ a_{m-1}\circ a_m = \theta_{i_m}(u_{i_0}u_{i_1}\cdots u_{i_{m-1}})(a_m),$$

where $a_k\in X_{i_k}$ for $k\leq m$. Thus, $a_0\circ a_1\circ\cdots\circ a_{m-1}\circ a_m=a_m$ if and only if $u_{i_0}u_{i_1}\cdots u_{i_{m-1}}\in H_{i_m}$. In particular, if $m<n$, then $a_0\circ a_1\circ\cdots\circ a_{m-1}\circ a_m=a_m$ is equivalent to the product $u_{i_0}u_{i_1}\cdots u_{i_{m-1}}$ being equal to either u_{i_m} or 1. If $u_{i_0}u_{i_1}\cdots u_{i_{m-1}}=u_{i_m}$, then $u_{i_0}\cdots u_{i_{j-1}}u_{i_{j+1}}\cdots u_{i_m}=u_{i_j}$, so that $a_0\circ\cdots\circ a_{j-1}\circ a_{j+1}\circ\cdots\circ a_m\circ a_j=a_j$, for all j. Thus, $U(1)\wedge U(2)\wedge\cdots\wedge U(m)\rightarrow U(0)$ is satisfied by $(a_0, a_1, \cdots, a_m)$ in this case. Assume that $u_{i_0}u_{i_1}\cdots u_{i_{m-1}}=1$. Then $m\geq 2$, and $u_{i_0}\cdots u_{i_{j-1}}u_{i_{j+1}}\cdots u_{i_m}=u_{i_j}u_{i_m}$ for all $j\leq m$. Moreover, the number of j such that $i_j\neq i_m$ is even, hence either 0 or ≥ 2. From this observation, it follows that $(a_0, a_1, \cdots, a_m)$ satisfies $U(1)\wedge U(2)\wedge\cdots\wedge U(m)\rightarrow U(0)$ in all cases. Assume that $m=n$. Choose $a_i\in X_i$

for all $i \leq n$. Then $a_{j+1} \circ a_{j+2} \circ \cdots \circ a_n \circ a_0 \circ \cdots \circ a_j = \theta_j(u_{j+1} u_{j+2} \cdots u_n u_0 \cdots u_{j-1})(a_j) = \theta_j(u_j w)(a_j)$. Hence $a_{j+1} \circ a_{j+2} \circ \cdots \circ a_j = a_j$ for $1 \leq j \leq n$, and $a_1 \circ a_2 \circ \cdots \circ a_n \circ a_0 \neq a_0$. Thus A does not satisfy $U(1) \wedge U(2) < \cdots \wedge U(n) \rightarrow U(0)$.

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