Ikeda, N. and Watanabe, S.
Osaka J. Math.
**14** (1977), 619–633

# A COMPARISON THEOREM FOR SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS AND ITS APPLICATIONS

NOBUYUKI IKEDA AND SHINZO WATANABE

(Received August 2, 1976)

**Introduction.** Comparison theorem for solutions of stochastic differential equations was discussed by A.V. Skorohod [9] and T. Yamada [10]. In §1, we will modify the main theorem of T. Yamada [10] so that it is more convenient for applications. As an application, we discuss in §2 some stochastic optimal control problem which was recently studied by V.E. Beneš [1] using different methods. In §3, we obtain some comparison theorem for one-dimensional projection of a diffusion process. As an example of applications, we see that Hashiminsky's test for explosion ([3], [7]) is obtained simply from a well known one-dimensional result.

## 1. A comparison theorem for one-dimensional Itô processes

The following theorem is a modification of Theorem 1.1 in T. Yamada [10].

**Theorem 1.1.** *Suppose we are given the following*:

(i) *a real continuous function* $\sigma(t, x)$ *defined on* $[0, \infty)\times R$ *such that*

$$|\sigma(t, x)-\sigma(t, y)| \leq \rho(|x-y|), \qquad x, y\in R, \quad t\geqq 0, \tag{1.1}$$

*where* $\rho(\xi)$ *is an increasing function on* $[0, \infty)$ *such that* $\rho(0)=0$ *and*

$$\int_{0+} \rho(\xi)^{-2}d\xi = \infty, \tag{1.2}$$

(ii) *real continuous functions* $b_1(t, x)$ *and* $b_2(t, x)$ *defined on* $[0, \infty)\times R$ *such that*

$$b_1(t, x)<b_2(t, x), \qquad t\geqq 0, \quad x\in R. \tag{1.3}$$

*Let* $(\Omega, \mathfrak{F}, P; \mathfrak{F}_t)$ *be a complete probability space with right continuous increasing family* $(\mathfrak{F}_t)_{t\geqq 0}$ *of sub* $\sigma$*-fields of* $\mathfrak{F}$ *each containing* $P$*-null sets and suppose we are given the following stochastic processes defined on it*:

(iii) *two real $\mathfrak{F}_t$-adapted continuous processes $x_1(t, \omega)$ and $x_2(t, \omega)$,*
(iv) *a one-dimensional $\mathfrak{F}_t$-Brownian motion $B(t, \omega)$ such that $B(0)=0$, a.s.,*
(v) *two real $\mathfrak{F}_t$-adapted well measurable processes $\beta_1(t, \omega)$ and $\beta_2(t, \omega)$.*

*We assume that they satisfy the following conditions with probability one*:

$$x_i(t)-x_i(0) = \int_0^t \sigma(s, x_i(s))dB(s)+\int_0^t \beta_i(s)ds\,, \quad i=1, 2\,, \tag{1.4}$$

$$x_1(0)\leq x_2(0)\,, \tag{1.5}$$

$$\beta_1(t)\leq b_1(t, x_1(t)), \qquad \textit{for all} \quad t\geqq 0\,, \tag{1.6}$$

$$\beta_2(t)\geq b_2(t, x_2(t)), \qquad \textit{for all} \quad t\geqq 0\,, \tag{1.7}$$

*Then, with probability one, we have*

$$x_1(t)\leq x_2(t), \qquad \textit{for all} \quad t\geqq 0\,. \tag{1.8}$$

*If, furthermore, the pathwise uniqueness* (cf. [11]) *of solutions holds for at least one of the following stochastic differential equations*

$$dX(t) = \sigma(t, X(t))dB(t)+b_i(t, X(t))dt\,, \quad i=1, 2\,, \tag{1.9}$$

*then, we have the same conclusion* (1.8) *by weakening* (1.3) *to*

$$b_1(t, x)\leq b_2(t, x)\,, \quad t\geq 0\,, \quad x\in R\,. \tag{1.3$'$}$$

Proof. In the following proof, it is more convenient to assume that $\sigma(t, x)$ and $b_i(t, x)$, $i=1, 2$, may depend on $\omega$ and $\omega \rightsquigarrow \sigma(t, x, \omega)$ and $\omega \rightsquigarrow b_i(t, x, \omega)$, $i=1, 2$, are $\mathfrak{F}_0$-measurable for any fixed $(t, x)$. Also, by a standard localization argument, we may assume that they are all bounded. First, we prove

$$P(\exists t>0;\ x_1(s)\leq x_2(s) \quad \text{for all} \quad s\in[0, t]) = 1\,, \tag{1.10}$$

under the above assumptions except that (1.5) is replaced by

$$x_1(0) = x_2(0)\,. \tag{1.5$'$}$$

For the proof, set

$$\tau = \inf\{s;\ b_2(s, x_2(s))<b_1(s, x_1(s))\}\,.$$

By (1.3) and (1.5)$'$, it is clear that $P\{\tau>0\}=1$. Let $\tilde{t}>0$ be fixed and set $t=\tilde{t}\wedge\tau$. Then

$$\begin{aligned} &E[x_2(t)-x_1(t)] \\ &= E\Big[\int_0^t \sigma(s, x_2(s))dB(s)-\int_0^t \sigma(s, x_1(s))dB(s)\Big] \\ &\qquad\qquad +E\Big[\int_0^t \beta_2(s)ds-\int_0^t \beta_1(s)ds\Big] \end{aligned} \tag{1.11}$$

$$= E\left[\int_0^t \{\beta_2(s) - \beta_1(s)\} ds\right].$$

Let $\varphi_n(u)$, $(n=1, 2, \cdots)$, be a non-negative continuous function such that its support is contained in $(a_n, a_{n-1})$, $\int_{a_n}^{a_{n-1}} \varphi_n(u) du = 1$ and $\varphi_n(u) \leq (2/n)\rho(u)^{-2}$ where the sequence $a_0 = 1 > a_1 > \cdots a_n > \cdots \rightarrow 0$ is defined by $\int_{a_n}^{a_{n-1}} \rho(u)^{-2} du = n$. Let

$$\psi_n(x) = \int_0^{|x|} dy \int_0^y \varphi_n(z) dz, \qquad n=1, 2, \cdots.$$

Then, $\psi_n \in C^2(R)$, $\psi_n(x) \uparrow |x|$ when $n \rightarrow \infty$ and $|\psi_n'(x)| \leq 1$. Using Itô's formula, we have

$$\begin{aligned}\psi_n(x_2(t) - x_1(t)) &= \int_0^t \psi_n'(x_2(s) - x_1(s)) \{\sigma(s, x_2(s)) - \sigma(s, x_1(s))\} dB(s) \\ &\quad + \int_0^t \psi_n'(x_2(s) - x_1(s)) \{\beta_2(s) - \beta_1(s)\} ds \\ &\quad + \frac{1}{2} \int_0^t \psi_n''(x_2(s) - x_1(s)) \{\sigma(s, x_2(s)) - \sigma(s, x_1(s))\}^2 ds \\ &= I_1 + I_2 + I_3, \quad \text{say.}\end{aligned}$$

Then, $E(I_1) = 0$ and

$$\begin{aligned}E(I_3) &\leq \frac{1}{2} E\left[\int_0^t \varphi_n(|x_2(s) - x_1(s)|) \rho^2(|x_2(s) - x_1(s)|) ds\right] \\ &\leq \frac{t}{n} \rightarrow 0, \quad \text{as} \quad n \rightarrow \infty.\end{aligned}$$

Since $t \leq \tau$, $\beta_2(s) - \beta_1(s) \geq b_2(s, x_2(s)) - b_1(s, x_1(s)) \geq 0$ for all $s \leq t$ by (1.6) and (1.7). Then, letting $n \rightarrow \infty$ and noting $|\psi_n'(x)| \leq 1$, we have

$$\begin{aligned}E|x_2(t) - x_1(t)| &= \lim_{n\rightarrow\infty} E[\psi_n(x_2(t) - x_1(t))] \leq \overline{\lim_{n\rightarrow\infty}} E(I_2) \\ &\leq E\left[\int_0^t |\beta_2(s) - \beta_1(s)| ds\right] = E\left[\int_0^t \{\beta_2(s) - \beta_1(s)\} ds\right].\end{aligned}$$

Combining this with (1.11), we have

$$E|x_2(t) - x_1(t)| \leq E(x_2(t) - x_1(t))$$

and this implies clearly that $x_2(t) \geq x_1(t)$, a.s.. This is true for all $t = \bar{t} \wedge \tau$ and, by the continuity of $x_i(s)$, we have

$$P\{t \in [0, \tau] \Rightarrow x_2(t) \geq x_1(t)\} = 1.$$

This implies (1.10).

Now, we prove the first assertion of the theorem. Let $\theta = \inf\{s; x_1(s) > x_2(s)\}$. In order to prove (1.8), it is sufficient to show that $\theta = \infty$, a.s.. Suppose, on the contrary, $P(\theta<\infty)>0$. Set $\tilde{\Omega}=\{\omega; \theta(\omega)<\infty\}$, $\tilde{\mathfrak{F}}_t=\mathfrak{F}_{t+\theta}|\tilde{\Omega}$, $\tilde{\mathfrak{F}}=\mathfrak{F}|\tilde{\Omega}$ and $\tilde{P}(A)=P(A)/P(\tilde{\Omega})$, $A\in\tilde{\mathfrak{F}}$. On the space $(\tilde{\Omega}, \tilde{\mathfrak{F}}, \tilde{P}; \tilde{\mathfrak{F}}_t)$, we set $\tilde{\sigma}(t, x)=\sigma(t+\theta, x)$, $\tilde{b}_i(t, x)=b_i(t+\theta, x)$, $\tilde{x}_i(t)=x_i(t+\theta)$, $\tilde{\beta}_i(t)=\beta_i(t+\theta)$, $i=1, 2$, and $\tilde{B}(t)=B(t+\theta)-B(\theta)$. Then, it is clear that $\tilde{x}_1(0)=x_1(\theta)=x_2(\theta)=\tilde{x}_2(0)$ a.s. and also, $\tilde{\beta}_1(t)\leq b_1(t, \tilde{x}_1(t))$, $\tilde{\beta}_2(t)\geq b_2(t, \tilde{x}_2(t))$ a.s.. Further,

$$\tilde{x}_i(t)-\tilde{x}_i(0)=\int_0^t \tilde{\sigma}(s, \tilde{x}_i(s))d\tilde{B}(s)+\int_0^t \tilde{\beta}_i(s)ds, \quad i=1, 2.$$

Therefore, we can apply (1.10) and obtain

$$\tilde{P}(\exists t>0; s\in[0, t] \Rightarrow \tilde{x}_1(s)\leq\tilde{x}_2(s))=1.$$

But this contradicts with the definition of $\theta$. Therefore, $\theta=\infty$ a.s. and hence (1.8) is proved.

To prove the second assertion, we assume that one of the stochastic differential equations (1.9), say, for $i=1$, the pathwise uniqueness of solutions holds. Let $X(t)$ be the solution of the equation

$$(1.12) \qquad \begin{cases} dX(t)=\sigma(t, X(t))dB(t)+b_1(t, X(t))dt, \\ X(0)=x_1(0). \end{cases}$$

Let, for $\varepsilon>0$, $X^{\pm\varepsilon}(t)$ be the solutions of

$$(1.13) \qquad \begin{cases} dX(t)=\sigma(t, X(t))dB(t)+[b_1(t, X(t))\pm\varepsilon]dt, \\ X(0)=x_1(0), \end{cases}$$

respectively. By what we have already proved,

$$X^{-\varepsilon}(t)\leq X(t)\leq X^{\varepsilon}(t), \quad \text{for all} \quad t>0,$$

and, by the continuity of $b_1(t, x)$ and the pathwise uniqueness of solutions of (1.12), we have

$$\lim_{\varepsilon\downarrow 0} X^{-\varepsilon}(t)=\lim_{\varepsilon\downarrow 0} X^{\varepsilon}(t)=X(t), \quad \text{for all} \quad t\geq 0.$$

Now, applying what we have already proved to $x_1(t)$ and $X_t^{\varepsilon}$, (note that $\beta_1(t)\leq b_1(t, x_1(t))$ a.s. and $b_1(t, x)<b_1(t, x)+\varepsilon$), we have $x_1(t)\leq X^{\varepsilon}(t)$ and hence, by letting $\varepsilon\downarrow 0$, $x_1(t)\leq X(t)$. Next, applying what we have already proved to $x_2(t)$ and $X_t^{-\varepsilon}$, (note that $\beta_2(t)\geq b_2(t, x_2(t))$ a.s. and $b_2(t, x)\geq b_1(t, x)>b_1(t, x)-\varepsilon$), we have $X^{-\varepsilon}(t)\leq x_2(t)$ and letting $\varepsilon\downarrow 0$, $X(t)\leq x_2(t)$. Therefore, $x_1(t)\leq X(t)\leq x_2(t)$.

q.e.d.

Remark 1.1. For applications, the following modification of the theorem 1.1 will be useful. Suppose that there exists an $\mathfrak{F}_t$-stopping time $S(\omega)>0$ such that $x_i(t, \omega)$ is defined for $t\in[0, S)$, $(i=1, 2)$, and (1.4), (1.6), (1.7) are satisfied for all $t\in[0, S)$. Then the conclusion (1.8) holds for all $t\in[0, S)$.

## 2. Application to some stochastic optimal control problem

Let $k(r)$ be a non-decreasing non-negative function defined on $[0, \infty)$. Let $(B_t, u_t)$ be a system of stochastic processes defined on a space $(\Omega, \mathfrak{F}, P; \mathfrak{F}_t)$ such that

(i) $B_t$, $(B_0=0$ a.s.), is an $n$-dimensional $\mathfrak{F}_t$-Brownian motion,

(ii) $u_t$ is an $n$-dimensional $\mathfrak{F}_t$-well measurable process such that $|u_t|\leq 1$ for any $t\geq 0$, a.s..

Such a system $(B_t, u_t)$ is called an *admissible system*. Let us consider the following optimization problem. Let $x\in R^n$ be given and fixed. For a given admissible system $(B_t, u_t)$, the *response* $X_t^u$ is defined by

$$X_t^u = x+B_t+\int_0^t u_s ds\,. \tag{2.1}$$

Now the problem is *to minimize the expectation* $E[k(|X_1^u|)]$ *among all possible admissible systems* $(B_t, u_t)$.

A solution is given as follows. Let $U(y)$ be defined by

$$U(y) = \begin{cases} -y/|y|, & y\in R^n\backslash\{0\}\,, \\ 0 \quad , & y = 0\in R^n\,. \end{cases} \tag{2.2}$$

Consider the following stochastic differential equation

$$\begin{cases} dX_t = dB_t+U(X_t)dt\,, \\ X_0 = x\,. \end{cases} \tag{2.3}$$

By the well known transformation of measures, a solution $(X_t^\circ, B_t^\circ)$ of (2.3) exists uniquely in the law sense. Set

$$u_t^\circ = U(X_t^\circ)\,. \tag{2.4}$$

Then the admissible system $(B_t^\circ, u_t^\circ)$ gives an optimal control: that is, for any admissible system $(B_t, u_t)$, we have

$$E[k(|X_1^\circ|)]\leq E[k(|X_1^u|)]\,. \tag{2.5}$$

In [1], V.E. Beneš obtained the above result using many techniques. Here we show that it is obtained most simply by applying our theorem 1.1. In fact, we can obtain the following theorem which asserts, in a sense, *pathwise optimality* of the response $X_t^\circ$.

**Theorem 2.1.** *Let $(B_t, u_t)$ be any given admissible system and, for a fixed $x\in R^n$, the response $\{X_t^u\}$ is defined by* (2.1).

*Then, on an appropriate probability space, we can construct $R^n$-valued stochastic* *processes* $\{\hat{X}_t^u\}$ *and* $\{\hat{X}_t^\circ\}$ *as follows*:

( i ) $\{\hat{X}_t^\circ\}$ *has the same law as the solution* $\{X_t^\circ\}$ *of* (2.3),
(ii) $\{\hat{X}_t^u\}$ *has the same law as the response* $\{X_t^u\}$,
(iii) *with probability one*, $|\hat{X}_t^\circ| \leq |\hat{X}_t^u|$ *for all* $t \geq 0$.

**Corollary.** *Let* $C^n = C([0, \infty) \to R^n)$ *be the space of all* $R^n$*-valued continuous paths and* $F(w)$ *be a non-negative Borel function defined on* $C^n$ *which is non-decreasing in the following sense*:

(2.6) *if* $w_1, w_2 \in C^n$ *and* $|w_1(t)| \leq |w_2(t)|$ *for all* $t \geq 0$ *then* $F(w_1) \leq F(w_2)$.

*Then, for any admissible system* $(B_t, u_t)$, *we have*

$$E(F(X_\cdot^\circ)) \leq E(F(X_\cdot^u)) . \tag{2.7}$$

*That is, the solution* $\{X_t^\circ\}$ *of* (2.3) *is optimal in the problem of minimizing the expectation of* $F(X_\cdot^u)$.

REMARK 2.1. The above problem is the case of $F(w) = k(|w(1)|)$ which clearly satisfies (2.6). Similarly, if $k(\xi_1, \xi_2, \cdots, \xi_m)$ is a non-negative function on $[0, \infty)^m$ which is non-decreasing in each of variables, then, for $0 \leq t_1 < t_2 < \cdots < t_m$,

$$F(w) = k(|w(t_1)|, \ |w(t_2)|, \ \cdots, \ |w(t_n)|)$$

or

$$F(w) = k\left(\int_0^{t_1} |(w(s)| ds, \ \int_{t_1}^{t_2} |w(s)| ds, \ \cdots, \ \int_{t_{m-1}}^{t_m} |w(s)| ds\right), \quad \text{etc.,}$$

satisfy (2.6).

First we shall prepare a lemma which is useful to realize several adapted processes on a same probability space without changing the law of processes.

**Lemma.** *Let* $(X_t, B_t)$ *be a pair of n-dimensional continuous adapted processes defined on a probability space* $(\Omega, \mathfrak{F}, P; \mathfrak{F}_t)$ *such that* $\{B_t\}$ *is an n-dimensional* $\mathfrak{F}_t$*-Brownian motion* ($B_0 = 0$ a.s.). *Let* $(Y_t, B_t')$ *be a similar pair defined on* $(\Omega', \mathfrak{F}', P'; \mathfrak{F}_t')$.

*Then we can construct a probability space* $(\hat{\Omega}, \hat{\mathfrak{F}}, \hat{P}; \hat{\mathfrak{F}}_t)$ *and a triple of n-dimensional continuous adaped processes* $(\hat{X}_t, \hat{Y}_t, \hat{B}_t)$ *such that*

( i ) $(X_t, B_t)$ *has the same law as* $(\hat{X}_t, \hat{B}_t)$,
(ii) $(Y_t, B_t')$ *has the same law as* $(\hat{Y}_t, \hat{B}_t)$,
(iii) $\{\hat{B}_t\}$ *is an n-dimensional* $\hat{\mathfrak{F}}_t$*-Brownian motion.*

Proof. The probability law of $\{B_t\}$ on $C^n$ is the $n$-dimensional Wiener measure starting at $0 \in R^n$ which we denote by $R(dw)$. Let $Q(dw_1 dw_2)$ be the probability law of $(X_t, B_t)$ on $C^n \times C^n$ and $Q'(dw_1 dw_2)$ be that of $(Y_t, B_t')$. Then the marginal distributions of $Q$ and $Q'$ with respect to $w_2$ coincide with $R(dw_2)$.

Let $Q^{w_3}(dw_1)=Q(dw_1dw_3)/R(dw_3)$ and $Q'^{w_3}(dw_2)=Q'(dw_2dw_3)/R(dw_3)$ be the regular conditional probability distributions (given $w_3$). Set $\hat{\Omega}=C^n\times C^n\times C^n$ (with compact uniform topology), $\hat{\mathfrak{F}}=$the completion by $\hat{P}$ of topological Borel field $\mathcal{B}(\hat{\Omega})$, $\hat{P}(dw_1dw_2dw_3)=Q^{w_3}(dw_1)Q'^{w_3}(dw_2)R(dw_3)$, $\hat{\mathfrak{F}}_t=\bigcap_{\varepsilon>0}(\mathcal{B}_{t+\varepsilon}\vee\mathcal{N})$ where $\mathcal{B}_t$ is the $\sigma$-field on $\hat{\Omega}$ generated by Borel cylinder sets up to time $t$ and $\mathcal{N}$ is the set of all $\hat{P}$-null sets, $\hat{X}(t,\omega)=w_1(t)$, $\hat{Y}(t,\omega)=w_2(t)$, $\hat{B}(t,\omega)=w_3(t)$ ($\omega=(w_1,w_2,w_3)\in\hat{\Omega}$). Then we can easily see that (i) (ii) (iii) are satisfied: a non-trivial point is only that $\{\hat{B}_t\}$ is $\hat{\mathfrak{F}}_t$-Brownian motion but it can be proved as in [8], pp. 73~74 or [11]. q.e.d.

Now we return to the proof of Theorem 2.1.

Proof of Theorem 2.1. Let an admissible system $(B_t, u_t)$ be given on some probability space and set $X_t^u=x+B_t+\int_0^t u_s ds$. Let $(X_t^\circ, B_t^\circ)$ be the solution of (2.3) defined on some other probability space. Let $p(x)=(p_{ij}(x))$ be a Borel function $x\in R^n \rightsquigarrow p(x)\in \mathrm{O}(n)$, where $\mathrm{O}(n)$ is the set of all $n\times n$ orthogonal matrices, such that

$$p_{1j}(x)=\begin{cases}x_j/|x|, & x=(x_1,x_2,\cdots,x_n)\neq 0\,,\\ \delta_{1j}\,, & x=0\,.\end{cases} \tag{2.8}$$

It is clear that we can choose one such $p(x)$. Set

$$\bar{B}_t=\int_0^t p(X_s^u)dB_s \quad \text{and} \quad \bar{B}'_t=\int_0^t p(X_s^\circ)dB_s^\circ\,.$$

Then we have

$$X_t^u=x+\int_0^t p^{-1}(X_s^u)d\bar{B}_s+\int_0^t u_s ds\,, \tag{2.9}$$

$$X_t^\circ=x+\int_0^t p^{-1}(X_s^\circ)d\bar{B}'_s+\int_0^t U(X_s^\circ)ds\,. \tag{2.10}$$

Now we apply the lemma to $(X_t^u, \bar{B}_t)$ and $(X_t^\circ, \bar{B}'_t)$. Then, on a probability space $(\hat{\Omega}, \hat{\mathfrak{F}}, \hat{P}; \hat{\mathfrak{F}}_t)$, we have a triple of $n$-dimensional continuous adapted processes $(\hat{X}_t^u, \hat{X}_t^\circ, \hat{B}_t)$ such that $\hat{B}_t$ is $\hat{\mathfrak{F}}_t$-Brownian motion and $(X_t^u, \bar{B}_t)\overset{\mathcal{L}}{\approx}(\hat{X}_t^u, \hat{B}_t)$, $(X_t^\circ, \bar{B}'_t)\overset{\mathcal{L}}{\approx}(\hat{X}_t^\circ, \hat{B}_t)$. (Here $\overset{\mathcal{L}}{\approx}$ denotes the coincidence of the law). Clearly there exists $\hat{\mathfrak{F}}_t$-well measurable $n$-dimensional process $\hat{u}_s$ such that $|\hat{u}_s|\leq 1$, $^\forall s\geq 0$, a.s. and

$$\hat{X}_t^u=x+\int_0^t p^{-1}(\hat{X}_s^u)d\hat{B}_s+\int_0^t \hat{u}_s ds\,.$$

Applying Itô's formula to $x_1(t)=|\hat{X}_t^\circ|^2$ and $x_2(t)=|\hat{X}_t^u|^2$, we have

$$\begin{aligned}dx_2(t)&=2\hat{X}_t^u\cdot p^{-1}(\hat{X}_t^u)d\hat{B}_t+2\hat{X}_t^u\cdot\hat{u}_t dt+n dt\\ &=2|\hat{X}_t^u|d\hat{B}_t^1+[2\hat{X}_t^u\cdot\hat{u}_t+n]dt\\ &=2\sqrt{x_2(t)}d\hat{B}_t^1+[2\hat{X}_t^u\cdot\hat{u}_t+n]dt\,,\end{aligned}$$

and

$$\begin{aligned} dx_1(t) &= 2\hat{X}_t^{\circ}\cdot p^{-1}(\hat{X}_t^{\circ})d\hat{B}_t+2\hat{X}_t^{\circ}\cdot U(\hat{X}_t^{\circ})dt+ndt \\ &= 2|\hat{X}_t^{\circ}|d\hat{B}_t^1+[-2|\hat{X}_t^{\circ}|+n]dt \\ &= 2\sqrt{x_1(t)}d\hat{B}_t^1+[-2\sqrt{x_1(t)}+n]dt\,, \end{aligned}$$

where $\hat{B}_t=(\hat{B}_t^1, \hat{B}_t^2, \cdots, \hat{B}_t^n)$. (Note that $[x\cdot p^{-1}(x)]_i=\sum_j x_j(p^{-1}(x))_{ji}=\sum_j x_j p_{ij}(x)=\delta_{i1}|x|$.) Set $\sigma(s, \xi)=2\sqrt{\xi\vee 0}$, $b_1(s, \xi)=b_2(s, \xi)=-2\sqrt{\xi\vee 0}+n$, $\beta_1(s)=-2\sqrt{x_1(s)}+n$, and $\beta_2(s)=2\hat{X}_s^u\cdot\hat{u}_s+n$. Then, cleary, $\beta_1(s)=b_1(s, x_1(s))$ and $\beta_2(s)\geqq -2|\hat{X}_s^u|+n=-2\sqrt{x_2(s)}+n=b_2(s, x_2(s))$. Furthermore, it is known (cf. [10], Example 1.2) that, for the stochastic differential equation

$$\begin{aligned} dX(s) &= \sigma(s, X(s))dB(s)+b_1(s, X(s))ds \\ &= 2(X(s)\vee 0)^{1/2}dB(s)+(-2(X(s)\vee 0)^{1/2}+n)ds\,, \end{aligned}$$

the pathwise uniqueness of solutions holds. Therefore, we can apply the second assertion of the theorem 1.1 and obtain

$$x_1(t)\leq x_2(t)\,, \qquad \text{for all} \quad t\geq 0\,, \qquad \text{a.s..}$$

This implies that

$$|\hat{X}_t^{\circ}|\leq|\hat{X}_t^u|, \qquad \text{for all} \quad t\geq 0\,, \qquad \text{a.s..} \qquad \text{q.e.d.}$$

REMARK 2.2. Slight generalizations of the above problem as are discussed in [1] are also covered by our method.

## 3. Comparison theorem for one dimensional projection of diffusion processes

In this section, we will apply Theorem 1.1 to obtain some comparison theorem for one dimensional projection of multi-dimensional diffusion processes. For simplicity, we consider the case of a diffusion process on the whole space $R^n$ but a similar result holds also for a minimal diffusion process on a domain $D$ in $R^n$.

Let $(\sigma_k^i(x))_{i,k=1}^n$ and $(b^i(x))_{i=1}^n$ be sufficiently smooth real functions on $R^n$ and let $X(t)$ be a diffusion process on $R^n$ defined by the solutions of the following stochastic differential equation

$$dX_t^i = \sum_{k=1}^n \sigma_k^i(X_t)dB_t^k+b^i(X_t)dt\,, \qquad i=1, 2, \cdots, n\,. \tag{3.1}$$

The diffusion $X(t)$ is defined up to the explosion time $e$:

$$e=\sup\{t;\ \sup_{s\in[0,t]}|X(s)|<\infty\}\,. \tag{3.2}$$

This is the diffusion process which corresponds to the differential operator

$$L = \frac{1}{2} \sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b^i(x) \frac{\partial}{\partial x_i} \tag{3.3}$$

where $a^{ij}(x) = \sum_{k=1}^{n} \sigma_k^i(x) \sigma_k^j(x)$.

Let $p(x)$ be a smooth real function on $R^n$ and let $I = \{\xi = p(x);\ x \in R^n\}$. Then $I$ is an interval in $R$. Let $S$ be the set (possibly empty) of all $x \in R^n$ such that $p(x)$ is an end point of $I$. Let $\mathring{I}$ be the maximal open interval contained in $I$ and $\bar{I}$ be the minimal closed interval in $[-\infty, \infty]$ which contains $I$. We assume that $|\nabla p(x)| = \sqrt{\sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial p}{\partial x_i}(x) \frac{\partial p}{\partial x_j}(x)} > 0$ for all $x \in R^n \backslash S$. Set

$$a(x) = \sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial p}{\partial x_i}(x) \frac{\partial p}{\partial x_j}(x), \tag{3.4}$$

$$\begin{aligned} b(x) &= \left[ \frac{1}{2} \sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial^2 p}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{n} b^i(x) \frac{\partial p}{\partial x_i}(x) \right] \Big/ a(x) \\ &= (Lp)(x)/a(x), \qquad x \in R^n \backslash S, \end{aligned} \tag{3.5}$$

$$a^+(\xi) = \sup_{x:\, p(x)=\xi} a(x), \qquad a^-(\xi) = \inf_{x:\, p(x)=\xi} a(x), \qquad \xi \in \mathring{I}, \tag{3.6}$$

$$b^+(\xi) = \sup_{x:\, p(x)=\xi} b(x), \qquad b^-(\xi) = \inf_{x:\, p(x)=\xi} b(x), \qquad \xi \in \mathring{I}. \tag{3.7}$$

We assume that $a^{\pm}(\xi)$ and $b^{\pm}(\xi)$ are finitely determined and define locally Lipschitz continuous functions on $\mathring{I}$. On the interval $\mathring{I}$, we consider the following four diffusion processes $(\xi^{\pm\pm}(t), P_\xi)_{\xi \in \mathring{I}}$ (for four possible combinations of $\pm\pm$) corresponding to the local generators $L^{\pm\pm}$ respectively, where

$$L^{++} = a^+(\xi) \left( \frac{1}{2} \frac{d^2}{d\xi^2} + b^+(\xi) \frac{d}{d\xi} \right), \tag{3.8}$$

$$L^{+-} = a^+(\xi) \left( \frac{1}{2} \frac{d^2}{d\xi^2} + b^-(\xi) \frac{d}{d\xi} \right), \tag{3.9}$$

$$L^{-+} = a^-(\xi) \left( \frac{1}{2} \frac{d^2}{d\xi^2} + b^+(\xi) \frac{d}{d\xi} \right), \tag{3.10}$$

$$L^{--} = a^-(\xi) \left( \frac{1}{2} \frac{d^2}{d\xi^2} + b^-(\xi) \frac{d}{d\xi} \right). \tag{3.11}$$

Each of the diffusions is uniquely determined before the first leaving time $\tau$ from $\mathring{I}$ and we set, if $\tau < \infty$, $\xi(t) = \lim_{s \uparrow \tau} \xi(s)$, (this limit exists in $\bar{I}$), for $t \geq \tau$ where $\xi(t)$ is any sample path of each of the diffusions. Thus, the sample paths $\xi^{\pm\pm}(t)$ are defined for all $t \geq 0$ as $\bar{I}$-valued continuous paths with $\bar{I} \backslash \mathring{I}$ as traps.

Let $X_t$ be a sample path of the above diffusion starting at $x_0 \in R^n \backslash S$ and set

$$(3.12) \qquad \zeta = \begin{cases} \inf\{t<e;\ X_t \in S\}\ , \\ e \qquad\qquad , \text{ if } \{\ \} = \phi\ . \end{cases}$$

We assume that, if $\zeta<\infty$, $\lim_{s\uparrow\zeta} p(X_s)$ exists in $\bar{I}$ and set $p(X_t)=\lim_{s\uparrow\zeta} p(X_s)$ for $t\geq\zeta$. Thus $p(X_t)$ is defined for all $t\geq 0$ as a $\bar{I}$-valued continuous path.

**Theorem 3.1.** *Let $x_0\in R^n\backslash S$ be fixed and $\xi_0=p(x_0)\in\mathring{I}$. Let $X_t$ be the above diffusion such that $X_0=x_0$. On a probability space, we can construct $\bar{I}$-valued continuous stochastic processes $\xi_t$, $\xi_t^{++}$, $\xi_t^{+-}$, $\xi_t^{-+}$, $\xi_t^{--}$ such that*

(i) *$\{\xi_t\}$ has the same law as $\{p(X_t)\}$.*

(ii) *$\{\xi^{\pm\pm}\}$ has the some law as $\{\xi^{\pm\pm}(t), P_{\xi_0}\}$ for each of four combinations of $\pm\pm$.*

(iii) *Set $\bar{\xi}_t=\max_{0\leq s\leq t}\xi_s$, $\underline{\xi}_t=\min_{0\leq s\leq t}\xi_s$ and $\bar{\xi}_t^{\pm\pm}=\max_{0\leq s\leq t}\xi_s^{\pm\pm}$, $\underline{\xi}_t^{\pm\pm}=\min_{0\leq s\leq t}\xi_s^{\pm\pm}$ for each of four combinations of $\pm\pm$. Then, with probability one, we have*

$$(3.13) \qquad \bar{\xi}_t^{--}\leq\bar{\xi}_t\leq\bar{\xi}_t^{++}, \qquad \textit{for all} \quad t\geq 0\,,$$

and

$$(3.14) \qquad \underline{\xi}_t^{+-}\leq\underline{\xi}_t\leq\underline{\xi}_t^{-+}, \qquad \textit{for all} \quad t\geq 0\,.$$

REMARK 3.1. If $a(x)$ depends only on $p(x)$, i.e., if there exists a function $\tilde{a}(\xi)$ defined on $I$ such that $a(x)=\tilde{a}(p(x))$, then $a^+(\xi)=a^-(\xi)=\tilde{a}(\xi)$ and therefore, we may assume that $\xi_t^{++}\equiv\xi_t^{-+}$ and $\xi_t^{+-}\equiv\xi_t^{--}$. In this case, we have

$$(3.15) \qquad \xi_t^{--}\leq\xi_t\leq\xi_t^{++}, \qquad \text{for all} \quad t\geq 0\,.$$

Proof. For simplicity, we assume that $a^+(\xi)\geq a^-(\xi)>0$ for every $\xi\in\mathring{I}$, $\zeta=\infty$ a.s. and $(\xi_t^{\pm\pm}, P_{\xi_0})$ are all conservative diffusion processes on $\mathring{I}$: general case can be proved with a slight modification.

Set $\varphi^+(t)=\int_0^t a(X_s)/a^+(p(X_s))ds$ and $\varphi^-(t)=\int_0^t a(X_s)/a^-(p(X_s))ds$. Then clearly,

$$\varphi^+(t)\leq t\leq\varphi^-(t)\,, \qquad \text{for all} \quad t\geq 0\,.$$

Let $\psi^+(t)$ and $\psi^-(t)$ be the inverse functions of $t \rightsquigarrow \varphi^+(t)$ and $t \rightsquigarrow \varphi^-(t)$, respectively and set $X_t^+=X(\psi^+(t))$, $X_t^-=X(\psi^-(t))$. Then, by the general theory of time change, we see that $X_t^+$ and $X_t^-$ satisfy the following stochastic differential equations with appropriate $n$-dimensional Wiener processes $B_t^+=((B_t^+)^1, (B_t^+)^2, \cdots, (B_t^+)^n)$, $(B_0^+=0)$ and $B_t^-=((B_t^-)^1, (B_t^-)^2, \cdots, (B_t^-)^n)$, $(B_0^-=0)$:

$$(3.16)\quad \begin{cases} d(X_t^+)^i = \sqrt{a^+(p(X_t^+))/a(X_t^+)}\sum_{k=1}^n \sigma_k^i(X_t^+)d(B_t^+)^k \\ \qquad +[a^+(p(X_t^+))/a(X_t^+)]b^i(X_t^+)dt\,, \\ X_0^+ = x_0\,, \qquad\qquad i=1,2,\cdots,n\,, \end{cases}$$

$$(3.17)\quad \begin{cases} d(X_t^-)^i = \sqrt{a^-(p(X_t^-))/a(X_t^-)}\sum_{k=1}^n \sigma_k^i(X_t^-)d(B_t^-)^k \\ \qquad +[a^-(p(X_t^-))/a(X_t^-)]b^i(X_t^-)dt\,, \\ X_0^- = x_0\,, \qquad\qquad i=1,2,\cdots,n\,. \end{cases}$$

Then, by Itô's formula,

$$(3.18)\quad \begin{cases} dp(X_t^+) = \sqrt{a^+(p(X_t^+))/a(X_t^+)}\sum_{i,k=1}^n \sigma_k^i(X_t^+)\dfrac{\partial p}{\partial x_i}(X_t^+)d(B_t^+)^k \\ \qquad +[a^+(p(X_t^+))/a(X_t^+)](Lp)(X_t^+)dt\,, \\ p(X_0^+) = \xi_0\,, \end{cases}$$

and

$$(3.19)\quad \begin{cases} dp(X_t^-) = \sqrt{a^-(p(X_t^-))/a(X_t^-)}\sum_{i,k=1}^n \sigma_k^i(X_t^-)\dfrac{\partial p}{\partial x_i}(X_t^-)d(B_t^-)^k \\ \qquad +[a^-(p(X_t^-))/a(X_t^-)](Lp)(X_t^-)dt\,, \\ p(X_0^-) = \xi_0\,. \end{cases}$$

Hence, if we set

$$\tilde{B}_t^\pm = \sum_{i,k=1}^n \int_0^t \sqrt{1/a(X_s^\pm)}\sigma_k^i(X_s^\pm)\frac{\partial p}{\partial x_i}(X_s^\pm)d(B_s^\pm)^k\,,$$

for each of $+$ and $-$, then $\tilde{B}_t^+$ and $\tilde{B}_t^-$ are 1-dimensional Wiener processes and we have

$$(3.20)\quad \begin{cases} dp(X_t^+) = \sqrt{a^+(p(X_t^+))}d\tilde{B}_t^+ + a^+(p(X_t^+))b(X_t^+)dt\,, \\ p(X_0^+) = \xi_0\,, \end{cases}$$

and

$$(3.21)\quad \begin{cases} dp(X_t^-) = \sqrt{a^-(p(X_t^-))}d\tilde{B}_t^- + a^-(p(X_t^-))b(X_t^-)dt\,, \\ p(X_t^-) = \xi_0\,. \end{cases}$$

Let $\xi_t^+ = p(X_t^+)$ and $\xi_t^- = p(X_t^-)$. Then, by (3.20) and (3.21), we have

$$(3.22)\quad \begin{cases} d\xi_t^+ = \sqrt{a^+(\xi_t^+)}d\tilde{B}_t^+ + a^+(\xi_t^+)b(X_t^+)dt\,, \\ \xi_0^+ = \xi_0\,, \end{cases}$$

and

$$(3.23)\qquad \begin{cases} d\xi_t^- = \sqrt{a^-(\xi_t^-)}d\tilde{B}_t^- + a^-(\xi_t^-)b(X_t^-)dt\,, \\ \xi_0^- = \xi_0\,. \end{cases}$$

Consider the following stochastic differential equation

$$(3.24)\qquad \begin{cases} d\xi_t^{++} = \sqrt{a^+(\xi_t^{++})}d\tilde{B}_t^+ + a^+(\xi_t^{++})b^+(\xi_t^{++})dt\,, \\ \xi_0^{++} = \xi_0\,. \end{cases}$$

Since we assumed that $a^{\pm}(\xi)$ and $b^{\pm}(\xi)$ are locally Lipschitz continuous, the pathwise uniquenss of solutions holds for the equation (3.24) and hence, by Theorem 1.1, we have

$$(3.25)\qquad \xi_t^+ \leq \xi_t^{++}\,, \qquad \text{for all} \quad t \geq 0\,, \text{ a.s.}\,.$$

(Take, in Theorem 1.1, $\sigma(t, \xi) = \sqrt{a^+(\xi)}$, $\beta_1(t) = b(X_t^+)a^+(\xi_t^+)$, $\beta_2(t) = b^+(\xi_t^{++})a^+(\xi_t^{++})$ and $b_1(t, \xi) = b_2(t, \xi) = b^+(\xi)a^+(\xi)$.) Similarly, if $\xi_t^{+-}$ is the solution of the stochastic differential equation

$$(3.26)\qquad \begin{cases} d\xi_t^{+-} = \sqrt{a^+(\xi_t^{+-})}d\tilde{B}_t^+ + a^+(\xi_t^{+-})b^-(\xi_t^{+-})dt\,, \\ \xi_0^{+-} = \xi_0\,, \end{cases}$$

then we have

$$(3.27)\qquad \xi_t^{+-} \leq \xi_t^+\,, \qquad \text{for all} \quad t \geq 0\,, \text{ a.s.}\,.$$

Also, if $\xi_t^{-+}$ is the solution of the stochastic differential equation

$$(3.28)\qquad \begin{cases} d\xi_t^{-+} = \sqrt{a^-(\xi_t^{-+})}d\tilde{B}_t^- + a^-(\xi_t^{-+})b^+(\xi_t^{-+})dt\,, \\ \xi_t^{-+} = \xi_0\,, \end{cases}$$

then we have

$$(3.29)\qquad \xi_t^- \leq \xi_t^{-+}\,, \qquad \text{for all} \quad t \geq 0\,, \text{ a.s.}\,,$$

and if $\xi_t^{--}$ is the solution of

$$(3.30)\qquad \begin{cases} d\xi_t^{--} = \sqrt{a^-(\xi_t^{--})}d\tilde{B}_t^- + a^-(\xi_t^{--})b^-(\xi_t^{--})dt\,, \\ \xi_0^{--} = \xi_0\,, \end{cases}$$

then

$$(3.31)\qquad \xi_t^{--} \leq \xi_t^-\,, \qquad \text{for all} \quad t \geq 0\,, \text{ a.s.}\,.$$

Finally, let $\xi_t = p(X_t)$. Then, since

$$\max_{0 \leq s \leq t} \xi_s^+ = \max_{0 \leq s \leq \psi^+(t)} \xi_s\,, \quad \min_{0 \leq s \leq t} \xi_s^+ = \min_{0 \leq s \leq \psi^+(t)} \xi_s$$

and $t \leq \psi^+(t)$, we have

$$\max_{0\le s\le t} \xi_s^+ \ge \max_{0\le s\le t} \xi_s \quad \text{and} \quad \min_{0\le s\le t} \xi_s^+ \le \min_{0\le s\le t} \xi_s . \tag{3.32}$$

Similarly, using $t\le\psi^-(t)$, we have

$$\max_{0\le s\le t} \xi_s^- \le \max_{0\le s\le t} \xi_s \quad \text{and} \quad \min_{0\le s\le t} \xi_s^- \ge \min_{0\le s\le t} \xi_s . \tag{3.33}$$

Then, by (3.25) and (3.31), we have

$$\bar{\xi}_t^{--} = \max_{0\le s\le t} \xi_s^{--} \le \max_{0\le s\le t} \xi_s^- \le \max_{0\le s\le t} \xi_s = \bar{\xi}_t ,$$

and

$$\bar{\xi}_t = \max_{0\le s\le t} \xi_s \le \max_{0\le s\le t} \xi_s^+ \le \max_{0\le s\le t} \xi_s^{++} = \bar{\xi}_t^{++} .$$

Similarly, by (3.27) and (3.29), we have

$$\underline{\xi}_t^{+-} \le \underline{\xi}_t \le \underline{\xi}_t^{-+} .$$

This proves the theorem.

EXAMPLE 3.1 (Hashiminsky's test for explosion, [3], [7]).

Let $X(t)$ be the diffusion process defined by the solution of (3.1). We assume that $\det(a^{ij}(x))>0$ for every $x\in R^n$ and $n>1$. Let $p(x)=\frac{1}{2}|x|^2=\frac{1}{2}\sum_{i=1}^n x_i^2$. Then $I=[0, \infty)$ and $S=\{0\}$. Let the explosion time $e$ be defined by (3.2). If $X(0)\neq 0$, $X(t)$ never visits the origin 0 and hence $\zeta=e$ a.s..

In this case

$$a(x)=\sum_{i,j=1}^n a^{ij}(x)x_ix_j , \quad b(x)=a(x)^{-1}\left\{\frac{1}{2}\sum_{i=1}^n a^{ii}(x)+\sum_{i=1}^n b^i(x)x_i\right\} ,$$
$$x\in R^n\setminus\{0\} ,$$

$$\left.\begin{aligned} a^+(r) &= \max_{|x|=\sqrt{2r}} a(x) , \quad a^-(r) = \min_{|x|=\sqrt{2r}} a(x) , \\ b^+(r) &= \max_{|x|=\sqrt{2r}} b(x) , \quad b^-(r) = \min_{|x|=\sqrt{2r}} b(x) , \end{aligned}\right. \quad r\in(0, \infty) .$$

$\{\xi^{++}(t), P_r\}_{s\in[0,\infty]}$ is the diffusion process on $[0, \infty]$ with the local generator $a^+(r)\left(\frac{1}{2}\frac{d^2}{dr^2}+b^+(r)\frac{d}{dr}\right)$ and the boundaries 0 and $\infty$ are traps. $\{\xi^{--}(t), P_r\}_{r\in[0,\infty]}$ is the diffusion process on $[0, \infty]$ with the local generator $a^-(r)\left(\frac{1}{2}\frac{d^2}{dr^2}+b^-(r)\frac{d}{dr}\right)$ and the boundaries 0 and $\infty$ are traps. It is easy to see that 0 is unattainable for both diffusions. Let $e^+=\inf\{t;\ \xi^{++}(t)=\infty\}$ and $e^-=\inf\{t;\ \xi^{--}(t)=\infty\}$. By Theorem 3.1, for a given $r>0$, we can realize, on a same probability space,

$X(t)$, $\xi^{++}(t)$ and $\xi^{--}(t)$ such that $\frac{1}{2}|X(0)|^2=\xi^{++}(0)=\xi^{--}(0)=r$ and

$$\max_{0\leq s\leq t} \xi^{--}(s) \leq \max_{0\leq s\leq t} \frac{1}{2}|X(s)|^2 \leq \max_{0\leq s\leq t} \xi^{++}(s)\,, \qquad \text{for all} \quad t\geq 0\,, \text{ a.s.}.$$

Then, clearly $e^+\leq e\leq e^-$ a.s.. Set

$$c^+(r) = \exp\int_1^r 2b^+(u)du \quad \text{and} \quad c^-(r) = \exp\int_1^r 2b^-(u)du\,.$$

By a well known result of Feller, (cf. [5]), if

$$\int_1^\infty [1/c^+(r)]\int_1^r [c^+(u)/a^+(u)]dudr = \infty\,,$$

then $e^+=\infty$ a.s. and *a fortiori* $e=\infty$ a.s.. On the otherhand, if

$$\int_1^\infty [1/c^-(r)]\int_1^r [c^-(u)/a^-(u)]dudr<\infty\,,$$

then $e^-<\infty$ a.s. and *a fortiori* $e<\infty$ a.s..

Similarly, we can prove Hashiminsky's test for regularity of boundary points [4] by using Theorem 3.1. Also, we would like to remark that such a comparison theorem is useful in the study of diffusion processes on a Riemannian manifold which has some interesting applications to analysis and geometry. For such topics, we refer to [6] and [2].

OSAKA UNIVERSITY
KYOTO UNIVERSITY

## References

[1] V.E. Beneš: *Composition and invariance methods for solving some stochastic control problems*, Advances Appl. Probability **7** (1975), 299–329.

[2] A. Debiard, B. Gaveau and E. Mazet: *Théorèmes de comparaison en géométrie riemannienne*, Publ. Res. Inst. Math. Sci. Kyoto Univ. **12** (1976), 391–425.

[3] R.Z. Hashiminsky: *Ergodic properties of recurrent diffusion processes and stabilization of solution of the Cauchy problem for parabolic equations*, Theor. Probability Appl. **5** (1960), 179–196.

[4] R.Z. Hashiminsky: *Diffusion processes and elliptic differential equations degenerating at the boundary of the domain*, Theor. Probability Appl. **3** (1958), 400–419.

[5] K. Itô and H.P. McKean, Jr.: Diffusion processes and their sample paths, Springer, 1965.

[6] P. Malliavin: *Asymptotic of the Green's function of a Riemannian manifold and Ito's stochastic integrals*, Proc. Nat. Acad. Sci. U.S.A. **71** (1974), 381–383.

[7] H.P. McKean, Jr.: Stochastic integrals, Academic Press, 1969.

[8] P. Priouret: *Processus de diffusion et équations différentielles stochastiques*, Ecole d'été de probabilités de Saint-Flour III-1973, Lecture Notes in Math. Vol. 390 (1974), 37–113.

[9] A.V. Skorohod: Studies in the theory of random process, Addison-Wesley, 1965.

[10] T. Yamada: *On a comparison theorem for solutions of stochastic differential equations and its applications*, J. Math. Kyoto Univ. **13** (1973), 497–512.

[11] T. Yamada and S. Watanabe: *On the uniqueness of solutions of stochastic differential equations*, J. Math. Kyoto Univ. **11** (1971), 155–167.