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ON THE ABSTRACT EVOLUTION EQUATION OF PARABOLIC TYPE

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1. Introduction

The object of the present paper is to prove the existence and uniqueness of the solution of the initial value problem for the abstract evolution equation of parabolic type

$$\text{(E)} \quad du/dt+A(t)u=f(t), \quad 0<t\leqslant T$$

in a Banach space X by constructing the evolution operator to (E). Here $u=u(\cdot)$ and $f(\cdot)$ are functions on $[0, T]$ to X and $A(\cdot)$ is one on $[0, T]$ to the set of linear operators, and '*of parabolic type*' means that each $-A(t)$ is the infinitesimal generator of an '*analytic*' semi-group.

Several papers have already been published concerning this problem, for instance, [1], [2] and [3]. The present author has also shown in [4] that

$$\|A(t)(\lambda-A(t))^{-1}dA(t)^{-1}/dt\|\leqslant N/|\lambda|^{\rho}, \ \rho\in(0, 1] \tag{1.1}$$

is one of sufficient conditions for the integrability of (E). In this article we will give a new sufficient condition weaker than (1.1). That is the estimation

$$\left.\begin{aligned} &\|\partial(\lambda-A(t))^{-1}/\partial t\|\leqslant N/|\lambda|^{\rho}, \ \rho\in(0, 1] \\ \text{and}\quad &\\ &\|A(t)(\lambda-A(t))^{-1}dA(t)^{-1}/dt-A(s)(\lambda-A(s))^{-1}dA(s)^{-1}/ds\| \\ &\quad\leqslant K\sum_{i=1}^{k}|\lambda|^{\alpha_i}|t-s|^{\beta_i}, \ (\alpha_i, \beta_i)\in\{(\alpha, \beta); \ -1\leqslant\alpha<\beta\leqslant 1\}, \end{aligned}\right\} \tag{1.2}$$

which is obviously weaker than (1.1) in view of

$$\partial(\lambda-A(t))^{-1}/\partial t=-A(t)(\lambda-A(t))^{-1}(dA(t)^{-1}/dt)A(t)(\lambda-A(t))^{-1}.$$

We also note that (1.2) is weaker than

$$\left.\begin{aligned} &\|\partial(\lambda-A(t))^{-1}/\partial t\|\leqslant N/|\lambda|^{\rho}, \ \rho\in(0, 1] \\ \text{and}\quad &\\ &\|dA(t)^{-1}/dt-dA(s)^{-1}/ds\|\leqslant K|t-s|^{\beta}, \ \beta\in(0, 1] \end{aligned}\right\} \tag{1.3}$$

which is made in [2], as is easily seen from

$$A(t)(\lambda-A(t))^{-1}dA(t)^{-1}/dt-A(s)(\lambda-A(s))^{-1}dA(s)^{-1}/ds$$
$$=\lambda\int_s^t(\partial(\lambda-A(\tau))^{-1}/\partial\tau)d\tau(dA(t)^{-1}/dt)$$
$$+A(s)(\lambda-A(s))^{-1}(dA(t)^{-1}/dt-dA(s)^{-1}/ds)\,.$$

As the previous paper [4] we follow the idea of [1] of establishing various estimates under the additional assumption that $A(t)$ is bounded and then approximating $A(t)$ by a sequence of bounded operators.

The author thanks Prof. M. Watanabe and Prof. H. Tanabe with all his heart. Prof. M. Watanabe informed him that (1.1) implies the condition

$$\|A(t)^{\delta}dA(t)^{-1}/dt\|\leqslant N,\ \delta\in(0,1] \tag{1.4}$$

in [3] because of the equality

$$A(t)^{\delta}dA(t)^{-1}/dt=\frac{1}{2\pi i}\int_{\Gamma}\lambda^{\delta-1}A(t)(\lambda-A(t))^{-1}(dA(t)^{-1}/dt)d\lambda$$

with $\delta\in(0,\rho)$. Thus (1.1) and (1.4) are essentially equivalent.

2. Construction of the evolution operator

Theorem 1. *Let* $\{A(t)\}_{0\leqq t\leqq T}$ *be a family of densely defined, closed linear operators acting in the Banach space* X. *Suppose that* $\{A(t)\}_{0\leqq t\leqq T}$ *satisfies the following conditions*:

(I) *For each* $t\in[0,T]$ *the resolvent set of* $A(t)$ *contains a fixed closed angular domain*

$$\sum=\{\lambda\in\boldsymbol{C};\ \arg\lambda\notin(-\theta,\theta)\},\ \theta\in(0,\pi/2)\,.$$

For any $t\in[0,T]$ *and* $\lambda\in\sum$ *the resolvent of* $A(t)$ *satisfies the inequality*

$$\|(\lambda-A(t))^{-1}\|\leqslant N_1/(|\lambda|+1)$$

with some constant N_1 *independent of* t *and* λ.

(II) $A(t)^{-1}$ (*and therefore* $(\lambda-A(t))^{-1}$ *also*) *is strongly continuously differentiable in* t, *and the derivative* $\partial(\lambda-A(t))^{-1}/\partial t$ *satisfies*

$$\|\partial(\lambda-A(t))^{-1}/\partial t\|\leqslant N_2/|\lambda|^{\rho}$$

for any $t\in[0,T]$ *and* $\lambda\in\sum$ *with some constants* N_2 *and* $\rho\in(0,1]$ *independent of* t *and* λ.

(III) *The estimation*

$$\|A(t)(\lambda-A(t))^{-1}dA(t)^{-1}/dt-A(s)(\lambda-A(s))^{-1}dA(s)^{-1}/ds\| \leqslant N_3\sum_{i=1}^{k}|\lambda|^{\alpha_i}|t-s|^{\beta_i}$$

holds for any $(t, s)\in[0, T]^2$ *and* $\lambda\in\sum$ *with some constant* N_3 *and some non-empty finite family* $\{(\alpha_i, \beta_i)\}_{1\leqslant i\leqslant k}$ *of the elements of* $\{(\alpha, \beta);\ -1\leqslant\alpha<\beta\leqslant 1\}$, N_3 *and the family being independent of* (t, s) *and* λ.

Then there exists a family $\{U(t, s)\}_{0\leqslant s\leqslant t\leqslant T}$ *of bounded operators on* X *having the following properties*:

1) $U(t, s)$ *is strongly continuous in* (t, s).

2) $U(t, r)U(r, s)=U(t, s)$ *for* $0\leqslant s\leqslant r\leqslant t\leqslant T$, *and* $U(s, s)=I$.

3) *For any* $t>s$, $R(U(t, s))\subset D(A(t))$. *The operator* $A(t)U(t, s)$ *defined for* $0\leqslant s<t\leqslant T$ *is strongly continuous in* (t, s) *and is estimated by*

$$\|A(t)U(t, s)\|\leqslant C_1(t-s)^{-1}$$

with some constant C_1 *determined by* $N_1, N_2, N_3, \theta, \rho, \{(\alpha_i, \beta_i)\}_{1\leqslant i\leqslant k}, T$ *and* $\sup_{0\leqslant t\leqslant T}\|dA(t)^{-1}/dt\|$.

4) $U(t, s)$ *is strongly differentiable in* $t\in(s, T]$, *and*

$$\partial U(t, s)/\partial t = -A(t)U(t, s)\,.$$

5) *For any* $t>s$ *and* $u_0\in D(A(s))$,

$$\lim_{h\to 0} h^{-1}\{U(t, s+h)-U(t, s)\}u_0 = U(t, s)A(s)u_0\,.$$

The proof consists of three steps.

The first step. Assuming in addition to (I), (II) and (III) that $A(t)$ is a bounded operator for each $t\in[0, T]$, we will establish various estimates concerning the evolution operator $U(t, s)$ using only the constants $N_1, N_2, N_3, \theta, \rho$, $\{(\alpha_i, \beta_i)\}_{1\leqslant i\leqslant k}$, T and $\sup_{0\leqslant t\leqslant T}\|dA(t)^{-1}/dt\|$ entering in the assumptions (I), (II) and (III).

Let $\{U(t, s)\}_{0\leqslant s\leqslant t\leqslant T}$ be the evolution operator to $\{A(t)\}_{0\leqslant t\leqslant T}$. Evidently $\{U(t, s)\}_{0\leqslant s\leqslant t\leqslant T}$ has the properties 1)$\sim$5). $U(t, s)$ is connected with the analytic semi-group $\exp(-(t-s)A(t))$ generated by $-A(t)$ as follows

$$\begin{aligned} &U(t, s)-\exp(-(t-s)A(t)) \\ &= -\int_s^t \frac{\partial}{\partial\tau}\{U(t, \tau)\exp(-(\tau-s)A(\tau))\}d\tau = \int_s^t U(t, \tau)P(\tau,s)d\tau\,, \qquad (2.1) \end{aligned}$$

where

$$P(t, s) = -(\partial/\partial t+\partial/\partial s) \exp(-(t-s)A(t)) . \tag{2.2}$$

Letting $A(t)$ operate on (2.1), we get

$$W(t, s) = R_1(t, s)+R_2(t, s)+\int_s^t W(t, \tau)P(\tau, s)d\tau , \tag{2.3}$$

where

$$W(t, s) = A(t)U(t, s)-A(t) \exp(-(t-s)A(t)) ,$$

$$R_1(t, s) = \int_s^t \{A(t) \exp(-(t-\tau)A(t)) - A(\tau) \exp(-(t-\tau)A(\tau))\} P(\tau, s)d\tau \tag{2.4}$$

and

$$R_2(t, s) = \int_s^t A(\tau) \exp(-(t-\tau)A(\tau))P(\tau, s)d\tau . \tag{2.5}$$

Regarding (2.1) and (2.3) as the integral equations for $U(t, s)$ and $W(t, s)$ respectively, we will estimate their kernels. It is well known that

$$\exp(-(t-s)A(t)) = \frac{1}{2\pi i}\int_\Gamma e^{-\lambda(t-s)}(\lambda-A(t))^{-1}d\lambda \tag{2.6}$$

with some smooth contour Γ running in $\sum$ from $\infty e^{i\theta}$ to $\infty e^{-i\theta}$. From this formula we easily conclude

$$\|\exp(-(t-s)A(t))\| \leqslant N_4 .$$

By the definition of $P(t, s)$ and (2.6) we have

$$P(t, s) = \frac{-1}{2\pi i}\int_\Gamma e^{-\lambda(t-s)}(\partial(\lambda-A(t))^{-1}/\partial t)d\lambda , \tag{2.7}$$

and using the assumption (II), we get

$$\|P(t, s)\| \leqslant N_5(t-s)^{\rho-1} . \tag{2.8}$$

In view of (2.8) and

$$\|A(t) \exp(-(t-s)A(t))-A(s) \exp(-(t-s)A(s))\| \leqslant N_6(t-s)^{\rho-1}$$

we have

$$\|R_1(t, s)\| \leqslant N_7(t-s)^{2\rho-1} .$$

The estimate of $R_2(t, s)$ is rather complicated. We express $R_2(t, s)$ as

$$R_2(t, s) = \sum_{j=1}^{5} F_j(t, s) , \tag{2.9}$$

where each summand on the right is defined below.

$$R_2(t, s) = (\int_r^t + \int_s^r) A(\tau) \exp(-(t-\tau)A(\tau))P(\tau, s)d\tau ,$$

where $r=(t+s)/2$. Define

$$F_1(t, s) = \int_s^r A(\sigma) \exp(-(t-\sigma)A(\sigma))P(\sigma, s)d\sigma .$$

Since

$$\begin{aligned} &\partial(\lambda - A(t))^{-1}/\partial t \\ &= -A(t)(\lambda - A(t))^{-1}(dA(t)^{-1}/dt)A(t)(\lambda - A(t))^{-1} \\ &= \{1-\lambda(\lambda - A(t))^{-1}\}(dA(t)^{-1}/dt)A(t)(\lambda - A(t))^{-1} , \end{aligned} \tag{2.10}$$

we can express $P(t, s)$ in the form

$$\begin{aligned} P(t, s) = &-(dA(t)^{-1}/dt)A(t) \exp(-(t-s)A(t)) \\ &-A(t)^{-1}\frac{1}{2\pi i}\int_\Gamma e^{-\lambda(t-s)}\lambda(\partial(\lambda - A(t))^{-1}/\partial t)d\lambda , \end{aligned}$$

hence

$$\begin{aligned} &\int_r^t A(\tau) \exp(-(t-\tau)A(\tau))P(\tau, s)d\tau \\ &= -\int_r^t A(\tau) \exp(-(t-\tau)A(\tau))(dA(\tau)^{-1}/d\tau)A(\tau) \exp(-(\tau-s)A(\tau))d\tau \\ &\quad -\int_r^t \exp(-(t-\tau)A(\tau))\frac{1}{2\pi i}\int_\Gamma e^{-\lambda(\tau-s)}\lambda(\partial(\lambda - A(\tau))^{-1}/\partial\tau)d\lambda\, d\tau . \end{aligned}$$

Define

$$F_2(t, s) = -\int_r^t \exp(-(t-\tau)A(\tau))\frac{1}{2\pi i}\int_\Gamma e^{-\lambda(\tau-s)}\lambda(\partial(\lambda - A(\tau))^{-1}/\partial\tau)d\lambda d\tau.$$

Next, define $F_3(t, s)$, $F_4(t, s)$ and $F_5(t, s)$ as follows

$$\begin{aligned} &-\int_r^t A(\tau) \exp(-(t-\tau)A(\tau))(dA(\tau)^{-1}/d\tau)A(\tau) \exp(-(\tau-s)A(\tau))d\tau \\ &= \int_r^t A(\tau) \exp(-(t-\tau)A(\tau))(dA(\tau)^{-1}/d\tau)\{A(t) \exp(-(\tau-s)A(t)) \\ &\quad -A(\tau) \exp(-(\tau-s)A(\tau))\}d\tau \\ &\quad +\int_r^t \{A(t) \exp(-(t-\tau)A(t))dA(t)^{-1}/dt \\ &\qquad -A(\tau) \exp(-(t-\tau)A(\tau))dA(\tau)^{-1}/d\tau\}A(t) \exp(-(\tau-s)A(t))d\tau \\ &\quad -\int_r^t A(t) \exp(-(t-\tau)A(t))(dA(t)^{-1}/dt)A(t) \exp(-(\tau-s)A(t))d\tau \\ &= F_3(t, s)+F_4(t, s)+F_5(t, s) . \end{aligned}$$

It is easily observed that

$$\|F_1(t, s)\| \leqslant N_8(t-s)^{\rho-1},\ \|F_2(t, s)\| \leqslant N_9(t-s)^{\rho-1} \quad \text{and}$$
$$\|F_3(t, s)\| \leqslant N_{10}(t-s)^{\rho-1}.$$

By (III) we obtain

$$\|A(t)\exp(-(t-s)A(t))dA(t)^{-1}/dt - A(s)\exp(-(t-s)A(s))dA(s)^{-1}/ds\| \leqslant N_{11}\sum_{i=1}^{k}(t-s)^{\beta_i-\alpha_i-1},$$

hence

$$\|F_4(t, s)\| \leqslant N_{12}(t-s)^{\delta-1}, \quad \text{with} \quad \delta = \min_{1\leqslant i\leqslant k}\{\beta_i-\alpha_i\}.$$

As for $F_5(t, s)$, we show

$$F_5(t, s) = P(t, r)\exp(-(r-s)A(t)). \tag{2.11}$$

In fact

$$\begin{aligned}
&\exp(-(t+h-s)A(t+h)) - \exp(-(t-s)A(t)) \\
&= \int_0^h \frac{\partial}{\partial\theta}\exp(-(t+\theta-s)A(t+h))d\theta \\
&\quad + \int_s^t \frac{\partial}{\partial\tau}\{\exp(-(t-\tau)A(t))\exp(-(\tau-s)A(t+h))\}d\tau \\
&= -\int_0^h A(t+h)\exp(-(t+\theta-s)A(t+h))d\theta \\
&\quad + \int_s^t A(t)\exp(-(t-\tau)A(t))(A(t+h)^{-1} \\
&\qquad\qquad - A(t)^{-1})A(t+h)\exp(-(\tau-s)A(t+h))d\tau,
\end{aligned}$$

therefore we conclude

$$\begin{aligned}
\frac{\partial}{\partial t}\exp(-(t-s)A(t)) &= -A(t)\exp(-(t-s)A(t)) \\
&\quad + \int_s^t A(t)\exp(-(t-\tau)A(t))(dA(t)^{-1}/dt)A(t)\exp(-(\tau-s)A(t))d\tau
\end{aligned}$$

or

$$\int_s^t A(t)\exp(-(t-\tau)A(t))(dA(t)^{-1}/dt)A(t)\exp(-(\tau-s)A(t))d\tau = -P(t, s).$$

(2.11) is obvious now. By (2.11)

$$\|F_5(t, s)\| \leqslant N_{13}(t-s)^{\rho-1}.$$

The second step. Let

$$A_n(t) = A(t)(1+n^{-1}A(t))^{-1}, \qquad n=1, 2, \cdots \tag{2.12}$$

be the Yosida approximation of $A(t)$. Since $A_n(t)$ is bounded, we can define $\{P_n(t, s)\}_{0\leq s<t\leq T}$, $\{R_{1,n}(t, s)\}_{0\leq s<t\leq T}$ and $\{R_{2,n}(t, s)\}_{0\leq s<t\leq T}$ by (2.2), (2.4) and (2.5) respectively replacing $A(t)$ by $A_n(t)$.

For the original family $\{A(t)\}_{0\leq t\leq T}$ we define $\{P(t, s)\}_{0\leq s<t\leq T}$ and $\{R_1(t, s)\}_{0\leq s<t\leq T}$ by (2.7) and (2.4) respectively and $\{R_2(t, s)\}_{0\leq s<t\leq T}$ by (2.9) noting that $F_1(t, s)$, $\cdots$, $F_5(t, s)$ are all meaningful, if $F_5(t, s)$ is defined by (2.11), even though $A(t)$ is unbounded.

The chief aim of this step is to show that for each $t>s$, $\{P_n(t, s)\}_{n=1,2,\dots}$, $\{R_{1,n}(t, s)\}_{n=1,2,\dots}$ and $\{R_{2,n}(t, s)\}_{n=1,2,\dots}$ are, roughly speaking, boundedly convergent to $P(t, s)$, $R_1(t, s)$ and $R_2(t, s)$, respectively, which implies the convergence of the solutions of the equations obtained by substituting $A_n(t)$ for $A(t)$ in (2.1) and (2.3).

In order to establish the uniform boundedness with respect to n, using the first step, we must show that $\{A_n(t)\}_{0\leq t\leq T}$ satisfies (I), (II) and (III) with some constants independent of n.

Since $\lambda\in\Sigma$ implies $n\lambda(n-\lambda)^{-1}\in\Sigma$, it is easily seen that the resolvent set of $A_n(t)$ contains Σ and for $\lambda\in\Sigma$ the resolvent is given by

$$(\lambda-A_n(t))^{-1}=(n-\lambda)^{-1}(n+A(t))(n\lambda(n-\lambda)^{-1}-A(t))^{-1} \tag{2.13}$$

or

$$=-(n-\lambda)^{-1}+n^2(n-\lambda)^{-2}(n\lambda(n-\lambda)^{-1}-A(t))^{-1}\,. \tag{2.14}$$

On the other hand we have the inequality

$$|\lambda-n|\geq\sin(\theta/2)(|\lambda|+n), \quad \text{for} \quad \lambda\in\Sigma\,. \tag{2.15}$$

(2.14) together with (2.15) gives

$$\|(\lambda-A_n(t))^{-1}\|\leq M_1/(|\lambda|+1) \tag{2.16}$$

for $\lambda\in\Sigma$ with some constant M_1 determined by N_1 and θ alone. If we take $\lambda=0$ in (2.14), then

$$A_n(t)^{-1}=n^{-1}+A(t)^{-1}\,,$$

hence $A_n(t)^{-1}$ is strongly differentiable in t and

$$dA_n(t)^{-1}/dt=dA(t)^{-1}/dt\,. \tag{2.17}$$

By (2.12), (2.13) and (2.17) and with the aid of (2.10), we get for $\lambda\in\Sigma$

$$\partial(\lambda-A_n(t))^{-1}/\partial t=n^2(n-\lambda)^{-2}\partial(n\lambda(n-\lambda)^{-1}-A(t))^{-1}/\partial t\,,$$

hence

$$\|\partial(\lambda-A_n(t))^{-1}/\partial t\|\leq M_2/|\lambda|^{\rho}$$

with some constant M_2 determined only by N_2, θ and ρ. Similarly by (2.12), (2.13) and (2.17) we get for $\lambda\in\sum$

$$\begin{aligned}&A_n(t)(\lambda-A_n(t))^{-1}dA_n(t)^{-1}/dt\\&=n(n-\lambda)^{-1}A(t)(n\lambda(n-\lambda)^{-1}-A(t))^{-1}dA(t)^{-1}/dt\,,\end{aligned}$$

hence

$$\begin{aligned}&\|A_n(t)(\lambda-A_n(t))^{-1}dA_n(t)^{-1}/dt-A_n(s)(\lambda-A_n(s))^{-1}dA_n(s)^{-1}/ds\|\\&\leqslant N_3\sum_{i=1}^{k}|n(n-\lambda)^{-1}|^{1+\alpha_i}|\lambda|^{\alpha_i}|t-s|^{\beta_i}\\&\leqslant M_3\sum_{i=1}^{k}|\lambda|^{\alpha_i}|t-s|^{\beta_i}\end{aligned}$$

with some constant M_3 determined by N_3, θ and $\{\alpha_i\}_{1\leqslant i\leqslant k}$ alone. Thus (I), (II) and (III) are fulfilled with the same constants except M_1, M_2 and M_3.

We have deduced, therefore, that

$$\begin{aligned}&\|\exp(-(t-s)A_n(t))\|\leqslant M_4,\ \|P_n(t,s)\|\leqslant M_5(t-s)^{\rho-1},\\&\|R_{1,n}(t,s)\|\leqslant M_6(t-s)^{2\rho-1}\quad\text{and}\\&\|R_{2,n}(t,s)\|\leqslant M_7\{(t-s)^{\rho-1}+(t-s)^{\delta-1}\}\quad\text{with}\quad\delta=\min_{1\leqslant i\leqslant k}\{\beta_i-\alpha_i\}\,.\end{aligned}$$

Next, we establish the convergence. The strong convergence of $\{\exp(-(t-s)A_n(t))\}_{n=1,2,\dots}$, $\{P_n(t,s)\}_{n=1,2,\dots}$ and $\{R_{1,n}(t,s)\}_{n=1,2,\dots}$ to $\exp(-(t-s)A(t))$, $P(t,s)$ and $R_1(t,s)$ respectively follows from that of $\{(\lambda-A_n(t))^{-1}\}_{n=1,2,\dots}$ to $(\lambda-A(t))^{-1}$ for each $\lambda\in\sum$ together with (2.16) and the repeated use of the theorem of bounded convergence. The strong convergence of $\{R_{2,n}(t,s)\}_{n=1,2,\dots}$ to $R_2(t,s)$ follows from that of each family $\{F_{j,n}(t,s)\}_{n=1,2,\dots}$ to $F_j(t,s)$.

Now, it would be natural to expect that the desired evolution operator $U(t,s)$ is the solution of the integral equation

$$\begin{aligned}&U(t,s)=\exp(-(t-s)A(t))+\int_s^t U(t,\tau)P(\tau,s)d\tau,\qquad 0\leqslant s<t\leqslant T\\&U(s,s)=\mathrm{I}\,.\end{aligned}\tag{2.18}$$

In order to derive the desired properies of $U(t,s)$ we need the operator valued function $W(t,s)$ which is the solution of the integral equation

$$W(t,s)=R_1(t,s)+R_2(t,s)+\int_s^t W(t,\tau)P(\tau,s)d\tau\,,\quad 0\leqslant s<t\leqslant T\,.\tag{2.19}$$

Since the inhomogeneous terms and kernel of (2.18) and (2.19) are strongly continuous and satisfy

$$\begin{aligned}&\|\exp(-(t-s)A(t))\|\leqslant M_8,\ \|P(t,s)\|\leqslant M_9(t-s)^{\rho-1},\\&\|R_1(t,s)\|\leqslant M_{10}(t-s)^{2\rho-1}\quad\text{and}\\&\|R_2(t,s)\|\leqslant M_{11}\{(t-s)^{\rho-1}+(t-s)^{\delta-1}\}\,,\end{aligned}$$

the solutions of (2.18) and (2.19) exist and are unique. As is easily seen, $U(t, s)$ and $W(t, s)$ are the strong limits of $U_n(t, s)$ and $W_n(t, s)$.

The final step. We prove $U(t, s)$ satisfies 1)~5). 1) is easily seen and 2) is a direct consequence of that for $U_n(t, s)$. The strong convergence of $W_n(t, s)$ to $W(t, s)$ together with

$$||W_n(t, s)|| \leqslant M_{12}\{(t-s)^{\rho-1}+(t-s)^{\delta-1}\}$$

implies 3) and 4). These arguments are similar to the proof of Theorem 1 of [4], and the detail is omitted. For the proof of 5) we use the following lemma (see P. 124 in [6]).

Lemma. *Let $\{V(t, s)\}_{0\leqslant s\leqslant t\leqslant T}$ be the family of bounded operators on X constructed by*

$$V(t, s) = \exp(-(t-s)A(s))+Z(t, s)$$

$$Z(t, s) = \int_s^t Q(t, \tau)\exp(-(\tau-s)A(s))d\tau\,,$$

where $\{Q(t, s)\}_{0\leqslant s<t\leqslant T}$ is the solution of

$$Q(t, s) = Q_1(t, s)+\int_s^t Q(t, \tau)Q_1(\tau, s)d\tau\,,$$

$$Q_1(t, s) = (\partial/\partial t+\partial/\partial s)\exp(-(t-s)A(s))\,.$$

Then $\{V(t, s)\}_{0\leqslant s\leqslant t\leqslant T}$ has 1) *and* 5).

The existence of $Q(t, s)$ (hence $Z(t, s)$ and $V(t, s)$) is proved by the analogous estimation

$$||Q_1(t, s)|| \leqslant M_{12}(t-s)^{\rho-1}$$

to (2.8). The proof of Lemma is not difficult, so we omit it.

It follows from 4) of $U(t, s)$ and 5) of $V(t, s)$ that $V(t, \tau)U(\tau,s)$ is strongly differentiable in $\tau\in(s, t)$ and $\partial V(t, \tau)U(\tau, s)/\partial\tau$ vanishes identically. Hence $V(t, \tau)U(\tau, s)$ is constant in $\tau\in(s, t)$. Letting $\tau\to s$ and $\tau\to t$, we get

$$V(t, s) = U(t, s), \qquad 0\leqslant s\leqslant t\leqslant T\,.$$

REMARK. If the following condition:

$$\text{(IV)} \qquad ||dA(t)^{-1}/dtA(t)(\lambda-A(t))^{-1}-dA(s)^{-1}/dsA(s)(\lambda-A(s))^{-1}|| \leqslant N_4\sum_{j=1}^{l}|\lambda|^{\alpha_j}|t-s|^{\beta_j}$$

holds for any $(t, s)\in[0, T]^2$ and $\lambda\in\sum$ with some constant N_4 and some non-

empty finite family $\{(\alpha_j, \beta_j)\}_{1\leqslant j\leqslant l}$ *of the elements of* $\{(\alpha, \beta);\ -1\leqslant\alpha<\beta\leqslant 1\}$, N_4 *and the family being independent of* (t, s) *and* λ.
is added to (I), (II) and (III), then 5) is strengthened as follows:
6) *For any* $t>s$ *the bounded extension* $\overline{U(t, s)A(s)}$ *of* $U(t, s)A(s)$ *exists and is strongly continuous in* $0\leqslant s<t\leqslant T$. *There exists some constant* C_2 *such that*

$$\|\overline{U(t, s)A(s)}\|\leqslant C_2(t-s)^{-1}.$$

7) $U(t, s)$ *is strongly differentiable in* $s\in[0, t)$, *and*

$$\partial U(t, s)/\partial s = \overline{U(t, s)A(s)}.$$

3. The initial value problem for (E)

Theorem 2. *Let* $\{A(t)\}_{0\leqslant t\leqslant T}$ *be a family of densely defined, closed linear operators in* X *satisfying* (I), (II) *and* (III), *and* $\{U(t, s)\}_{0\leqslant s\leqslant t\leqslant T}$ *be its evolution operator. Then the following statements hold*:
i) *If* f *is continuous, then any strict solution* $u\in C([0, T]; X)\cap C^1((0, T]; X)$ *of* (E) *on* $[0, T]$ *with its initial value* $u_0\in X$ *can be expressed in the form*

$$u(t) = U(t, 0)u_0+\int_0^t U(t, \tau)f(\tau)d\tau. \tag{3.1}$$

ii) *If* f *is Hölder continuous and* u_0 *is an arbitrary element of* X, *then the function* u *defined by* (3.1) *is the strict solution of* (E) *on* $[0, T]$.

The proof of i) and ii) is almost the same as the proof of 4) and 5) in [4], therefore, it is omitted. But we will give, here, another proof of i) without using the differentiability in s of $U(t, s)$.

For any $\varepsilon>0$, $U_n(t, \tau)u(\tau)$ is continuously differentiable in $\tau\in[\varepsilon, t]$, and

$$\partial U_n(t, \tau)u(\tau)/\partial\tau = U_n(t, \tau)f(\tau) + U_n(t, \tau)\{(1+n^{-1}A(\tau))^{-1}-I\}A(\tau)u(\tau). \tag{3.2}$$

Integrating (3.2) on $[\varepsilon, t]$, we have

$$u(t)-U_n(t, \varepsilon)u(\varepsilon) = \int_\varepsilon^t U_n(t, \tau)f(\tau)d\tau + \int_s^t U_n(t, \tau)\{(1+n^{-1}A(\tau))^{-1}-I\}A(\tau)u(\tau)d\tau. \tag{3.3}$$

Since $U_n(t, s)$ is uniformly bounded, the second integral of (3.3) vanishes when n tends to infinity. Thus we have

$$u(t)-U(t, \varepsilon)u(\varepsilon) = \int_\varepsilon^t U(t, \tau)f(\tau)d\tau.$$

Letting ε tend to zero, we obtain the desired equality (3.1).

4. Application

In this section we entirely follow the terminology and notations in §7 of [2], and some of them may be used without precise definitions. Let H and K be Hilbert spaces with the inner products $(\cdot, \cdot)$ and $((\cdot, \cdot))$ respectively such that $K \subset H$ algebraically and topologically. Let $\{a(t; \cdot, \cdot)\}_{0 \leq t \leq T}$ be a family of continuous sesquilinear forms on $K \times K$, and let $\{V(t)\}_{0 \leq t \leq T}$ be a family of closed subspaces of K. We define the operator $A(t)$ for each $t \in [0, T]$ in the following manner:
$u \in V(t)$ belongs to $D(A(t))$ and $A(t)u = f \in H$ if and only if $a(t; u, v) = (f, v)$ for any $v \in V(t)$.

Theorem 3. *Assume* (K. 1), (K. 2), (K. 3) *and* (K. 4) *which are stated in* [2] *and, instead of* (K. 5) *and* (K. 6), *the following*:
(K. 5)′ *There exist some constants M and $\alpha \in (0, 1]$ such that for any $(t, s) \in [0, T]^2$ and $u \in K$*

$$|(\dot{P}(t) - \dot{P}(s))u| \leq M|t-s|^{\alpha}\|u\|$$

and

$$|(\dot{Q}(t) - \dot{Q}(s))u| \leq M|t-s|^{\alpha}\|u\| .$$

(K. 6)′ $\lim_{h \to 0} \sup_{(u,v) \in B^2} |\dot{a}(t+h; u, v) - \dot{a}(t; u, v)| = 0 ,$

where $B = \{u \in K; \|u\| \leq 1\}$.

Then (I), (II) *and* (III) *of $\{-A(t)\}_{0 \leq t \leq T}$ hold with $\rho = 1/2$ and $\{(\alpha_i, \beta_i)\}_{1 \leq i \leq k} = \{(-1/2, 0), (0, \alpha), (1/2, 1)\}$.*

By Lemma 7.1 and Theorem 7.1 in [2], we have only to show (III).

Taking $g = A(t)^*(\bar{\lambda} + A(t)^*)^{-1}g'$ in the formula (7.13) in [2], we have the expression

$$\begin{aligned}(A(t)(\lambda + A(t))^{-1}dA(t)^{-1}/dtf, g) &= -\dot{a}(t; A(t)^{-1}f, (\bar{\lambda} + A(t)^*)^{-1}g) \\ &+ (\dot{P}(t)A(t)^{-1}f, A(t)^*(\bar{\lambda} + A(t)^*)^{-1}g) + (f, \dot{Q}(t)(\bar{\lambda} + A(t)^*)^{-1}g) \\ &- a(t; A(t)^{-1}f, \dot{Q}(t)(\bar{\lambda} + A(t)^*)^{-1}g) - a(t; \dot{P}(t)A(t)^{-1}f, (\bar{\lambda} + A(t)^*)^{-1}g) .\end{aligned}$$

From this we get for $\lambda \in \Sigma$

$$\begin{aligned}&(\{A(t)(\lambda + A(t))^{-1}dA(t)^{-1}/dt - A(s)(\lambda + A(s))^{-1}dA(s)^{-1}/ds\} f, g) \\ &= I(t, s; f, g) + R(t, s; f, g),\end{aligned}$$

where

$$\begin{aligned}I(t, s; f, g) = &(\dot{P}(t)A(t)^{-1}f, A(t)^*(\bar{\lambda} + A(t)^*)^{-1}g) \\ &- (\dot{P}(s)A(s)^{-1}f, A(s)^*(\bar{\lambda} + A(s)^*)^{-1}g)\end{aligned}$$

and

$$R(t, s; f, g) = \textit{the sum of the remaining terms.}$$

By (7.15) and (7.16) in [2] we easily conclude

$$|R(t, s; f, g)| \leqslant M_1 |\lambda|^{-1/2} |f| |g|$$

with some constant M_1.

$$\begin{aligned} |I| \leqslant & |(\{\dot{P}(t) - \dot{P}(s)\} A(t)^{-1} f, A(t)^*(\bar{\lambda} + A(t)^*)^{-1} g)| \\ & + |(\dot{P}(s)\{A(t)^{-1} - A(s)^{-1}\} f, A(t)^*(\bar{\lambda} + A(t)^*)^{-1} g)| \\ & + |(\dot{P}(s) A(s)^{-1} f, \bar{\lambda} \int_s^t (\partial(\bar{\lambda} + A(\tau)^*)^{-1} / \partial\tau) g d\tau)| , \end{aligned}$$

in view of (K. 5)′ and (II) of $\{-A(t)^*\}_{0 \leqslant t \leqslant T}$, we finally conclude

$$|I(t, s; f, g)| \leqslant M_2 \{|t-s|^\alpha + |\lambda|^{1/2} |t-s|\} |f| |g|$$

with some constant M_2.

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